Structure theorems for compact Kähler manifolds

Jean-Pierre Demailly
joint work with Frédéric Campana & Thomas Peternell

Institut Fourier, Université de Grenoble I, France
& Académie des Sciences de Paris

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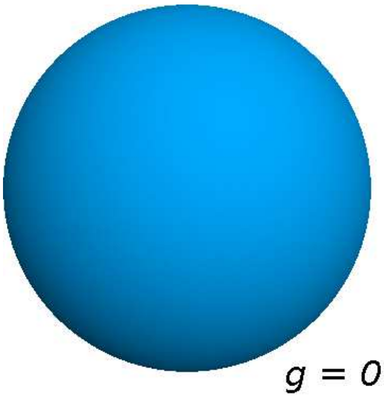
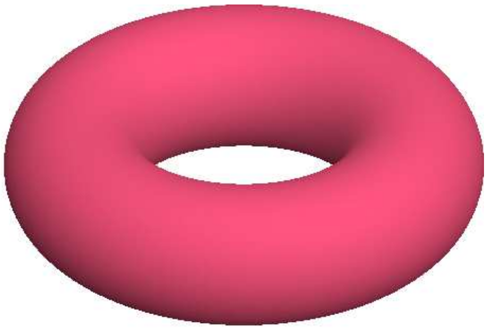
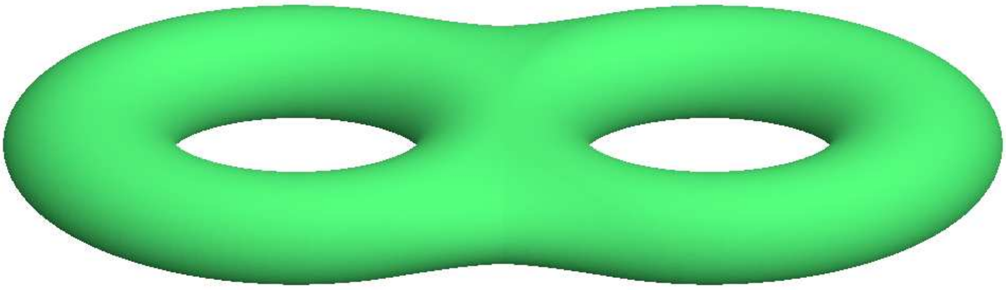
Goals

- Analyze the geometric structure of projective or compact Kähler manifolds
- As is well known since the beginning of the XX$^{\text{th}}$ century at least, the geometry depends on the sign of the curvature of the canonical line bundle
$$K_X = \Lambda^n T_X^*, \qquad n = \dim_{\mathbb{C}} X.$$
- $L \to X$ is pseudoeffective (psef) if $\exists h = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$, (possibly singular) such that $\Theta_{L,h} = -dd^c \log h \geq 0$ on X, in the sense of currents [for X projective: $c_1(L) \in \overline{\rm Eff}$].
- $L \to X$ is positive (semipositive) if $\exists h = e^{-\varphi}$ smooth s.t. $\Theta_{L,h} = -dd^c \log h > 0 \ \ (\geq 0)$ on X.
- L is nef if $\forall \varepsilon > 0$, $\exists h_\varepsilon = e^{-\varphi_\varepsilon}$ smooth such that $\Theta_{L,h_\varepsilon} = -dd^c \log h_\varepsilon \geq -\varepsilon\omega$ on X [for X projective: $L \cdot C \geq 0$, $\forall C$ alg. curve].

Complex curves ($n = 1$) : genus and curvature







$K_X = \Lambda^n T_X^*$, $\deg(K_X) = 2g - 2$

Comparison of positivity concepts

Recall that for a line bundle

$> 0 \Leftrightarrow$ ample $\Rightarrow$ semiample $\Rightarrow$ semipositive $\Rightarrow$ nef $\Rightarrow$ psef

but none of the reverse implications in red holds true.

