Monge-Ampère functionals for the curvature tensor of a holomorphic vector bundle

Jean-Pierre Demailly

Institut Fourier, Université Grenoble Alpes & Académie des Sciences de Paris

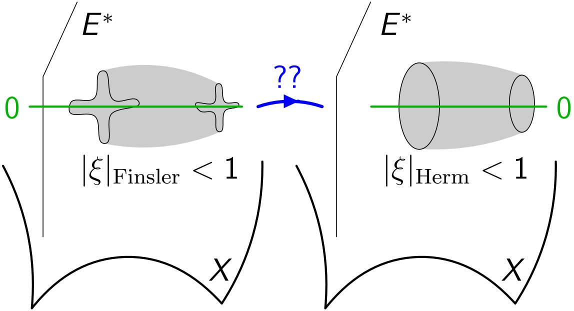
Conference on Complex Geometric Analysis
in honor of Kang-Tae Kim for his 65th birthday
January 14, 2022, 17:00 – 17:50

Plan of the talk

1. Positivity concepts for holomorphic vector bundles
2. Monge-Ampère functionals for vector bundles
3. Chern class inequalities for Monge-Ampère volumes

Plan of the talk

1. Positivity concepts for holomorphic vector bundles
2. Monge-Ampère functionals for vector bundles
3. Chern class inequalities for Monge-Ampère volumes
4. A Hermitian-Yang-Mills approach to the Griffiths conjecture



5. Further results by Siarhei Finski

Positive and ample vector bundles

Let X be a projective n-dimensional manifold and $E \to X$ a holomorphic vector bundle of rank $r \geq 1$.

Positive and ample vector bundles

Let X be a projective n-dimensional manifold and $E \to X$ a holomorphic vector bundle of rank $r \geq 1$.

Ample vector bundles

$E \to X$ is said to be ample in the sense of Hartshorne if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the hyperplane bundle $\mathbb{P}(E)$ is ample.

Positive and ample vector bundles

Let X be a projective n-dimensional manifold and $E \to X$ a holomorphic vector bundle of rank $r \geq 1$.

Ample vector bundles

$E \to X$ is said to be ample in the sense of Hartshorne if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the hyperplane bundle $\mathbb{P}(E)$ is ample.

By Kodaira (1954), this is equivalent to the existence of a smooth hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with positive curvature (equivalently, a negatively curved Finsler metric on E^*).

Positive and ample vector bundles

Let X be a projective n-dimensional manifold and $E \to X$ a holomorphic vector bundle of rank $r \geq 1$.

Ample vector bundles

$E \to X$ is said to be ample in the sense of Hartshorne if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the hyperplane bundle $\mathbb{P}(E)$ is ample.

By Kodaira (1954), this is equivalent to the existence of a smooth hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with positive curvature (equivalently, a negatively curved Finsler metric on E^*).

Chern curvature tensor of a hermitian bundle (E, h)

This is $\Theta_{E,h} = i\nabla^2_{E,h} \in C^\infty(\Lambda^{1,1}T^*_X \otimes \mathrm{Hom}(E, E))$, which can be written

$$\Theta_{E,h} = i \sum_{1\leq j,k\leq n,\, 1\leq\lambda,\mu\leq r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu$$

in terms of an orthonormal frame $(e_\lambda)_{1\leq\lambda\leq r}$ of E.

Griffiths positivity concept for vector bundles

Definition

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu.$$

Griffiths positivity concept for vector bundles

Definition

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu.$$

Then E is said to be Griffiths positive (Griffiths 1969) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall 0 \neq \xi \in T_{X,z},\ \forall 0 \neq v \in E_z$$

Griffiths positivity concept for vector bundles

Definition

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu.$$

Then E is said to be Griffiths positive (Griffiths 1969) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall 0 \neq \xi \in T_{X,z}, \; \forall 0 \neq v \in E_z$$

Well known fact

E Griffiths $> 0 \Rightarrow E$ ample.

Griffiths positivity concept for vector bundles

Definition

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu.$$

Then E is said to be Griffiths positive (Griffiths 1969) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall 0 \neq \xi \in T_{X,z},\ \forall 0 \neq v \in E_z$$

Well known fact

E Griffiths $> 0 \Rightarrow E$ ample.

Proof. E Griffiths $> 0 \Rightarrow \mathcal{O}_{\mathbb{P}(E)}(1) > 0 \underset{\text{Kodaira}}{\Longleftrightarrow} \mathcal{O}_{\mathbb{P}(E)}(1)$ ample.

Griffiths positivity concept for vector bundles

Definition

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1 \leq j,k \leq n,\, 1 \leq \lambda,\mu \leq r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu.$$

Then E is said to be Griffiths positive (Griffiths 1969) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall 0 \neq \xi \in T_{X,z},\ \forall 0 \neq v \in E_z$$

Well known fact

E Griffiths $> 0 \Rightarrow E$ ample.

Proof. E Griffiths $> 0 \Rightarrow \mathcal{O}_{\mathbb{P}(E)}(1) > 0 \underset{\text{Kodaira}}{\Longleftrightarrow} \mathcal{O}_{\mathbb{P}(E)}(1)$ ample.

Griffiths conjecture [unsolved, except for $n = 1$ (Umemura 1973)]

Is it true that E ample $\Rightarrow E$ Griffiths > 0 ? (If so, both are $\Leftrightarrow$).

Nakano / dual Nakano positivity concepts

The curvature tensor yields a natural hermitian form on $T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu}, \quad \tau \in T_{X,z}\otimes E_z.$$

Nakano / dual Nakano positivity concepts

The curvature tensor yields a natural hermitian form on $T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\, 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu}, \quad \tau \in T_{X,z} \otimes E_z.$$

Definition of Nakano positivity

E is Nakano positive (Nakano 1955) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\, 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0, \;\; \forall\tau \in T_{X,z} \otimes E_z,\; \tau \neq 0.$$

Nakano / dual Nakano positivity concepts

The curvature tensor yields a natural hermitian form on $T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\, 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu}, \quad \tau \in T_{X,z} \otimes E_z.$$

Definition of Nakano positivity

E is Nakano positive (Nakano 1955) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\, 1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0, \quad \forall\tau \in T_{X,z} \otimes E_z, \ \tau \neq 0.$$

Curvature tensor of the dual bundle E^*

$$\Theta_{E^*,h^*} = -{}^T\Theta_{E,h} = -\sum_{1\le j,k\le n,\, 1\le\lambda,\mu\le r} c_{jk\mu\lambda} dz_j \wedge d\overline{z}_k \otimes (e^*_\lambda)^* \otimes e^*_\mu.$$

Nakano / dual Nakano positivity concepts

The curvature tensor yields a natural hermitian form on $T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu}, \quad \tau \in T_{X,z}\otimes E_z.$$

Definition of Nakano positivity

E is Nakano positive (Nakano 1955) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0, \quad \forall\tau \in T_{X,z}\otimes E_z,\ \tau\neq 0.$$

Curvature tensor of the dual bundle E^*

$$\Theta_{E^*,h^*} = -{}^T\Theta_{E,h} = -\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}dz_j\wedge d\overline{z}_k\otimes(e^*_\lambda)^*\otimes e^*_\mu.$$

Definition of dual Nakano positivity

E is dual Nakano positive if E^* is Nakano < 0, i.e.

$$-\widetilde{\Theta}_{E^*,h^*}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0, \quad \forall\tau \in T_{X,z}\otimes E^*_z,\ \tau\neq 0.$$

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.
- Griffiths and dual Nakano Nakano positivity are preserved by taking quotients: $E > 0 \Rightarrow$ any quotient $Q = E/S$ is also > 0.

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.
- Griffiths and dual Nakano Nakano positivity are preserved by taking quotients: $E > 0 \Rightarrow$ any quotient $Q = E/S$ is also > 0. This is wrong for Nakano positivity.

