Entire curves in complex projective varieties and differential equations

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Introduction and goals

Let X be a complex projective manifold, $\dim_{\mathbb{C}} X = n$. Our goal is to study the existence and distribution of entire curves, i.e. non constant holomorphic curves $f : \mathbb{C} \to X$.

Conjecture (Green-Griffiths-Lang)

Assume that X is of general type, i.e. $\kappa(X) = n = \dim X$ where

$$\kappa(X) := \limsup_{m \to +\infty} \frac{\log h^0(X, K_X^{\otimes m})}{\log m}.$$

Then $\exists Y \subsetneq X$ algebraic containing all entire curves $f : \mathbb{C} \to X$.

Definition. The smallest algebraic subvariety above will be denoted $Y = \mathrm{Exc}(X) =$ exceptional locus of X.

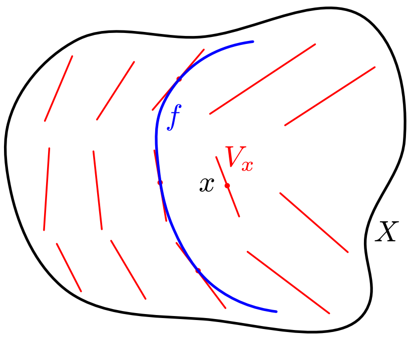
Arithmetic counterpart (Lang 1987) – very optimistic ?

For X projective defined over a number field $\mathbb{K}_0$, the exceptional locus $Y = \mathrm{Exc}(X)$ in GGL's conjecture equals $\mathrm{Mordell}(X) =$ smallest Y such that $X(\mathbb{K}) \smallsetminus Y$ is finite, $\forall \mathbb{K}$ number field $\supset \mathbb{K}_0$.

Category of directed varieties

More generally, we are interested in entire curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$, where V is a (possibly singular) linear subspace of X, i.e. a closed irreducible analytic subspace such that $\forall x \in X$, $V_x := V \cap T_{X,x}$ is linear.

$f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$



Definition (Category of directed varieties)

– Objects : pairs (X, V), X manifold/$\mathbb{C}$ and $V \subset T_X$
– Arrows $\psi : (X, V) \to (Y, W)$ holomorphic s.t. $d\psi(V) \subset W$
– "Absolute case" (X, T_X), i.e. $V = T_X$
– "Relative case" $(X, T_{X/S})$ where $X \to S$
– "Integrable case" when $[\mathcal{O}(V), \mathcal{O}(V)] \subset \mathcal{O}(V)$ (foliations)

Canonical sheaf of a directed variety (X,V)

Canonical sheaf of a directed manifold (X, V)

When V is nonsingular, i.e. a subbundle, one simply sets

$$K_V = \det(V^*) \quad \text{(as a line bundle).}$$

When V is singular, we first introduce the rank 1 sheaf ${}^b\mathcal{K}_V$ of sections of $\det V^*$ that are locally bounded with respect to a smooth ambient metric on T_X. One can show that ${}^b\mathcal{K}_V$ is equal to the integral closure of the image of the natural morphism

$$\mathcal{O}(\Lambda^r T_X^*) \to \mathcal{O}(\Lambda^r V^*) \to \mathcal{L}_V := \text{invert. sheaf } \mathcal{O}(\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V \otimes \mathcal{J}_V$, $\mathcal{J}_V \subset \mathcal{O}_X$,

$$ {}^b\mathcal{K}_V = \mathcal{L}_V \otimes \overline{\mathcal{J}}_V, \quad \overline{\mathcal{J}}_V = \text{integral closure of } \mathcal{J}_V.$$

Caution

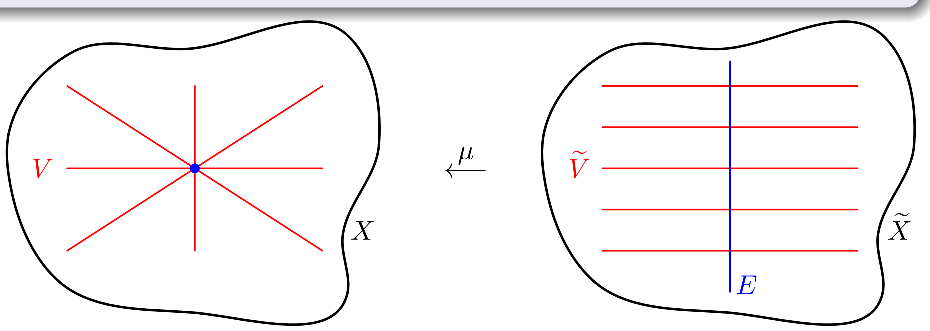
One may have to first blow up X, otherwise ${}^b\mathcal{K}_V$ need not always provide the appropriate geometric information.