Example

Let X be the rational surface obtained by blowing up $\mathbb{P}^2$ in 9 distinct points $\{p_i\}$ on a smooth (cubic) elliptic curve $C \subset \mathbb{P}^2$, $\mu : X \to \mathbb{P}^2$ and $\hat{C}$ the strict transform of C. Then

$$K_X = \mu^* K_{\mathbb{P}^2} \otimes \mathcal{O}(\textstyle\sum E_i) \Rightarrow -K_X = \mu^* \mathcal{O}_{\mathbb{P}^2}(3) \otimes \mathcal{O}(-\textstyle\sum E_i),$$

thus

$$-K_X = \mu^* \mathcal{O}_{\mathbb{P}^2}(C) \otimes \mathcal{O}(-\textstyle\sum E_i) = \mathcal{O}_X(\hat{C}).$$

One has

$$-K_X \cdot \Gamma = \hat{C} \cdot \Gamma \geq 0 \qquad \text{if } \Gamma \neq \hat{C},$$
$$-K_X \cdot \hat{C} = (-K_X)^2 = (\hat{C})^2 = C^2 - 9 = 0 \quad \Rightarrow \quad -K_X \text{ nef.}$$

Rationally connected manifolds

In fact

$$G := (-K_X)_{|\hat{C}} \simeq \mathcal{O}_{\mathbb{P}^2|C}(3) \otimes \mathcal{O}_C(-\sum p_i) \in \mathrm{Pic}^0(C)$$

If G is a torsion point in $\mathrm{Pic}^0(C)$, then one can show that $-K_X$ is semi-ample, but otherwise it is not semi-ample.

Brunella has shown that $-K_X$ is C^∞ semipositive if $c_1(G)$ satisfies a diophantine condition found by T. Ueda, but that otherwise it may not be semipositive (although nef).

$\mathbb{P}^2 \# 9$ points is an example of rationally connected manifold:

Definition

Recall that a compact complex manifold is said to be rationally connected (or RC for short) if any 2 points can be joined by a chain of rational curves

Remark. $X = \mathbb{P}^2$ blown-up in ≥ 10 points is RC but $-K_X$ not nef.

Ex. of compact Kähler manifolds with $-K_X \geq 0$

(Recall: By Yau, $-K_X \geq 0 \Leftrightarrow \exists\omega$ Kähler with $\mathrm{Ricci}(\omega) \geq 0$.)

- Ricci flat manifolds
  - – Complex tori $T = \mathbb{C}^q/\Lambda$
  - – Holomorphic symplectic manifolds S (also called hyperkähler): $\exists\sigma \in H^0(S, \Omega_S^2)$ symplectic
  - – Calabi-Yau manifolds Y: $\pi_1(Y)$ finite and some multiple of K_Y is trivial (may assume $\pi_1(Y) = 1$ and K_Y trivial by passing to some finite étale cover)
- the rather large class of rationally connected manifolds Z with $-K_Z \geq 0$
- all products $T \times \prod S_j \times \prod Y_k \times \prod Z_\ell$.

Main result. Essentially, this is a complete list !

Structure theorem for manifolds with $-K_X \geq 0$

Theorem

[Campana, D, Peternell, 2012] Let X be a compact Kähler manifold with $-K_X \geq 0$. Then

(a) $\exists$ holomorphic and isometric splitting

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod Y_j \times \prod S_k \times \prod Z_\ell$$

where $Y_j =$ Calabi-Yau (holonomy $\mathrm{SU}(n_j)$), $S_k =$ holomorphic symplectic (holonomy $\mathrm{Sp}(n'_k/2)$), and $Z_\ell =$ RC with $-K_{Z_\ell} \geq 0$ (holonomy $\mathrm{U}(n''_\ell)$).

(b) There exists a finite étale Galois cover $\widehat{X} \to X$ such that the Albanese map $\alpha : \widehat{X} \to \mathrm{Alb}(\widehat{X})$ is an (isometrically) locally trivial holomorphic fiber bundle whose fibers are products $\prod Y_j \times \prod S_k \times \prod Z_\ell$, as described in (a).

