L^2 extension theorems and applications to algebraic geometry

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Second lecture

Second lecture: extension with optimal L^2 estimates

Setup. Let $L \to X$ be a holomorphic line bundle, equipped with a singular hermitian metric $h = h_0 e^{-\varphi}$, φ quasi-psh. Let $\psi \in L^1_{\rm loc}$ such that $\varphi + \psi$ is quasi-psh, and $Y \subset X$ the subvariety defined by the conductor ideal $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

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For a section $f \in H^0(Y, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, the goal is to get an "extension" $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$,

$$\textit{via}\ \ \mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi}), \quad F \mapsto f,$$

with an explicit L^2 estimate of F on X in terms of a suitable L^2 integral of f on the subvariety Y.

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We first define the Ohsawa residual measure associated with f. As for f, this will be a measure supported on Y.

The Ohsawa residual measure

Given $f \in H^0(U, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$, there exists a Stein covering (U_i) of X and liftings $\widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h))$ of f on U_i via $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$. We obtain in this way a C^∞ extension $\widetilde{f} = \sum \xi_i \widetilde{f}_i$ where (ξ_i) is a partition of unity.

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Definition of the Ohsawa residual measure

For $g \in C_c(Y)$, $g \geq 0$, and $0 \leq \widetilde{g} \in C_c(X)$ extending g, we set

$$\int_Y g\, dV_Y[f^2, h, \psi] := \inf_{\widetilde{g}} \limsup_{t \to -\infty} \int_{\{t<\psi<t+1\}} \widetilde{g}\, |\widetilde{f}|^2_{\omega,h} e^{-\psi} dV_{X,\omega}.$$

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Proposition

$dV_Y[f^2, h, \psi]$ is independent of the choice of $\widetilde{f}$ as well as of ω, and defines a positive measure on Y (but not necessarily locally finite).

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$dV_Y[f^2, h, \psi]$ is independent of the choice of $\widetilde{f}$ as well as of ω, and defines a positive measure on Y (but not necessarily locally finite).

Proof. When $\delta \widetilde{f}_i \in H^0(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(he^{-\psi}))$, then $|\delta \widetilde{f}_i|^2_{\omega,h} e^{-\psi} \in L^1_{\mathrm{loc}}(X)$ and the lim sup $\to 0$ for $\mathrm{Supp}(\widetilde{g}) \subset U$.

The Ohsawa residual measure (2)

Example 1. Take $\psi(z) = r \log |s(z)|^2_{h_E}$, where $s \in H^0(X, E)$ and $r = \mathrm{rank}(E)$. Assume that $Y = s^{-1}(0)$ is of codimension r, that s is generically transverse to 0 on Y and $h \in C^\infty$. Then

$$dV_Y[f^2, h, \psi] = c_{n,r} \frac{|f|^2_{\omega,h}\, dV_{Y,\omega}}{|\Lambda^r(ds)|^2_{\omega,h_E}} \quad \text{on } Y \smallsetminus \{\Lambda^r(ds) = 0\}.$$

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Proof. Near a regular point z_0 be can pick a holomorphic frame $(e_\lambda)_{1\le\lambda\le r}$ of E and coordinates $(z_1, \ldots, z_n)$ such that (e_λ) is h-orthornormal and $(\partial/\partial z_j)$ is ω-orthonormal at z_0, and $s(z) = \sum_{1\le j\le r} \lambda_j z_j e_j$, $\lambda_j \neq 0$. Then $\omega \sim i\sum dz_j \wedge d\overline{z}_j$ and $\psi(z) \sim r\log(|\lambda_1|^2|z_1|^2 + \ldots + |\lambda_r|^2|z_r|^2)$. This is an easy calculation of integrals on ellipsoids.

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Example 2. Take now $\psi(z) = \sum c_j \log|s_{D_j}|^2_{h_j}$ where $D = \sum c_j D_j$ is a simple normal crossing divisor, $c_j > 0$, and h_j is a C^∞ metric on $\mathcal{O}_X(D_j)$. Also assume $h \in C^\infty$.