Canonical sheaf of a directed variety (X,V) [sequel]

Blow up process for a directed variety

If $\mu : \widetilde{X} \to X$ is a modification, then $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V} = \overline{\widetilde{\mu}^{-1}(V_{|X'})}$, where $X' \subset X$ is a Zariski open set over which μ is a biholomorphism.

$$V : z \mapsto \sum z_j \frac{\partial}{\partial z_j}$$



Observation

One always has

$$^b\mathcal{K}_V \subset \mu_*(^b\mathcal{K}_{\widetilde{V}}) \subset \mathcal{L}_V = \mathcal{O}(\det V^*)^{**},$$

and $\mu_*(^b\mathcal{K}_{\widetilde{V}})$ "increases" with μ (taking $\widetilde{\widetilde{X}} \to \widetilde{X} \to X$).

Canonical sheaf of a directed variety (X,V) [sequel]

By Noetherianity, one can define a sequence of rank 1 sheaves

$$\mathcal{K}_V^{[m]} = \lim_{\mu} \uparrow \mu_*(^b\mathcal{K}_{\widetilde{V}})^{\otimes m}, \quad \mu_*(^b\mathcal{K}_V)^{\otimes m} \subset \mathcal{K}_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

which we call the pluricanonical sheaf sequence of (X, V).

Remark

The blow-up μ for which the limit is attained may depend on m. We do not know if there is a μ that works for all m.

This generalizes the concept of reduced singularities of foliations, (which is known to work in that form only for surfaces).

Definition

We say that (X, V) is of general type if the pluricanonical sheaf sequence $\mathcal{K}_V^{[\bullet]}$ is big, i.e. $H^0(X, \mathcal{K}_V^{[m]})$ provides a generic embedding of X for a suitable $m \gg 1$.

Generalized Green-Griffiths-Lang conjecture

Generalized GGL conjecture

If (X, V) is directed manifold of general type, i.e. $K_V^{[\bullet]}$ is big, then there exists an algebraic locus $Y \subsetneq X$ such that for every $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$, one has $f(\mathbb{C}) \subset Y$.

Remark 1. Elementary by Ahlfors-Schwarz if $r = \operatorname{rank} V = 1$.
$t \mapsto \log \|f'(t)\|_{V,h}$ is strictly subharmonic if $r = 1$ and (V^*, h^*) big.

Remark 2. The above statement is possibly too optimistic. It might be safer to add a suitable (semi)stability condition on V.

Basic strategy

Show that the entire curves $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ must satisfy nontrivial algebraic differential equations $P(f\,;\,f', f'', \ldots, f^{(k)}) = 0$, and actually, many such equations.

Definition of algebraic differential operators

Let $(\mathbb{C}, T_\mathbb{C}) \to (X, V)$, $t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a curve, $f(0) = x$, and pick local holomorphic coordinates $(z_1, \ldots, z_n)$ centered at x on a coordinate open set $U \simeq U' \times U'' \subset \mathbb{C}^r \times \mathbb{C}^{n-r}$ such that $\pi' : U \to U'$ induces an isomorphism $d\pi' : V \to U \times \mathbb{C}^r$. Then f is determined by the Taylor expansion

$$\pi' \circ f(t) = t\xi_1 + \ldots + t^k \xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!}\nabla^s f(0),$$

where ∇ is the trivial connection on $V \simeq U \times \mathbb{C}^r$.

