L^2 extension theorems and applications to algebraic geometry

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Log canonical thresholds

The goal is to explain a proof of the strong openness conjecture for log canonical thresholds. Let Ω be a domain in $\mathbb{C}^n$, $f \in \mathcal{O}(\Omega)$ a holomorphic function, and $\varphi \in \mathrm{PSH}(\Omega)$ a psh function on Ω.

The log canonical threshold $c_{z_0}(\varphi) \in \,]0, +\infty]$ (or complex singularity exponent) is defined to be

$$c_{z_0}(\varphi) = \sup\left\{c > 0\,;\; e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\right\}.$$

A well known theorem of Skoda asserts that

$$\frac{1}{n}\nu(\varphi, z_0) \le c_{z_0}(\varphi)^{-1} \le \nu(\varphi, z_0).$$

For every holomorphic function f on Ω, we also introduce the weighted log canonical threshold $c_{f,z_0}(\varphi) \in \,]0, +\infty]$ of φ with weight f at z_0 to be

$$c_{f,z_0}(\varphi) = \sup\left\{c > 0\,;\; |f|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\right\}.$$

Semi-continuity theorem / strong openness

Theorem (Guan-Zhou 2013, version due to Pham H. Hiep 2014))

Let f be a holomorphic function on an open set Ω in $\mathbb{C}^n$ and let φ be a psh function on Ω.

(i) ("Semicontinuity theorem") Assume that $\int_{\Omega'} e^{-2c\varphi} dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$ and let $z_0 \in \Omega'$. Then there exists $\delta = \delta(c, \varphi, \Omega', z_0) > 0$ such that for every $\psi \in \mathrm{PSH}(\Omega')$, $\|\psi - \varphi\|_{L^1(\Omega')} \leq \delta$ implies $c_{z_0}(\psi) > c$. Moreover, as ψ converges to φ in $L^1(\Omega')$, the function $e^{-2c\psi}$ converges to $e^{-2c\varphi}$ in L^1 on every relatively compact open subset $\Omega'' \Subset \Omega'$.

(ii) ("Strong effective openness") Assume that $\int_{\Omega'} |f|^2 e^{-2c\varphi} dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$. When $\psi \in \mathrm{PSH}(\Omega')$ converges to φ in $L^1(\Omega')$ with $\psi \leq \varphi$, the function $|f|^2 e^{-2c\psi}$ converges to $|f|^2 e^{-2c\varphi}$ in L^1 norm on every relatively compact open subset $\Omega'' \Subset \Omega'$.

Consequences of the semi-continuity theorem

Corollary 1 (Strong openness, Guan-Zhou 2013)

For any plurisubharmonic function φ on a neighborhood of a point $z_0 \in \mathbb{C}^n$, the set $\{c > 0 : \ |f|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\}$ is an open interval $]0, c_{f,z_0}(\varphi)[$.

Proof. After subtracting a large constant to φ, we can assume $\varphi \le 0$. Then Cor. 1 is a consequence of assertion (ii) of the main theorem by taking Ω' small enough and $\psi = (1+\delta)\varphi$ with $\delta \searrow 0$.

Application to multiplier ideal sheaves (Guan-Zhou 2013)

Let $h = e^{-\varphi}$ a singular hermitian metric with φ quasi-psh. The "upper semicontinuous regularization" of $\mathcal{I}(h)$ is defined to be

$$\mathcal{I}_+(h) = \lim_{\varepsilon \to 0} \mathcal{I}(h^{1+\varepsilon}) = \lim_{\varepsilon \to 0} \mathcal{I}((1+\varepsilon)\varphi) = \lim_{k \to +\infty} \mathcal{I}((1+1/k)\varphi)$$

(by Noetherianity, this increasing sequence is stationary on all compact subsets). Then $\mathcal{I}_+(h) = \mathcal{I}(h)$.

Convergence from below / idea of the proof

Corollary 2 (Convergence from below)

If $\psi \le \varphi$ converges to φ in a neighborhood of $z_0 \in \mathbb{C}^n$, then $c_{f,z_0}(\psi) \le c_{f,z_0}(\varphi)$ converges to $c_{f,z_0}(\varphi)$.

Proof. We have by definition $c_{f,z_0}(\psi) \le c_{f,z_0}(\varphi)$ for $\psi \le \varphi$, but again (ii) shows that $c_{f,z_0}(\psi)$ becomes $\ge c$ for any given value $c \in (0, c_{f,z_0}(\varphi))$, when $\|\psi - \varphi\|_{L^1(\Omega')}$ is sufficiently small.

