On the locus of higher order jets of entire curves in complex projective varieties

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Abstract. For a given complex projective variety, the existence of entire curves is strongly constrained by the positivity properties of the cotangent bundle. The Green-Griffiths-Lang conjecture stipulates that entire curves drawn on a variety of general type should all be contained in a proper algebraic subvariety. We present here new results on the existence of differential equations that strongly restrain the locus of entire curves in the general context of foliated or directed varieties, under appropriate positivity conditions.

Keywords. Projective variety, directed variety, entire curve, jet differential, Green-Griffiths bundle, Semple bundle, exceptional locus, algebraic differential operator, holomorphic Morse inequalities.

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in memory of Professor C.S. Seshadri

1. Introduction and goals

Let X be a complex projective manifold, $\dim_{\mathbb{C}} X = n$. Our aim is to study the existence and distribution of entire curves, namely, of non constant holomorphic curves $f : \mathbb{C} \to X$. The global geometry of X plays a fundamental role in this context, and especially the positivity properties of the canonical bundle $K_X = \Lambda^n T_X^*$. One of the major open problems of the domain is the following conjecture due to Green-Griffiths [GGr80] and Lang [Lan87].

1.1. GGL conjecture. *Assume that X is of general type, namely that $\kappa(X) = \dim X$ where $\kappa(X) := \limsup_{m\to+\infty} \log h^0(X, K_X^{\otimes m})/\log m$. Then there exists an algebraic subvariety $Y \subsetneq X$ containing all entire curves $f : \mathbb{C} \to X$.*

1.2. Definition. *The smallest algebraic subvariety above will be denoted $Y = \mathrm{Exc}(X)$ and called the exceptional locus of X.*

When X is an arithmetic variety, the exceptional locus is expected to carry a strong arithmetic significance. Especially, one can (very optimistically) hope for the following result, which is a slight variation of a conjecture made by Lang [Lan86]: for a projective variety X defined over a number field $\mathbb{K}_0$, the exceptional locus $Y = \mathrm{Exc}(X)$ in the GGL conjecture coincides with the Mordell locus, where $\mathrm{Mordell}(X)$ is the smallest complex subvariety Y such that $X(\mathbb{K}) \smallsetminus Y$ is finite for all number fields $\mathbb{K} \supset \mathbb{K}_0$.

The GGL conjecture unfortunately seems out of reach at this point. In the present work, we obtain a number of weaker results that still provide strong restrictions on the distribution of entire curves in higher order jet bundles. Among these results, we prove for instance the following statement.

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1.3. Theorem. *Let X be a nonsingular projective variety of general type. Assume that $\Lambda^2 T_X^*$ is strongly big on X, in the sense that for a given ample line bundle $A \in \mathrm{Pic}(X)$ the symmetric powers $S^m(\Lambda^2 T_X^*) \otimes \mathcal{O}_X(-A)$ are generated by their global sections on a Zariski open set $X \smallsetminus Y$, $Y \subsetneq X$, when $m > 0$ is large (if $\Lambda^2 T_X^*$ is ample, we can take $Y = \emptyset$). Then there exist finitely many rank 1 foliations $\mathcal{F}_\alpha$ on subvarieties $Z_\alpha \subset X_k$ of a suitable k-jet Semple bundle of X, such that all entire curves $f : \mathbb{C} \to X$ are either contained in Y or have a k-jet lifting $f_{[k]}$ that is contained in some Z_α and tangent to $\mathcal{F}_\alpha$. In particular, the latter curves are supported by the parabolic leaves of these foliations, which can be parametrized as a subspace of a finite dimensional variety.*

Theorem 1.3 generalizes a result that has been known for a long time for surfaces of general type, which obviously satisfy the hypotheses (see [GGr80]). By the work of Etesse [Ete19], another class of examples of projective manifolds possessing ample exterior powers $\Lambda^p T_X^*$ are general complete intersections $X = H_1 \cap \ldots \cap H_c$ of sufficient high degree in complex projective space $\mathbb{P}^N_{\mathbb{C}}$, when the codimension c is at least equal to $N/(p+1)$ ([Ete19] provides an explicit bound for the required degree of the H_j's).

The locus Y described in Theorem certainly includes all abelian and rationally connected subvarieties $Y \subset X$, and in these cases, the space of entire curves is infinite dimensional as soon as $\dim Y \geq 2$, even modulo reparametrization. Our approach is based on existence theorems for jet differentials, using holomorphic Morse inequalities ([Dem85], [Dem11]), and involves a finer study of the geometry of jet bundles. Especially, the proof of Theorem 1.3 relies on the use of certain new tautological morphisms related to induced directed structures on subvarieties of the higher jet bundles.

A complete solution of the GGL conjecture still appears rather elusive. The techniques used in sections 6 and 7 suggest that one should try to make a better use of the (semi-)stability properties of the cotangent bundle, possibly in connection with Ahlfors currents and related versions of the vanishing theorem, following for instance the ideas of McQuillan [McQ98]. We would like to celebrate here the pioneering work of Professor C.S. Seshadri in the study of the positivity and stability properties of vector bundles, which underlie much of our approach.

2. Category of directed varieties

We are interested in entire curves $f : \mathbb{C} \to X$ such that $f'(\mathbb{C}) \subset V$, where V is a (possibly singular) linear subspace of X, i.e. a closed irreducible analytic subspace such that the fiber $V_x := V \cap T_{X,x}$ is a vector subspace of the tangent space $T_{X,x}$ for all $x \in X$. We briefly recall below some of the relevant concepts, and refer to [Dem11], [Dem20] for further details.

2.1. Definition of the category of directed varieties.

(a) *Objects are pairs (X, V) where X is a complex manifold and $V \subset T_X$ a linear subspace of T_X.*

(b) *Arrows $\psi : (X, V) \to (Y, W)$ are holomorphic maps $X \to Y$ such that $d\psi(V) \subset W$*

(c) *The* absolute case *refers to the case $V = T_X$, i.e. of pairs (X, T_X).*

(d) *The* relative case *refers to pairs $(X, T_{X/S})$ where $X \to S$ is a fibration.*

(e) *We say that we are in the* Integrable case *the sheaf of sections $\mathcal{O}(V)$ is stable by Lie brackets. This corresponds to holomorphic (and possibly singular) foliations.*

We now define the canonical sheaf of a directed manifold (X,V). When V is nonsingular, i.e. is a subbundle, we simply set $K_V=\det(V^*)$ (this is a line bundle, i.e. an invertible sheaf). When V is singular, we first introduce the rank 1 sheaf ${}^b\mathcal{K}_V$ of sections of $\det V^*$ that are locally bounded with respect to a smooth ambient metric on T_X. One can show that ${}^b\mathcal{K}_V$ is equal to the integral closure of the image of the natural morphism

$$\mathcal{O}(\Lambda^r T_X^*)\to\mathcal{O}(\Lambda^r V^*)\to\mathcal{L}_V:=\text{invertible sheaf }\mathcal{O}(\Lambda^r V^*)^{**}$$

that is, if the image is $\mathcal{L}_V\otimes\mathcal{J}_V$, $\mathcal{J}_V\subset\mathcal{O}_X$,

$$(2.2)\qquad {}^b\mathcal{K}_V=\mathcal{L}_V\otimes\overline{\mathcal{J}}_V,\qquad \overline{\mathcal{J}}_V=\text{integral closure of }\mathcal{J}_V.$$

However, one may have to first blow up X as follows to ensure that ${}^b\mathcal{K}_V$ provides the appropriate geometric information.