One considers the Green-Griffiths bundle $E^{\mathrm{GG}}_{k,m} V^*$ of polynomials of weighted degree m, written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \xi_s \in V.$$

These can also be viewed as algebraic differential operators

$$\begin{aligned} P(f_{[k]}) &= P(f\,;\,f', f'', \ldots, f^{(k)}) \\ &= \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\; f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}. \end{aligned}$$

Definition of algebraic differential operators [sequel]

Here $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet, and $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are supposed to holomorphic functions on X.

The reparametrization action : $f \mapsto f \circ \varphi_\lambda$, $\varphi_\lambda(t) = \lambda t$, $\lambda \in \mathbb{C}^*$ yields $(f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$, whence a $\mathbb{C}^*$-action

$$\lambda \cdot (\xi_1, \xi_1, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k).$$

$E_{k,m}^{\mathrm{GG}}$ is precisely the set of polynomials of weighted degree m, corresponding to coefficients $a_{\alpha_1\ldots\alpha_k}$ with $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$.

Direct image formula

If $J_k^{\mathrm{nc}} V$ is the set of non constant k-jets, one defines the Green-Griffiths bundle to be $X_k^{\mathrm{GG}} = J_k^{\mathrm{nc}} V/\mathbb{C}^*$ and $\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$ to be the associated tautological rank 1 sheaf. Then we have

$$\pi_k : X_k^{\mathrm{GG}} \to X, \qquad E_{k,m}^{\mathrm{GG}} V^* = (\pi_k)_* \mathcal{O}_{X_k^{\mathrm{GG}}}(m).$$

Main cohomology estimates

As an application of holomorphic Morse inequalities, one can get the following fundamental estimates.

Theorem (D-, 2010)

Let (X, V) be a directed manifold, $A \to X$ an ample $\mathbb{Q}$-line bundle, (V, h) and (A, h_A) hermitian, $\Theta_{A,h_A} > 0$. Define

$$L_k = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^* \mathcal{O}\Big(-\frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big)A\Big),$$

$$\eta = \Theta_{\det V^*, \det h^*} - \Theta_{A,h_A}.$$

Then for all $q \geq 0$ and all $m \gg k \gg 1$ such that m is sufficiently divisible, we have upper and lower bounds [$q = 0$ is most useful!]

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q)} (-1)^q \eta^n + \frac{C}{\log k}\Big)$$

$$h^q(X_k^{\mathrm{GG}}, \mathcal{O}(L_k^{\otimes m})) \geq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n!\,(k!)^r}\Big(\int_{X(\eta,q,\,q\pm1)} (-1)^q \eta^n - \frac{C}{\log k}\Big).$$

Holomorphic Morse inequalities: main statement

The q-index set of a real $(1,1)$-form θ is defined to be

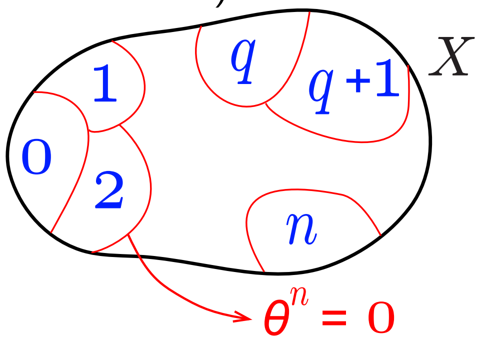
$$X(\theta, q) = \{x \in X \mid \theta(x) \text{ has signature } (n-q, q)\}$$

(exactly q negative eigenvalues and $n-q$ positive ones)

Set also $X(\theta, \leq q) = \bigcup_{0\leq j\leq q} X(\theta, j)$.

$X(\theta, q)$ and $X(\theta, \leq q)$ are open sets.

$\operatorname{sign}(\theta^n) = (-1)^q$ on $X(\theta, q)$.