Phams's theorem is proved by induction on n ($n = 0, 1$ are easy).

Aassume that the theorem holds for dimension $n-1$. Let $f \in \mathcal{O}(\Delta^n_R)$ be holomorphic on a n-dimensional polydisc, such that $\int_{\Delta^n_R} |f(z)|^2 e^{-2c\varphi(z)} dV_{2n}(z)$ converges. The idea is to restrict f to a generic hyperplane $z_n = w_n$. By induction, the integral of the restriction still converges after increasing c to $c + \varepsilon$ (shrinking R). By the Ohsawa-Takegoshi theorem, the restriction can be extended to a function F and one proceeds by comparing f and F.

Key lemma in Pham's proof

Lemma (Pham)

Let $\varphi \le 0$ be psh and f be holomorphic on the polydisc Δ_R^n of center 0 and (poly)radius $R > 0$ in $\mathbb{C}^n$, such that for some $c > 0$

$$\int_{\Delta_R^n} |f(z)|^2 e^{-2c\,\varphi(z)} dV_{2n}(z) < +\infty.$$

Let $\psi_j \le 0$, $j \in \mathbb{N}$, be psh functions on Δ_R^n with $\psi_j \to \varphi$ in $L^1_{\rm loc}(\Delta_R^n)$, and assume that $f \equiv 1$ or $\psi_j \le \varphi$ for all $j \ge 1$. Then for every $r < R$ and $\varepsilon \in]0, \frac{1}{2}r]$, there exist a value $w_n \in \Delta_\varepsilon \smallsetminus \{0\}$ (in a set of measure > 0), an index $j_0 = j_0(w_n)$, a constant $\tilde c = \tilde c(w_n) > c$ and holomorphic functions F_j on Δ_r^n, $j \ge j_0$, such that $F_j(z) = f(z) + (z_n - w_n) \sum a_{j,\alpha} z^\alpha$ with $|w_n||a_{j,\alpha}| \le r^{-|\alpha|}\varepsilon$ for all $\alpha \in \mathbb{N}^n$, $\underline{\rm IM}(F_j) \le \underline{\rm IM}(f)$, and

$$\int_{\Delta_r^n} |F_j(z)|^2 e^{-2\tilde c\,\psi_j(z)} dV_{2n}(z) \le \frac{\varepsilon^2}{|w_n|^2} < +\infty, \quad \forall j \ge j_0.$$

[Here $\underline{\rm IM}(F)$ = Initial Monomial in lexicographic order at 0].

Idea of proof of the key lemma

By Fubini's theorem we have

$$\int_{\Delta_R}\left[\int_{\Delta_R^{n-1}}|f(z',z_n)|^2e^{-2c\,\varphi(z',z_n)}dV_{2n-2}(z')\right]dV_2(z_n)<+\infty.$$

Since the integral extended to a small disc $z_n\in\Delta_\eta$ tends to 0 as $\eta\to 0$, it will become smaller than any preassigned value, say $\varepsilon_0^2>0$, for $\eta\le\eta_0$ small enough. Therefore we can choose a set of positive measure of values $w_n\in\Delta_\eta\smallsetminus\{0\}$ such that

$$\int_{\Delta_R^{n-1}}|f(z',w_n)|^2e^{-2c\,\varphi(z',w_n)}dV_{2n-2}(z')\le\frac{\varepsilon_0^2}{\pi\eta^2}<\frac{\varepsilon_0^2}{|w_n|^2}.$$

Since the main theorem is assumed to hold for $n-1$, for any $\rho<R$ there exist $j_0=j_0(w_n)$ and $\tilde c=\tilde c(w_n)>c$ such that

$$\int_{\Delta_\rho^{n-1}}|f(z',w_n)|^2e^{-2\tilde c\,\psi_j(z',w_n)}dV_{2n-2}(z')<\frac{\varepsilon_0^2}{|w_n|^2},\quad\forall j\ge j_0.$$

Idea of proof of the key lemma (2)