2.3. Blow up process for a directed variety. *If $\mu:\widetilde{X}\to X$ is a modification, then $\widetilde{X}$ is equipped with the pull-back directed structure $\widetilde{V}=\overline{\widetilde{\mu}^{-1}(V_{|X'})}$, where $X'\subset X$ is a Zariski open set over which μ is a biholomorphism.*

2.4. Observation. *One always has ${}^b\mathcal{K}_V\subset\mu_*({}^b\mathcal{K}_{\widetilde{V}})\subset\mathcal{L}_V=\mathcal{O}(\det V^*)^{**}$, and $\mu_*({}^b\mathcal{K}_{\widetilde{V}})$ "increases" with μ (taking successive blow-ups $\widetilde{\widetilde{X}}\to\widetilde{X}\to X$).*

By Noetherianity, one can define a sequence of rank 1 sheaves

$$(2.5)\qquad \mathcal{K}_V^{[m]}=\lim_{\mu}\uparrow\mu_*({}^b\mathcal{K}_{\widetilde{V}})^{\otimes m},\quad \mu_*({}^b\mathcal{K}_V)^{\otimes m}\subset\mathcal{K}_V^{[m]}\subset\mathcal{L}_V^{\otimes m}$$

which we call the *pluricanonical sheaf sequence* of (X,V). Remark that the blow-up μ for which the limit is attained may depend on m. We do not know if there exists a modification μ that works for all m. This generalizes the concept of *reduced singularities* of foliations (which is known to work in that form only for surfaces).

2.6. Definition. *We say that (X,V) is of* general type *if the pluricanonical sheaf sequence $\mathcal{K}_V^{[\bullet]}$ is big, that is, if $H^0(X,\mathcal{K}_V^{[m]})$ provides a generic embedding of X for sufficiently large powers $m\gg 1$.*

The Green-Griffiths-Lang conjecture can be generalized to directed varieties as follows.

2.7. Generalized GGL conjecture. *If (X,V) is a directed manifold of general type, i.e. if $\mathcal{K}_V^{[\bullet]}$ is big, there exists an algebraic locus $Y\subsetneq X$ such that for every entire curve $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$, one has $f(\mathbb{C})\subset Y$.*

When $r=\operatorname{rank}V=1$, the generalized GGL conjecture is an elementary consequence of the Ahlfors-Schwarz lemma. In fact, the function $t\mapsto\log\|f'(t)\|_{V,h}$ is strictly subharmonic whenever (V^*,h^*) is assumed to be big.

2.8. Remark. The directed form of the GGL conjecture as stated above is possibly too optimistic. It might be safer to add a suitable (semi-)stability condition on V. In the absolute case $V=T_X$, such a semistability property is automatically satisfied when K_X is ample, as a consequence of the existence of a Kähler-Einstein metric.

3. Definition of algebraic differential operators.

The basic strategy to attack the Green-Griffiths-Lang conjecture is to show that all entire curves $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$ must satisfy nontrivial algebraic differential equations $P(f\,;\,f', f'', \ldots, f^{(k)}) = 0$, and actually, as we will see, many such equations. Following [GGr80] and [Dem97], we now introduce the useful concept of jet differential operator. Let $(\mathbb{C}, 0) \to (X, V)$, $t \mapsto f(t) = (f_1(t), \ldots, f_n(t))$ be a germ of curve such that $f(0) = x \in X$. Pick local holomorphic coordinates $(z_1, \ldots, z_n)$ centered at x on a coordinate open set $U \simeq U' \times U'' \subset \mathbb{C}^r \times \mathbb{C}^{n-r}$ such that $\pi' : U \to U'$ induces an isomorphism $d\pi' : V \to U \times \mathbb{C}^r$. Then f is determined by the Taylor expansion

$$\pi' \circ f(t) = t\xi_1 + \ldots + t^k \xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!}\nabla^s f(0), \tag{3.1}$$

where ∇ is the trivial connection on $V \simeq U \times \mathbb{C}^r$. One considers the Green-Griffiths bundle $E^{\mathrm{GG}}_{k,m}V^*$ of polynomials of weighted degree m, written locally in coordinate charts as

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(x)\xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \xi_s \in V.$$

These can also be viewed as algebraic differential operators

$$P(f_{[k]}) = P(f\,;\,f', f'', \ldots, f^{(k)}) = \sum a_{\alpha_1\alpha_2\ldots\alpha_k}(f(t))\ f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}. \tag{3.2}$$

Here $t \mapsto z = f(t)$ is a curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet, and $a_{\alpha_1\alpha_2\ldots\alpha_k}(z)$ are supposed to holomorphic functions on X. The reparametrization action: $f \mapsto f \circ \varphi_\lambda$, $\varphi_\lambda(t) = \lambda t$, $\lambda \in \mathbb{C}^*$ yields $(f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$, whence a $\mathbb{C}^*$-action

$$\lambda \cdot (\xi_1, \xi_1, \ldots, \xi_k) = (\lambda\xi_1, \lambda^2\xi_2, \ldots, \lambda^k\xi_k). \tag{3.3}$$

$E^{\mathrm{GG}}_{k,m}$ is precisely the set of polynomials of weighted degree m, corresponding to coefficients $a_{\alpha_1\ldots\alpha_k}$ with $m = |\alpha_1| + 2|\alpha_2| + \ldots + k|\alpha_k|$. An important fact is the

3.4. Direct image formula. *Let $J^{\mathrm{nc}}_k V \subset J_k V$ be the set of non constant k-jets. One defines the* Green-Griffiths *bundle to be $X^{\mathrm{GG}}_k = J^{\mathrm{nc}}_k V/\mathbb{C}^*$ and $\mathcal{O}_{X^{\mathrm{GG}}_k}(1)$ to be the associated tautological rank* 1 *sheaf. Let $\pi_k : X^{\mathrm{GG}}_k \to X$ be the natural projection. Then we have the direct image formula*

$$E^{\mathrm{GG}}_{k,m}V^* = (\pi_k)_*\mathcal{O}_{X^{\mathrm{GG}}_k}(m).$$

4. Consequences of the holomorphic Morse inequalities

Given a real $(1,1)$-form θ on X, the q-index set of θ is defined to be

$$X(\theta, q) = \{x \in X \mid \theta(x) \text{ has signature } (n-q, q)\} \tag{4.1}$$

(exactly q negative eigenvalues and $n - q$ positive ones). We also set

$$X(\theta, \leq q) = \bigcup_{0 \leq j \leq q} X(\theta, j). \tag{4.2}$$

Then $X(\theta,q)$ and $X(\theta,\le q)$ are open sets. and $\mathrm{sign}(\theta^n)=(-1)^q$ on $X(\theta,q)$. The general statement of holomorphic Morse inequalities [Dem85] asserts that for any hermitian line bundle (L,h) of curvature form $\theta=\Theta_{L,h}$ on X, and any coherent sheaf $\mathcal{F}$ of rank $r=\mathrm{rank}\,\mathcal{F}$, one has

$$\sum_{j=0}^{q}(-1)^{q-j}h^j(X,L^{\otimes m}\otimes\mathcal{F})\le r\,\frac{m^n}{n!}\int_{X(\theta,\le q)}(-1)^q\theta^n+o(m^n)\qquad\text{as } m\to+\infty. \tag{4.3}$$

An application of these inequalities yields the following fundamental estimates [Dem11].