Theorem (D-, 1985)

Let (L, h) be a hermitian line bundle on X, $\mathcal{F}$ a coherent sheaf, $\theta = \Theta_{L,h}$ and $r = \operatorname{rank} \mathcal{F}$. Then, as $m \to +\infty$

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes \mathcal{F}) \leq r \frac{m^n}{n!} \int_{X(\theta, \leq q)} (-1)^q \theta^n + o(m^n).$$

1^{st} step: define a Finsler metric on k-jet bundles

Let $J_k V$ be the bundle of k-jets of curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$
Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on $J^k V$ by taking $p = k!$ and

$$\Psi_{h_k}(f) := \Big(\sum_{1\leq s\leq k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s}\Big)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \cdots \gg \varepsilon_k.$$

Letting $\xi_s = \nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k := \mathcal{O}_{X_k^{\mathrm{GG}}}(1)$, with curvature form $(x, \xi_1, \ldots, \xi_k) \mapsto$

$$\Theta_{L_k, h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1\leq s\leq k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor Θ_{V^*,h^*} and $\omega_{\mathrm{FS},k}$ is the vertical Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}} \to X$.

The expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \quad u_s = \xi_s / |\xi_s|_h = \nabla^s f(0) / |\nabla^s f(0)|.$$

2^{nd} step: probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$.

The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \ge 0$, $\sum x_s = 1$.

This is essentially a sum of the form $\sum \frac{1}{s} Q(u_s)$ where $Q(u) = \langle \Theta_{V^*,h^*} u, u\rangle$ and u_s are random points of the sphere, and so as $k \to +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) \int_{u\in SV} Q(u)\, du.$$

As Q is quadratic, $\displaystyle\int_{u\in SV} Q(u)\, du = \frac{1}{r}\mathrm{Tr}(Q) = \frac{1}{r}\mathrm{Tr}(\Theta_{V^*,h^*}) = \frac{1}{r}\Theta_{\det V^*}.$

Fundamental vanishing theorem and diff. equations

Passing to a "singular version" of holomorphic Morse inequalities to accommodate singular metrics ([Bonavero, 1996]), one gets

Corollary: existence of global jet differentials (D-, 2010)

Let (X, V) be of general type, i.e. ${}^b\mathcal{K}_V^{\otimes p}$ big rank 1 sheaf, and let

$$L_{k,\varepsilon} = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}(-\delta_k \varepsilon A), \quad \delta_k = \frac{1}{kr}\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big),$$

with A ample. Then there exist many nontrivial global sections

$$P \in H^0(X_k^{\mathrm{GG}}, L_{k,\varepsilon}^{\otimes m}) \simeq H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-m\delta_k\varepsilon A))$$

for $m \gg k \gg 1$ and $\varepsilon \in \mathbb{Q}_{>0}$ small.

Fundamental vanishing theorem $\Rightarrow$ differential equations

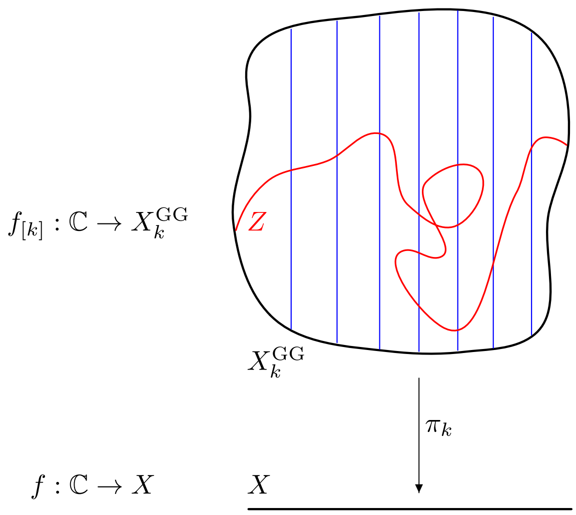
[Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]

For all global differential operators $P \in H^0(X, E_{k,m}^{\mathrm{GG}} V^* \otimes \mathcal{O}(-qA))$, $q \in \mathbb{N}^*$, and all $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, one has $P(f_{[k]}) \equiv 0$.

The base locus problem

Geometrically, this can interpreted by stating that the image $f_{[k]}(\mathbb{C})$ of the k-jet curve lies in the base locus

$$Z = \bigcap_{m\in\mathbb{N}^*} \bigcap_{\sigma\in H^0(X_k^{\rm GG}, L_{k,\varepsilon}^{\otimes m})} \sigma^{-1}(0) \quad \subset \quad X_k^{\rm GG}.$$



To prove the GGL conjecture, we would need to get $\pi_k(Z) \subsetneq X$.