By Ohsawa-Takegoshi, there exists a holomorphic function F_j on $\Delta_\rho^{n-1} \times \Delta_R$ such that $F_j(z', w_n) = f(z', w_n)$ for all $z' \in \Delta_\rho^{n-1}$, and

$$\int_{\Delta_\rho^{n-1}\times\Delta_R} |F_j(z)|^2 e^{-2\tilde{c}\,\psi_j(z)} dV_{2n}(z)$$
$$\leq C_n R^2 \int_{\Delta_\rho^{n-1}} |f(z', w_n)|^2 e^{-2\tilde{c}\,\psi_j(z', w_n)} dV_{2n-2}(z') \leq \frac{C_n R^2 \varepsilon_0^2}{|w_n|^2},$$

where C_n is a constant which only depends on n (the constant is universal for $R = 1$ and is rescaled by R^2 otherwise).
Taking $\rho = \frac{1}{2}(r + R)$, the mean value inequality implies

$$\|F_j\|_{L^\infty(\Delta_r^n)} \leq \frac{2^n C_n^{\frac{1}{2}} R\varepsilon_0}{\pi^{\frac{n}{2}}(R-r)^n |w_n|}.$$

Since $F_j(z', w_n) - f(z', w_n) = 0$, $\forall z' \in \Delta_r^{n-1}$, we can write $F_j(z) = f(z) + (z_n - w_n) g_j(z)$ for some holomorphic function $g_j(z) = \sum_{\alpha\in\mathbb{N}^n} a_{j,\alpha} z^\alpha$ on $\Delta_r^{n-1} \times \Delta_R$. Then analyze $\underline{\mathrm{IM}}(F_j)$ ...

Volume and numerical dimension of currents

Definition

let (X, ω) be a compact Kähler manifold, and $T \geq 0$ a closed $(1,1)$-current on X. The positive intersection $\langle T^p \rangle \in H^{p,p}_{\geq 0}(X)$ (in the sense of Boucksom) is

$$\lim_{\varepsilon \to 0} \Big(\limsup (\mu_{m,\varepsilon})_* (\beta^p_{m,\varepsilon}) \Big), \qquad \mu_{m,\varepsilon} : \widetilde{X}_{m,\varepsilon} \to X$$

for the Zariski decomposition $\mu^*_{m,\varepsilon} T_{m,\varepsilon} = \beta_{m,\varepsilon} + [E_{m,\varepsilon}]$ of Bergman approximations $T_{m,\varepsilon}$ of $T + \varepsilon\omega$. The volume is $\mathrm{Vol}(T) = \langle T^n \rangle$.

Numerical dimension of a current

$$\mathrm{nd}(T) = \max \big\{ p \in \mathbb{N}\,;\ \langle T^p \rangle \neq 0 \ \text{ in } H^{p,p}_{\geq 0}(X) \big\}.$$

Numerical dimension of a hermitian line bundle (L, h)

If $\Theta_{L,h} \geq 0$, one defines $\mathrm{nd}(L, h) = \mathrm{nd}(\Theta_{L,h})$.

Generalized Nadel vanishing theorem

Theorem (Junyan Cao, PhD thesis 2012)

Let X be compact Kähler, and (L,h) be s.t. $\Theta_{L,h} \geq 0$ on X. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0 \quad \text{for } q \geq n - \mathrm{nd}(L,h) + 1,$$

Moreover we have in fact $\mathcal{I}_+(h) = \mathcal{I}(h)$ by Guan-Zhou.

Remark 1. There is also a concept of numerical dimension of a class $\alpha \in H^{1,1}(X)$: one defines $\mathrm{nd}(L)$ to be $-\infty$ if L is not psef, and

$$\mathrm{nd}(L) = \max\{p \in \mathbb{N}\,;\ \lim_{\varepsilon \to 0} \sup_{\{T \in C_1(L),\ T \geq -\varepsilon\omega\}} \langle (T+\varepsilon\omega)^p \rangle \neq 0$$

when L is psef. In general, we have $\mathrm{nd}(L,h) \leq \mathrm{nd}(L)$, but it may happen that $\sup_{\{h,\ \Theta_{L,h} \geq 0\}} \mathrm{nd}(L,h) < \mathrm{nd}(L)$.

Remark 2. In the projective case, one can use a hyperplane section argument, using Tsuji's algebraic expression of $\mathrm{nd}(L,h)$:

$$\mathrm{nd}(L,h) = \max\left\{p \in \mathbb{N}\,;\ \exists Y^p \subset X,\ h^0(Y, (L^{\otimes m} \otimes \mathcal{I}(h^m))_{|Y}) \geq c m^p\right\}.$$

Proof of generalized Nadel vanishing (projective case)

Hyperplane section argument (projective case). Take $A =$ very ample divisor, $\omega = \Theta_{A,h_A} > 0$, and $Y = A_1 \cap \ldots \cap A_{n-p}$, $A_j \in |A|$. Then

$$\langle \Theta^p_{L,h} \rangle \cdot Y = \int_X \langle \Theta^p_{L,h} \rangle \cdot Y = \int_X \langle \Theta^p_{L,h} \rangle \wedge \omega^{n-p} > 0.$$

From this one concludes that $(\Theta_{L,h})_{|Y}$ is big.