4.4. Main cohomology estimates. *Let (X,V) be a directed manifold, A an ample line bundle over X, (V,h) and (A,h_A) hermitian structures such that $\Theta_{A,h_A}>0$. Define*

(a) $$L_{k,\varepsilon}=\mathcal{O}_{X_k^{\mathrm{GG}}}(1)\otimes\pi_k^*\mathcal{O}\Big(-\frac{1}{kr}\Big(1+\frac{1}{2}+\ldots+\frac{1}{k}\Big)\varepsilon A\Big),$$

(b) $$\eta_\varepsilon=\Theta_{\det V^*,\det h^*}-\varepsilon\,\Theta_{A,h_A}.$$

Then all $m\gg k\gg 1$ such that m is sufficiently divisible and for all $q\ge 0$ we have upper and lower bounds

(c) $$h^q(X_k^{\mathrm{GG}},\mathcal{O}(L_{k,\varepsilon}^{\otimes m}))\le\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta,q)}(-1)^q\eta_\varepsilon^n+\frac{C}{\log k}\bigg),$$

(d) $$h^q(X_k^{\mathrm{GG}},\mathcal{O}(L_{k,\varepsilon}^{\otimes m}))\ge\frac{m^{n+kr-1}}{(n+kr-1)!}\frac{(\log k)^n}{n!\,(k!)^r}\bigg(\int_{X(\eta,\{q,\,q\pm1\})}(-1)^q\eta_\varepsilon^n-\frac{C}{\log k}\bigg).$$

The case $q=0$ is the most useful one, as it gives estimates for the number of holomorphic sections. We now explain the essential ideas involved in the proof of the estimates.

1st *step: construction of a Finsler metric on k-jet bundles*

Let J_kV be the bundle of k-jets of curves $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$. Assuming that V is equipped with a hermitian metric h, one defines a "weighted Finsler metric" on J^kV by taking $p=k!$ and

$$\Psi_{h_k}(f):=\Big(\sum_{1\le s\le k}\big(\varepsilon_s\|\nabla^s f(0)\|_{h(x)}\big)^{2p/s}\Big)^{1/p},\quad 1=\varepsilon_1\gg\varepsilon_2\gg\cdots\gg\varepsilon_k. \tag{4.5}$$

Letting $\xi_s=\nabla^s f(0)$, this can actually be viewed as a metric h_k on $L_k:=\mathcal{O}_{X_k^{\mathrm{GG}}}(1)$. Some error terms of the form $O(\varepsilon_s/\varepsilon_t)_{t<s}$ appear, but they are negligible in the limit, and the leading term of the curvature form is $(z,\xi_1,\ldots,\xi_k)\mapsto\Theta^0_{L_k,h_k}(z,\xi)$ with

$$\Theta^0_{L_k,h_k}(z,\xi)=\omega_{\mathrm{FS},k}(\xi)+\frac{i}{2\pi}\sum_{1\le s\le k}\frac{1}{s}\frac{|\xi_s|^{2p/s}}{\sum_t|\xi_t|^{2p/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}(z)\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j, \tag{4.6}$$

where $(c_{ij\alpha\beta}(s))$ are the coefficients of the curvature tensor Θ_{V^*,h^*} at point z, and $\omega_{\mathrm{FS},k}(\xi)$ is the vertical Fubini-Study metric on the fibers of $X_k^{\mathrm{GG}}\to X$. The expression gets simpler by using polar coordinates

$$x_s=\frac{|\xi_s|_h^{2p/s}}{\sum_t|\xi_t|_h^{2p/t}},\quad u_s=\frac{\xi_s}{|\xi_s|_h}=\frac{\nabla^s f(0)}{|\nabla^s f(0)|},\quad 1\le s\le k. \tag{4.7}$$

2^{nd} *step: probabilistic interpretation of the curvature*
In such polar coordinates, one gets the formula

$$\Theta_{L_k,h_k}(z,\xi) \simeq \omega_{\mathrm{FS},p,k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j \tag{4.8}$$

where $\omega_{\mathrm{FS},k}(\xi)$ is positive definite in ξ. The other terms are a weighted average of the values of the curvature tensor $\Theta_{V,h}$ on vectors u_s in the unit sphere bundle $SV \subset V$. The weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2p/s} = 1$, so we can take here $x_s \ge 0$, $\sum x_s = 1$. This is essentially a sum of the form $\sum \frac{1}{s} Q(u_s)$ where $Q(u) = \langle \Theta_{V^*,h^*} u, u\rangle$ and u_s are random points of the sphere, and so as $k \to +\infty$ this can be estimated by a "Monte-Carlo" integral

$$\Big(1 + \frac{1}{2} + \ldots + \frac{1}{k}\Big) \int_{u\in SV} Q(u)\, du. \tag{4.9}$$

As Q is quadratic, we have

$$\int_{u\in SV} Q(u)\, du = \frac{1}{r}\,\mathrm{Tr}(Q) = \frac{1}{r}\,\mathrm{Tr}(\Theta_{V^*,h^*}) = \frac{1}{r}\Theta_{\det V^*,\det h^*}. \tag{4.10}$$

The above equality show that the Monte Carlo approximation only depends on $\det V^*$ and that for the unitary invariant probability measure taken on $(SV)^k \ni (u_s)$, we have an expected value

$$\mathbf{E}(\Theta_{L_k,h_k}) = \omega_{\mathrm{FS},p,k}(\xi) + \sum_{1\le s\le k} \frac{1}{s}\frac{1}{kr}\Theta_{\det V^*,\det h^*}(z) \tag{4.11}$$

(The factor $\frac{1}{k}$ comes from the fact that the expected value of x_s on the $(k-1)$-simplex $\sum_{1\le s\le k} x_s = 1$ is $\frac{1}{k}$). Formula (4.11) is the main reason why the leading term of the cohomology estimates only involves $\det V^*$. Of course, a complete proof requires an estimate of the standard deviation occurring in the Monte Carlo process, as is done in [Dem11]. □

By passing to a "singular version" of holomorphic Morse inequalities to accommodate singular metrics (Bonavero, [Bon93]) and subtracting the terms involving $\varepsilon\,\Theta_{A,h_A}$, one gets

4.12. Corollary: existence of global jet differentials ([Dem11). *Assume that* (X,V) *is of general type, namely that* ${}^b\mathcal{K}_V$ *is a* big rank 1 sheaf, *and let*

$$L_{k,\varepsilon} = \mathcal{O}_{X_k^{\mathrm{GG}}}(1) \otimes \pi_k^*\mathcal{O}(-\delta_k\varepsilon A), \quad \delta_k = \frac{1}{kr}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)$$

with $A \in \mathrm{Pic}(X)$ *ample. Then there exist many nontrivial global sections*

$$P \in H^0(X_k^{\mathrm{GG}}, L_{k,\varepsilon}^{\otimes m}) \simeq H^0(X, E_{k,m}^{\mathrm{GG}}V^* \otimes \mathcal{O}(-m\delta_k\varepsilon A))$$

for $m \gg k \gg 1$ *and* $\varepsilon \in \mathbb{Q}_{>0}$ *small.*

The fact that entire curves satisfy differential equations is now a consequence of the following result.

4.13. Fundamental vanishing theorem ([GGr80], [Dem97], [SYe97]). *For all global differential operators*

$$P \in H^0(X, E^{\mathrm{GG}}_{k,m}V^* \otimes \mathcal{O}(-qA)), \quad q \in \mathbb{N}^*,$$

and all $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X, V)$, *one has* $P(f_{[k]}) \equiv 0$.

Geometrically, this can interpreted by stating that the image $f_{[k]}(\mathbb{C})$ of the k-jet curve lies in the base locus

$$Z = \bigcap_{m\in\mathbb{N}^*} \bigcap_{\sigma\in H^0(X^{\mathrm{GG}}_k, L^{\otimes m}_{k,\varepsilon})} \sigma^{-1}(0) \quad \subset \quad X^{\mathrm{GG}}_k. \tag{4.14}$$

To prove the GGL conjecture, we would need to check that $\pi_k(Z) \subsetneq X$. This turns out to be a hard problem.