(c) $\pi_1(\widehat{X}) \simeq \mathbb{Z}^{2q} \rtimes \Gamma$, Γ finite ("almost abelian" group).

Criterion for rational connectedness

Criterion

Let X be a projective algebraic n-dimensional manifold. The following properties are equivalent.

(a) X is rationally connected.

(b) For every invertible subsheaf $\mathcal{F} \subset \Omega^p_X := \mathcal{O}(\Lambda^p T^*_X)$, $1 \leq p \leq n$, $\mathcal{F}$ is not psef.

(c) For every invertible subsheaf $\mathcal{F} \subset \mathcal{O}((T^*_X)^{\otimes p})$, $p \geq 1$, $\mathcal{F}$ is not psef.

(d) For some (resp. for any) ample line bundle A on X, there exists a constant $C_A > 0$ such that

$$H^0(X, (T^*_X)^{\otimes m} \otimes A^{\otimes k}) = 0 \quad \forall m,\ k \in \mathbb{N}^* \text{ with } m \geq C_A k.$$

Proof of the RC criterion

Proof (essentially from Peternell 2006)

(a) $\Rightarrow$ (d) is easy (RC implies there are many rational curves on which T_X, so $T_X^* < 0$), (d) $\Rightarrow$ (c) and (c) $\Rightarrow$ (b) are trivial.

Thus the only thing left to complete the proof is (b) $\Rightarrow$ (a).

Consider the MRC quotient $\pi : X \to Y$, given by the "equivalence relation $x \sim y$ if x and y can be joined by a chain of rational curves.

Then (by definition) the fibers are RC, maximal, and a result of Graber-Harris-Starr (2002) implies that Y is not uniruled.

By BDPP (2004), Y not uniruled $\Rightarrow K_Y$ psef. Then $\pi^* K_Y \hookrightarrow \Omega_X^p$ where $p = \dim Y$, which is a contradiction unless $p = 0$, and therefore X is RC.

Generalized holonomy principle

Generalized holonomy principle

Let $(E, h) \to X$ be a hermitian holomorphic vector bundle of rank r over X compact/$\mathbb{C}$. Assume that

$$\Theta_{E,h} \wedge \frac{\omega^{n-1}}{(n-1)!} = B\,\frac{\omega^n}{n!}, \quad B \in \mathrm{Herm}(E, E), \quad B \geq 0 \text{ on } X.$$

Let H the restricted holonomy group of (E, h). Then

(a) If there exists a psef invertible sheaf $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$, then $\mathcal{L}$ is flat and invariant under parallel transport by the connection of $(E^*)^{\otimes m}$ induced by the Chern connection ∇ of (E, h); moreover, H acts trivially on $\mathcal{L}$.

(b) If H satisfies $H = \mathrm{U}(r)$, then none of the invertible sheaves $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$ can be psef for $m \geq 1$.

Proof. $\mathcal{L} \subset \mathcal{O}((E^*)^{\otimes m})$ which has trace of curvature ≤ 0 while $\Theta_{\mathcal{L}} \geq 0$, use Bochner formula. □

Generically nef vector bundles

Definition

Let X compact Kähler manifold, $\mathcal{E} \to X$ torsion free sheaf.

(a) $\mathcal{E}$ is generically nef with respect to the Kähler class ω if

$$\mu_\omega(\mathcal{S}) \geq 0$$

for all torsion free quotients $\mathcal{E} \to \mathcal{S} \to 0$.
If $\mathcal{E}$ is ω-generically nef for all ω, we simply say that $\mathcal{E}$ is generically nef.

(b) Let

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = \mathcal{E}$$

be a filtration of $\mathcal{E}$ by torsion free coherent subsheaves such that the quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ are ω-stable subsheaves of $\mathcal{E}/\mathcal{E}_i$ of maximal rank. We call such a sequence a refined Harder-Narasimhan (HN) filtration w.r.t. ω.

