From the Ohsawa-Takegoshi theorem to asymptotic cohomology estimates

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dedicated to Prof. Takeo Ohsawa and Tetsuo Ueda
on the occasion of their sixtieth birthday

July 20, 2011 / Kyoto Symposium SCV XIV

The Ohsawa-Takegoshi extension theorem

Theorem (Ohsawa-Takegoshi 1987), (Manivel 1993)

Let (X, ω) be a Kähler manifold, which is compact or weakly pseudoconvex, $n = \dim_{\mathbb{C}} X$, $L \to X$ a hermitian line bundle, E a hermitian holomorphic vector bundle, and $s \in H^0(X, E)$ s. t. $Y = \big\{x \in X\,;\ s(x) = 0, \Lambda^r ds(x) \neq 0\big\}$ is dense in $\overline{Y} = \{s(x) = 0\}$, so that $p = \dim \overline{Y} = n - r$. Assume that $\exists \alpha(x) \geq 1$ continuous s. t.

(i) $\quad i\Theta_L + r\, i\,\partial\overline{\partial} \log |s|^2 \geq \max\left(0, \alpha^{-1}\dfrac{\{i\Theta_E s, s\}}{|s|^2}\right)$

(ii) $\quad |s| \leq e^{-\alpha}$. Then

$\forall f \in H^0(Y, (K_X \otimes L)_{|Y})$ s. t. $\int_Y |f|^2 |\Lambda^r(ds)|^{-2} dV_{Y,\omega} < +\infty$,
$\exists F \in H^0(X, K_X \otimes L)$ s. t. $F_{|Y} = f$ and

$$\int_X \frac{|F|^2}{|s|^{2r}(-\log|s|)^2}\, dV_{X,\omega} \leq C_r \int_Y \frac{|f|^2}{|\Lambda^r(ds)|^2} dV_{Y,\omega}.$$

The Ohsawa-Takegoshi extension theorem (II)

Theorem (Ohsawa-Takegoshi 1987)

Let $X = \Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex set, φ a plurisubharmonic function on Ω and $Y = \Omega \cap S$ where S is an affine linear subspace of $\mathbb{C}^n$ of any codimension r.

For every $f \in H^0(Y, \mathcal{O}_Y)$ such that $\int_Y |f|^2 e^{-\varphi} dV_Y < +\infty$, there exists $F \in H^0(\Omega, \mathcal{O}_\Omega)$ s. t. $F_{|Y} = f$ and

$$\int_\Omega |F|^2 e^{-\varphi}\, dV_\Omega \le C_r (\mathrm{diam}\,\Omega)^{2r} \int_Y |f|^2 e^{-\varphi} dV_Y.$$

Even the case when $Y = \{z_0\}$ is highly non trivial, thanks to the L^2 estimate : $\forall z_0 \in \Omega$, $\exists F \in H^0(\Omega, \mathcal{O}_\Omega)$, such that $F(z_0) = C_n^{-1/2} (\mathrm{diam}\,\Omega)^{-n} e^{\varphi(z_0)/2}$ and

$$\|F\|^2 = \int_\Omega |F|^2 e^{-\varphi}\, dV_\Omega \le 1.$$

Local approximation of plurisubharmonic functions

Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded pseudoconvex set, φ a plurisubharmonic function on Ω. Consider the Hilbert space

$$\mathcal{H}(\Omega, m\varphi) = \Big\{f \in \mathcal{O}(\Omega)\,;\ \int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty\Big\}.$$

One defines an "approximating sequence" of φ by putting

$$\varphi_m(z) = \frac{1}{2m} \log \sum_{j\in\mathbb{N}} |g_{j,m}(z)|^2$$

where $(g_{j,m})$ is a Hilbert basis of $\mathcal{H}(\Omega, m\varphi)$ (Bergman kernel procedure).

