Monge-Ampère functionals for the curvature tensor of a holomorphic vector bundle

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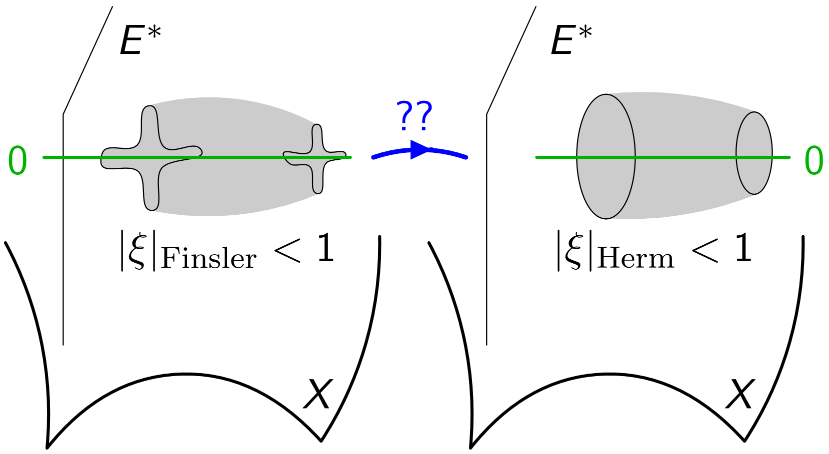
Plan of the talk

1. Positivity concepts for holomorphic vector bundles
2. Monge-Ampère functionals for vector bundles
3. Chern class inequalities for Monge-Ampère volumes
4. A Hermitian-Yang-Mills approach to the Griffiths conjecture



5. Further results by Siarhei Finski

Positive and ample vector bundles

Let X be a projective n-dimensional manifold and $E \to X$ a holomorphic vector bundle of rank $r \geq 1$.

Ample vector bundles

$E \to X$ is said to be ample in the sense of Hartshorne if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on the hyperplane bundle $\mathbb{P}(E)$ is ample.

By Kodaira (1954), this is equivalent to the existence of a smooth hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with positive curvature (equivalently, a negatively curved Finsler metric on E^*).

Chern curvature tensor of a hermitian bundle (E, h)

This is $\Theta_{E,h} = i\nabla^2_{E,h} \in C^\infty(\Lambda^{1,1}T^*_X \otimes \mathrm{Hom}(E,E))$, which can be written

$$\Theta_{E,h} = i \sum_{1\leq j,k\leq n,\, 1\leq\lambda,\mu\leq r} c_{jk\lambda\mu} dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu$$

in terms of an orthonormal frame $(e_\lambda)_{1\leq\lambda\leq r}$ of E.

Griffiths positivity concept for vector bundles

Definition

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1\leq j,k\leq n,\, 1\leq\lambda,\mu\leq r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu.$$

Then E is said to be Griffiths positive (Griffiths 1969) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall 0 \neq \xi \in T_{X,z},\ \forall 0 \neq v \in E_z$$

Well known fact

E Griffiths $> 0 \Rightarrow E$ ample.

Proof. E Griffiths $> 0 \Rightarrow \mathcal{O}_{\mathbb{P}(E)}(1) > 0 \underset{\text{Kodaira}}{\Longleftrightarrow} \mathcal{O}_{\mathbb{P}(E)}(1)$ ample.

Griffiths conjecture [unsolved, except for $n = 1$ (Umemura 1973)]

Is it true that E ample $\Rightarrow E$ Griffiths > 0 ? (If so, both are $\Leftrightarrow$).