General problem concerning base loci

Let (L, h) be a hermitian line bundle over X. If we assume that $\theta = \Theta_{L,h}$ satisfies $\int_{X(\theta,\leq 1)} \theta^n > 0$, then we know that L is big, i.e. that $h^0(X, L^{\otimes m}) \geq c\, m^n$, for $m \geq m_0$ and $c > 0$, but this does not tell us anything about the base locus $\mathrm{Bs}(L) = \bigcap_{\sigma\in H^0(X,L^{\otimes m})} \sigma^{-1}(0)$.

Definition

The "iterated base locus" $\mathrm{IBs}(L)$ is obtained by picking inductively $Z_0 = X$ and $Z_k =$ zero divisor of a section σ_k of $L^{\otimes m_k}$ over the normalization of Z_{k-1}, and taking $\bigcap_{k,m_1,\ldots,m_k,\sigma_1,\ldots,\sigma_k} Z_k$.

Unsolved problem

Find a condition, e.g. in the form of Morse integrals (or analogs) for $\theta = \Theta_{L,h}$, ensuring for instance that $\mathrm{codim}\,\mathrm{IBs}(L) > p$.

We would need for instance to be able to check the positivity of Morse integrals $\int_{Z(\theta_{|Z},\leq 1)} \theta^{n-p}$ for Z irreducible, $\mathrm{codim}\, Z = p$.

A new result on the base locus of jet differentials

Theorem (D-, 2021)

Let (X,V) be a directed variety of general type. Then there exists $k_0\in\mathbb{N}$ and $\delta>0$ with the following properties.
Let $Z\subset X_k^{\rm GG}$ be an irreducible algebraic subvariety that is a component of a complete intersection of irreducible hypersurfaces

$$\bigcap_{1\le j\le\ell}\big\{k\text{-jets } f_{[k]}\in X_k^{\rm GG}\,;\,P_j(f)=0\big\},\quad P_j\in H^0(X,E^{\rm GG}_{s_j,m_j}V^*\otimes G_j)$$

with $k\ge k_0$, ${\rm ord}(P_j)=s_j$, $1\le s_1<\cdots<s_\ell\le k$, $\sum_{1\le j\le\ell}\frac{1}{s_j}\le\delta\log k$, and $G_j\in{\rm Pic}(X)$. Then the Morse integrals $\int_{Z(L_{k,\varepsilon},\le 1)}\Theta^{\dim Z}_{L_{k,\varepsilon}}$ of

$$L_{k,\varepsilon}=\mathcal{O}_{X_k^{\rm GG}}(1)\otimes\pi_k^*\mathcal{O}_X\Big(-\frac{1}{kr}\Big(1+\frac12+\cdots+\frac1k\Big)\varepsilon A\Big)$$

are positive for $\varepsilon>0$ small, hence $H^0(Z,L_{k,\varepsilon}^{\otimes m})\ge c\,m^{\dim Z}$ for $m\gg 1$.

Unfortunately, this seems insufficient to prove the GGL conjecture.

Further geometric structures: Semple jet bundles

- **Functor "1-jet" :** $(X,V)\mapsto(\tilde X,\tilde V)$ where :
  $\tilde X=P(V)=$ bundle of projective spaces of lines in V
  $\pi:\tilde X=P(V)\to X,\quad (x,[v])\mapsto x,\quad v\in V_x$
  $\tilde V_{(x,[v])}=\big\{\xi\in T_{\tilde X,(x,[v])}\,;\,\pi_*\xi\in\mathbb{C}v\subset T_{X,x}\big\}$
- For every entire curve $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$ tangent to V
  f lifts as $\begin{cases} f_{[1]}(t):=(f(t),[f'(t)])\in P(V_{f(t)})\subset\tilde X\\ f_{[1]}:(\mathbb{C},T_{\mathbb{C}})\to(\tilde X,\tilde V)\end{cases}$ (projectivized $1^{\rm st}$-jet)
- **Definition.** *Semple jet bundles :*
  – $(X_k,V_k)=k$-th iteration of functor $(X,V)\mapsto(\tilde X,\tilde V)$
  – $f_{[k]}:(\mathbb{C},T_{\mathbb{C}})\to(X_k,V_k)$ is the projectivized k-jet of f.
- **Basic exact sequences.** On $X_k=P(V_{k-1})$, one has
  $0\to T_{X_k/X_{k-1}}\to V_k\xrightarrow{d\pi_k}\mathcal{O}_{X_k}(-1)\to 0\quad\Rightarrow\text{rank } V_k=r$
  $0\to\mathcal{O}_{X_k}\to\pi_k^\star V_{k-1}\otimes\mathcal{O}_{X_k}(1)\to T_{X_k/X_{k-1}}\to 0$ (Euler)