Ohsawa residual measure for s.n.c. singularities

By a change of coordinates, we are reduced to computing $dV_Y[f^2, h, \psi]$ for $\psi(z) = \sum c_j \log |z_j|^2 + u(z)$, $u \in C^\infty$. However

$$dV_Y[f^2, h, \psi + u] = e^{-u}\, dV_Y[f^2, h, \psi],$$

thus we may assume $u = 0$. At a regular point of $D_j \smallsetminus \bigcup_{k \neq j} D_k$, (and $j = 1$, say) we apply the Fubini theorem with $z = (z_1, z')$, $z' = (z_2, \ldots, z_n)$. We have to compute limits of the form

$$\lim_{t \to -\infty} \int_{e^t < |z_1|^{2c_1} < e^{t+1}} \frac{\widetilde{g}(z) |\widetilde{f}(z)|^2}{|z_1|^{2c_1}}\, i dz_1 \wedge d\overline{z}_1 = \frac{2\pi}{m_1}\, g(0, z') |\widetilde{h}(0, z')|^2$$

when $c_1 = m_1 \in \mathbb{N}^*$ and $\widetilde{f}(z) = z_1^{m_1 - 1} \widetilde{h}(z)$. However, if $c_j < 1$, we get 0, and in general, if $c_j \notin \mathbb{N}^*$ and $c_j > 1$, we can get only 0 or ∞ values, according to the divisibility of f by $z_j^{m_j - 1}$, $m_j = \lfloor c_j \rfloor \in \mathbb{N}^*$.

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As a consequence, we can capture an interesting (i.e. locally finite, non zero) residual measure $dV_Y[f^2, h, \psi]$ only in the case where one of the coefficients c_j is an integer.

Ohsawa residual measure for analytic singularities

One general case of interests is when ψ has analytic singularities, i.e. locally $\psi(z) = c \log \sum |g_j(z)|^2 + u(z)$, $g_j \in \mathcal{O}_X(V)$, $u \in C^\infty(V)$.

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By Hironaka, we know that there exists a composition of blow-ups $\mu : \widetilde{X} \to X$ such that the pull-back ideal $\mu^*(g_j) = (g_j \circ \mu)$ is an invertible ideal sheaf $\mathcal{O}_{\widetilde{X}}(-\sum m_j D_j)$ associated with a simple normal crossing divisor.

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$$\mathcal{I}(e^{-s\psi}) = \mu_*(K_{\widetilde{X}/X} \otimes \mathcal{I}(e^{-s\,\psi\circ\mu})) = \mu_* \mathcal{O}_{\widetilde{X}}\Big(\sum (a_j - \lfloor s m_j \rfloor) D_j\Big)$$

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where $K_{\widetilde{X}/X} = \mathcal{O}_{\widetilde{X}}(\sum a_j D_j)$. This implies that $\mathcal{I}(e^{-s\psi})$ “jumps” precisely for a discrete sequence of rational numbers $0 = s_0 < s_1 < \ldots < s_k < \ldots$ such that $s_k m_j \in \mathbb{N}$ for some j.

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For $f \in \mathcal{I}(e^{-s_{k-1}\psi})$, the measure $dV_Y[f^2, h, s_k\psi]$ will be interesting.

Restricted multiplier ideals

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Definition of the restricted multiplier ideal

For $x \in Y$, we define $\mathcal{I}'_\psi(h)_x \subset \mathcal{I}(h)_x$ to be the ideal of germs of functions $\widetilde{f} \in \mathcal{I}(h)_x$ associated with $f = \widetilde{f} \mod \mathcal{I}(he^{-\psi})_x$ in $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})_x$, for which $dV[f^2, h, \psi]$ is locally finite near x on Y.