5. Investigation of the base locus

We start by formulating the base locus problem in a very general context. Let (L, h) be a hermitian line bundle over X. If we assume that $\theta = \Theta_{L,h}$ satisfies $\int_{X(\theta,\le 1)} \theta^n > 0$, then we know that L is big, i.e. that $h^0(X, L^{\otimes m}) \ge c\,m^n$, for $m \ge m_0$ and $c > 0$, but this does not tell us anything about the base locus $\mathrm{Bs}(L) = \bigcap_{\sigma\in H^0(X,L^{\otimes m})} \sigma^{-1}(0)$.

5.1. Definition. *The "iterated base locus"* $\mathrm{IBs}(L)$ *is obtained by picking inductively* $Z_0 = X$ *and* $Z_k =$ *zero divisor of a section* σ_k *of* $L^{\otimes m_k}$ *over the normalization of* Z_{k-1}, *and taking* $\bigcap_{k,m_1,\ldots,m_k,\sigma_1,\ldots,\sigma_k} Z_k$.

5.2. Unsolved problem. *Find a condition, e.g. in the form of Morse integrals* (*or analogous integrals*) *for* $\theta = \Theta_{L,h}$, *ensuring that* $\mathrm{codim}\,\mathrm{IBs}(L) > p$.

For instance, there might exist a way of deriving from such conditions the positivity of Morse integrals $\int_{Z(\theta_{|Z},\le 1)} \theta^{n-p}$ for arbitrary irreducible subvarieties, $\mathrm{codim}\, Z = p$, that are obtained themselves as iterated base loci. In the specific case of the Green-Griffiths tautological line bundle L_k, one can get the following statement (without loss of generality, we can exclude the case of directed structures of rank $r = 1$, since the GGL conjecture is then trivial).

5.3. Theorem. *Let* (X, V) *be a directed variety of* general type, *with* $r = \mathrm{rank}\, V \ge 2$. *Then there exist* $k_0 \in \mathbb{N}$ *and* $\delta > 0$ *with the following properties. Let* $Z \subset X^{\mathrm{GG}}_k$ *be an algebraic subvariety that is a complete intersection of irreducible hypersurfaces*

$$Z = \bigcap_{1\le j\le \ell} \big\{k\text{-jets } f_{[k]} \in X^{\mathrm{GG}}_k\,;\, P_j(f) = 0\big\}, \quad P_j \in H^0(X, E^{\mathrm{GG}}_{s_j,m_j}V^* \otimes G_j),$$

with $k \ge k_0$, $\mathrm{ord}(P_j) = s_j$, $1 \le s_1 < \cdots < s_\ell \le k$, $\sum_{1\le j\le \ell} \frac{1}{s_j} \le \delta \log k$, *and* $G_j \in \mathrm{Pic}(X)$. *Then* $\dim Z = n + kr - 1 - \ell$ *and the Morse integrals*

$$\int_{Z(L_{k,\varepsilon},\le 1)} \Theta^{\dim Z}_{L_{k,\varepsilon}}$$

of $L_{k,\varepsilon} = \mathcal{O}_{X_k^{\rm GG}}(1) \otimes \pi_k^*\mathcal{O}_X\big(-\frac{1}{kr}\big(1+\frac{1}{2}+\cdots+\frac{1}{k}\big)\varepsilon A\big)$ *are positive for* $\varepsilon > 0$ *small, hence* $H^0(Z, L_{k,\varepsilon}^{\otimes m}) \ge c\, m^{\dim Z}$ *for* $m \gg 1$*, with* $c > 0$*.*

At the expense of a slightly more involved statement, it would be possible to allow some repetitions in the degrees s_j of the polynomials P_j. Unfortunately, none of the extended versions we have been able to reach seems sufficient to prove the GGL conjecture, which would follow if we could cut down Z to a subvariety of dimension $\dim Z < n$, thus of very high codimension in the jet bundle.

Proof. We come back to the calculations made in [Dem11], especially those detailed in § 2. The main point is an integration on the fiber of $X_k^{\rm GG} \to X$, which is reduced in polar coordinates to an integration on a product of unit spheres $(SV)^k$ with the $(k-1)$-dimensional simplex. Here, we integrate on the fibers Z_z of $Z \to X$, namely on a rather complicated subvariety of the weighted projective space $P_\bullet(V^k) = (V^k \smallsetminus \{0\})/\mathbb{C}^*$ of the form

$$P_j(z, \xi_1, \ldots, \xi_{s_j}) = 0, \quad (z \in X \text{ fixed}), \tag{5.4}$$

where the P_j's are weighted homogeneous polynomials in $(\xi_1, \ldots, \xi_{s_j})$ of positive degree in ξ_{s_j}. The choice of our Finsler metric can be combined with a rescaling of the form $\xi_s = \varepsilon_s^{-1}\tilde\xi_s$ (see [Dem11, Lemma 2.12]). Since $\varepsilon_s \ll \varepsilon_t$ for $t < s$, the rescaled subvariety (5.4) actually converges to the subvariety defined in the new coordinates $\tilde\xi = (\tilde\xi_s)$ by the equations

$$Q_j(z, \tilde\xi_{s_j}) = P_j(z, 0, \ldots, 0, \tilde\xi_{s_j}) = 0, \tag{5.5}$$

at least, for all x in the full measure Zariski open set of X where the leading coefficients of $Q_j(z, \bullet)$ do not vanish. These polynomials depend on non overlapping variables ξ_{s_j}, hence the limit subvariety can be seen as a product of cones over a product subvariety in $P(V)^k$, where some of the factors $P(V)$ are replaced by hypersurfaces. The calculation of the Morse integrals on Z requires computing the integrals

$$\int_{Z_z} x^\alpha \omega_{{\rm FS},p,k}(\xi)^N, \quad x_s = \frac{|\xi_s|_h^{2p/s}}{\sum_t |\xi_t|_h^{2p/t}}, \quad N = \dim Z_z = kr - 1 - \ell, \quad |\alpha| \le n,$$

and an evaluation of the standard deviation via the Cauchy-Schwarz inequality involves the same type of integrals with $|\alpha| \le 2n-2$ (see [Dem11]). By the Lelong-Poincaré formula and the Fubini theorem, the above integrals are equal to

$$\int_{P_\bullet(V^k)} x^\alpha \omega_{{\rm FS},p,k}(\xi)^N \wedge \bigwedge_{1\le j\le \ell} \frac{i}{2\pi}\,\partial\overline\partial \log |Q_j(z, \xi_{s_j})|^2.$$

