L^2 extension theorems and applications to algebraic geometry

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General plan of the lectures

(1) First lecture: a general qualitative extension theorem

– Setup and general statement

– Main ideas of the proof

(2) Second lecture: extension with optimal L^2 estimates

– Ohsawa residual measure

– Log canonical case, case of higher order jets

– Main L^2 estimate; solution of the Suita conjecture

– Approximation of quasi-psh functions and currents

(3) Third lecture: applications

– Solution of the strong openness conjecture (Guan and Zhou)

– Pham's strong semicontinuity theorem

– Generalized Nadel vanishing theorem by Junyan Cao

– Hard Lefschetz theorem with psef coefficients
(and a complement by Xiaojun Wu)

First lecture

First lecture: notation and main concepts

Let (X, ω) be a possibly noncompact n-dimensional Kähler manifold, and L a holomorphic line bundle on X, with a possibly singular hermitian metric $h = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$. The curvature current is

$$\Theta_{L,h} = i\,\partial\overline{\partial} \log h^{-1} = i\partial\overline{\partial}\varphi$$

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Very often, one needs positivity assumptions for L.

Definition

- L is positive if $\exists h \in C^\infty$ such that $\Theta_{L,h} > 0$ ($\Leftrightarrow L$ ample);
- L is nef if $\forall \varepsilon > 0$, $\exists h_\varepsilon \in C^\infty$ such that $\Theta_{L,h_\varepsilon} \geq -\varepsilon\omega$;
- L is pseudoeffective (psef) if $\exists h$ singular such that $\Theta_{L,h} \geq 0$.

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Now, let $\mathcal{J} \subset \mathcal{O}_X$ a coherent ideal sheaf, $Y = V(\mathcal{J})$ its zero variety and $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}$. Here Y may be non reduced, i.e. $\mathcal{O}_Y$ may have nilpotent elements.

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Surjectivity / extension problem

Under which conditions on X, $Y = V(\mathcal{J})$ and (L, h) is

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a surjective restriction morphism?

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Equivalent injectivity problem

When is $H^{q+1}(X, K_X \otimes L \otimes \mathcal{J}) \to H^{q+1}(X, K_X \otimes L)$ injective ?

Multiplier ideal sheaves

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$$\mathcal{I}(e^{-\varphi})_{x_0} = \Big\{ f \in \mathcal{O}_{X,x_0} \,;\ \exists U \ni x_0\,,\ \int_U |f|^2 e^{-\varphi} d\lambda < +\infty \Big\}$$

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Moreover, $\mathcal{I}(e^{-\varphi})$ is always integrally closed.

One says that a quasi-psh function φ has *analytic singularities*, i.e. locally on a neighborhood V of an arbitrary point $x_0 \in X$ we have

$$\varphi(z) = c \log \sum |g_j(z)|^2 + u(z),\quad g_j \in \mathcal{O}_X(V),\ c > 0,\ u \in C^\infty(V),$$

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Example: $\varphi(z) = c \log |s(z)|^2_{h_E},\ c > 0,\ s \in H^0(X,E),\ h_E \in C^\infty.$

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Corollary

Assume instead that $(*)$ $\Theta_{L,h} + i\partial\overline{\partial}\psi \geq \alpha\omega$ for some quasi-psh function ψ on X. Then $H^q(X, K_X \otimes L \otimes \mathcal{I}(he^{-\psi})) = 0$ for $q \geq 1$, and for all $q \geq 0$, we have a surjective restriction morphism

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However, one would like to relax the strict positivity assumption $(*)$.

Motivation: abundance conjecture and MMP

One potential application would be to study the Minimal Model Program (MMP) for arbitrary projective – or even Kähler – varieties, whereas only the case of general type varieties is known.

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For a line bundle L, one defines the Kodaira-Iitaka dimension $\kappa(L) = \limsup_{m\to+\infty} \log\dim H^0(X, L^{\otimes m})/\log m$ and the numerical dimension $\mathrm{nd}(L) =$ maximum exponent p of non zero "positive intersections" $\langle T^p\rangle$ of a positive current $T \in c_1(L)$ when L is psef (pseudoeffective), and $\mathrm{nd}(L) = -\infty$ otherwise. They always satisfy

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The fundamental abundance conjecture can be stated: for each nonsingular klt pair (X, Δ) the $\mathbb{Q}$-line bundle $K_X + \Delta$ is abundant.

Generalized base point free theorem ?

One can try to investigate the abundance of $L = K_X + \Delta$ by induction on the dimension $n = \dim X$, by extending sections of $K_X + L_m$, $L_m = (m-1)K_X + m\Delta$ from subvarieties (noticing that $K_X + \Delta$ psef implies L_m psef, and even $L_m - \Delta$ psef).