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Typical case of application. Assume that $h = e^{-\varphi}$ and ψ have analytic singularities, and that $s_k = 1$ is one of jumping values for $s \mapsto \mathcal{I}(e^{-s\psi})$ (case of log canonical singularities: $s_1 = 1$).

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Then $\mathcal{I}'_\psi(h) \subset \mathcal{I}(he^{-s_{k-1}\psi})$ on X, and $\mathcal{I}'_\psi(h) = \mathcal{I}(he^{-s_{k-1}\psi})$ on a Zariski open subset $X_0 = X \smallsetminus Z$, $Z \subsetneq Y$ (however, the ideals may differ on Z).

Use of more "flexible" weights

The next issue is that we need special and rather flexible weights. Let $\alpha \in \,]0,1[$ and $A = \sup_X \psi \in \,]-\infty, +\infty]$. We consider functions $\rho : [-\infty, A] \to \mathbb{R}_+^*$, such as

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The L^2 estimates will involve integrals of the form $\int_X |F|^2_{\omega,h} e^{-\psi} |\rho'(\psi)|\,dV_{X,\omega}$, where $|\rho'(\psi)| = (C-\psi)^{-2}$ in the above example, so that $e^{-\psi}|\rho'(\psi)|$ is locally sommable when ψ has log canonical singularities.

General L^2 extension theorem

Theorem (X. Zhou-L. Zhu 2019)

Let (X,ω) be a weakly pseudoconvex Kähler manifold, L a holomorphic line bundle with a hermitian metric $h = h_0 e^{-\varphi}$, $h_0 \in C^\infty$, φ quasi-psh on X, and $\psi \in L^1_{\rm loc}(X)$.

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$$\Theta_{L,h} + (1+\nu\alpha) i\partial\overline{\partial}\psi \geq 0 \quad \text{on } X, \quad \nu = 0, 1.$$

Then, for every $f \in H^0(Y, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}'_\psi(h)/\mathcal{I}(he^{-\psi}))$ s.t.

$$\int_Y dV_Y[f^2, h, \psi] < +\infty,$$

there exists $F \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}'_\psi(h)$ that is mapped to f by the morphism $\mathcal{I}'_\psi(h) \to \mathcal{I}'_\psi(h)/\mathcal{I}(he^{-\psi})$, such that

$$\int_X |F|^2_{\omega,h} e^{-\psi} |\rho'(\psi)|\, dV_{X,\omega} \leq \rho(-\infty) \int_Y dV_Y[f^2, h, \psi].$$

(1) Construction of a smooth extension

Every section $f \in H^0(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$ admits a C^∞ lifting

$$\widetilde{f} = \sum \xi_i \widetilde{f}_i, \quad \widetilde{f}_i \in H^0\big(U_i, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)\big)$$

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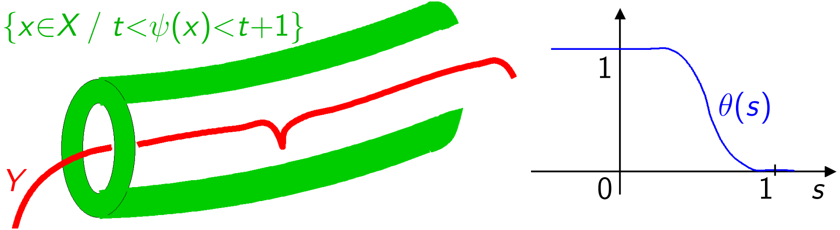
It will be enough to get estimates on X_c, and then let $c \to +\infty$.

(2) Solving the $\overline{\partial}$ equation

The next idea is to truncate $\widetilde{f}$ by multiplying $\widetilde{f}$ with a cut-off function $\theta(\psi - t)$ equal to 1 near $Y \subset \psi^{-1}(-\infty)$.