In view of the unitary invariance of $x^\alpha \omega_{{\rm FS},p,k}(\xi)^N$ in each ξ_s, the Crofton formula implies that the previous integral is equal to

$$\int_{P_\bullet(V^k)} x^\alpha \omega_{{\rm FS},p,k}(\xi)^N \wedge \bigwedge_{1\le j\le \ell} d_j\,\frac{i}{2\pi}\,\partial\overline\partial \log |\xi_{s_j}|_h^2 \tag{5.6}$$

where $d_j = \deg_{\xi_{s_j}} Q_j(z, \xi_{s_j})$. Here the Fubini-Study metric is derived from the Finsler metric $(\sum_s |\xi_s|_h^{2p/s})^{1/p}$. If we make a change of variable $\xi_{s,\lambda} \mapsto \xi_{s,\lambda}^s$ for each component of ξ_s and apply the comparison principle of [Dem87], we see that (5.6) is equivalent, up to factors controlled by constants $C_n^{\pm 1}$ depending only on the dimension of X, to the integral

$$(5.7) \qquad \frac{1}{(k!)^r} \prod_{1\le j\le \ell} d_j s_j \int_{P(\mathbb{C}^{kr})} x^\alpha \omega_{\rm FS}(\xi)^N \wedge \bigwedge_{1\le j\le \ell} \frac{i}{2\pi}\partial\overline{\partial}\log|\xi_{s_j}|^2$$

where $|\xi|^2 = \sum_s |\xi_s|^2$, $\omega_{\rm FS}(\xi) = \frac{i}{2\pi}\partial\overline{\partial}\log|\xi|^2$ and $x_s = \frac{|\xi_s|^2}{|\xi|^2}$. Additionally, the correcting factor is equal to 1 if $\alpha = 0$, and in that case the precise value of our integrals is $\frac{1}{(k!)^r}\prod_{1\le j\le \ell} d_j s_j$, where the factor $\frac{1}{(k!)^r}$ comes from the weighted Fubini-Study metric, and $\prod_{1\le j\le \ell} d_j s_j$ is the relative degree of Z_z in the weighted projective space. Again by the Crofton formula, we can replace each factor $\frac{i}{2\pi}\partial\overline{\partial}\log|\xi_{s_j}|^2$ by the current of integration on the hyperplane $\xi_{s_j,1} = 0$. This shows that our integrals are equal to

$$(5.8) \qquad \frac{1}{(k!)^r} \prod_{1\le j\le \ell} d_j s_j \int_{P((\mathbb{C}^r)^{k-\ell}\times(\mathbb{C}^{r-1})^\ell)} x^\alpha \omega_{\rm FS}(\xi)^N,$$

and these can be computed by means of the formulas given in [Dem11]. In particular, the calculation of the expected value $\mathbf{E}(\Theta_{L_k,h_k})$ depends on the integrals (5.7) with $|\alpha| = 1$, i.e. $x^\alpha = x_s$. There are 2 possible values, one for $s \in \{s_1, \ldots, s_\ell\}$, namely $I' = \frac{r-1}{kr-\ell}$ and another for s taken in the complement, equal to $I'' = \frac{r}{kr-\ell}$, so that $\ell I' + (k-\ell)I'' = 1$. The ratio I'/I'' belongs to $[1-1/r, 1] \subset [1/2, 1]$. Our assumption $\sum_{1\le j\le\ell} \frac{1}{s_j} \le \delta \log k$ implies $\log \ell = \delta \log k + O(1)$, hence $\ell = O(k^\delta) \ll k$. Therefore, if we are concerned with estimates only up to universal constants, we may consider I', I'' to be of the same order of magnitude $\frac{1}{k}$. Then, most of the arguments employed in [Dem11] remain unchanged. The integration is performed on a certain codimension ℓ subvariety of $\mathbb{C}^{kr}$ which is a ramified cover over a product $\mathbb{C}^{kr-\ell} = (\mathbb{C}^r)^{k-\ell} \times (\mathbb{C}^{r-1})^\ell$, and the equality (4.10) applies to each of the $(k-\ell)$ factors $\mathbb{C}^r$, where the polar coordinate change yields an integral on the whole unit sphere. The ℓ remaining factors possibly lead to terms of unknown signature, but their sum has a relative size $O(\sum \frac{1}{s_j} / \sum \frac{1}{s}) = O(\delta)$. Therefore the expected value of the curvature $\mathbf{E}(\Theta_{L_{k,\varepsilon},h_k})$, which can be derived from (4.11) (see also [Dem11, 2.18]), has an horizontal term equal to $\frac{1}{kr}\log k\,(\eta_\varepsilon \pm O(\delta))$. Our estimate follows. □

5.9. Remark. The above calculations even give an evaluation of the leading constant c, namely

$$c = \frac{\prod_{1\le j\le\ell} d_j s_j}{(n+kr-1)!}\frac{(\log k)^n}{n!(k!)^r}\Big(\int_{X(\eta_\varepsilon,\le 1)} \eta_\varepsilon^n - O((\log k)^{-1}) - O(\delta)\Big)$$

where $\eta_\varepsilon = \Theta_{\det V^*,\det h^*} - \varepsilon\,\Theta_{A,h_A}$. If Z is taken to be just one irreducible component of a complete intersection, the quantity $\prod_{1\le j\le\ell} d_j s_j$ has to be replaced by the relative degree over X of that component.

6. Semple jet bundles and induced directed structures

Semple jet bundles provide further geometric information that appear very useful in the investigation of the base locus. We briefly recall the relevant notation and concepts from

[Dem97]. To begin with, we assume that X is non singular and connected, and that V is non singular, i.e. that V is a subbundle of T_X, and we set $\dim X = n$, $\operatorname{rank} V = r$.

6.1. 1-jet functor. *This is the functor* $(X,V) \mapsto (\tilde X, \tilde V)$ *on the category of directed varieties defined by*

(a) $\tilde X = P(V) =$ *bundle of projective spaces of lines in* V,

(b) $\pi : \tilde X = P(V) \to X, \quad (x,[v]) \mapsto x, \quad v \in V_x\,,$

(c) $\tilde V_{(x,[v])} = \big\{\xi \in T_{\tilde X,(x,[v])}\,;\ \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\big\}.$

By taking tangents in $\tilde V = P(V)$, every entire curve $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X,V)$ lifts as a curve $f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \to (\tilde X, \tilde V)$ by putting $f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde X$.

6.2. Definition of Semple jet bundles.

(a) *We define* (X_k, V_k) *to be the* k*-th iteration of the functor* $(X,V) \mapsto (\tilde X, \tilde V)$.

(b) *In this way, every curve* $f : (\mathbb{C}, T_{\mathbb{C}}) \to (X,V)$ *gives rise to a projectivized* k*-jet lifting* $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \to (X_k, V_k)$.

6.3. Basic exact sequences. By construction we have $X_k = P(V_{k-1})$ and a tautological line bundle $\mathcal{O}_{X_k}(1)$ on $X_k = P(V_{k-1})$. One can also check that there are exact sequences

$$\text{(a)} \qquad 0 \to T_{X_k/X_{k-1}} \to V_k \xrightarrow{d\pi_k} \mathcal{O}_{X_k}(-1) \to 0,$$

$$\text{(b)} \qquad 0 \to \mathcal{O}_{X_k} \to \pi_k^\star V_{k-1} \otimes \mathcal{O}_{X_k}(1) \to T_{X_k/X_{k-1}} \to 0.$$

The sequence (a) is equivalent to the definition of V_k, and (b) is just the Euler exact sequence for $X_k = P(V_{k-1})$. These sequences imply that $\operatorname{rank} V_k = \operatorname{rank} V$ is constant. Letting $n = \dim X$ and $r = \operatorname{rank} V$, we obtain morphisms

$$\pi_{k,\ell} = \pi_{\ell-1} \circ \cdots \circ \pi_{k-1} \circ \pi_k : X_k \to X_\ell$$

and a *tower of* $\mathbb{P}^{r-1}$*-bundles*

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \to \cdots \to X_1 \xrightarrow{\pi_1} X_0 = X \tag{6.4}$$

with $\dim X_k = n + k(r-1)$, $\operatorname{rank} V_k = r$. In the sequel, we introduce the weighted invertible sheaves

$$\mathcal{O}_{X_k}(\underline{a}) = \pi_{k,1}^*\mathcal{O}_{X_1}(a_1) \otimes \cdots \otimes \pi_{k,k-1}^*\mathcal{O}_{X_{k-1}}(a_{k-1}) \otimes \mathcal{O}_{X_k}(a_k) \tag{6.5}$$

for every k-tuple $\underline{a} = (a_1, ..., a_k) \in \mathbb{Z}^k$, and let $\underline{1} = (1, ..., 1) \in \mathbb{Z}^k$. Then

$$\det V_k = \det T_{X_k/X_{k-1}} \otimes \mathcal{O}_{X_k}(-1), \quad \det T_{X_k/X_{k-1}} = \pi_k^* \det V_{k-1} \otimes \mathcal{O}_{X_k}(r), \tag{6.6}$$

and we infer inductively

$$\det V_k = \pi_k^* \det V_{k-1} \otimes \mathcal{O}_{X_k}(r-1), \tag{6.6$'$}$$

$$= \pi_{k,0}^\star \det V \otimes \mathcal{O}_{X_k}((r-1)\underline{1}). \tag{6.6$''$}$$