Nakano / dual Nakano positivity concepts

The curvature tensor yields a natural hermitian form on $T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu}, \quad \tau \in T_{X,z}\otimes E_z.$$

Definition of Nakano positivity

E is Nakano positive (Nakano 1955) if at every point $z \in X$

$$\widetilde{\Theta}_{E,h}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0, \quad \forall\tau \in T_{X,z}\otimes E_z,\ \tau\neq 0.$$

Curvature tensor of the dual bundle E^*

$$\Theta_{E^*,h^*} = -{}^T\Theta_{E,h} = -\sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}dz_j\wedge d\overline{z}_k\otimes(e_\lambda^*)^*\otimes e_\mu^*.$$

Definition of dual Nakano positivity

E is dual Nakano positive if E^* is Nakano <0, i.e.

$$-\widetilde{\Theta}_{E^*,h^*}(\tau) = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\mu\lambda}\tau_{j\lambda}\overline{\tau}_{k\mu} > 0, \quad \forall\tau \in T_{X,z}\otimes E_z^*,\ \tau\neq 0.$$

Known results

- Nakano and dual Nakano positivity imply Griffiths positivity.
- Griffiths and dual Nakano Nakano positivity are preserved by taking quotients: $E>0 \Rightarrow$ any quotient $Q=E/S$ is also >0.
  This is wrong for Nakano positivity.
- E ample $\not\Rightarrow$ E Nakano >0.
  For instance, $T_{\mathbb{P}^n}$ is ample and even Griffiths >0 for the Fubini-Study metric, but it is not Nakano >0. Otherwise the Nakano vanishing theorem would imply

$$H^{n-1,n-1}(\mathbb{P}^n,\mathbb{C}) = H^{n-1}(\mathbb{P}^n,\Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n}\otimes T_{\mathbb{P}^n}) = 0 \ \ !!!$$

- E ample $\not\Rightarrow$ E dual Nakano >0.
  For instance, any compact quotient $X=\mathbb{B}^n/\Gamma$ has T^*_X ample and even Griffiths >0 for the hyperbolic metric, but T^*_X is not dual Nakano >0, otherwise T_X would be Nakano <0 and

$$H^{1,0}(X,\mathbb{C}) = H^0(X,\Omega^1_X\otimes T_X) = H^0(X,\mathrm{Hom}(T_X,T_X)) \ni \mathrm{Id}_{T_X}$$

  would contradict the (dual) Nakano vanishing theorem.

Positivity thresholds

There are subtle relations between the various positivity concepts.

Theorem (Berndtsson 2009)

E ample $\Rightarrow S^m E \otimes \det E$ Nakano > 0 for every $m \in \mathbb{N}$.

Theorem (Liu-Sun-Yang 2013)

E ample $\Rightarrow S^m E \otimes \det E$ dual Nakano > 0 for every $m \in \mathbb{N}$.

This leads in a natural way to the following definition.

Definition

Let $P = A, G, N, N^*$ mean the Ampleness / Griffiths / Nakano / dual Nakano positivity concepts. Let $E \to X$ be a vector bundle such that $\det E$ is ample. We let

$$\tau_P(E) = \inf \left\{ t \in \mathbb{R}\,;\ E \otimes (\det E)^t >_P 0 \right\}.$$

Remark. $\Theta_{E\otimes(\det E)^t} = \Theta_E + t\,\Theta_{\det E} \otimes \mathrm{Id}_E, \quad \Theta_{\det E} = \mathrm{Tr}_E\,\Theta_E.$

Simple facts about positivity thresholds

Notice that Nakano and dual Nakano positivity are stronger than Griffiths positivity, the latter being itself stronger than ampleness, hence we always have

$$\tau_N(E) \geq \tau_G(E) \geq \tau_A(E), \quad \tau_{N^*}(E) \geq \tau_G(E) \geq \tau_A(E).$$

Moreover, since $E \otimes (\det E)^{-1/r}$ has trivial determinant, we also have $\tau_A(E) \geq -1/r$.

Proposition

One has $\tau_A(E) = -1/r \Leftrightarrow F = E \otimes (\det E)^{-1/r}$ is numerically flat (i.e. F, F^* both nef), so that $E = F \otimes L$ where $L = (\det E)^{1/r}$ is ample: we say that E is projectively numerically flat. Then

$$\tau_N(E) = \tau_{N^*}(E) = \tau_G(E) = \tau_A(E) = -\tfrac{1}{r}.$$

Remark

The Griffiths conjecture is equivalent to: E ample $\Rightarrow \tau_G(E) < 0$.