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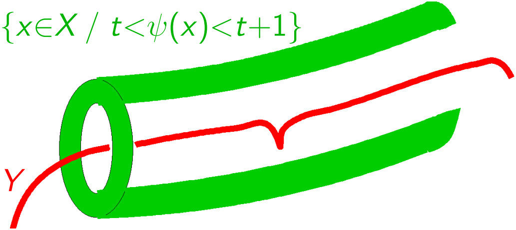

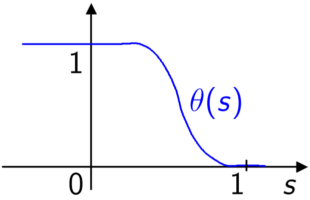

We next solve the approximate $\overline{\partial}$-equation

$$(*) \qquad \overline{\partial} u_{t,\varepsilon} = v_t + w_{t,\varepsilon}$$

with $v_t := \overline{\partial}(\theta(\psi - t) \cdot \widetilde{f}) = \theta(\psi - t) \cdot \overline{\partial}\widetilde{f} + \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f}$.

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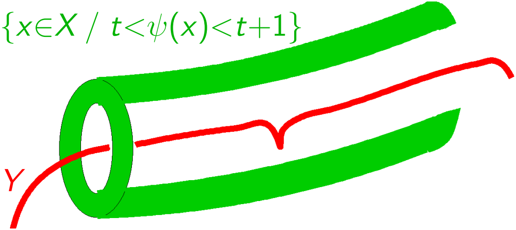

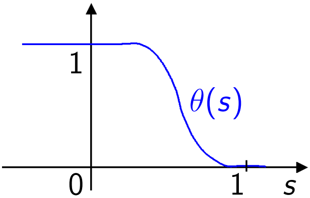

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It the weights ψ and φ of $h = h_0 e^{-\varphi}$ are not smooth, we use regularizations $\varphi_\delta \downarrow \varphi$, $\psi_\delta \downarrow \psi$ and complete Kähler metrics $\omega_\delta \downarrow \omega$ on $X \smallsetminus Z_\delta$. (We omit details here).

(3) L^2 estimates for solution and error term

The existence theorem with twisting factors $\eta_{t,\varepsilon}$, $\lambda_{t,\varepsilon}$ yields

$$\int_{X_c} (\eta_{t,\varepsilon} + \lambda_{t,\varepsilon})^{-1} |u_{t,\varepsilon}|^2_{\omega,h_0} e^{-\varphi-\psi} dV_{X,\omega} + \frac{1}{\varepsilon} \int_{X_c} |w_{t,\varepsilon}|^2_{\omega,h_0} e^{-\varphi-\psi} dV_{X,\omega}$$
$$\leq 4 \int_{X_c \cap \{\psi < t+1\}} |\overline{\partial}\widetilde{f}|^2_{\omega,h_0} e^{-\varphi-\psi} dV_\omega$$
$$+ 4 \int_{X_c \cap \{t < \psi < t+1\}} \langle (B_t + \varepsilon\,\mathrm{Id})^{-1} \overline{\partial}\psi \wedge \widetilde{f}, \overline{\partial}\psi \wedge \widetilde{f} \rangle_{\omega,h_0} e^{-\varphi-\psi}.$$

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The first integral in the right hand side tends to 0 as $t\to-\infty$.

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$$\zeta(u) = \log\frac{\rho(-\infty)}{\rho(u)},\quad \chi(u) = \frac{\int_u^A \rho(v)dv + \frac{1}{\alpha\rho(A)}}{\rho(u)},\quad \beta = \frac{(\chi')^2}{\chi\zeta''-\chi''},$$

$$\sigma_{t,\varepsilon}(u) = \max{}_\varepsilon(u,t),\quad \eta_{t,\varepsilon} = \chi(\sigma_{t,\varepsilon}(\psi)),\quad \lambda_{t,\varepsilon} = \beta(\sigma_{t,\varepsilon}(\psi)).$$

Extension from hypersurface (Stein case)

In the hypersurface case, one gets the following simpler statement.