6.7. Proposition ([Dem97])**.** *When (X,V) is non singular, the Semple bundle X_k is a smooth compactification of the quotient $X_k^{\mathrm{GG,reg}}/\mathbb{G}_k = J_k^{\mathrm{GG,reg}}/\mathbb{G}_k$, where $\mathbb{G}_k$ is the group of k-jets of germs of biholomorphisms of $(\mathbb{C},0)$, acting on the right by reparametrization: $(f,\varphi)\mapsto f\circ\varphi$, and where J_k^{reg} is the space of k-jets of regular curves.*

In the absolute case $V=T_X$, we obtain what we call the "absolute Semple tower" (X_k^a,V_k^a) associated with $(X_0^a,V_0^a)=(X,T_X)$. When X is nonsingular, all stages (X_k^a,V_k^a) of the Semple bundles are nonsingular. This allows to extend the definition of (X_k,V_k) in case V is singular (but X non singular). Indeed, let $X'\subset X$ be the Zariski open set on which $V'=V_{|X'}$ is a subbundle of T_X. By functoriality, we get a Semple tower (X'_k,V'_k) and injections $(X'_k,V'_k)\hookrightarrow(X_k^a,V_k^a)$. We then simply define (X_k,V_k) to be the pair obtained by taking the closure of X'_k and V'_k in X_k^a and V_k^a respectively.

6.8. Proposition (Direct image formula for invariant differential operators, [Dem97])**.** *Let $E_{k,m}V^*$ be the sheaf of algebraic differential operators $f\mapsto P(f_{[k]})$ acting on germs of curves $f:(\mathbb{C},T_{\mathbb{C}})\to(X,V)$ such that $P((f\circ\varphi)_{[k]})=\varphi'^mP(f_{[k]})\circ\varphi$. Then*

$$(\pi_{k,0})_*\mathcal{O}_{X_k}(m)\simeq E_{k,m}V^*.$$

6.9. Some tautological morphisms. *Let (X,V) be a directed structure on a nonsingular projective variety X. For every $p=1,\ldots,r$, there is a tautological morphism*

$$\Phi_{k,p}^{X,V}:\pi_{k,0}^*\Lambda^pV^*\otimes\mathcal{O}_{X_k}(-(p-1)\underline{1})\to\Lambda^pV_k^*.$$

Proof. First assume that (X,V) is nonsingular. For every integer $p\geq 1$, the exact sequence (6.3 b) gives rise to a surjective contraction morphism by the Euler vector field

$$\Lambda^p\big(\pi_k^*V_{k-1}\otimes\mathcal{O}_{X_k}(1)\big)^*\to\Lambda^{p-1}T^*_{X_k/X_{k-1}},\tag{6.10}$$

while (6.3 a) and the dual sequence $0\to\mathcal{O}_{X_k}(1)\to V_k^*\to T^*_{X_k/X_{k-1}}\to 0$ induce an injective wedge multiplication morphism

$$\Lambda^{p-1}T^*_{X_k/X_{k-1}}\otimes\mathcal{O}_{X_k}(1)\to\Lambda^pV_k^*.\tag{6.11}$$

A composition of these two arrows yields a tautological morphism

$$\pi_k^*\Lambda^pV_{k-1}^*\otimes\mathcal{O}_{X_k}(-(p-1))\to\Lambda^pV_k^*,\tag{6.12}$$

which turns out to be dual to the isomorphism (6.6) in case $p=r=\operatorname{rank}V$. There also exists a (more naive) composed morphism

$$\pi_k^*\Lambda^pV_{k-1}\otimes\mathcal{O}_{X_k}(p)\to\Lambda^pT_{X_k/X_{k-1}}\to\Lambda^pV_k$$

of dual

$$\Lambda^pV_k^*\to\pi_k^*\Lambda^pV_{k-1}^*\otimes\mathcal{O}_{X_k}(-p),\tag{6.12$'$}$$

but it is less interesting for us (the composition of (6.12) and (6.12′) is equal to 0, and (6.12′) vanishes for $p=r$; these morphisms can be seen to be dual modulo exchange of p

and $r-p$ and contraction with the determinants). We can iterate (6.12) for all stages of the tower, and get in this way the desired tautological morphism

$$\Phi_{k,p}^{X,V} : \pi_{k,0}^*\Lambda^p V^* \otimes \mathcal{O}_{X_k}(-(p-1)\underline{1}) \to \Lambda^p V_k^*,$$

By what we have seen, there is in particular an absolute tautological morphism $\Phi_{k,p}^{X^a,V^a}$ defined at the level of the absolute Semple tower (X_k^a, V_k^a), and it is clear that $\Phi_{k,p}^{X,V}$ is obtained by restricting $\Phi_{k,p}^{X^a,V^a}$ to (X_k, V_k). Therefore, our morphisms are also well defined in case V is singular, by taking the Zariski closure of the regular part in (X_k^a, V_k^a). □

Now, let Z be an irreducible algebraic subset of some Semple k-jet bundle X_k over X (k being arbitrary). We define an *induced directed structure* $(Z, W) \hookrightarrow (X_k, V_k)$ by taking the linear subspace $W \subset T_Z \subset T_{X_k|Z}$ to be the closure of $T_{Z'} \cap V_k$ taken on a suitable Zariski open set $Z' \subset Z_{\mathrm{reg}}$ where the intersection has constant rank and is a subbundle of $T_{Z'}$. Alternatively, one could also take W to be the closure of $T_{Z'} \cap V_k$ in the absolute k-th bundle V_k^a. This produces an *induced directed subvariety*

$$(Z, W) \subset (X_k, V_k) \subset (X_k^a, V_k^a). \tag{6.13}$$

6.14. Observation. One can see that the hypotheses $Z \subsetneq X_k$ and $\pi_{k,k-1}(Z) = X_{k-1}$ imply $\operatorname{rank} W < \operatorname{rank} V_k$. Otherwise, we would have $W = (V_k)_{|Z'} \supset T_{X_k/X_{k-1}|Z'}$ over a Zariski open, hence Z would contain a dense open subset of X_k, contradiction.