If $\mathrm{ev}_z : \mathcal{H}(\Omega, m\varphi) \to \mathbb{C}$ is the evaluation linear form, one also has

$$\varphi_m(z) = \frac{1}{m} \log \|\,\mathrm{ev}_z\,\| = \frac{1}{m} \sup_{f\in\mathcal{H}(\Omega,m\varphi),\ \|f\|\le 1} \log |f(z)|.$$

Local approximation of psh functions (II)

The Ohsawa-Takegoshi approximation theorem implies

$$\varphi_m(z) \geq \varphi(z) - \frac{C_1}{m}$$

In the other direction, the mean value inequality gives

$$\varphi_m(z) \leq \sup_{B(z,r)} \varphi + \frac{n}{m} \log \frac{C_2}{r}, \qquad \forall B(z,r) \subset \Omega$$

Corollary 1 (“strong psh approximation”)

One has $\lim \varphi_m = \varphi$ and the Lelong-numbers satisfy

$$\nu(\varphi, z) - \frac{n}{m} \leq \nu(\varphi_m, z) \leq \nu(\varphi, z).$$

Corollary 2 (new proof of Siu’s Theorem, 1974)

The Lelong-number sublevel sets
$E_c(\varphi) = \{z \in \Omega\,;\ \nu(\varphi, z) \geq c\}$, $c > 0$ are analytic subsets.

Approximation of global closed (1,1) currents

Let (X, ω) be a compact Kähler manifold and $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ a cohomology class given by a smooth representative α.

Let $T \in \{\alpha\}$ be an almost positive current, i.e. a closed $(1,1)$-current such that

$$T = \alpha + i\partial\overline{\partial}\varphi, \qquad T \geq \gamma$$

where γ is a continuous $(1,1)$-form (e.g. $\gamma = 0$ in case $T \geq 0$).

One can write $T = \alpha + i\partial\overline{\partial}\varphi$ for some quasi-psh potential φ on X, with $i\partial\overline{\partial}\varphi \geq \gamma - \alpha$. Then use a finite covering $(B_j)_{1\leq j\leq N}$ of X by coordinate balls, a partition of unity (θ_j), and set

$$\varphi_m(z) = \sum_{j=1}^{N} \theta_j(z)\Big(\psi_{j,m} + \sum_{k=1}^{n} \lambda_{j,k}|z_k^{(j)}|^2\Big)$$

where $\psi_{j,m}$ are Bergman approximations of $\psi_j(z) := \varphi(z) - \sum \lambda_{j,k}|z_k^{(j)}|^2$ (coordinates $z^{(j)}$ and coefficients $\lambda_{j,k}$ are chosen so that ψ_j is psh on B_j).

Approximation of global closed (1,1) currents (II)

Approximation thorem ([D − 1992])

Let (X, ω) be a compact Kähler manifold and
$T = \alpha + i\partial\overline{\partial}\varphi \geq \gamma$ a quasi-positive closed $(1,1)$-currents.
Then $T = \lim T_m$ weakly where

(i) $T_m = \alpha + i\partial\overline{\partial}\varphi_m \geq \gamma - \varepsilon_m\omega, \qquad \varepsilon_m \to 0$

(ii) $\nu(T, z) - \dfrac{n}{m} \leq \nu(T_m, z) \leq \nu(T, z)$

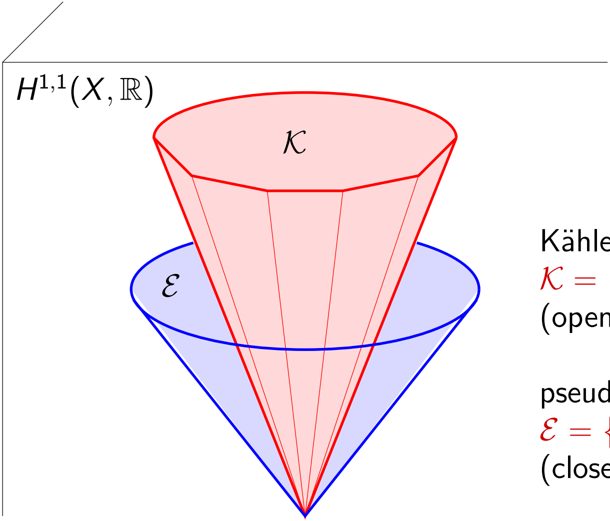
(iii) the potentials φ_m have only analytic singularities of the form $\frac{1}{2m}\log\sum_j |g_{j,m}|^2 + C^\infty$

(iv) The local coherent ideal sheaves $(g_{j,m})$ glue together into a global ideal $\mathcal{J}_m$ = multiplier ideal sheaf $\mathcal{I}(m\varphi)$.