Theorem

Let X be a Stein manifold of dimension n. Let φ and ψ be plurisubharmonic functions on X. Assume that w is a holomorphic function on X such that $\sup_X(\psi + 2\log|w|) \leq 0$ and dw does not vanish identically on any branch of $w^{-1}(0)$.

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Then for any holomorphic $(n-1)$-form f on Y_0 satisfying

$$\int_{Y_0} \mathrm{e}^{-\varphi-\psi} i^{(n-1)^2} f \wedge \bar{f} < +\infty,$$

there exists a holomorphic n-form F on X satisfying $F_{|Y_0} = dw \wedge f$ and an optimal estimate

$$\int_X \mathrm{e}^{-\varphi} i^{n^2} F \wedge \bar{F} \leq 2\pi \int_{Y_0} \mathrm{e}^{-\varphi-\psi} i^{(n-1)^2} f \wedge \bar{f}.$$

The Suita conjecture

The Suita conjecture was posed originally on open Riemann surfaces in 1972. The motivation was to answer a question posed by Sario and Oikawa about the relation between the Bergman kernel B_Ω for holomorphic $(1,0)$ forms on an open Riemann surface Ω which admits a Green function G_Ω.

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Recall that the logarithmic capacity $c_\beta(z)$ is locally defined by

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The Suita conjecture holds true (planar case: Błocki 2013; general case: Guan-Zhou 2014). Moreover (Guan-Zhou 2014), equality holds iff Ω biholomorphic to disc minus a closed polar set.

Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed (1,1)-current T such that $T \geq \delta\omega$ for a smooth $(1,1)$ form $\omega > 0$ and $\delta \ll 1$.

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Easy observation

$\alpha \in \mathcal{E}^\circ$ (interior of $\mathcal{E}$) $\iff$ $\alpha = \{T\}$, $T =$ a Kähler current.

We say that $\mathcal{E}^\circ$ is the cone of big $(1,1)$-classes.

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Theorem on approximate Zariski decomposition (D, 1992)

Any Kähler current can be written $T = \lim T_m$ where $T_m \in \{T\}$ has analytic singularities & logarithmic poles, i.e. $\exists$ modification $\mu_m : \widetilde{X}_m \to X$ such that $\mu_m^\star T_m = [E_m] + \beta_m$, where $E_m \geq 0$ is a $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coeff. in $\frac{1}{m}\mathbb{Z}$ and β_m is a Kähler form on $\widetilde{X}_m$.

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Moreover (Boucksom), $\mathrm{Vol}(\beta_m) = \int_{\widetilde{X}_m} \beta_m^n \to \mathrm{Vol}(T)$ as $m \to +\infty$.

Proof of the analytic Zariski decomposition

- Write locally on any coordinate ball $\Omega \subset X$

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where $(g_{\ell,m})$ is a Hilbert basis of the space

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The mean value inequality implies

$$|f(z)|^2 \le \frac{1}{\pi^n r^{2n}/n!}\sup_{B(z,r)} e^{2m\varphi(z)} \Rightarrow \varphi_m(z) \le \sup_{B(z,r)}\varphi + \frac{n}{m}\log\frac{C}{r}.$$

Use of the pointwise Ohsawa-Takegoshi theorem

- The Ohsawa-Takegoshi L^2 extension theorem (extension from a single isolated point) implies that for every $z_0 \in \Omega$, there exists $f \in \mathcal{O}(\Omega)$ such that $f(z_0) = c\, e^{m\varphi(z_0)}$ ($c > 0$ small), such that

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This implies Siu's analyticity result for Lelong upper level sets $E_c(T)$.

• The case of a global current $T = \alpha + dd^c\varphi$ is obtained by using a covering of X by balls Ω_j, and gluing the local approximations $\varphi_{j,m}$ of φ into a global one φ_m by a partition of unity.