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.
- Griffiths and dual Nakano Nakano positivity are preserved by taking quotients: $E > 0 \Rightarrow$ any quotient $Q = E/S$ is also > 0. This is wrong for Nakano positivity.
- E ample $\not\Rightarrow$ E Nakano > 0.

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.
- Griffiths and dual Nakano Nakano positivity are preserved by taking quotients: $E > 0 \Rightarrow$ any quotient $Q = E/S$ is also > 0. This is wrong for Nakano positivity.
- E ample $\not\Rightarrow$ E Nakano > 0.

  For instance, $T_{\mathbb{P}^n}$ is ample and even Griffiths > 0 for the Fubini-Study metric, but it is not Nakano > 0.

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.
- Griffiths and dual Nakano Nakano positivity are preserved by taking quotients: $E > 0 \Rightarrow$ any quotient $Q = E/S$ is also > 0. This is wrong for Nakano positivity.
- E ample $\not\Rightarrow$ E Nakano > 0. For instance, $T_{\mathbb{P}^n}$ is ample and even Griffiths > 0 for the Fubini-Study metric, but it is not Nakano > 0. Otherwise the Nakano vanishing theorem would imply

$$H^{n-1,n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0 \quad !!!$$

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.
- Griffiths and dual Nakano Nakano positivity are preserved by taking quotients: $E > 0 \Rightarrow$ any quotient $Q = E/S$ is also > 0. This is wrong for Nakano positivity.
- E ample $\not\Rightarrow$ E Nakano > 0.
  For instance, $T_{\mathbb{P}^n}$ is ample and even Griffiths > 0 for the Fubini-Study metric, but it is not Nakano > 0. Otherwise the Nakano vanishing theorem would imply

$$H^{n-1,n-1}(\mathbb{P}^n, \mathbb{C}) = H^{n-1}(\mathbb{P}^n, \Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n} \otimes T_{\mathbb{P}^n}) = 0 \quad !!!$$

- E ample $\not\Rightarrow$ E dual Nakano > 0.

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.
- Griffiths and dual Nakano Nakano positivity are preserved by taking quotients: $E > 0 \Rightarrow$ any quotient $Q = E/S$ is also > 0. This is wrong for Nakano positivity.
- E ample $\not\Rightarrow$ E Nakano > 0.
  For instance, $T_{\mathbb{P}^n}$ is ample and even Griffiths > 0 for the Fubini-Study metric, but it is not Nakano > 0. Otherwise the Nakano vanishing theorem would imply

$$H^{n-1,n-1}(\mathbb{P}^n,\mathbb{C}) = H^{n-1}(\mathbb{P}^n,\Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n}\otimes T_{\mathbb{P}^n}) = 0 \quad !!!$$

- E ample $\not\Rightarrow$ E dual Nakano > 0.
  For instance, any compact quotient $X = \mathbb{B}^n/\Gamma$ has T^*_X ample and even Griffiths > 0 for the hyperbolic metric, but T^*_X is not dual Nakano > 0, otherwise T_X would be Nakano < 0 and

$$H^{1,0}(X,\mathbb{C}) = H^0(X,\Omega^1_X\otimes T_X) = H^0(X,\mathrm{Hom}(T_X,T_X)) \ni \mathrm{Id}_{T_X}$$

would contradict the (dual) Nakano vanishing theorem.

Positivity thresholds

There are subtle relations between the various positivity concepts.

Positivity thresholds

There are subtle relations between the various positivity concepts.

Theorem (Berndtsson 2009)

E ample $\Rightarrow S^m E \otimes \det E$ Nakano > 0 for every $m \in \mathbb{N}$.

Positivity thresholds

There are subtle relations between the various positivity concepts.

Theorem (Berndtsson 2009)

E ample $\Rightarrow S^m E \otimes \det E$ Nakano > 0 for every $m \in \mathbb{N}$.

Theorem (Liu-Sun-Yang 2013)

E ample $\Rightarrow S^m E \otimes \det E$ dual Nakano > 0 for every $m \in \mathbb{N}$.

Positivity thresholds

There are subtle relations between the various positivity concepts.

Theorem (Berndtsson 2009)

E ample $\Rightarrow S^m E \otimes \det E$ Nakano > 0 for every $m \in \mathbb{N}$.

Theorem (Liu-Sun-Yang 2013)

E ample $\Rightarrow S^m E \otimes \det E$ dual Nakano > 0 for every $m \in \mathbb{N}$.

This leads in a natural way to the following definition.

Definition

Let $P = A, G, N, N^*$ mean the Ampleness / Griffiths / Nakano / dual Nakano positivity concepts. Let $E \to X$ be a vector bundle such that $\det E$ is ample. We let

$$\tau_P(E) = \inf\big\{t \in \mathbb{R}\,;\ E \otimes (\det E)^t >_P 0\big\}.$$

Positivity thresholds

There are subtle relations between the various positivity concepts.

Theorem (Berndtsson 2009)

E ample $\Rightarrow S^m E \otimes \det E$ Nakano > 0 for every $m \in \mathbb{N}$.

Theorem (Liu-Sun-Yang 2013)

E ample $\Rightarrow S^m E \otimes \det E$ dual Nakano > 0 for every $m \in \mathbb{N}$.

This leads in a natural way to the following definition.

Definition

Let $P = A, G, N, N^*$ mean the Ampleness / Griffiths / Nakano / dual Nakano positivity concepts. Let $E \to X$ be a vector bundle such that $\det E$ is ample. We let

$$\tau_P(E) = \inf\big\{t \in \mathbb{R}\,;\ E \otimes (\det E)^t >_P 0\big\}.$$

Remark. $\Theta_{E\otimes(\det E)^t} = \Theta_E + t\,\Theta_{\det E} \otimes \mathrm{Id}_E, \quad \Theta_{\det E} = \mathrm{Tr}_E\,\Theta_E.$

Simple facts about positivity thresholds

Notice that Nakano and dual Nakano positivity are stronger than Griffiths positivity, the latter being itself stronger than ampleness, hence we always have

$$\tau_N(E) \geq \tau_G(E) \geq \tau_A(E), \quad \tau_{N^*}(E) \geq \tau_G(E) \geq \tau_A(E).$$

Simple facts about positivity thresholds

Notice that Nakano and dual Nakano positivity are stronger than Griffiths positivity, the latter being itself stronger than ampleness, hence we always have

$$\tau_N(E) \geq \tau_G(E) \geq \tau_A(E), \quad \tau_{N^*}(E) \geq \tau_G(E) \geq \tau_A(E).$$

Moreover, since $E \otimes (\det E)^{-1/r}$ has trivial determinant, we also have $\tau_A(E) \geq -1/r$.

Simple facts about positivity thresholds

Notice that Nakano and dual Nakano positivity are stronger than Griffiths positivity, the latter being itself stronger than ampleness, hence we always have

$$\tau_N(E) \geq \tau_G(E) \geq \tau_A(E), \quad \tau_{N^*}(E) \geq \tau_G(E) \geq \tau_A(E).$$

Moreover, since $E \otimes (\det E)^{-1/r}$ has trivial determinant, we also have $\tau_A(E) \geq -1/r$.

Proposition

One has $\tau_A(E) = -1/r \Leftrightarrow F = E \otimes (\det E)^{-1/r}$ is numerically flat (i.e. F, F^* both nef), so that $E = F \otimes L$ where $L = (\det E)^{1/r}$ is ample: we say that E is projectively numerically flat.

Simple facts about positivity thresholds

Notice that Nakano and dual Nakano positivity are stronger than Griffiths positivity, the latter being itself stronger than ampleness, hence we always have

$$\tau_N(E) \geq \tau_G(E) \geq \tau_A(E), \quad \tau_{N^*}(E) \geq \tau_G(E) \geq \tau_A(E).$$

Moreover, since $E \otimes (\det E)^{-1/r}$ has trivial determinant, we also have $\tau_A(E) \geq -1/r$.