The OT theorem implies that (φ_{2^m}) is decreasing, i.e. that the singularities of φ_{2^m} increase to those of φ by "subadditivity":

$$\mathcal{I}(\varphi + \psi) \subset \mathcal{I}(\varphi) + \mathcal{I}(\psi) \quad \Rightarrow \quad \mathcal{I}(2^{m+1}\varphi) \subset (\mathcal{I}(2^m\varphi))^2.$$

Kähler (red) cone and pseudoeffective (blue) cone



Kähler classes:
$\mathcal{K} = \{ \{\alpha\} \ni \omega \}$
(open convex cone)

pseudoeffective classes:
$\mathcal{E} = \{ \{\alpha\} \ni T \geq 0 \}$
(closed convex cone)

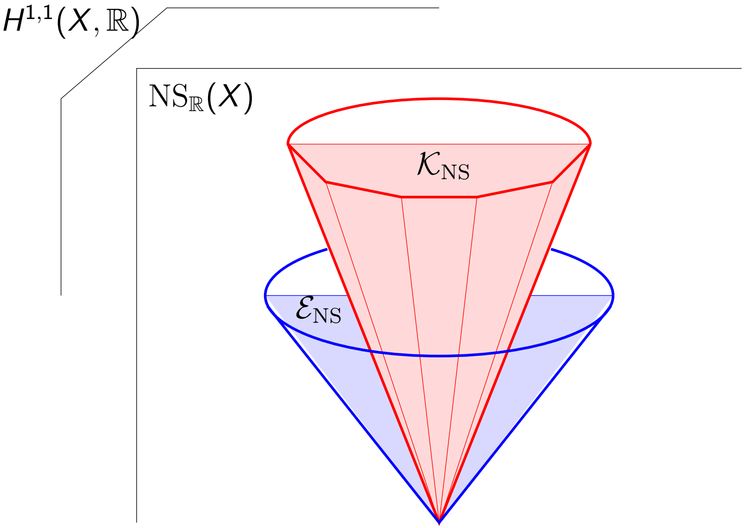
Neron Severi parts of the cones

In case X is projective, it is interesting to consider the "algebraic part" of our "transcendental cones" $\mathcal{K}$ and $\mathcal{E}$, which consist of suitable integral divisor classes. Since the cohomology classes of such divisors live in $H^2(X,\mathbb{Z})$, we are led to introduce the Neron-Severi lattice and the associated Neron-Severi space

$$\begin{aligned} \mathrm{NS}(X) &:= H^{1,1}(X,\mathbb{R}) \cap \big(H^2(X,\mathbb{Z})/\{\text{torsion}\}\big), \\ \mathrm{NS}_{\mathbb{R}}(X) &:= \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}, \\ \mathcal{K}_{\mathrm{NS}} &:= \mathcal{K} \cap \mathrm{NS}_{\mathbb{R}}(X) = \text{cone of ample divisors}, \\ \mathcal{E}_{\mathrm{NS}} &:= \mathcal{E} \cap \mathrm{NS}_{\mathbb{R}}(X) = \overline{\text{cone of effective divisors}}. \end{aligned}$$

The interior $\mathcal{E}^\circ$ is by definition the cone of big classes.

Neron Severi parts of the cones



Approximation of Kähler currents

Definition

On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.

Observation

$\alpha \in \mathcal{E}^\circ \Leftrightarrow \alpha = \{T\}, \ \ T =$ a Kähler current.

Consequence of approximation theorem

Any Kähler current T can be written $T = \lim T_m$ where $T_m \in \alpha = \{T\}$ has logarithmic poles, i.e.
$\exists$ a modification $\mu_m : \widetilde{X}_m \to X$ such that $\mu_m^\star T_m = [E_m] + \beta_m$
where : E_m effective $\mathbb{Q}$-divisor and β_m Kähler form on $\widetilde{X}_m$.