Lemma (J. Cao)

When (L, h) is big, i.e. $\langle \Theta^n_{L,h} \rangle > 0$, there exists a metric $\widetilde{h}$ such that $\mathcal{I}(\widetilde{h}) = \mathcal{I}_+(h)$ with $\Theta_{L,\widetilde{h}} \geq \varepsilon\omega$ [Riemann-Roch].

Then Nadel $\Rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0$ for $q \geq 1$.

Conclude by induction on $\dim X$ and the exact cohomology sequence for the restriction to a hyperplane section.

Proof of generalized Nadel vanishing (Kähler case)

Kähler case. By the regularization theorem, one finds an approximation $\tilde{h}_\varepsilon = h_0 e^{-\tilde{\varphi}_\varepsilon}$ with analytic singularities of the metric h of L, such that $\Theta_{L,\tilde{h}_\varepsilon} \geq -\frac{1}{2}\varepsilon\omega$.

Then, by blowing-up X to achieve divisorial singularities for $\tilde{h}_\varepsilon$ and using Yau's theorem, one solves on X a singular Monge-Ampère equation: $\exists h_\varepsilon = h_0 e^{-\varphi_\varepsilon}$ with logarithmic poles, such that

$$(\Theta_{L,h_\varepsilon} + \varepsilon\omega)^n = C_\varepsilon \omega^n.$$

where $C_\varepsilon \geq \binom{n}{p}\langle\Theta^p_{L,h}\rangle \cdot (\varepsilon\omega)^{n-p} \sim C\varepsilon^{n-p}$, $p = \mathrm{nd}(L,h)$.

Another important fact is that one can ensure the equalities $\mathcal{I}_+(h) = \mathcal{I}(h^{1+\varepsilon}) = \mathcal{I}(h_\varepsilon)$ (looking deeper in the regularization).

Ch. Mourougane argument (PhD thesis 1996). Let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of $\Theta_{L,h} + \varepsilon\omega$ with respect to ω at each point $x \in X$. Then

$$\lambda_1 \ldots \lambda_n = C_\varepsilon \geq \text{Const } \varepsilon^{n-p}.$$

Final step: use Bochner-Kodaira formula

Moreover

$$\int_X \lambda_{q+1}\ldots\lambda_n\,\omega^n = \int_X \Theta_{L,h}^{n-q}\wedge\omega^q \leq \text{Const},\quad \forall q\geq 1,$$

so $\lambda_{q+1}\ldots\lambda_n \leq C$ on a large open set $U\subset X$ and

$$\lambda_q^q \geq \lambda_1\ldots\lambda_q \geq c\varepsilon^{n-p} \quad\Rightarrow\quad \lambda_q \geq c\varepsilon^{(n-p)/q} \ \text{ on } U,$$

$$\Rightarrow \sum_{j=1}^q(\lambda_j-\varepsilon) \geq \lambda_q - q\varepsilon \geq c\varepsilon^{(n-p)/q} - q\varepsilon > 0 \ \text{ for } q > n-p.$$

$\lambda_j =$ eigenvalues of $(\Theta_{L,h_\varepsilon}+\varepsilon\omega) \Rightarrow$ (eigenvalues of $\Theta_{L,h_\varepsilon}) = \lambda_j - \varepsilon$

and the Bochner-Kodaira formula yields

$$\|\overline{\partial}u\|_\varepsilon^2 + \|\overline{\partial}^* u\|_\varepsilon^2 \geq \int_U \Big(\sum_{j=1}^q(\lambda_j-\varepsilon)\Big)|u|^2 e^{-\varphi_\varepsilon}dV_\omega.$$

The fact that U has almost full volume allows to take the limit as $\varepsilon\to 0$ and conclude that $u=0$. QED

Hard Lefschetz theorem with psef coefficients

Hard Lefschetz theorem (D-Peternell-Schneider 2001)