Proposition

One has $\tau_A(E) = -1/r \Leftrightarrow F = E \otimes (\det E)^{-1/r}$ is numerically flat (i.e. F, F^* both nef), so that $E = F \otimes L$ where $L = (\det E)^{1/r}$ is ample: we say that E is projectively numerically flat. Then

$$\tau_N(E) = \tau_{N^*}(E) = \tau_G(E) = \tau_A(E) = -\tfrac{1}{r}.$$

Simple facts about positivity thresholds

Notice that Nakano and dual Nakano positivity are stronger than Griffiths positivity, the latter being itself stronger than ampleness, hence we always have

$$\tau_N(E) \geq \tau_G(E) \geq \tau_A(E), \quad \tau_{N^*}(E) \geq \tau_G(E) \geq \tau_A(E).$$

Moreover, since $E \otimes (\det E)^{-1/r}$ has trivial determinant, we also have $\tau_A(E) \geq -1/r$.

Proposition

One has $\tau_A(E) = -1/r \Leftrightarrow F = E \otimes (\det E)^{-1/r}$ is numerically flat (i.e. F, F^* both nef), so that $E = F \otimes L$ where $L = (\det E)^{1/r}$ is ample: we say that E is projectively numerically flat. Then

$$\tau_N(E) = \tau_{N^*}(E) = \tau_G(E) = \tau_A(E) = -\frac{1}{r}.$$

Remark

The Griffiths conjecture is equivalent to: E ample $\Rightarrow \tau_G(E) < 0$.

Monge-Ampère functionals for vector bundles

Definition of the functionals, $\Theta_{E,h} \mapsto$ volume (n, n)-form on X :

Monge-Ampère functionals for vector bundles

Definition of the functionals, $\Theta_{E,h} \mapsto$ volume (n,n)-form on X :

– If $E >_N 0$, we set $\Phi_N(\Theta_{E,h}) := \det_{T_X \otimes E}(\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_N(\Theta_{E,h}) := \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

Monge-Ampère functionals for vector bundles

Definition of the functionals, $\Theta_{E,h} \mapsto$ volume (n,n)-form on X :

– If $E >_N 0$, we set $\Phi_N(\Theta_{E,h}) := \det_{T_X \otimes E}(\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_N(\Theta_{E,h}) := \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

– If $E >_{N^*} 0$, we set $\Phi_{N^*}(\Theta_{E,h}) := \det_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_{N^*}(\Theta_{E,h}) := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

Monge-Ampère functionals for vector bundles

Definition of the functionals, $\Theta_{E,h} \mapsto$ volume (n,n)-form on X :

– If $E >_N 0$, we set $\Phi_N(\Theta_{E,h}) := \det_{T_X\otimes E}(\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_N(\Theta_{E,h}) := \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

– If $E >_{N^*} 0$, we set $\Phi_{N^*}(\Theta_{E,h}) := \det_{T_X\otimes E^*}({}^T\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_{N^*}(\Theta_{E,h}) := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

– If $E >_G 0$, we set

$$\Phi_G(\Theta_{E,h}) := \inf_{|v|_h=1} \langle \Theta_{E,h} \cdot v, v\rangle^n \qquad \text{(not differentiable)},$$

Monge-Ampère functionals for vector bundles

Definition of the functionals, $\Theta_{E,h} \mapsto$ volume (n,n)-form on X :

– If $E >_N 0$, we set $\Phi_N(\Theta_{E,h}) := \det_{T_X\otimes E}(\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_N(\Theta_{E,h}) := \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1\wedge d\overline{z}_1\wedge\ldots\wedge idz_n\wedge d\overline{z}_n.$$

– If $E >_{N^*} 0$, we set $\Phi_{N^*}(\Theta_{E,h}) := \det_{T_X\otimes E^*}({}^T\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_{N^*}(\Theta_{E,h}) := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1\wedge d\overline{z}_1\wedge\ldots\wedge idz_n\wedge d\overline{z}_n.$$

– If $E >_G 0$, we set

$$\Phi_G(\Theta_{E,h}) := \inf_{|v|_h=1}\langle\Theta_{E,h}\cdot v, v\rangle^n \qquad \text{(not differentiable)},$$

$$\Phi_{G,s}(\Theta_{E,h}) := \left(\int_{|v|_h=1}(\langle\Theta_{E,h}\cdot v, v\rangle^n)^{-s}\,d\sigma(v)\right)^{-1/s} \underset{s\to+\infty}{\longrightarrow} \Phi_G(\Theta_{E,h}).$$

Monge-Ampère functionals for vector bundles

Definition of the functionals, $\Theta_{E,h} \mapsto$ volume (n,n)-form on X :

– If $E >_N 0$, we set $\Phi_N(\Theta_{E,h}) := \det_{T_X \otimes E}(\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_N(\Theta_{E,h}) := \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

– If $E >_{N^*} 0$, we set $\Phi_{N^*}(\Theta_{E,h}) := \det_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_{N^*}(\Theta_{E,h}) := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

– If $E >_G 0$, we set

$$\Phi_G(\Theta_{E,h}) := \inf_{|v|_h=1} \langle \Theta_{E,h} \cdot v, v\rangle^n \quad \text{(not differentiable)},$$

$$\Phi_{G,s}(\Theta_{E,h}) := \left(\int_{|v|_h=1} (\langle \Theta_{E,h} \cdot v, v\rangle^n)^{-s}\, d\sigma(v)\right)^{-1/s} \underset{s\to+\infty}{\longrightarrow} \Phi_G(\Theta_{E,h}).$$

These (n,n)-forms are intrinsic: they do not depend on the choice of coordinates (z_j) on X, nor on the choice of the orthonormal frame (e_λ) on E.

Main properties of the Monge-Ampère functionals

Coercivity of the Φ_P functionals

For $P = N, N^*$ or $P = (G, s)$, $s \in [r-1, \infty]$,
$\Phi_P(\bullet)$ prevents degeneration of positivity, i.e.

$$\Theta_{E,h} \geq_P 0 \text{ and } \Phi_P(\Theta_{E,h}) > 0 \text{ on } X \implies \Theta_{E,h} >_P 0.$$

Main properties of the Monge-Ampère functionals

Coercivity of the Φ_P functionals

For $P = N, N^*$ or $P = (G, s)$, $s \in [r-1, \infty]$,
$\Phi_P(\bullet)$ prevents degeneration of positivity, i.e.

$$\Theta_{E,h} \geq_P 0 \text{ and } \Phi_P(\Theta_{E,h}) > 0 \text{ on } X \implies \Theta_{E,h} >_P 0.$$

Chern class inequality for Monge-Ampère volumes

For any P, we define Monge-Ampère volumes for vector bundles by

$$\mathrm{MAVol}_P(E) = \sup_{h,\,\Theta_{E,h} >_P 0} \frac{1}{(2\pi)^n} \int_X \Phi_P(\Theta_{E,h}).$$

Main properties of the Monge-Ampère functionals

Coercivity of the Φ_P functionals

For $P = N, N^*$ or $P = (G, s)$, $s \in [r-1, \infty]$,

$\Phi_P(\bullet)$ prevents degeneration of positivity, i.e.

$$\Theta_{E,h} \geq_P 0 \text{ and } \Phi_P(\Theta_{E,h}) > 0 \text{ on } X \implies \Theta_{E,h} >_P 0.$$

Chern class inequality for Monge-Ampère volumes

For any P, we define Monge-Ampère volumes for vector bundles by

$$\mathrm{MAVol}_P(E) = \sup_{h,\,\Theta_{E,h} >_P 0} \frac{1}{(2\pi)^n} \int_X \Phi_P(\Theta_{E,h}).$$