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Standard base point free theorem

Let (X, Δ) be a projective klt pair, and L be a nef line bundle such that $L - (K_X + \Delta)$ is nef and big. Then L is semiample, i.e. $|mL|$ is base point free for some $m > 0$.

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Question (weak positivity variant of the BPF property ?)

Assume that X is not uniruled, i.e. that K_X is pseudoeffective, and let L be a line bundle such that $L - \varepsilon K_X$ is pseudoeffective for some $0 < \varepsilon \ll 1$. Does there exist $G \in \mathrm{Pic}^0(X)$ such that $L + G$ is abundant ?

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Then, for all $q \geq 0$, the following restriction map is surjective:

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Remark. Here $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ is supported on the subvariety $(Y, \mathcal{O}_Y)$ where $\mathcal{O}_Y = \mathcal{O}_X/\mathcal{J}_Y$ and $\mathcal{J}_Y$ is the conductor ideal:

$$\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h) \underset{\mathrm{def}}{=} \{f \in \mathcal{O}_X\,;\ f \cdot \mathcal{I}(h) \subset \mathcal{I}(he^{-\psi})\}.$$

Simple algebraic corollary

Assume that X is projective (or $\exists$ projective morphism $X \to S$ over S affine algebraic). Let $Y = \sum m_j Y_j$ be a simple normal crossing divisor, and $\mathcal{O}_Y = \mathcal{O}_X / \mathcal{O}_X(-Y)$.

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$$\psi(z) = \sum c_j \log |\sigma_{Y_j}|^2_{h_j}, \quad c_j > 0 \text{ such that } \lfloor c_j \rfloor = m_j,$$

for any choice of smooth hermitian metrics h_j on $\mathcal{O}_X(Y_j)$.
We have $i\partial\overline{\partial}\psi = \sum c_j(2\pi[Y_j] - \Theta_{\mathcal{O}(Y_j),h_j})$.

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Assume $\exists\,(G_\nu)_{\nu=0,1}$ semiample $\mathbb{Q}$-divisors such that

$$(**) \quad L - (1+\nu\alpha)\sum c_j Y_j \equiv G_\nu \mod \mathrm{Pic}^0(X), \quad c_j > 0, \ \alpha > 0.$$

Then, for $Y = \sum m_j Y_j$, $m_j = \lfloor c_j \rfloor$, there is a surjective morphism

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The case where ψ has analytic singularities can in fact always be reduced to the divisorial case by blowing up.

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Consequence. Coboundary spaces are closed in cocycle spaces.

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Assume X to be holomorphically convex. By the Cartan-Remmert theorem, this is the case iff X admits a proper holomorphic map $p : X \to S$ only a Stein complex space S.

Observation : cohomology is then always Hausdorff

Let X be a holomorphically convex complex space and $\mathcal{F}$ a coherent analytic sheaf over X. Then all cohomology groups $H^q(X, \mathcal{F})$ are Hausdorff with respect to their natural topology (local uniform convergence of holomorphic Čech cochains)

Proof. $H^q(X, \mathcal{F}) \simeq H^0(S, R^q p_* \mathcal{F})$ is a Fréchet space.

Consequence. Coboundary spaces are closed in cocycle spaces.

Corollary

To solve an equation $\overline{\partial} u = v$ on a holomorphically convex manifold X, it is enough to solve it approximately:

$$\overline{\partial} u_\varepsilon = v + w_\varepsilon, \qquad w_\varepsilon \to 0 \ \text{ as } \varepsilon \to 0.$$

(2) Twisted Bochner-Kodaira-Nakano inequality (Ohsawa-Takegoshi)

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For every compacted supported section $u \in \mathcal{C}^\infty_c(X, \Lambda^{p,q} T^*_X \otimes L)$ with values in a hermitian line bundle (L, h), one has

$$\|(\eta+\lambda)^{\frac{1}{2}}\overline{\partial}^* u\|^2 + \|\eta^{\frac{1}{2}}\overline{\partial} u\|^2 + \|\lambda^{\frac{1}{2}}\partial u\|^2 + 2\|\lambda^{-\frac{1}{2}}\partial\eta \wedge u\|^2 \geq \int_X \langle B^{p,q}_{L,h,\omega,\eta,\lambda} u, u\rangle dV_{X,\omega}$$

where $dV_{X,\omega} = \frac{1}{n!}\omega^n$ is the Kähler volume element and $B^{p,q}_{L,h,\omega,\eta,\lambda}$ is the Hermitian operator on $\Lambda^{p,q} T^*_X \otimes L$ such that

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In the sequel, we will apply this to the case of (n, q)-forms $(p = n)$, and choose $\eta, \lambda > 0$ so that $B^{p,q}_{L,h,\omega,\eta,\lambda}$ is ≥ 0 (or close).