Monge-Ampère functionals for vector bundles

Definition of the functionals, $\Theta_{E,h} \mapsto$ volume (n,n)-form on X :

– If $E >_N 0$, we set $\Phi_N(\Theta_{E,h}) := \det_{T_X\otimes E}(\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_N(\Theta_{E,h}) := \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1\wedge d\overline{z}_1\wedge\ldots\wedge idz_n\wedge d\overline{z}_n.$$

– If $E >_{N^*} 0$, we set $\Phi_{N^*}(\Theta_{E,h}) := \det_{T_X\otimes E^*}({}^T\Theta_{E,h})^{1/r}$, i.e.

$$\Phi_{N^*}(\Theta_{E,h}) := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1\wedge d\overline{z}_1\wedge\ldots\wedge idz_n\wedge d\overline{z}_n.$$

– If $E >_G 0$, we set

$$\Phi_G(\Theta_{E,h}) := \inf_{|v|_h=1}\langle\Theta_{E,h}\cdot v,v\rangle^n \quad \text{(not differentiable)},$$

$$\Phi_{G,s}(\Theta_{E,h}) := \Big(\int_{|v|_h=1}(\langle\Theta_{E,h}\cdot v,v\rangle^n)^{-s}\,d\sigma(v)\Big)^{-1/s} \underset{s\to+\infty}{\longrightarrow} \Phi_G(\Theta_{E,h}).$$

These (n,n)-forms are intrinsic: they do not depend on the choice of coordinates (z_j) on X, nor on the choice of the orthonormal frame (e_λ) on E.

Main properties of the Monge-Ampère functionals

Coercivity of the Φ_P functionals

For $P = N, N^*$ or $P = (G,s)$, $s\in[r-1,\infty]$,
$\Phi_P(\bullet)$ prevents degeneration of positivity, i.e.

$$\Theta_{E,h}\geq_P 0 \text{ and } \Phi_P(\Theta_{E,h}) > 0 \text{ on } X \implies \Theta_{E,h} >_P 0.$$

Chern class inequality for Monge-Ampère volumes

For any P, we define Monge-Ampère volumes for vector bundles by

$$\mathrm{MAVol}_P(E) = \sup_{h,\,\Theta_{E,h}>_P 0}\frac{1}{(2\pi)^n}\int_X \Phi_P(\Theta_{E,h}).$$

Then

$$\mathrm{MAVol}_P(E) \leq \tfrac{1}{n!\,r^n}\, c_1(E)^n.$$

The equality occurs, with the supremum being a maximum, if and only if E is projectively flat.

Conjecture

Equality occurs for the sup iff E is numerically projectively flat.

Proof of the Chern class inequality

Take h with $\Theta_{E,h} >_P 0$, set $\omega = \Theta_{\det E,h} = \mathrm{Tr}_E\,\Theta_{E,h} > 0$, and let $(\lambda_j)_{1\le j\le nr}$ = eigenvalues of $\widetilde\Theta_{E,h}$ with respect to $\omega\otimes h$ on $T_X\otimes E$.

The proof is a consequence of the inequality $(\prod\lambda_j)^{1/nr} \le \frac{1}{nr}\sum\lambda_j$ between geometric and arithmetic means. For Φ_N, we get

$$\frac{1}{(2\pi)^n}\int_X \Phi_N(\Theta_{E,h}) = \int_X \Big(\prod\lambda_j\Big)^{1/r}\frac{\omega^n/n!}{(2\pi)^n} \le \int_X\Big(\frac{1}{nr}\sum\lambda_j\Big)^n\frac{\omega^n/n!}{(2\pi)^n}$$
$$\le \int_X \frac{1}{n!\,r^n}\Big(\frac{1}{n}\mathrm{Tr}_\omega(\mathrm{Tr}_E\,\Theta_{E,h})\Big)^n\frac{\omega^n}{(2\pi)^n} = \frac{1}{n!\,r^n}\,c_1(E)^n.$$

Equality occurs iff all eigenvalues λ_j are equal (and then equal to $1/r$), which means that E is projectively flat.