Then

$$\mathrm{MAVol}_P(E) \leq \tfrac{1}{n!\, r^n}\, c_1(E)^n.$$

Main properties of the Monge-Ampère functionals

Coercivity of the Φ_P functionals

For $P = N, N^*$ or $P = (G, s)$, $s \in [r-1, \infty]$,
$\Phi_P(\bullet)$ prevents degeneration of positivity, i.e.

$$\Theta_{E,h} \geq_P 0 \text{ and } \Phi_P(\Theta_{E,h}) > 0 \text{ on } X \implies \Theta_{E,h} >_P 0.$$

Chern class inequality for Monge-Ampère volumes

For any P, we define Monge-Ampère volumes for vector bundles by

$$\mathrm{MAVol}_P(E) = \sup_{h,\,\Theta_{E,h} >_P 0} \frac{1}{(2\pi)^n} \int_X \Phi_P(\Theta_{E,h}).$$

Then

$$\mathrm{MAVol}_P(E) \leq \tfrac{1}{n!\, r^n}\, c_1(E)^n.$$

The equality occurs, with the supremum being a maximum,
if and only if E is projectively flat.

Main properties of the Monge-Ampère functionals

Coercivity of the Φ_P functionals

For $P = N, N^*$ or $P = (G, s)$, $s \in [r-1, \infty]$,
$\Phi_P(\bullet)$ prevents degeneration of positivity, i.e.

$$\Theta_{E,h} \geq_P 0 \text{ and } \Phi_P(\Theta_{E,h}) > 0 \text{ on } X \implies \Theta_{E,h} >_P 0.$$

Chern class inequality for Monge-Ampère volumes

For any P, we define Monge-Ampère volumes for vector bundles by

$$\mathrm{MAVol}_P(E) = \sup_{h,\, \Theta_{E,h} >_P 0} \frac{1}{(2\pi)^n} \int_X \Phi_P(\Theta_{E,h}).$$

Then

$$\mathrm{MAVol}_P(E) \leq \frac{1}{n!\, r^n}\, c_1(E)^n.$$

The equality occurs, with the supremum being a maximum, if and only if E is projectively flat.

Conjecture

Equality occurs for the sup iff E is numerically projectively flat.

Proof of the Chern class inequality

Take h with $\Theta_{E,h} >_P 0$, set $\omega = \Theta_{\det E,h} = \mathrm{Tr}_E\,\Theta_{E,h} > 0$, and let $(\lambda_j)_{1\le j\le nr}$ = eigenvalues of $\widetilde{\Theta}_{E,h}$ with respect to $\omega \otimes h$ on $T_X \otimes E$.

Proof of the Chern class inequality

Take h with $\Theta_{E,h} >_P 0$, set $\omega = \Theta_{\det E,h} = \mathrm{Tr}_E\,\Theta_{E,h} > 0$, and let $(\lambda_j)_{1\le j\le nr}$ = eigenvalues of $\widetilde{\Theta}_{E,h}$ with respect to $\omega\otimes h$ on $T_X\otimes E$.

The proof is a consequence of the inequality $(\prod\lambda_j)^{1/nr} \le \frac{1}{nr}\sum\lambda_j$ between geometric and arithmetic means. For Φ_N, we get

$$\frac{1}{(2\pi)^n}\int_X \Phi_N(\Theta_{E,h}) = \int_X \Big(\prod\lambda_j\Big)^{1/r}\frac{\omega^n/n!}{(2\pi)^n} \le \int_X \Big(\frac{1}{nr}\sum\lambda_j\Big)^n\frac{\omega^n/n!}{(2\pi)^n}$$

$$\le \int_X \frac{1}{n!\,r^n}\Big(\frac{1}{n}\mathrm{Tr}_\omega(\mathrm{Tr}_E\,\Theta_{E,h})\Big)^n\frac{\omega^n}{(2\pi)^n} = \frac{1}{n!\,r^n}\,c_1(E)^n.$$

Proof of the Chern class inequality

Take h with $\Theta_{E,h} >_P 0$, set $\omega = \Theta_{\det E,h} = \mathrm{Tr}_E\,\Theta_{E,h} > 0$, and let $(\lambda_j)_{1\le j\le nr}$ = eigenvalues of $\widetilde\Theta_{E,h}$ with respect to $\omega\otimes h$ on $T_X\otimes E$.

The proof is a consequence of the inequality $(\prod\lambda_j)^{1/nr} \le \frac{1}{nr}\sum\lambda_j$ between geometric and arithmetic means. For Φ_N, we get

$$\frac{1}{(2\pi)^n}\int_X \Phi_N(\Theta_{E,h}) = \int_X \Big(\prod\lambda_j\Big)^{1/r}\frac{\omega^n/n!}{(2\pi)^n} \le \int_X \Big(\frac{1}{nr}\sum\lambda_j\Big)^n\frac{\omega^n/n!}{(2\pi)^n}$$

$$\le \int_X \frac{1}{n!\,r^n}\Big(\frac{1}{n}\mathrm{Tr}_\omega(\mathrm{Tr}_E\,\Theta_{E,h})\Big)^n\frac{\omega^n}{(2\pi)^n} = \frac{1}{n!\,r^n}\,c_1(E)^n.$$

Equality occurs iff all eigenvalues λ_j are equal (and then equal to $1/r$), which means that E is projectively flat.

Proof of the Chern class inequality

Take h with $\Theta_{E,h} >_P 0$, set $\omega = \Theta_{\det E,h} = \mathrm{Tr}_E\,\Theta_{E,h} > 0$, and let $(\lambda_j)_{1\le j\le nr}$ = eigenvalues of $\widetilde\Theta_{E,h}$ with respect to $\omega\otimes h$ on $T_X\otimes E$.

The proof is a consequence of the inequality $(\prod\lambda_j)^{1/nr} \le \frac{1}{nr}\sum\lambda_j$ between geometric and arithmetic means. For Φ_N, we get

$$\frac{1}{(2\pi)^n}\int_X \Phi_N(\Theta_{E,h}) = \int_X \Big(\prod\lambda_j\Big)^{1/r}\frac{\omega^n/n!}{(2\pi)^n} \le \int_X \Big(\frac{1}{nr}\sum\lambda_j\Big)^n\frac{\omega^n/n!}{(2\pi)^n}$$

$$\le \int_X \frac{1}{n!\,r^n}\Big(\frac{1}{n}\mathrm{Tr}_\omega(\mathrm{Tr}_E\,\Theta_{E,h})\Big)^n\frac{\omega^n}{(2\pi)^n} = \frac{1}{n!\,r^n}\,c_1(E)^n.$$

Equality occurs iff all eigenvalues λ_j are equal (and then equal to $1/r$), which means that E is projectively flat.

The proof for Φ_{N^*} is the same.

Proof of the Chern class inequality

Take h with $\Theta_{E,h} >_P 0$, set $\omega = \Theta_{\det E,h} = \mathrm{Tr}_E\,\Theta_{E,h} > 0$, and let $(\lambda_j)_{1\le j\le nr}$ = eigenvalues of $\widetilde\Theta_{E,h}$ with respect to $\omega\otimes h$ on $T_X\otimes E$.

The proof is a consequence of the inequality $(\prod\lambda_j)^{1/nr} \le \frac{1}{nr}\sum\lambda_j$ between geometric and arithmetic means. For Φ_N, we get

$$\frac{1}{(2\pi)^n}\int_X \Phi_N(\Theta_{E,h}) = \int_X \Big(\prod\lambda_j\Big)^{1/r}\frac{\omega^n/n!}{(2\pi)^n} \le \int_X \Big(\frac{1}{nr}\sum\lambda_j\Big)^n\frac{\omega^n/n!}{(2\pi)^n}$$

$$\le \int_X \frac{1}{n!\,r^n}\Big(\frac{1}{n}\mathrm{Tr}_\omega(\mathrm{Tr}_E\,\Theta_{E,h})\Big)^n\frac{\omega^n}{(2\pi)^n} = \frac{1}{n!\,r^n}c_1(E)^n.$$

Equality occurs iff all eigenvalues λ_j are equal (and then equal to $1/r$), which means that E is projectively flat.