(3) L^2 approximate solutions of $\overline{\partial}$-equations

L^2 existence theorem "with error term"

Let (X, ω) be a Kähler manifold possessing a complete Kähler metric let (E, h_E) be a Hermitian vector bundle over X. Assume that $B = B^{n,q}_{E,h,\omega,\eta,\lambda}$ satisfies $B + \varepsilon\,\mathrm{Id} > 0$ for some $\varepsilon > 0$ (so that B can be just semi-positive or slightly negative, e.g. $B \geq -\frac{\varepsilon}{2}\,\mathrm{Id}$).

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$$M(\varepsilon) := \int_X \langle (B + \varepsilon\,\mathrm{Id})^{-1} v, v\rangle \, dV_{X,\omega} < +\infty.$$

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Then there exists an approximate solution $u_\varepsilon \in L^2(X, \Lambda^{n,q-1}T^*_X \otimes E)$ and a correction term $w_\varepsilon \in L^2(X, \Lambda^{n,q}T^*_X \otimes E)$ such that

$$\overline{\partial} u_\varepsilon = v + w_\varepsilon \quad \text{and}$$

$$\int_X (\eta + \lambda)^{-1} |u_\varepsilon|^2\, dV_{X,\omega} + \frac{1}{\varepsilon} \int_X |w_\varepsilon|^2\, dV_{X,\omega} \leq M(\varepsilon).$$

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Moreover, notice that $\varepsilon\, M(\varepsilon)$ involves $\varepsilon(B + \varepsilon\,\mathrm{Id})^{-1} \leq 2\,\mathrm{Id}$.

(4) Represent cohomology classes as Čech cocycles

Every cohomology class in

$$H^q(X, \mathcal{O}_X(K_X \otimes L) \otimes \mathcal{I}(h)/\mathcal{I}(he^{-\psi}))$$

is represented by a holomorphic Čech q-cocycle with respect to a Stein covering $\mathcal{U} = (U_i)$, say $(c_{i_0 \ldots i_q})$,

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By the standard sheaf theoretic isomorphism with Dolbeault cohomology, this class is represented by a smooth (n, q)-form

$$f = \sum_{i_0,\ldots,i_q} c_{i_0\ldots i_q}\, \xi_{i_0} \overline{\partial}\xi_{i_1} \wedge \ldots \overline{\partial}\xi_{i_q}$$

by means of a partition of unity (ξ_i) subordinate to (U_i). This form is to be interpreted as a form on the (non necessarily reduced) analytic subvariety Y associated with the conductor ideal sheaf $\mathcal{J}_Y = \mathcal{I}(he^{-\psi}) : \mathcal{I}(h)$.

(5) Smooth lifting and associated $\overline{\partial}$ equation

We get an extension of f as a smooth (no longer $\overline{\partial}$-closed) (n,q)-form $\widetilde{f}$ on X by taking a lifting via $\mathcal{I}(h) \to \mathcal{I}(h)/\mathcal{I}(he^{-\psi})$

$$\widetilde{f} = \sum_{i_0,\ldots,i_q} \widetilde{c}_{i_0\ldots i_q} \xi_{i_0} \overline{\partial}\xi_{i_1} \wedge \ldots \overline{\partial}\xi_{i_q},$$

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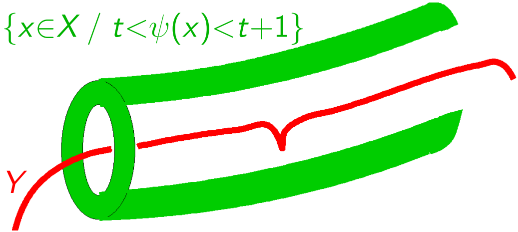

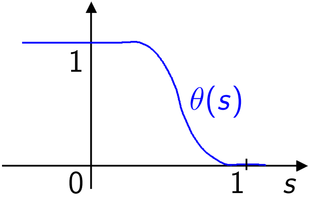

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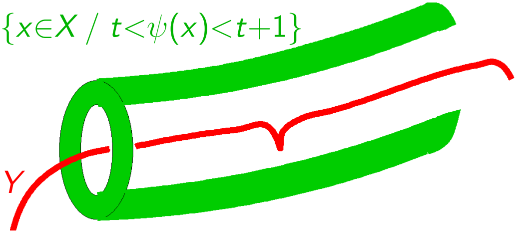