The proof for Φ_{N^*} is the same.

The proof for Φ_G is based on the concavity of the function $A\mapsto(\det A)^{1/n}$ on $(n\times n)$-hermitian matrices.

Further remarks

- In the split case $E=\bigoplus_{1\le j\le r}L_j$ and $h=\bigoplus_{1\le j\le r}h_j$, the inequality reads
$$\Big(\prod_{1\le j\le r}c_1(L_j)^n\Big)^{1/r}\le r^{-n}c_1(E)^n,$$
with equality iff $c_1(L_1)=\cdots=c_1(L_r)$.
- In the split case, it seems natural to conjecture that
$$\mathrm{MAVol}_P(E)=\Big(\prod_{1\le j\le r}c_1(L_j)^n\Big)^{1/r},$$
i.e. that the supremum is reached for split metrics $h=\bigoplus h_j$.
- We also conjecture that $\inf_{h,\,\Theta_{E,h}>_P0}\frac{1}{(2\pi)^n}\int_X\Phi_P(\Theta_{E,h})=0$. (true in the split case).
- The Euler-Lagrange equation for the maximizer is complicated (4th order!). It somehow generalizess the 4th order differential equation characterizing cscK metrics.

Approach by Hermitian Yang-Mills equations

Let $E \to X$ be a holomorphic vector bundle such that det E is ample.

Use of coercivity + continuity method, with "time" parameter t

Assigning for the unknown h a generalized Monge-Ampère equation

$$(*) \qquad \Phi_P(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E) = f_t > 0$$

where f_t is a positive (n,n)-form, may enforce the P-positivity of $\Theta_{E\otimes(\det E)^t,h}$, if that assignment is combined with a continuity technique from an initial time value $t = t_0$ for which the existence of a P-positively curved metric h is known.

We then try to decrease t to 0, until we reach $\Theta_{E,h} >_P 0$.

Case $r = \mathrm{rank}\, E = 1$: reduction to Yau's theorem

When E is a line bundle and $h = h_0 e^{-\varphi}$, $(*)$ is equivalent to the standard Monge-Ampère equation $(\omega_0 + i\partial\overline{\partial}\varphi)^n = \widetilde{f}_t = (1+t)^{-n} f_t$ where $\omega_0 = \Theta_{E,h_0}$, which is solvable provided $(2\pi)^{-n}\int_X \widetilde{f}_t = c_1(E)^n$.

Recovering an exactly determined differential system

Problem: underdeterminacy of the equation $(*)$

For $r = \mathrm{rank}\, E > 1$, the equation $(*)$ amounts for only 1 scalar equation, while there are r^2 functions $(h_{\lambda\mu})_{1\le\lambda,\mu\le r}$ to determine. Solutions might still exist, but lack uniqueness and a priori bounds.

Mitigation of the problem

In order to recover a well determined system of equations, one needs an additional "matrix equation" of rank $r^2 - 1$.

Use of a Hermite-Einstein equation (Donaldson / Uhlenbeck-Yau)

Let ω be a Kähler metric on X and $\log h$ the logarithm of the endomorphism h with respect to a fixed metric h_0 on E. Let u° the trace free part of a hermitian endomorphism u. Then $\exists! h$ such that $\det_{h_0}(h) = 1$ and $\omega^{n-1} \wedge \Theta^\circ_{E,h} = -\varepsilon \log h\, \omega^n \in \mathrm{Herm}^\circ_h(E,E)$.

This is an equation of $\mathrm{rank}_{\mathbb{R}}\, r^2 - 1$, always solvable for $\varepsilon > 0$ ...