The proof for Φ_{N^*} is the same.

The proof for Φ_G is based on the concavity of the function $A\mapsto(\det A)^{1/n}$ on $(n\times n)$-hermitian matrices.

Further remarks

- In the split case $E = \bigoplus_{1 \le j \le r} L_j$ and $h = \bigoplus_{1 \le j \le r} h_j$, the inequality reads
$$\Big(\prod_{1 \le j \le r} c_1(L_j)^n\Big)^{1/r} \le r^{-n} c_1(E)^n,$$

Further remarks

- In the split case $E = \bigoplus_{1\le j\le r} L_j$ and $h = \bigoplus_{1\le j\le r} h_j$, the inequality reads
$$\Big(\prod_{1\le j\le r} c_1(L_j)^n\Big)^{1/r} \le r^{-n} c_1(E)^n,$$
with equality iff $c_1(L_1) = \cdots = c_1(L_r)$.

Further remarks

- In the split case $E = \bigoplus_{1\le j\le r} L_j$ and $h = \bigoplus_{1\le j\le r} h_j$, the inequality reads
$$\Big(\prod_{1\le j\le r} c_1(L_j)^n\Big)^{1/r} \le r^{-n} c_1(E)^n,$$
with equality iff $c_1(L_1) = \cdots = c_1(L_r)$.
- In the split case, it seems natural to conjecture that
$$\mathrm{MAVol}_P(E) = \Big(\prod_{1\le j\le r} c_1(L_j)^n\Big)^{1/r},$$
i.e. that the supremum is reached for split metrics $h = \bigoplus h_j$.

Further remarks

- In the split case $E = \bigoplus_{1\le j\le r} L_j$ and $h = \bigoplus_{1\le j\le r} h_j$, the inequality reads
$$\Big(\prod_{1\le j\le r} c_1(L_j)^n\Big)^{1/r} \le r^{-n} c_1(E)^n,$$
with equality iff $c_1(L_1) = \cdots = c_1(L_r)$.
- In the split case, it seems natural to conjecture that
$$\mathrm{MAVol}_P(E) = \Big(\prod_{1\le j\le r} c_1(L_j)^n\Big)^{1/r},$$
i.e. that the supremum is reached for split metrics $h = \bigoplus h_j$.
- We also conjecture that $\inf\limits_{h,\,\Theta_{E,h}>_P 0} \dfrac{1}{(2\pi)^n}\displaystyle\int_X \Phi_P(\Theta_{E,h}) = 0.$ (true in the split case).

Further remarks

- In the split case $E = \bigoplus_{1\le j\le r} L_j$ and $h = \bigoplus_{1\le j\le r} h_j$, the inequality reads
$$\Big(\prod_{1\le j\le r} c_1(L_j)^n\Big)^{1/r} \le r^{-n} c_1(E)^n,$$
with equality iff $c_1(L_1) = \cdots = c_1(L_r)$.
- In the split case, it seems natural to conjecture that
$$\mathrm{MAVol}_P(E) = \Big(\prod_{1\le j\le r} c_1(L_j)^n\Big)^{1/r},$$
i.e. that the supremum is reached for split metrics $h = \bigoplus h_j$.
- We also conjecture that $\displaystyle \inf_{h,\,\Theta_{E,h}>_P 0} \frac{1}{(2\pi)^n}\int_X \Phi_P(\Theta_{E,h}) = 0.$
(true in the split case).
- The Euler-Lagrange equation for the maximizer is complicated (4th order!). It somehow generalizess the 4th order differential equation characterizing cscK metrics.

Approach by Hermitian Yang-Mills equations

Let $E \to X$ be a holomorphic vector bundle such that det E is ample.

Approach by Hermitian Yang-Mills equations

Let $E \to X$ be a holomorphic vector bundle such that det E is ample.

Use of coercivity + continuity method, with "time" parameter t

Assigning for the unknown h a generalized Monge-Ampère equation

$$(*) \qquad \Phi_P(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E) = f_t > 0$$

where f_t is a positive (n,n)-form,

Approach by Hermitian Yang-Mills equations

Let $E \to X$ be a holomorphic vector bundle such that $\det E$ is ample.

Use of coercivity + continuity method, with "time" parameter t

Assigning for the unknown h a generalized Monge-Ampère equation

$$(*) \qquad \Phi_P(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E) = f_t > 0$$

where f_t is a positive (n,n)-form, may enforce the P-positivity of $\Theta_{E\otimes(\det E)^t,h}$, if that assignment is combined with a continuity technique from an initial time value $t = t_0$ for which the existence of a P-positively curved metric h is known.

Approach by Hermitian Yang-Mills equations

Let $E \to X$ be a holomorphic vector bundle such that $\det E$ is ample.

Use of coercivity + continuity method, with "time" parameter t

Assigning for the unknown h a generalized Monge-Ampère equation

$$(*) \qquad \Phi_P(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E) = f_t > 0$$

where f_t is a positive (n,n)-form, may enforce the P-positivity of $\Theta_{E\otimes(\det E)^t,h}$, if that assignment is combined with a continuity technique from an initial time value $t = t_0$ for which the existence of a P-positively curved metric h is known.

We then try to decrease t to 0, until we reach $\Theta_{E,h} >_P 0$.

Approach by Hermitian Yang-Mills equations

Let $E \to X$ be a holomorphic vector bundle such that det E is ample.

Use of coercivity + continuity method, with "time" parameter t

Assigning for the unknown h a generalized Monge-Ampère equation

$$(*) \qquad \Phi_P(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E) = f_t > 0$$

where f_t is a positive (n, n)-form, may enforce the P-positivity of $\Theta_{E\otimes(\det E)^t,h}$, if that assignment is combined with a continuity technique from an initial time value $t = t_0$ for which the existence of a P-positively curved metric h is known.

We then try to decrease t to 0, until we reach $\Theta_{E,h} >_P 0$.

Case $r = \mathrm{rank}\, E = 1$: reduction to Yau's theorem

When E is a line bundle and $h = h_0 e^{-\varphi}$, $(*)$ is equivalent to the standard Monge-Ampère equation $(\omega_0 + i\partial\overline{\partial}\varphi)^n = \widetilde{f}_t = (1+t)^{-n} f_t$ where $\omega_0 = \Theta_{E,h_0}$, which is solvable provided $(2\pi)^{-n}\int_X \widetilde{f}_t = c_1(E)^n$.

Recovering an exactly determined differential system

Problem: underdeterminacy of the equation $(*)$

For $r = \operatorname{rank} E > 1$, the equation $(*)$ amounts for only 1 scalar equation, while there are r^2 functions $(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ to determine.

Recovering an exactly determined differential system

Problem: underdeterminacy of the equation $(*)$

For $r = \mathrm{rank}\, E > 1$, the equation $(*)$ amounts for only 1 scalar equation, while there are r^2 functions $(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ to determine.
Solutions might still exist, but lack uniqueness and a priori bounds.

Recovering an exactly determined differential system

Problem: underdeterminacy of the equation $(*)$

For $r = \mathrm{rank}\, E > 1$, the equation $(*)$ amounts for only 1 scalar equation, while there are r^2 functions $(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ to determine. Solutions might still exist, but lack uniqueness and a priori bounds.

Mitigation of the problem

In order to recover a well determined system of equations, one needs an additional "matrix equation" of rank $r^2 - 1$.