Direct image formula for Semple bundles

For $n = \dim X$ and $r = \operatorname{rank} V$, one gets a tower of $\mathbb{P}^{r-1}$-bundles

$$\pi_{k,0} : X_k \overset{\pi_k}{\to} X_{k-1} \to \cdots \to X_1 \overset{\pi_1}{\to} X_0 = X$$

with $\dim X_k = n + k(r-1)$, $\operatorname{rank} V_k = r$,
and tautological line bundles $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$.

Theorem

X_k is a smooth compactification of $X_k^{\mathrm{GG,reg}}/\mathbb{G}_k = J_k^{\mathrm{GG,reg}}/\mathbb{G}_k$, where $\mathbb{G}_k$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C}, 0)$, acting on the right by reparametrization: $(f, \varphi) \mapsto f \circ \varphi$, and J_k^{reg} is the space of k-jets of regular curves.

Direct image formula for invariant differential operators

$E_{k,m}V^* := (\pi_{k,0})_* \mathcal{O}_{X_k}(m) =$ sheaf of algebraic differential operators $f \mapsto P(f_{[k]})$ acting on germs of curves $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ such that $P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi$.

Induced directed structure on a subvariety

Let Z be an irreducible algebraic subset of some Semple k-jet bundle X_k over X (k arbitrary).

We define an induced directed structure $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$.

Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the k-th stage (X_k^a, V_k^a) of the "absolute Semple tower" associated with $(X_0^a, V_0^a) = (X, T_X)$
(so as to deal only with nonsingular ambient Semple bundles).

This produces an induced directed subvariety

$$(Z, W) \subset (X_k, V_k).$$

It is easy to show that
$\pi_{k,k-1}(Z) = X_{k-1} \Rightarrow \operatorname{rank} W < \operatorname{rank} V_k = \operatorname{rank} V$.

Some tautological morphisms

Denote $\mathcal{O}_{X_k}(\underline{a}) = \pi_{k,1}^*\mathcal{O}_{X_1}(a_1) \otimes \cdots \otimes \pi_{k,k-1}^*\mathcal{O}_{X_{k-1}}(a_{k-1}) \otimes \mathcal{O}_{X_k}(a_k)$ for every k-tuple $\underline{a} = (a_1, \ldots, a_k) \in \mathbb{Z}^k$, and let $\underline{1} = (1, \ldots, 1) \in \mathbb{Z}^k$.

Absolute and induced tautological morphisms

- For all $p = 1, \ldots, n$, there is a tautological morphism
$$\Phi_{k,p}^X : \pi_{k,0}^*\Lambda^p T_X^* \to \Lambda^p(V_k^a)^* \otimes \mathcal{O}_{X_k^a}((p-1)\underline{1})$$
- Let Z be an irreducible subvariety of X_k such that $\pi_{k,0}(Z) = X$. Consider the induced directed structure $(Z, W) \subset (X_k, V_k)$ and set $r' = \operatorname{rank} W$. Then there is over Z a subsheaf $W_0 \subset \pi_{k,0}^* V$ of rank $r_0 \geq r'$, and there exist nonzero tautological morphisms derived from $\Phi_{k,p}^X$, of the form
$$\Phi_k^{Z,W} : {}^b\Lambda^{r_0} W_0^* \to {}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(\underline{a})_{|Z}$$
where ${}^b\mathcal{K}_W \subset (\Lambda^{r'} W^*)^{**}$, ${}^b\Lambda^{r_0} W_0^*$ is a quotient of the sheaf $\pi_{k,0}^*{}^b\Lambda^{r_0} V^*$ of bounded r_0-forms on V, and $\underline{a} \in \mathbb{N}^k$.