Proof of the consequence

Since $T \geq \delta\omega$, the main approximation theorem implies

$$T_m = i\partial\overline{\partial}\frac{1}{2m}\log\sum_j |g_{j,m}|^2 (\mathrm{mod}\ C^\infty) \geq \frac{\delta}{2}\omega, \quad m \geq m_0$$

and $\mathcal{J}_m = \mathcal{I}(m\varphi)$ is a global coherent sheaf. The modification $\mu_m : \widetilde{X}_m \to X$ is obtained by blowing-up this ideal sheaf, so that

$$\mu_m^\star \mathcal{J}_m = \mathcal{O}(-mE_m)$$

for some effective $\mathbb{Q}$-divisor E_m with normal crossings on $\widetilde{X}_m$. If h is a generator of $\mathcal{O}(-mE_m)$, and we see that

$$\beta_m = \mu_m^* T_m - [E_m] = \frac{1}{2m}\log\sum_j |g_{j,m}\circ\mu_m/h|^2 \quad \text{locally on } \widetilde{X}_m$$

hence β_m is a smooth semi-positive form on $\widetilde{X}_m$ which is > 0 on $\widetilde{X}_m \smallsetminus \mathrm{Supp}\, E_m$. By a perturbation argument using transverse negativity of exceptional divisors, β_m can easily be made Kähler.

Analytic Zariski decomposition

Theorem

For every class $\{\alpha\} \in \mathcal{E}$, there exists a positive current $T_{\min} \in \{\alpha\}$ with minimal singularities.

Proof. Take $T = \alpha + i\partial\overline{\partial}\varphi_{\min}$ where

$$\varphi_{\min}(x) = \max\{\varphi(x)\,;\ \varphi \leq 0 \text{ and } \alpha + i\partial\overline{\partial}\varphi \geq 0\}.$$

Theorem

Let X be compact Kähler and let $\{\alpha\} \in \mathcal{E}^\circ$ be a big class and $T_{\min} \geq 0$ be a current with minimal singularities. Then $T_{\min} = \lim T_m$ where T_m are Kähler currents such that

(i) $\exists$ modification $\mu_m : \widetilde{X}_m \to X$ with $\mu_m^\star T_m = [E_m] + \beta_m$, where E_m is a $\mathbb{Q}$-divisor and β_m a Kähler form on $\widetilde{X}_m$.

(ii) $\int_{\widetilde{X}_m} \beta_m^n$ is an increasing sequence converging to

$$\mathrm{Vol}(X, \{\alpha\}) := \int_X (T_{\min})^n_{\mathrm{ac}} = \sup_{T\in\{\alpha\},\mathrm{anal.sing}} \int_{X \smallsetminus \mathrm{Sing}(T)} T^n.$$

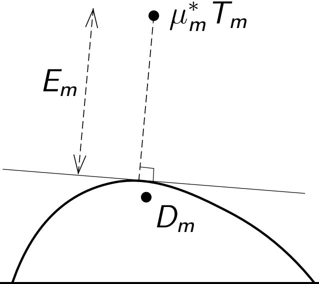
Orthogonality estimate

Theorem (Boucksom-Demailly-Păun-Peternell 2004)

Assume X projective and $\{\alpha\} \in \mathcal{E}^{\circ}_{\mathrm{NS}}$. Then $\beta_m = [D_m]$ is an ample $\mathbb{Q}$-divisor such that

$$(D_m^{n-1} \cdot E_m)^2 \le 20\,(C\omega)^n\big(\mathrm{Vol}(\alpha) - D_m^n\big)$$

where $\omega = c_1(H)$ is a fixed polarization and $C \ge 0$ is a constant such that $\pm\alpha$ is dominated by $C\omega$ (i.e., $C\omega \pm \alpha$ nef).



Proof similar to projection of a point onto a convex set, using elementary case of Morse inequalities:

$\mathrm{Vol}(\beta - \gamma) \ge \beta^n - n\beta^{n-1} \cdot \gamma$
$\forall \beta, \gamma$ ample classes

Duality between $\mathcal{E}_{\rm NS}$ and $\mathcal{M}_{\rm NS}$

Theorem (BDPP, 2004)

For X projective, a class α is in $\mathcal{E}_{\rm NS}$ (pseudo-effective) if and only if $\alpha \cdot C_t \geq 0$ for all mobile curves, i.e. algebraic curves which can be deformed to fill the whole of X.
In other words, $\mathcal{E}_{\rm NS}$ is the dual cone of the cone $\mathcal{M}_{\rm NS}$ of mobile curves with respect to Serre duality.