Let (L, h) be a psef line bundle on a compact n-dimensional Kähler manifold (X, ω), $\Theta_{L,h} \geq 0$. Then, the Lefschetz map : $u \mapsto \omega^q \wedge u$ induces a surjective morphism :

$$\Phi^q_{\omega,h} : H^0(X, \Omega^{n-q}_X \otimes L \otimes \mathcal{I}(h)) \longrightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h)).$$

The proof is based on using approximated metrics $h_\nu = h_0 e^{-\varphi_\nu}$, $\varphi_\nu \downarrow \varphi$, that are smooth on $X \smallsetminus Z_\nu$, with an increasing sequence of analytic sets Z_ν, such that $\Theta_{L,h_\nu} \geq -\varepsilon_\nu \omega$. We also consider Kähler metrics $\omega_\nu \downarrow \omega$ that are complete on $X \smallsetminus Z_\nu$.

Any cohomology class $\{u\}$ is represented by a (ω_ν, h_ν)-harmonic (n, q) form u_ν with values in $K_X \otimes L \otimes \mathcal{I}(h_\nu)$. One gets a unique $(n-q, 0)$-form v_ν s.t. $\omega^q_\nu \wedge v_\nu = u_\nu$, and a Bochner type formula

$$\|\overline{\partial} u\|^2 + \|\overline{\partial}^*_{h_\nu} u\|^2 = \|\overline{\partial} v\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_{\nu,j}\Big) |u_{IJ}|^2 e^{-\varphi_\nu} dV_{\omega_\nu}.$$

Proof of the Hard Lefschetz theorem

Here the $\lambda_{\nu,j}$ are the curvature eigenvalues of Θ_{L,h_ν}, so $\lambda_{\nu,j} \ge -\varepsilon_\nu$.
Taking $u_\nu =$ harmonic representative, we get $\overline{\partial} u_\nu = \overline{\partial}^*_{h_\nu} u_\nu = 0$, hence

$$\|\overline{\partial} v_\nu\|^2 = \int_X |\overline{\partial} v_\nu|_{\omega_\nu} e^{-\varphi_\nu} dV_{\omega_\nu} \le q\varepsilon_\nu \int_X |u_\nu|^2_{\omega_\nu} e^{-\varphi_\nu} dV_{\omega_\nu}$$
$$\le q\varepsilon_\nu \int_X |u|^2_{\omega_\nu} e^{-\varphi_\nu} dV_{\omega_\nu} \le q\varepsilon_\nu \int_X |u|^2_{\omega} e^{-\varphi} dV_{\omega}.$$

We need the following consequence of the Ohsawa-Takegoshi theorem:

Equisingular approximation theorem

Writing $h = h_0 e^{-\varphi}$, there exists a decreasing sequence $\varphi_\nu \downarrow \varphi$
$\Rightarrow$ $h = \lim h_\nu$ with $h_\nu = h_0 e^{-\varphi_\nu}$, such that

- $\varphi_\nu \in C^\infty(X \smallsetminus Z_\nu)$,
  where Z_ν is an increasing sequence of analytic sets,
- $\mathcal{I}(h_\nu) = \mathcal{I}(h), \quad \forall \nu$,
- $\Theta_{L,h_\nu} \ge -\varepsilon_\nu \omega$.

Important complement by Xiaojun Wu

Theorem (Xiaojun Wu, PhD thesis 2020)

Let (L, h) be a psef line bundle on a compact Kähler manifold (X, ω), $\Theta_{L,h} \geq 0$. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces an isomorphism

$$H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \cap \mathrm{Ker}(\partial_h) \longrightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}(h)).$$

Moreover, each section $v \in H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathcal{I}(h)) \cap \mathrm{Ker}(\partial_h)$ is ∇_h-parallel, and gives rise to a holomorphic foliation of X by considering the subsheaf $\mathcal{F}_v = \{\xi \in \mathcal{O}(T_X)\,;\ i_\xi v = 0\} \subset \mathcal{O}(T_X)$.

Proof. In fact, with $c_q = i^{(n-q+1)^2}$, a formal integration by parts gives

$$\int_X |\partial_h v|_h^2 dV_\omega = \int_X c_q \{\partial_h v, \partial_h v\}_h \wedge \omega^{q-1} = -\int_X c_q \{i\overline{\partial}\partial_h v, v\}_h \wedge \omega^{q-1}$$

$$= -\int_X c_q \{\Theta_{L,h} v, v\}_h \wedge \omega^{q-1} \leq 0 \quad \Rightarrow \quad \partial_h v = 0.$$

One can check that this is meaningful in the sense of distributions.

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