6.15. Induced tautological morphisms. Let Z be an irreducible subvariety of X_k such that $\pi_{k,0}(Z) = X$, and (Z, W_k) the induced directed structure. Consider the vertical part $W_k^v \subset W_k$ which is defined to be $W_k^v = \overline{(W_k \cap T_{X_k/X_{k-1}})_{|Z'}}$ where Z' is the Zariski open set where the intersection has minimal rank (since $\pi_{k,0}(Z) = X$, we can assume $Z' \subset \pi_{k,0}^{-1}(X')$, where X' is a Zariski open set where $V_{|X'}$ is a subbundle of T_X). Then W_k^v has corank at most 1 in W_k (since $T_{X_k/X_{k-1}}$ has corank 1 in V_k). We define $W_{k-1} \subset \pi_{k,k-1}^* V_{k-1|Z}$ to be the closure of the preimage of $(W_k^v \otimes \mathcal{O}_{X_k}(-1))_{|Z'}$ by the restriction to Z' of the morphism

$$\pi_{k,k-1}^* V_{k-1} \to T_{X_k/X_{k-1}} \otimes \mathcal{O}_{X_k}(-1)$$

induced by (6.3 b). Notice that $\operatorname{rank} W_{k-1} = \operatorname{rank} W_k^v + 1$ is equal to $\operatorname{rank} W_k$ or $\operatorname{rank} W_k + 1$. We then take

$$W_{k-1}^v = \overline{(W_{k-1} \cap \pi_{k,k-1}^* T_{X_{k-1}/X_{k-2}})_{|Z''}}$$

for a suitable Zariski open set $Z'' \subset Z$, and obtain in this way a preimage $W_{k-2} \subset \pi_{k,k-2}^* V_{k-2}$ of $W_{k-1}^v \otimes \pi_{k,k-1}^* \mathcal{O}_{X_{k-1}}(-1)$. Inductively, we get a linear subspace $W_\ell \subset \pi_{k,\ell}^* V_\ell$, and finally $W_0 \subset \pi_{k,0}^* V_0 = \pi_{k,0}^* V$, with

$$\operatorname{rank} W_0 \geq \operatorname{rank} W_1 \geq \ldots \geq \operatorname{rank} W_{k-1} \geq \operatorname{rank} W_k. \tag{6.15 a}$$

Set $r_\ell = \operatorname{rank} W_\ell$. The contraction by the Euler vector field (6.10) induces by restriction a generically surjective morphism

$$\Lambda^p W_{k-1}^* \otimes \mathcal{O}_{X_k}(-p)_{|Z} \to \Lambda^{p-1} (W_k^v)^*.$$

Now, we also have a sequence $\mathcal{O}_{X_k}(1) \to W_k^* \to (W_k^v)^*$, which either gives an isomorphism $\Lambda^{p-1}W_k^* = \Lambda^{p-1}(W_k^v)^*$ when $W_k^v = W_k$ (in that case, the arrow $\mathcal{O}_{X_k}(1) \to W_k^*$ is equal to 0), or a generically surjective morphism

$$\Lambda^{p-1}(W_k^v)^* \otimes \mathcal{O}_{X_k}(1) \to \Lambda^p W_k^*$$

induced by (6.11) when $\operatorname{rank} W_k^v = \operatorname{rank} W_k - 1$. Therefore we obtain by composition generically surjective morphisms

$$\begin{aligned} &\Lambda^p W_{k-1}^* \otimes \mathcal{O}_{X_k}(-p)_{|Z} \to \Lambda^{p-1} W_k^* \quad \text{when } r_{k-1} = r_k + 1, \\ &\Lambda^p W_{k-1}^* \otimes \mathcal{O}_{X_k}(1-p)_{|Z} \to \Lambda^p W_k^* \quad \text{when } r_{k-1} = r_k. \end{aligned}$$

Replacing p by $r_{k-1} - p$, we obtain in both cases a generically surjective morphism

$$\Lambda^{r_{k-1}-p} W_{k-1}^* \to \Lambda^{r_k - p} W_k^* \otimes \mathcal{O}_{X_k}(2r_{k-1} - r_k - p - 1)_{|Z},$$

and inductively, we obtain for every $p \geq 0$ a generically surjective morphism

(6.15 b) $$\Lambda^{r_0 - p} W_0^* \to \Lambda^{r_k - p} W_k^* \otimes \mathcal{O}_{X_k}(\underline{a} - p\,\underline{1})_{|Z}, \quad \underline{a} = (a_\ell), \quad a_\ell = 2r_{\ell-1} - r_\ell - 1.$$

For $p = 0$, we get in particular a generically surjective morphism

(6.15 c) $$\Lambda^{r_0} W_0^* \to \Lambda^{r_k} W_k^* \otimes \mathcal{O}_{X_k}(\underline{a})_{|Z},$$

which is especially interesting since we are in rank 1. Since $W_0 \subset (\pi_{k,0}^* V)_{|Z} \subset (\pi_{k,0}^* T_X)_{|Z}$, we also have generically surjective morphisms

$$(\pi_{k,0}^* \Lambda^{r_0} T_X^*)_{|Z} \to (\pi_{k,0}^* \Lambda^{r_0} V^*)_{|Z} \to \Lambda^{r_0} W_0^*.$$

Also, all of these morphisms are obtained by taking either restrictions of forms or contractions by the Euler vector fields of the various stages. With respect to smooth hermitian metrics on the (nonsingular) absolute Semple tower (X_k^a, V_k^a), bounded forms are certainly mapped to bounded forms. From this we conclude :

6.16. Corollary. *Let X be a nonsingular variety, (X, V) a directed structure, and $Z \subset X_k$ an irreducible subvariety in the k-th stage of the Semple tower (X_k, V_k) such that $\pi_{k,0}(Z) = X$. Then the induced directed variety (Z, W) has the following property: there exists $r_0 \geq r_k := \operatorname{rank} W$, a weight $\underline{a} \in \mathbb{N}^k$ and a generically surjective sheaf morphism*

$$(\pi_{k,0}^* \Lambda^{r_0} T_X^*)_{|Z} \to (\pi_{k,0}^* {}^b\Lambda^{r_0} V^*)_{|Z} \to {}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(\underline{a})_{|Z}.$$

7. Tautological morphisms and differential equations

The tautological morphisms give a potential technique for controlling inductively the positivity of the canonical sheaf ${}^b\mathcal{K}_W$ for all induced directed structures (Z, W). We rely on the following statement that was already observed in [Dem14].

7.1. Proposition. *Let X be a nonsingular variety, (X, V) a directed structure, and (Z, W) the induced directed structure on an irreducible subvariety $Z \subset X_k$. Assume that* *there exists a weight* $\underline{a} \in \mathbb{Q}_+^k$ *such that* ${}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(\underline{a})_{|Z}$ *is big. Then there exists* $Y \subsetneq Z$ *and finitely many irreducible directed subvarieties* $(Z'_{\ell,\alpha}, W'_{\ell,\alpha}) \subset (X_{k+\ell}, V_{k+\ell})$ *contained in the Semple tower* (Z_ℓ, W_ℓ) *of* (Z, W), *with* $\operatorname{rank} W'_{\ell,\alpha} < \operatorname{rank} W$, *such that every entire curve* $f : (\mathbb{C}, T_\mathbb{C}) \to (Z, W)$ *is either contained in* Y, *or has a lifting* $f_{[\ell]} : (\mathbb{C}, T_\mathbb{C}) \to (Z'_{\ell,\alpha}, W'_{\ell,\alpha})$ *for some* α.

Idea of the proof. One can take $Y \subset Z$ to be a subvariety such that $Z \smallsetminus Y \subset Z_{\rm reg}$, $W_{|Z\smallsetminus Y}$ is non singular, and for some ample divisor A' on Z and $m_0 \gg 1$, the sheaf $({}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(\underline{a})_{|Z})^{\otimes m_0} \otimes \mathcal{O}_Z(-A')$ is generated by sections over $Z \smallsetminus Y$. The assumption on ${}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(\underline{a})_{|Z}$ allows to use again holomorphic Morse inequalities to construct a nontrivial section

$$\sigma \in H^0\big((\tilde{Z}_\ell, \mathcal{O}_{\tilde{Z}_\ell}(\underline{b}) \otimes \tilde{\pi}^*_{\ell,0}(\mu^*\mathcal{O}_{X_k}(p\,\underline{a})_{|Z} \otimes \mathcal{O}_{\tilde{Z}}(-A))\big)$$

on some nonsingular modification $\mu : \tilde{Z} \to Z$. This method produces a new differential equation $\sigma = 0$ that yields the desired subvariety $Z'_\ell = \bigcup Z'_{\ell,\alpha} \subsetneq Z_\ell$. □

We will also make use of the following observation due to Laytimi and Nahm [LNa99], valid for any holomorphic vector bundle $E \to X$ on a projective manifold (one can more generally consider torsion free coherent sheaves).