Recovering an exactly determined differential system

Problem: underdeterminacy of the equation $(*)$

For $r = \mathrm{rank}\, E > 1$, the equation $(*)$ amounts for only 1 scalar equation, while there are r^2 functions $(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ to determine. Solutions might still exist, but lack uniqueness and a priori bounds.

Mitigation of the problem

In order to recover a well determined system of equations, one needs an additional "matrix equation" of rank $r^2 - 1$.

Use of a Hermite-Einstein equation (Donaldson / Uhlenbeck-Yau)

Let ω be a Kähler metric on X and $\log h$ the logarithm of the endomorphism h with respect to a fixed metric h_0 on E. Let u° the trace free part of a hermitian endomorphism u. Then $\exists! h$ such that $\det_{h_0}(h) = 1$ and $\omega^{n-1} \wedge \Theta^\circ_{E,h} = -\varepsilon \log h\, \omega^n \in \mathrm{Herm}^\circ_h(E,E)$.

Recovering an exactly determined differential system

Problem: underdeterminacy of the equation $(*)$

For $r = \mathrm{rank}\, E > 1$, the equation $(*)$ amounts for only 1 scalar equation, while there are r^2 functions $(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ to determine. Solutions might still exist, but lack uniqueness and a priori bounds.

Mitigation of the problem

In order to recover a well determined system of equations, one needs an additional "matrix equation" of rank $r^2 - 1$.

Use of a Hermite-Einstein equation (Donaldson / Uhlenbeck-Yau)

Let ω be a Kähler metric on X and $\log h$ the logarithm of the endomorphism h with respect to a fixed metric h_0 on E. Let u° the trace free part of a hermitian endomorphism u. Then $\exists! h$ such that $\det_{h_0}(h) = 1$ and $\omega^{n-1} \wedge \Theta^\circ_{E,h} = -\varepsilon \log h\, \omega^n \in \mathrm{Herm}^\circ_h(E, E)$.

This is an equation of $\mathrm{rank}_{\mathbb{R}}\ r^2 - 1$, always solvable for $\varepsilon > 0$ ...

Setup of the Yang-Mills differential system

In view of the above, we are led to considering a Yang-Mills differential system denoted (YM_t), $t \in \,]t_{\inf}, t_0]$, consisting of a scalar Monge-Ampère type equation

Setup of the Yang-Mills differential system

In view of the above, we are led to considering a Yang-Mills differential system denoted (YM_t), $t \in \,]t_{\inf}, t_0]$, consisting of a scalar Monge-Ampère type equation

$$(YM_t^\Phi) \qquad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E\big) = f_t \left(\frac{\Omega}{\omega_h^n}\right)^{\beta} \Omega,$$

where Ω is a fixed volume form on X, $\omega_h = \Theta_{\det E,h}$, $f_t \in C^\infty(X,\mathbb{R})$, $f_t > 0$, $\beta \in \mathbb{R}$;

Setup of the Yang-Mills differential system

In view of the above, we are led to considering a Yang-Mills differential system denoted (YM_t), $t \in \,]t_{\inf}, t_0]$, consisting of a scalar Monge-Ampère type equation

$$(YM_t^\Phi) \qquad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E\,\big) = f_t\left(\frac{\Omega}{\omega_h^n}\right)^\beta \Omega,$$

where Ω is a fixed volume form on X, $\omega_h = \Theta_{\det E,h}$, $f_t \in C^\infty(X,\mathbb{R})$, $f_t > 0$, $\beta \in \mathbb{R}$; we add a matrix trace free Hermite-Einstein equation

$$(YM_t^\circ) \qquad \omega_h^{n-1} \wedge \Theta^\circ_{E,h} = g_t\,\omega_h^n, \quad g_t \in C^\infty(X, \mathrm{Herm}_h^\circ(E,E)).$$

Setup of the Yang-Mills differential system

In view of the above, we are led to considering a Yang-Mills differential system denoted (YM_t), $t \in \,]t_{\inf}, t_0]$, consisting of a scalar Monge-Ampère type equation

$$(YM_t^\Phi) \qquad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E\big) = f_t \left(\frac{\Omega}{\omega_h^n}\right)^\beta \Omega,$$

where Ω is a fixed volume form on X, $\omega_h = \Theta_{\det E,h}$, $f_t \in C^\infty(X,\mathbb{R})$, $f_t > 0$, $\beta \in \mathbb{R}$; we add a matrix trace free Hermite-Einstein equation

$$(YM_t^\circ) \qquad \omega_h^{n-1} \wedge \Theta^\circ_{E,h} = g_t\,\omega_h^n, \quad g_t \in C^\infty(X, \mathrm{Herm}^\circ_h(E,E)).$$

The reason for introducing a factor $(\frac{\Omega}{\omega_h^n})^\beta$ comes from the following

Theorem 1 (D, 2021 – essentially linear algebra!)

There exist explicit distortion functions $\beta_{P,h,t}$ in $C^0(X,\mathbb{R}_+)$ s.t. for any metric h on E satisfying $\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E >_P 0$ and any $\beta > \beta_0 = \sup_X \beta_{P,h,t}$,

Setup of the Yang-Mills differential system

In view of the above, we are led to considering a Yang-Mills differential system denoted (YM_t), $t \in \,]t_{\inf}, t_0]$, consisting of a scalar Monge-Ampère type equation

$$(YM_t^\Phi) \qquad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E\big) = f_t \left(\frac{\Omega}{\omega_h^n}\right)^\beta \Omega,$$

where Ω is a fixed volume form on X, $\omega_h = \Theta_{\det E,h}$, $f_t \in C^\infty(X,\mathbb{R})$, $f_t > 0$, $\beta \in \mathbb{R}$; we add a matrix trace free Hermite-Einstein equation

$$(YM_t^\circ) \qquad \omega_h^{n-1} \wedge \Theta^\circ_{E,h} = g_t\,\omega_h^n, \quad g_t \in C^\infty(X, \mathrm{Herm}^\circ_h(E,E)).$$

The reason for introducing a factor $(\frac{\Omega}{\omega_h^n})^\beta$ comes from the following

Theorem 1 (D, 2021 – essentially linear algebra!)

There exist explicit distortion functions $\beta_{P,h,t}$ in $C^0(X,\mathbb{R}_+)$ s.t. for any metric h on E satisfying $\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E >_P 0$ and any $\beta > \beta_0 = \sup_X \beta_{P,h,t}$, the system of differential equations (YM_t) possesses an elliptic linearization in a C^2 neighborhood of h.

Expression of the distortion functions

Letting $\theta_t(h) = \Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E$ and
$\theta_t(h))^{\rm cof} =$ cofactor matrix of $\widetilde{\theta}_t(h) \in \mathrm{Herm}(T_X \otimes E)$,
the distortion functions are given explicitly at each point of X by

Expression of the distortion functions

Letting $\theta_t(h) = \Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E$ and $\theta_t(h))^{\mathrm{cof}} =$ cofactor matrix of $\widetilde{\theta}_t(h) \in \mathrm{Herm}(T_X \otimes E)$, the distortion functions are given explicitly at each point of X by

$$\beta_{N,h,t} = \frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|\theta_t(h)^{\mathrm{cof}}|}{\det\theta_t(h)}$$

Expression of the distortion functions

Letting $\theta_t(h) = \Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E$ and
$\theta_t(h))^{\mathrm{cof}} =$ cofactor matrix of $\widetilde{\theta}_t(h) \in \mathrm{Herm}(T_X \otimes E)$,
the distortion functions are given explicitly at each point of X by

$$\beta_{N,h,t} = \frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|\theta_t(h)^{\mathrm{cof}}|}{\det\theta_t(h)}$$

$$\beta_{N^*,h,t} = \frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|({}^T\theta_t(h))^{\mathrm{cof}}|}{\det({}^T\theta_t(h))},$$