Geometric use of the tautological morphisms

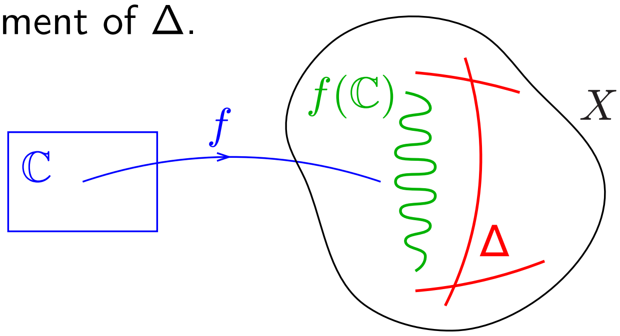
Theorem (D-, 2021)

Let (X, V) be a directed variety. Assume that ${}^b\Lambda^p V^*$ is strongly big for some $p \leq r = \operatorname{rank} V$, in the sense that for $A \in \operatorname{Pic}(X)$ ample, the symmetric powers $S^m({}^b\Lambda^p V^*) \otimes \mathcal{O}(-A)$ are generated by their sections over a Zariski open set of X, for $m \gg 1$.

- If $p = 1$, (X, V) satisfies the generalized GGL conjecture.
- If $p \geq 2$, there exists a subvariety $Y \subsetneq X$ and finitely many induced directed subvarieties $(Z_\alpha, W_\alpha) \subset (X_k, V_k)$ with $\operatorname{rank} W_\alpha \leq p-1$, such that all curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ satisfy either $f(\mathbb{C}) \subset Y$ or $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to \bigcup (Z_\alpha, W_\alpha)$.
- In particular, if $p = 2$, all entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ are either contained in $Y \subsetneq X$, or they are tangent to a rank 1 foliation on a subvariety $Z \subset X_k$. This implies that the latter curves are parametrized by a finite dimensional space.

Logarithmic version

More generally, if $\Delta = \sum \Delta_j$ is a reduced normal crossing divisor in X, we want to study entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$ drawn in the complement of Δ.



At a point where $\Delta = \{z_1 \ldots z_p = 0\}$ one defines the cotangent logarithmic sheaf $T^*_{X\langle\Delta\rangle}$ to be generated by $\frac{dz_1}{z_1}, \ldots, \frac{dz_p}{z_p}, dz_{p+1}, \ldots, dz_n$.

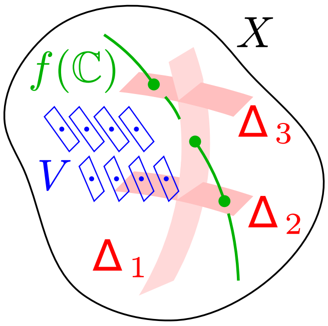
Theorem (D-, 2021)

If $\Lambda^2 T^*_{X\langle\Delta\rangle}$ is strongly big on X, there exists a subvariety $Y \subsetneq X$ and a rank 1 foliation $\mathcal{F}$ on some k-jet bundle X_k, such that all entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$ are contained in Y or tangent to $\mathcal{F}$.

Logarithmic/orbifold directed versions

(Work in progress with F. Campana, L. Darondeau & E. Rousseau)

There are also more general versions dealing with entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ and avoiding a normal crossing divisor Δ transverse to V ("logarithmic case"), or meeting $\Delta = \sum(1 - \frac{1}{\rho_j})\Delta_j$ with multiplicities $\geq \rho_j$ along Δ_j ("orbifold case").



At this step, positivity is to be expressed for a sequence of orbifold cotangent bundles $V^*\langle\Delta^{(s)}\rangle$, $\Delta^{(s)} = \sum_j \left(1 - \frac{s}{\rho_j}\right)_+ \Delta_j$.

In all cases, proving the GGL conjecture with optimal positivity conditions (i.e. only assuming bigness of the logarithmic/orbifold canonical sheaf) seems to require a better use of stability properties.

Thank you for your attention!