7.2. Definition. *We say that* $E \to X$ is strongly big *if for any* $A \in \operatorname{Pic}(X)$ *ample, the symmetric powers* $S^mE \otimes \mathcal{O}(-A)$ *are generated by their sections over a Zariski open set of* X, *for a sufficiently large integer* $m \gg 1$.

7.3. Lemma. *For every* $p \geq 0$, *if* Λ^pE *is big, then* $\Lambda^{p+1}E$ *is also big.*

Sketch of proof. This is a consequence of the fact, observed by [LNa99, Lemma 2.4], that for $k \gg 1$, the Schur irreducible components of $(\Lambda^{k+1}E)^{\otimes kp}$ all appear in $(\Lambda^kE)^{(k+1)p}$. We thank L. Manivel for pointing out this simple argument. □

From this, one can now derive the following statement.

7.4. Theorem. *Let* (X, V) *be a directed variety. Assume that* ${}^b\Lambda^pV^*$ *is a strongly big sheaf for some* $p \in \mathbb{N}^*$, $p \leq r = \operatorname{rank} V$.

(a) *If* $p = 1$, (X, V) *satisfies the generalized GGL conjecture, i.e., there exists a subvariety* $Y \subsetneq X$ *containing all entire curves* $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$.

(b) *If* $p \geq 2$, *there exists a subvariety* $Y \subsetneq X$, *an integer* $k \in \mathbb{N}$ *and a finite collection of induced directed subvarieties* $(Z_\alpha, W_\alpha) \subset (X_k, V_k)$ *with* Z_α *irreducible,* $\operatorname{rank} W_\alpha \leq p - 1$, *such that all entire curves* $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ *satisfy either* $f(\mathbb{C}) \subset Y$ *or have a* k*-jet lifting* $f_{[k]} : (\mathbb{C}, T_\mathbb{C}) \to (Z_\alpha, W_\alpha)$ *for some* α.

(c) *In particular, if* $p = 2$, *all entire curves* $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ *are either contained in* $Y \subsetneq X$, *or they are tangent to a* rank 1 foliation *on a subvariety* $Z = \bigcup Z_\alpha \subset X_k$. *This implies that the latter curves are supported by the parabolic leaves of the above foliations, which can be parametrized as a subspace of a finite dimensional variety.*

(d) *The subvariety* Y *described in* (a), (b) *or* (c) *can be taken to be any subvariety such that* $S^{m_0}({}^b\Lambda^pV^*) \otimes \mathcal{O}_X(-A)$ *is generated by sections over* $X \smallsetminus Y$, *for a suitable* $A \in \operatorname{Pic}(X)$ *ample and* $m_0 \gg 1$. *In particular, if* ${}^b\Lambda^pV^*$ *is ample, one can take* $Y = \emptyset$.

Proof. (a) As we already observed, the rank 1 case is an easy consequence of the Ahlfors-Schwarz lemma. Also, (c) is a particular case of (b), so we only have to check (b) and (d).

(b) For $p = r$, the statement is a consequence of Corollary 4.12. In general, we decompose all occurring subvarieties into irreductible components (Z_α, W_α) and apply descending induction on $r_\alpha = \operatorname{rank} W_\alpha$ for each of them. As long as $r_\alpha \geq p$, The assumption on V combined with Corollary 6.16 implies that there exists a weight $\underline{a} \in \mathbb{Q}_+^k$ such that ${}^b\mathcal{K}_{W_\alpha} \otimes \mathcal{O}_{X_k}(\underline{a})_{|Z_\alpha}$ is big (for this we use the fact that $\mathcal{O}_{X_k}(1)$ is relatively big with respect to $X_k \to X$). Then Proposition 7.1 either produces a subvariety $Y' \subsetneq Z_\alpha$ (in which case we consider the irreducible components Y'_β and apply descending induction on $\dim Y'_\beta$, if the rank of the induced structure does not decrease right away), or we get irreducible directed subvarieties $(Z'_{\ell,\beta}, W'_{\ell,\beta})$ in the ℓ-th stage of the Semple tower of (Z_α, W_α), with $r'_{\ell,\beta} = \operatorname{rank} W'_{\ell,\beta} < r_\alpha$. The induction hypothesis applies as long as $p \leq r'_{\ell,\beta} < r_\alpha$. Therefore we end up with $r'_{\ell,\beta} < p$ after finitely many iterations.

(d) is a consequence of the technique of proof of [Dem11] and [Dem14]. In fact we use singular hermitian metrics on V satisfying suitable positivity properties, and Y can be taken to be their set of poles. □

7.5. Remark. It would be interesting to know if the rather restrictive bigness hypothesis for ${}^b\Lambda^p V^*$ can be relaxed, assuming instead suitable semistability conditions. In fact, if none of the algebraic hypersurfaces contains all of the entire curves, one can show that there exists an Ahlfors current that defines a mobile bidegree $(n-1, n-1)$ class, and one could try to use the Harder-Narasimhan filtration of V with respect to such mobile classes.

7.6. Logarithmic and orbifold directed versions. More generally, let $\Delta = \sum \Delta_j$ be a reduced normal crossing divisor in X. We want to study entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$ drawn in the complement of Δ. At a point where $\Delta = \{z_1 \ldots z_p = 0\}$ one defines the *logarithmic cotangent sheaf* $T^*_{X\langle\Delta\rangle}$ to be generated by $\frac{dz_1}{z_1}, ..., \frac{dz_p}{z_p}, dz_{p+1}, ..., dz_n$. The results stated above can easily be extended to the logarithmic case. In particular, we obtain the following statement.

7.7. Theorem. *If $\Lambda^2 T^*_{X\langle\Delta\rangle}$ is strongly big on X, there exists a subvariety $Y \subsetneq X$ and rank 1 foliations $\mathcal{F}_\alpha$ on some subvarieties $Z_\alpha \subset X_k$ of the k-jet bundle, such that all entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$ are either contained in Y or have a k-jet lifting $f_{[k]}$ that is contained in some Z_α and tangent to $\mathcal{F}_\alpha$. When $\Lambda^2 T^*_{X\langle\Delta\rangle}$ is ample, we can take $Y = \emptyset$.*

One can obtain even more general versions dealing with entire curves $f : (\mathbb{C}, T_\mathbb{C}) \to (X, V)$ that are tangent to V and avoid a normal crossing divisor Δ transverse to V (*logarithmic case*), or meet $\Delta = \sum (1 - \frac{1}{\rho_j})\Delta_j$ with multiplicities $\geq \rho_j$ along Δ_j (*orbifold case*). Such statements are the subject of a work in progress with F. Campana, L. Darondeau and E. Rousseau. In this setting, the positivity conditions have to be expressed not just for the orbifold cotangent bundle, but also for the "*derived orbifold cotangent bundles*" of higher order $V^*\langle\Delta^{(s)}\rangle$, associated with the divisors $\Delta^{(s)} = \sum_j \big(1 - \frac{s}{\rho_j}\big)_+ \Delta_j$, $1 \leq s \leq k$.

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