Setup of the Yang-Mills differential system

In view of the above, we are led to considering a Yang-Mills differential system denoted (YM_t), $t \in\,]t_{\rm inf}, t_0]$, consisting of a scalar Monge-Ampère type equation

$$(YM_t^\Phi) \qquad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h}\otimes \mathrm{Id}_E\big) = f_t\left(\frac{\Omega}{\omega_h^n}\right)^\beta \Omega,$$

where Ω is a fixed volume form on X, $\omega_h = \Theta_{\det E,h}$, $f_t \in C^\infty(X,\mathbb{R})$, $f_t > 0$, $\beta \in \mathbb{R}$; we add a matrix trace free Hermite-Einstein equation

$$(YM_t^\circ) \qquad \omega_h^{n-1}\wedge \Theta^\circ_{E,h} = g_t\,\omega_h^n, \quad g_t \in C^\infty(X, \mathrm{Herm}^\circ_h(E,E)).$$

The reason for introducing a factor $(\frac{\Omega}{\omega_h^n})^\beta$ comes from the following

Theorem 1 (D, 2021 – essentially linear algebra!)

There exist explicit distortion functions $\beta_{P,h,t}$ in $C^0(X,\mathbb{R}_+)$ s.t. for any metric h on E satisfying $\Theta_{E,h} + t\,\Theta_{\det E,\det h}\otimes \mathrm{Id}_E >_P 0$ and any $\beta > \beta_0 = \sup_X \beta_{P,h,t}$, the system of differential equations (YM_t) possesses an elliptic linearization in a C^2 neighborhood of h.

Expression of the distortion functions

Letting $\theta_t(h) = \Theta_{E,h} + t\,\Theta_{\det E,\det h}\otimes \mathrm{Id}_E$ and $\theta_t(h))^{\rm cof}$ = cofactor matrix of $\widetilde{\theta}_t(h) \in \mathrm{Herm}(T_X\otimes E)$, the distortion functions are given explicitly at each point of X by

$$\beta_{N,h,t} = \frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|\theta_t(h)^{\rm cof}|}{\det\theta_t(h)}$$

$$\beta_{N^*,h,t} = \frac{\sqrt{n-1}+1}{r}\,\frac{|\Theta^\circ_{E,h}|\,|({}^T\theta_t(h))^{\rm cof}|}{\det({}^T\theta_t(h))},$$

$$\begin{aligned}\beta_{G,s,h,t} = {} & (\sqrt{n-1}+1)\,|\Theta^\circ_{E,h}| \\ & \times\left(\int_{v\in E,\ |v|_h=1}\frac{d\sigma(v)}{((\langle\theta_t(h)\cdot v,v\rangle_h)^n)^s}\right)^{-1} \\ & \times\int_{v\in E,\ |v|_h=1}\frac{n\,(\langle\theta_t(h)\cdot v,v\rangle_h)^{n-1}\wedge\omega_h\,d\sigma(v)}{((\langle\theta_t(h)\cdot v,v\rangle_h)^n)^{s+1}}\end{aligned}$$

where $\omega_h = \Theta_{\det E,\det h}$.

but we need ellipticity _and_ local invertibility . . .

Local invertibility of the linearized elliptic operator is needed to apply the implicit function theorem and get openness for solutions.

Theorem 2 (D, 2021 – local openness of existence for solutions)

Consider the more specific Yang-Mills system (YM_t), $t \in\,]t_{\min}, t_0]$

$$(YM_t^\Phi) \qquad \Phi_P\big(\Theta_{E,h} + t\,\Theta_{\det E,\det h} \otimes \mathrm{Id}_E\big) = \Big(\frac{\det h_{t_0}}{\det h}\Big)^\lambda \Big(\frac{\Omega}{\omega_h^n}\Big)^\beta \Omega,$$

$$(YM_t^\circ) \qquad \omega_h^{-n}\,(\omega_h^{n-1} \wedge \Theta_{E,h}^\circ) = -\varepsilon\, A(\det h)\,(\log h)^\circ,$$

where $A > 0$ is any C^∞ functional, and $\log h$ is computed with respect to the initial metric h_{t_0}.

Then there exist bounds $\beta_0 := \sup_X \beta_{P,t,h}$, $\varepsilon_0(A,\beta)$ and $\lambda_0(\beta)$ such that for any choice of constants

$$\beta > \beta_0,\ \varepsilon > \varepsilon_0(A,\beta) \text{ and } \lambda > \lambda_0(\beta),$$

the system (YM_t) possesses an invertible elliptic linearization.