Proof. We want to show that $\mathcal{E}_{\rm NS} = \mathcal{M}_{\rm NS}^\vee$. By obvious positivity of the integral pairing, one has in any case

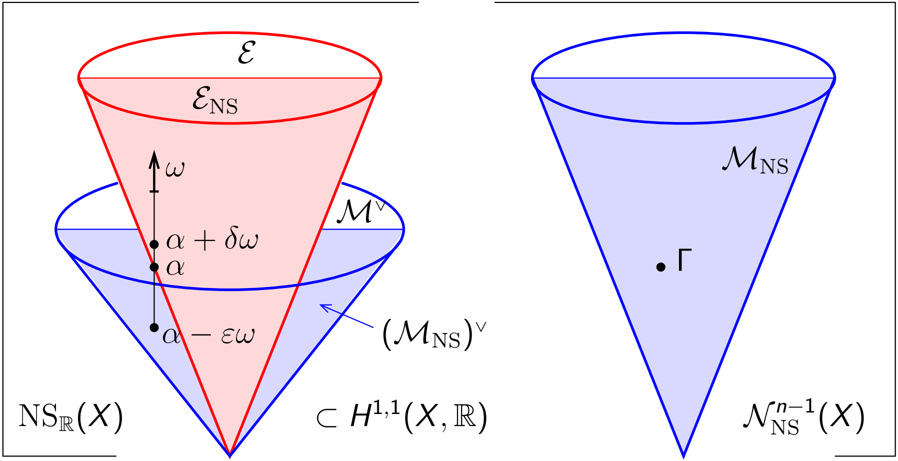
$$\mathcal{E}_{\rm NS} \subset (\mathcal{M}_{\rm NS})^\vee.$$

If the inclusion is strict, there is an element $\alpha \in \partial\mathcal{E}_{\rm NS}$ on the boundary of $\mathcal{E}_{\rm NS}$ which is in the interior of $\mathcal{N}_{\rm NS}^\vee$. Hence

$$\alpha \cdot \Gamma \geq \varepsilon\omega \cdot \Gamma$$

for every moving curve Γ, while $\langle \alpha^n \rangle = \mathrm{Vol}(\alpha) = 0$.

Schematic picture of the proof



Then use approximate Zariski decomposition of $\{\alpha + \delta\omega\}$ and orthogonality relation to contradict $(*)$ with $\Gamma = \langle \alpha^{n-1} \rangle$.

Characterization of uniruled varieties

Recall that a projective variety is called uniruled if it can be covered by a family of rational curves $C_t \simeq \mathbb{P}^1_{\mathbb{C}}$.

Theorem (Boucksom-Demailly-Paun-Peternell 2004)

A projective manifold X has its canonical bundle K_X pseudo-effective, i.e. $K_X \in \mathcal{E}_{\mathrm{NS}}$, if and only if X is not uniruled.

Proof (*of the non trivial implication*). If $K_X \notin \mathcal{E}_{\mathrm{NS}}$, the duality pairing shows that there is a moving curve C_t such that $K_X \cdot C_t < 0$. The standard "bend-and-break" lemma of Mori then implies that there is family Γ_t of rational curves with $K_X \cdot \Gamma_t < 0$, so X is uniruled.

Note: Mori's proof uses characteristic p, so it is hard to extend to the Kähler case !

Asymptotic cohomology functionals

Definition

Let X be a compact complex manifold and let $L \to X$ be a holomorphic line bundle.

(i) $\widehat{h}^q(X, L) := \limsup_{k \to +\infty} \frac{n!}{m^n} h^q(X, L^{\otimes m})$

(ii) *(asymptotic Morse partial sums)*

$\widehat{h}^{\leq q}(X, L) := \limsup_{m \to +\infty} \frac{n!}{m^n} \sum_{0 \leq j \leq q} (-1)^{q-j} h^j(X, L^{\otimes m})$.

Conjecture

$\widehat{h}^q(X, L)$ *and* $\widehat{h}^{\leq q}(X, L)$ *depend only on* $c_1(L) \in H^{1,1}_{\mathrm{BC}}(X, \mathbb{R})$.

Theorem (Küronya, 2005), (D, 2010)

This is true if $c_1(L)$ belongs to the "divisorial Neron-Severi group" $\mathrm{DNS}_{\mathbb{R}}(X)$ generated by divisors.