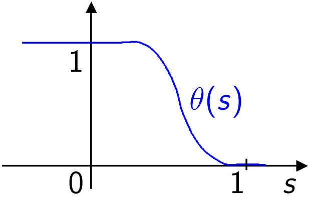

Now, truncate $\widetilde{f}$ as $\theta(\psi - t)\cdot\widetilde{f}$ on the green hollow tubular neighborhood, and solve an approximate $\overline{\partial}$-equation

$$(*) \qquad \overline{\partial}u_{t,\varepsilon} = \overline{\partial}(\theta(\psi - t)\cdot\widetilde{f}) + w_{t,\varepsilon}, \quad 0 \le \theta \le 1, \quad |\theta'| \le 1 + \varepsilon.$$

(6) L^2 bound and regularization of the metrics

Here we have

$$\overline{\partial}(\theta(\psi - t) \cdot \widetilde{f}) = \theta(\psi - t) \cdot \overline{\partial}\widetilde{f} + \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f}$$

where the second term vanishes near Y.

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Moreover the image of $\overline{\partial}\widetilde{f}$ in $\mathcal{I}(h)/\mathcal{I}(he^{-\psi})$ is $\overline{\partial}f = 0$, thus $\overline{\partial}\widetilde{f}$ has coefficients in $\mathcal{I}(he^{-\psi})$. Hence $\overline{\partial}\widetilde{f} \in L^2_{\mathrm{loc}}(he^{-\psi}) = L^2_{\mathrm{loc}}(h_0 e^{-\varphi-\psi})$.

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Truncate $p : X \to S$ by taking $X' = p^{-1}(S')$, $S' \Subset S$ Stein.

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Truncate $p : X\to S$ by taking $X' = p^{-1}(S')$, $S'\Subset S$ Stein. There are quasi-psh regularizations $\varphi_\delta\downarrow\varphi$, $\psi_\delta\downarrow\psi$ with analytic singularities, smooth on $X'\smallsetminus Z_\delta$, Z_δ analytic, and a complete Kähler metric ω_δ on $X'\smallsetminus Z_\delta$ such that

$$\int_{X'\smallsetminus Z_\delta}|\overline{\partial}\widetilde{f}|^2_{\omega_\delta,h_0}e^{-\varphi_\delta-\psi_\delta}dV_{\omega_\delta}\le\int_{X'}|\overline{\partial}\widetilde{f}|^2_{\omega,h_0}e^{-\varphi-\psi}dV_\omega < +\infty,$$

and we have an arbitrary small loss $O(\delta)$ of positivity in the curvature assumptions. Since ε errors are permitted, we take $\delta\ll\varepsilon$ and are reduced to the case where φ and ψ are smooth on X'.

(7) Bound of the error term in the $\overline{\partial}$-equation

We obtain an approximate L^2 solution $u_{t,\varepsilon}$ of the $\overline{\partial}$-equation

$\overline{\partial} u_{t,\varepsilon} = v_t + w_{t,\varepsilon}$, $\;v_t := \theta(\psi - t) \cdot \overline{\partial}\widetilde{f} + \theta'(\psi - t)\overline{\partial}\psi \wedge \widetilde{f}$, with

$$\int_{X'} |w_{t,\varepsilon}|^2_{\omega,h_0}\, e^{-\varphi-\psi} dV_{X,\omega} \leq 4 \int_{X' \cap \{\psi < t+1\}} |\overline{\partial}\widetilde{f}|^2_{\omega,h_0}\, e^{-\varphi-\psi} dV_\omega$$
$$+ 4 \int_{X' \cap \{t < \psi < t+1\}} \varepsilon \langle (B_t + \varepsilon\,\mathrm{Id})^{-1} \overline{\partial}\psi \wedge \widetilde{f}, \overline{\partial}\psi \wedge \widetilde{f} \rangle_{\omega,h_0}\, e^{-\varphi-\psi}.$$

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The main point is to choose ad hoc factors $\eta=\eta_t$, $\lambda=\lambda_t$ in the twisted Bochner identity to get the last integral to converge to 0. As $X'\Subset X$, we can assume α constant and $\psi<0$. For $u<0$, set

$$\zeta(u) = \log\frac{\frac{1}{\alpha}+1}{\frac{1}{\alpha}+1-e^u},\;\; \chi(u) = \frac{\frac{1}{\alpha^2}-1+e^u-(\frac{1}{\alpha}+1)u}{\frac{1}{\alpha}+1-e^u},\;\; \beta = \frac{(\chi')^2}{\chi\zeta''-\chi''}.$$

One checks that $\varepsilon = e^{2t}$, $\sigma_t(u) = \log(e^u+e^t)$, $\eta_t = \chi(\sigma_t(\psi))$, $\lambda_t = \beta(\sigma_t(\psi))$ and $h_0\mapsto h_t = h_0 e^{-\zeta(\sigma_t(\psi))}$ yield an $O(e^t)$ bound.