Expression of the distortion functions

Letting $\theta_t(h) = \Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E$ and
$\theta_t(h))^{\mathrm{cof}} =$ cofactor matrix of $\widetilde{\theta}_t(h) \in \mathrm{Herm}(T_X \otimes E)$,
the distortion functions are given explicitly at each point of X by

$$\beta_{N,h,t} = \frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|\theta_t(h)^{\mathrm{cof}}|}{\det\theta_t(h)}$$

$$\beta_{N^*,h,t} = \frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|({}^T\theta_t(h))^{\mathrm{cof}}|}{\det({}^T\theta_t(h))},$$

$$\beta_{G,s,h,t} = (\sqrt{n-1}+1)\,|\Theta^\circ_{E,h}|$$
$$\times\left(\int_{v\in E,\ |v|_h=1}\frac{d\sigma(v)}{\big((\langle\theta_t(h)\cdot v,v\rangle_h)^n\big)^s}\right)^{-1}$$
$$\times\int_{v\in E,\ |v|_h=1}\frac{n\,(\langle\theta_t(h)\cdot v,v\rangle_h)^{n-1}\wedge\omega_h\,d\sigma(v)}{\big((\langle\theta_t(h)\cdot v,v\rangle_h)^n\big)^{s+1}}$$

where $\omega_h = \Theta_{\det E,\det h}$.

but we need ellipticity _and_ local invertibility . . .

Local invertibility of the linearized elliptic operator is needed to apply the implicit function theorem and get openness for solutions.

but we need ellipticity _and_ local invertibility . . .

Local invertibility of the linearized elliptic operator is needed to apply the implicit function theorem and get openness for solutions.

Theorem 2 (D, 2021 – local openness of existence for solutions)

Consider the more specific Yang-Mills system (YM_t), $t \in\,]t_{\min}, t_0]$

$$(YM_t^\Phi) \quad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E\big) = \left(\frac{\det h_{t_0}}{\det h}\right)^\lambda \left(\frac{\Omega}{\omega_h^n}\right)^\beta \Omega,$$

but we need ellipticity _and_ local invertibility . . .

Local invertibility of the linearized elliptic operator is needed to apply the implicit function theorem and get openness for solutions.

Theorem 2 (D, 2021 – local openness of existence for solutions)

Consider the more specific Yang-Mills system (YM_t), $t \in\,]t_{\min}, t_0]$

$$(YM_t^\Phi) \quad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E\big) = \left(\frac{\det h_{t_0}}{\det h}\right)^\lambda \left(\frac{\Omega}{\omega_h^n}\right)^\beta \Omega,$$

$$(YM_t^\circ) \quad \omega_h^{-n}\,(\omega_h^{n-1} \wedge \Theta_{E,h}^\circ) = -\varepsilon\, A(\det h)\,(\log h)^\circ,$$

where $A > 0$ is any C^∞ functional, and $\log h$ is computed with respect to the initial metric h_{t_0}.

but we need ellipticity _and_ local invertibility . . .

Local invertibility of the linearized elliptic operator is needed to apply the implicit function theorem and get openness for solutions.

Theorem 2 (D, 2021 – local openness of existence for solutions)

Consider the more specific Yang-Mills system (YM_t), $t \in \,]t_{\min}, t_0]$

$$(YM_t^\Phi) \quad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E\big) = \left(\frac{\det h_{t_0}}{\det h}\right)^{\lambda} \left(\frac{\Omega}{\omega_h^n}\right)^{\beta} \Omega,$$

$$(YM_t^\circ) \quad \omega_h^{-n}\,(\omega_h^{n-1} \wedge \Theta_{E,h}^\circ) = -\varepsilon\, A(\det h)\,(\log h)^\circ,$$

where $A > 0$ is any C^∞ functional, and $\log h$ is computed with respect to the initial metric h_{t_0}.

Then there exist bounds $\beta_0 := \sup_X \beta_{P,t,h}$, $\varepsilon_0(A,\beta)$ and $\lambda_0(\beta)$ such that for any choice of constants

$$\beta > \beta_0,\ \varepsilon > \varepsilon_0(A,\beta) \text{ and } \lambda > \lambda_0(\beta),$$

the system (YM_t) possesses an invertible elliptic linearization.

Very rough sketch of proof of ellipticity/invertibility

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\overline{\partial}(h^{-1}\partial h) = i\overline{\partial}(\widetilde{h}^{-1}\partial_{H_0}\widetilde{h}),$$

where $\partial_{H_0} s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection on $\mathrm{Hom}(E,E)$ associated with $H_0 = h_{t_0}$ on E.

Very rough sketch of proof of ellipticity/invertibility

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\overline{\partial}(h^{-1}\partial h) = i\overline{\partial}(\widetilde{h}^{-1}\partial_{H_0}\widetilde{h}),$$

where $\partial_{H_0}s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection on $\mathrm{Hom}(E,E)$ associated with $H_0 = h_{t_0}$ on E.

Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \to C^\infty(W), \quad f \mapsto P(f) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\,\xi^\alpha \in \mathrm{Hom}(V,W)$$

whenever $0 \neq \xi \in T^*_{X,x}$.

Very rough sketch of proof of ellipticity/invertibility

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\overline{\partial}(h^{-1}\partial h) = i\overline{\partial}(\widetilde{h}^{-1}\partial_{H_0}\widetilde{h}),$$

where $\partial_{H_0} s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection on $\mathrm{Hom}(E,E)$ associated with $H_0 = h_{t_0}$ on E.

Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \to C^\infty(W), \quad f \mapsto P(f) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\,\xi^\alpha \in \mathrm{Hom}(V,W)$$

whenever $0 \neq \xi \in T^*_{X,x}$.

For instance, on the torus $\mathbb{R}^n/\mathbb{Z}^n$, $f \mapsto P_\lambda(f) = -\Delta f + \lambda f$ has an invertible symbol $\sigma_{P_\lambda}(x,\xi) = -|\xi|^2$, but P_λ is invertible only when λ avoids the eigenvalues of Δ, e.g. when $\lambda > 0$.

Important remaining points . . .

- We have been able to set-up a Yang-Mills differential system (YM_t) that is elliptic invertible, and ensures the existence of an open time interval $]t_1, t_0]$ for which we have uniqueness of the solution.

Important remaining points . . .

- We have been able to set-up a Yang-Mills differential system (YM_t) that is elliptic invertible, and ensures the existence of an open time interval $]t_1, t_0]$ for which we have uniqueness of the solution.
- We somehow know that the solution persists unless some distortion occurs (in the sense that $\sup_X \beta_{P,h,t} \to +\infty$, or the trace free part ratio $|\Theta^\circ_{E,h}|/(1 + |\log h|)$ explodes at t_1).

Important remaining points . . .

- We have been able to set-up a Yang-Mills differential system (YM_t) that is elliptic invertible, and ensures the existence of an open time interval $]t_1, t_0]$ for which we have uniqueness of the solution.
- We somehow know that the solution persists unless some distortion occurs (in the sense that $\sup_X \beta_{P,h,t} \to +\infty$, or the trace free part ratio $|\Theta^\circ_{E,h}|/(1 + |\log h|)$ explodes at t_1).
- The latter point might possibly be used (as in the work of Uhlenbeck-Yau) to get suitable destabilizing subsheaves, that would e.g. contradict the ampleness assumption if $P = G$ and $t_1 \geq 0$.

Important remaining points . . .