Holomorphic Morse inequalities

Theorem (D, 1985)

Let $L \to X$ be a holomorphic line bundle on a compact complex manifold. Then

(i) $\widehat{h}^q(X,L) \le \inf_{u\in c_1(L)} \int_{X(u,q)} (-1)^q u^n$

(ii) $\widehat{h}^{\le q}(X,L) \le \inf_{u\in c_1(L)} \int_{X(u,\le q)} (-1)^q u^n$

where $X(u,q)$ is the q-index set of the $(1,1)$-form u and $X(u,\le q) = \bigcup_{0\le j\le q} X(u,j)$.

Question (or Conjecture !)

Are these inequalities always equalities ?

If the answer is yes, then $\widehat{h}^q(X,L)$ and $\widehat{h}^{\le q}(X,L)$ actually only depend only on $c_1(L)$ and can be extended to $H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$, e.g.

$$h^{\le q}_{\mathrm{tr}}(X,\alpha) := \inf_{u\in\alpha} \int_{X(u,\le q)} (-1)^q u^n, \quad \forall\alpha \in H^{1,1}_{\mathrm{BC}}(X,\mathbb{R})$$

Converse of Andreotti-Grauert theorem

Theorem (D, 2010) / related result S.-I. Matsumura, 2011

Let X be a projective variety. Then

(i) *the conjectures are true for $q = 0$:*
$\widehat{h}^0(X, L) = \mathrm{Vol}(X, c_1(L)) = \inf_{u \in c_1(L)} \int_{X(u,0)} u^n$

(ii) *The conjectures are true for $\dim X \le 2$*

The limsup's are limits in all of these cases.

Observation 1. The question is invariant by Serre duality :

$$\widehat{h}^q(X, L) = \widehat{h}^{n-q}(X, -L)$$

Observation 2. (Birational invariance). If $\mu : \widetilde{X} \to X$ is a modification, then $\widehat{h}^q(X, L) = \widehat{h}^q(\widetilde{X}, \mu^* L)$ by the Leray spectral sequence and

$$\inf_{u \in \alpha} \int_{X(u, \le q)} (-1)^q u^n = \inf_{\widetilde{u} \in \mu^* \alpha} \int_{\widetilde{X}(\widetilde{u}, \le q)} (-1)^q \widetilde{u}^n.$$

Main idea of the proof

It is enough to consider the case of a big line bundle L. Then use approximate Zariski decomposition:

$$\forall\delta > 0,\ \exists\mu = \mu_\delta : \widetilde{X} \to X, \quad \mu^* L = E + A$$

where E is $\mathbb{Q}$-effective and A $\mathbb{Q}$-ample, and

$$\mathrm{Vol}(X, L) - \delta \le A^n \le \mathrm{Vol}(X, L), \quad E \cdot A^{n-1} \le C\delta^{1/2},$$

the latter inequality by the orthogonality estimate.

Take $\omega \in c_1(A)$ a Kähler form and a metric h on $\mathcal{O}(E)$ such that

$$\Theta_{\mathcal{O}(E),h} \wedge \omega^{n-1} = c_\delta \omega^n, \qquad c_\delta = O(\delta^{1/2}).$$

The last line is obtained simply by solving a Laplace equation, thanks to the orthogonality estimate.

End of the proof

$$\mu^* L = E + A \quad \Rightarrow \quad \widetilde{u} = \Theta_{\mathcal{O}(E),h} + \omega \in c_1(\mu^* L).$$

If $\lambda_1 \le \ldots \le \lambda_n$ are the eigenvalues of $\Theta_{\mathcal{O}(E),h}$ with respect to ω, then $\sum \lambda_i = \text{trace} \le C\delta^{1/2}$. We have

$$\widetilde{u}^n = \prod (1+\lambda_i)\omega^n \le \Big(1 + \frac{1}{n}\sum \lambda_i\Big)^n \omega^n \le (1 + O(\delta^{1/2})\omega^n,$$

therefore

$$\int_{\widetilde{X}(u,0)} \widetilde{u}^n \le (1 + O(\delta^{1/2}) \int_X \omega^n \le (1 + O(\delta^{1/2}) \,\mathrm{Vol}(X, L).$$

As $\delta \to 0$ we find

$$\inf_{u \in c_1(L)} \int_{X(u,0)} u^n = \inf_{\mu} \inf_{\widetilde{u} \in c_1(\mu^* L)} \int_{\widetilde{X}(u,0)} \widetilde{u}^n \le \mathrm{Vol}(X, L). \quad \text{QED}$$