Very rough sketch of proof of ellipticity/invertibility

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\overline\partial(h^{-1}\partial h) = i\overline\partial(\widetilde h^{-1}\partial_{H_0}\widetilde h),$$

where $\partial_{H_0}s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection on $\mathrm{Hom}(E,E)$ associated with $H_0 = h_{t_0}$ on E.

Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \to C^\infty(W), \quad f \mapsto P(f) = \textstyle\sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x,\xi) = \textstyle\sum_{|\alpha|=m} a_\alpha(x)\,\xi^\alpha \in \mathrm{Hom}(V,W)$$

whenever $0 \ne \xi \in T^*_{X,x}$.

For instance, on the torus $\mathbb{R}^n/\mathbb{Z}^n$, $f \mapsto P_\lambda(f) = -\Delta f + \lambda f$ has an invertible symbol $\sigma_{P_\lambda}(x,\xi) = -|\xi|^2$, but P_λ is invertible only when λ avoids the eigenvalues of Δ, e.g. when $\lambda > 0$.

Important remaining points . . .

- We have been able to set-up a Yang-Mills differential system (YM_t) that is elliptic invertible, and ensures the existence of an open time interval $]t_1, t_0]$ for which we have uniqueness of the solution.
- We somehow know that the solution persists unless some distortion occurs (in the sense that $\sup_X \beta_{P,h,t} \to +\infty$, or the trace free part ratio $|\Theta^\circ_{E,h}|/(1 + |\log h|)$ explodes at t_1).
- The latter point might possibly be used (as in the work of Uhlenbeck-Yau) to get suitable destabilizing subsheaves, that would e.g. contradict the ampleness assumption if $P = G$ and $t_1 \geq 0$.
- A natural question is whether one can arrange that the infimum $t_{\rm inf}$ of times t for which (YM_t) has a solution coincides with the positivity threshold $\tau_P(E)$, in the case of P-positivity. For this, we would probably need uniform a priori estimates . . .

On the Fulton Lazarsfeld inequalities

A fundamental result due to Fulton-Lazarsfeld asserts that if $E \to X$ is an ample vector bundle, then all Schure polynomials $P(c_\bullet(E))$ in the Chern classes are numerically positive, i.e.

$$\int_Y P(c_\bullet(E)) > 0$$

for all irreducible cycles Y of the appropriate dimension in X.

Recently, Siarhei Finski has shown

Theorem (Finski 2020)

If (E, h) is a (dual) Nakano positive vector bundle, then all Schur polynomials $P(c_\bullet(E, h))$ in the Chern forms are pointwise positive (k, k)-forms (in the sense of the weak positivity of forms).

This is a compelling motivation to investigate the various types of positivity for vector bundles.

Further recent results by Siarhei Finski

When $E \to X$ is an ample vector bundle, the symmetric powers $S^m E$ have enough sections to generate 1-jets for $m \geq m_0 \gg 1$, and one can immediately derive from there that

$$E \text{ ample} \Rightarrow S^m E \text{ dual-Nakano positive for } m \geq m_0 \gg 1.$$

Then it makes sense to wonder whether there is an asymptotic formula for the monge-Ampère volume $\mathrm{MAVol}_P(S^m E)$.
S. Finski obtained more generally an asymptotic formula for the Monge-Ampère volume of direct images $E_m = \pi_*(L^m \otimes G)$ by any proper morphism $\pi : Y \to X$ of any line bundle $(L, h_L) > 0$ on Y.

Theorem (S. Finski 2020)

Given any volume form $d\nu$ on X, the direct images satisfy

$$\mathrm{MAVol}_{N^*}(E_m, h_{E_m}) \sim m^{\dim X} \int_X \exp\left(\frac{\int_Y \log\left(\omega_H^{\dim X}/\pi^*\nu\right)\omega^{\dim Y}}{\int_Y c_1(L)^{\dim Y}}\right) d\nu,$$

where $\omega = \Theta_{L,h_L} > 0$ on Y, and ω_H is its horizontal part.

The end

Best wishes Kang-Tae !

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