- We have been able to set-up a Yang-Mills differential system (YM_t) that is elliptic invertible, and ensures the existence of an open time interval $]t_1, t_0]$ for which we have uniqueness of the solution.
- We somehow know that the solution persists unless some distortion occurs (in the sense that $\sup_X \beta_{P,h,t} \to +\infty$, or the trace free part ratio $|\Theta^\circ_{E,h}|/(1 + |\log h|)$ explodes at t_1).
- The latter point might possibly be used (as in the work of Uhlenbeck-Yau) to get suitable destabilizing subsheaves, that would e.g. contradict the ampleness assumption if $P = G$ and $t_1 \geq 0$.
- A natural question is whether one can arrange that the infimum $t_{\inf}$ of times t for which (YM_t) has a solution coincides with the positivity threshold $\tau_P(E)$, in the case of P-positivity.

Important remaining points . . .

- We have been able to set-up a Yang-Mills differential system (YM_t) that is elliptic invertible, and ensures the existence of an open time interval $]t_1, t_0]$ for which we have uniqueness of the solution.
- We somehow know that the solution persists unless some distortion occurs (in the sense that $\sup_X \beta_{P,h,t} \to +\infty$, or the trace free part ratio $|\Theta^\circ_{E,h}|/(1 + |\log h|)$ explodes at t_1).
- The latter point might possibly be used (as in the work of Uhlenbeck-Yau) to get suitable destabilizing subsheaves, that would e.g. contradict the ampleness assumption if $P = G$ and $t_1 \geq 0$.
- A natural question is whether one can arrange that the infimum $t_{\inf}$ of times t for which (YM_t) has a solution coincides with the positivity threshold $\tau_P(E)$, in the case of P-positivity. For this, we would probably need uniform a priori estimates . . .

On the Fulton Lazarsfeld inequalities

A fundamental result due to Fulton-Lazarsfeld asserts that if $E \to X$ is an ample vector bundle, then all Schure polynomials $P(c_\bullet(E))$ in the Chern classes are numerically positive, i.e.

$$\int_Y P(c_\bullet(E)) > 0$$

for all irreducible cycles Y of the appropriate dimension in X.

On the Fulton Lazarsfeld inequalities

A fundamental result due to Fulton-Lazarsfeld asserts that if $E \to X$ is an ample vector bundle, then all Schure polynomials $P(c_\bullet(E))$ in the Chern classes are numerically positive, i.e.

$$\int_Y P(c_\bullet(E)) > 0$$

for all irreducible cycles Y of the appropriate dimension in X.

Recently, Siarhei Finski has shown

Theorem (Finski 2020)

If (E, h) is a (dual) Nakano positive vector bundle, then all Schur polynomials $P(c_\bullet(E, h))$ in the Chern forms are pointwise positive (k, k)-forms (in the sense of the weak positivity of forms).

On the Fulton Lazarsfeld inequalities

A fundamental result due to Fulton-Lazarsfeld asserts that if $E \to X$ is an ample vector bundle, then all Schure polynomials $P(c_\bullet(E))$ in the Chern classes are numerically positive, i.e.

$$\int_Y P(c_\bullet(E)) > 0$$

for all irreducible cycles Y of the appropriate dimension in X.

Recently, Siarhei Finski has shown

Theorem (Finski 2020)

If (E, h) is a (dual) Nakano positive vector bundle, then all Schur polynomials $P(c_\bullet(E, h))$ in the Chern forms are pointwise positive (k, k)-forms (in the sense of the weak positivity of forms).

This is a compelling motivation to investigate the various types of positivity for vector bundles.

Further recent results by Siarhei Finski

When $E \to X$ is an ample vector bundle, the symmetric powers $S^m E$ have enough sections to generate 1-jets for $m \geq m_0 \gg 1$, and one can immediately derive from there that

$$E \text{ ample} \Rightarrow S^m E \text{ dual-Nakano positive for } m \geq m_0 \gg 1.$$

Further recent results by Siarhei Finski

When $E \to X$ is an ample vector bundle, the symmetric powers $S^m E$ have enough sections to generate 1-jets for $m \geq m_0 \gg 1$, and one can immediately derive from there that

$$E \text{ ample} \Rightarrow S^m E \text{ dual-Nakano positive for } m \geq m_0 \gg 1.$$

Then it makes sense to wonder whether there is an asymptotic formula for the monge-Ampère volume $\mathrm{MAVol}_P(S^m E)$.

Further recent results by Siarhei Finski

When $E \to X$ is an ample vector bundle, the symmetric powers $S^m E$ have enough sections to generate 1-jets for $m \geq m_0 \gg 1$, and one can immediately derive from there that

$$E \text{ ample} \Rightarrow S^m E \text{ dual-Nakano positive for } m \geq m_0 \gg 1.$$

Then it makes sense to wonder whether there is an asymptotic formula for the monge-Ampère volume $\mathrm{MAVol}_P(S^m E)$.
S. Finski obtained more generally an asymptotic formula for the Monge-Ampère volume of direct images $E_m = \pi_*(L^m \otimes G)$ by any proper morphism $\pi : Y \to X$ of any line bundle $(L, h_L) > 0$ on Y.

Further recent results by Siarhei Finski

When $E \to X$ is an ample vector bundle, the symmetric powers $S^m E$ have enough sections to generate 1-jets for $m \geq m_0 \gg 1$, and one can immediately derive from there that

$$E \text{ ample} \Rightarrow S^m E \text{ dual-Nakano positive for } m \geq m_0 \gg 1.$$

Then it makes sense to wonder whether there is an asymptotic formula for the monge-Ampère volume $\mathrm{MAVol}_P(S^m E)$.
S. Finski obtained more generally an asymptotic formula for the Monge-Ampère volume of direct images $E_m = \pi_*(L^m \otimes G)$ by any proper morphism $\pi : Y \to X$ of any line bundle $(L, h_L) > 0$ on Y.

Theorem (S. Finski 2020)

Given any volume form $d\nu$ on X, the direct images satisfy

$$\mathrm{MAVol}_{N^*}(E_m, h_{E_m}) \sim m^{\dim X} \int_X \exp\left(\frac{\int_Y \log\left(\omega_H^{\dim X} / \pi^*\nu\right) \omega^{\dim Y}}{\int_Y c_1(L)^{\dim Y}} \right) d\nu,$$

where $\omega = \Theta_{L,h_L} > 0$ on Y, and ω_H is its horizontal part.

Best wishes Kang-Tae !

References

[Ber09] Berndtsson B.: *Curvature of vector bundles associated to holomorphic fibrations*, Annals of Math. **169** (2009), 531—560.

[Dem20] Demailly J.-P: *Hermitian-Yang-Mills approach to the conjecture of Griffiths on the positivity of ample vector bundles*, Mat. Sbornik, **212** Number 3 (2021) 39–53, `https://doi.org/10.4213/sm9387`.

[Dem21] Demailly J.-P: *Monge-Ampère functionals for the curvature tensor of a holomorphic vector bundle*, to appear in Analysis Math. 2022, vol. in honor of László Lempert.

[Fin20a] Finski, S. *On characteristic forms of positive vector bundles, mixed discriminants and pushforward identities*, arXiv:2009.13107.

References (continued)

[Fin20b] Finski, S. *On Monge-Ampère volumes of direct images* arXiv:2010.01839.

[LSY13] Liu K., Sun X., Yang, X.: *Positivity and vanishing theorems for ample vector bundles*, J. Alg. Geom. **22** (2013) 303—331.

[Pin20] Pingali, V.P.: *A vector bundle version of the Monge-Ampère equation*, Adv. in Math. **360** (2020), 40 pages, https://doi.org/10.1016/j.aim.2019.106921.

[UhY86] Uhlenbeck, K., Yau, S.T.: *On the existence of Hermitian-Yang-Mills connections in stable vector bundles*, Comm. Pure and Appl. Math. **39** (1986) 258–293.

[Ume73] Umemura, H.: *Some results in the theory of vector bundles*, Nagoya Math. J. **52** (1973), 97–128.