Characterization of generically nef vector bundles

It is a standard fact that refined HN-filtrations always exist, moreover

$$\mu_\omega(\mathcal{E}_i/\mathcal{E}_{i-1}) \geq \nu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i)$$

for all i.

Proposition

Let (X,ω) be a compact Kähler manifold and $\mathcal{E}$ a torsion free sehaf on X. Then $\mathcal{E}$ is ω-generically nef if and only if

$$\mu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0$$

for some refined HN-filtration.

Proof. Easy arguments on filtrations. □

A result of J. Cao about manifolds with $-K_X$ nef

Theorem

(Junyan Cao, 2013) Let X be a compact Kähler manifold with $-K_X$ nef. Then the tangent bundle T_X is ω-generically nef for all Kähler classes ω.

Proof. use the fact that $\forall\varepsilon > 0$, $\exists$ Kähler metric with $\mathrm{Ricci}(\omega_\varepsilon) \geq -\varepsilon\,\omega_\varepsilon$ (Yau, DPS 1995).

From this, one can deduce

Theorem

Let X be a compact Kähler manifold with nef anticanonical bundle. Then the bundles $T_X^{\otimes m}$ are ω-generically nef for all Kähler classes ω and all positive integers m. In particular, the bundles $S^k T_X$ and $\bigwedge^p T_X$ are ω-generically nef.

A lemma on sections of contravariant tensors

Lemma

Let (X, ω) be a compact Kähler manifold with $-K_X$ nef and

$$\eta \in H^0(X, (\Omega^1_X)^{\otimes m} \otimes \mathcal{L})$$

where $\mathcal{L}$ is a numerically trivial line bundle on X.
Then the filtered parts of η w.r.t. the refined HN filtration are parallel w.r.t. the Bando-Siu metric in the 0 slope parts, and the < 0 slope parts vanish.

Proof. By Cao's theorem there exists a refined HN-filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \ldots \subset \mathcal{E}_s = T_X^{\otimes m}$$

with ω-stable quotients $\mathcal{E}_{i+1}/\mathcal{E}_i$ such that $\mu_\omega(\mathcal{E}_{i+1}/\mathcal{E}_i) \geq 0$ for all i. Then we use the fact that any section in a (semi-)negative slope reflexive sheaf $\mathcal{E}_{i+1}/\mathcal{E}_i \otimes \mathcal{L}$ is parallel w.r.t. its Bando-Siu metric (resp. vanishes). □

Smoothness of the Albanese morphism (after Cao)

Theorem (J.Cao 2013, D-Peternell, 2014)

Non-zero holomorphic p-forms on a compact Kähler manifold X with $-K_X$ nef vanish only on the singular locus of the refined HN filtration of T^*X.

This implies the following result essentially due to J.Cao.

Corollary

Let X be a compact Kähler manifold with nef anticanonical bundle. Then the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a submersion on the complement of the HN filtration singular locus in X [$\Rightarrow \alpha$ surjects onto $\mathrm{Alb}(X)$ (Paun 2012)].

Proof. The differential $d\alpha$ is given by $(d\eta_1, \ldots, d\eta_q)$ where $(\eta_1, \ldots, \eta_q)$ is a basis of 1-forms, $q = \dim H^0(X, \Omega^1_X)$.

Cao's thm $\Rightarrow$ rank of $(d\eta_1, \ldots, d\eta_q)$ is $= q$ generically. □

Emerging general picture

Conjecture (known for X projective!)

Let X be compact Kähler, and let $X \to Y$ be the MRC fibration (after taking suitable blow-ups to make it a genuine morphism). Then K_Y is psef.

Proof ? Take the part of slope > 0 in the HN filtration of T_X w.r.t. to classes in the dual of the psef cone, and apply duality.

Remaining problems

- Develop the theory of singular Calabi-Yau and singular holomorphic symplectic manifolds.
- Show that the "slope 0" part corresponds to blown-up tori, singular Calabi-Yau and singular holomorphic symplectic manifolds (as fibers and targets).
- The rest of T_X (slope < 0) yields a general type quotient.

Thank you for your attention!