

## Recent results on jet differentials and entire holomorphic curves in algebraic varieties

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## Kobayashi hyperbolicity and entire curves

### Definition

A complex space  $X$  is said to be **Kobayashi hyperbolic** if the Kobayashi pseudodistance  $d_{\text{Kob}} : X \times X \rightarrow \mathbb{R}_+$  is a distance (i.e. everywhere non degenerate).

By an **entire curve** we mean a non constant holomorphic map  $f : \mathbb{C} \rightarrow X$  into a complex  $n$ -dimensional manifold.

### Theorem (Brody, 1978)

For a **compact** complex manifold  $X$ ,  $\dim_{\mathbb{C}} X = n$ , TFAE:

- (i)  $X$  is **Kobayashi hyperbolic**
- (ii)  $X$  is **Brody hyperbolic**, i.e.  $\nexists$  entire curves  $f : \mathbb{C} \rightarrow X$
- (iii) The Kobayashi **infinitesimal pseudometric**  $\mathbf{k}_x$  is everywhere non degenerate

Our interest is the study of hyperbolicity for **projective varieties**.  
In  $\dim n = 1$ ,  $X$  is hyperbolic iff genus  $g \geq 2$ .

# Main conjectures

## Conjecture of General Type (CGT)

- A compact complex variety  $X$  is **volume hyperbolic**  $\iff X$  is of **general type**, i.e.  $K_X$  is big [implication  $\Leftarrow$  is well known].
- $X$  **Kobayashi (or Brody) hyperbolic** should imply  $K_X$  **ample**.

## Green-Griffiths-Lang Conjecture (GGL)

Let  $X$  be a projective variety/ $\mathbb{C}$  of general type. Then  $\exists Y \subsetneq X$  algebraic such that all entire curves  $f : \mathbb{C} \rightarrow X$  satisfy  $f(\mathbb{C}) \subset Y$ .

## Arithmetic counterpart (Lang 1987) – very optimistic ?

If  $X$  is projective and defined over a number field  $\mathbb{K}_0$ , the smallest locus  $Y = \text{GGL}(X)$  in GGL's conjecture is also the **smallest**  $Y$  such that  $X(\mathbb{K}) \setminus Y$  is finite  $\forall \mathbb{K}$  number field  $\supset \mathbb{K}_0$ .

## Consequence of CGT + GGL

A compact complex manifold  $X$  should be Kobayashi hyperbolic iff it is projective and every subvariety  $Y$  of  $X$  is of **general type**.

# Kobayashi conjecture on generic hyperbolicity

## Kobayashi conjecture (1970)

- Let  $X^n \subset \mathbb{P}^{n+1}$  be a (very) generic hypersurface of degree  $d \geq d_n$  large enough. Then  $X$  is Kobayashi hyperbolic.
- By a result of M. Zaidenberg, the optimal bound must satisfy  $d_n \geq 2n + 1$ , and one expects  $d_n = 2n + 1$ .

Using “jet technology” and **deep results of McQuillan** for curve foliations on surfaces, the following has been proved:

## Theorem (D., El Goul, 1998)

A very generic surface  $X^2 \subset \mathbb{P}^3$  of **degree  $d \geq 21$**  is hyperbolic. Independently McQuillan got  $d \geq 35$ .

This has been improved to  $d \geq 18$  (Păun, 2008).

In 2012, Yum-Tong Siu announced a proof of the case of **arbitrary dimension  $n$ , with a non explicit  $d_n$**  (and a rather involved proof).

## Results on the generic Green-Griffiths conjecture

By a combination of an algebraic existence theorem for jet differentials and of Y.T. Siu's technique of "slanted vector fields" (itself derived from ideas of H. Clemens, L. Ein and C. Voisin), the following was proved:

Theorem (S. Diverio, J. Merker, E. Rousseau, 2009)

A generic hypersurface  $X^n \subset \mathbb{P}^{n+1}$  of degree  $d \geq d_n := 2^{n^5}$  satisfies the GGL conjecture. Bound then improved to  $d_n = O(\exp(n^{1+\varepsilon}))$  :

$$d_n = \left\lfloor \frac{n^4}{3} (n \log(n \log(24n)))^n \right\rfloor \quad (\text{D-, 2012}),$$
$$d_n = (5n)^2 n^n \quad (\text{Darondeau, 2015}).$$

Theorem (S. Diverio, S. Trapani, 2009)

Additionally, a generic hypersurface  $X^3 \subset \mathbb{P}^4$  of degree  $d \geq 593$  is hyperbolic.

## Recent proof of the Kobayashi conjecture

In 2016, Brotbek gave a shorter and more geometric proof of Y.T. Siu's result on the Kobayashi conjecture, using again jet techniques.

Theorem (Brotbek, April 2016)

Let  $Z$  be a projective  $n + 1$ -dimensional projective manifold and  $A \rightarrow Z$  a very ample line bundle. Let  $\sigma \in H^0(Z, dA)$  be a generic section. Then, for  $d \gg 1$  large, the hypersurface  $X_\sigma = \sigma^{-1}(0)$  is **hyperbolic**.

The initial proof of Brotbek did not provide effective bounds. Through various improvements, Deng Ya showed in his PhD thesis:

Theorem (Y. Deng, May 2016)

In the above setting, a generic hypersurface  $X_\sigma = \sigma^{-1}(0)$  is hyperbolic as soon as

$$d \geq d_n = (n + 1)^{n+2} (n + 2)^{2n+7} = O(n^{3n+9}).$$

# Solution of a conjecture of Debarre (2005)

In the same vein, the following results have also been proved.

## Solution of Debarre's conjecture (Brotbek-Darondeau & Xie, 2015)

Let  $Z$  be a projective  $n + c$ -dimensional projective manifold and  $A \rightarrow Z$  a very ample line bundle. Let  $\sigma_j \in H^0(Z, d_j A)$  be generic sections,  $1 \leq j \leq c$ . Then, for  $c \geq n$  and  $d_j \gg 1$  large, the  $n$ -dimensional complete intersection  $X_\sigma = \bigcap \sigma_j^{-1}(0) \subset Z$  has an ample cotangent bundle  $T_{X_\sigma}^*$ .

In particular, such a generic complete intersection is hyperbolic.

Xie got the sufficient lower bound  $d_j \geq d_{n,c} = N^{N^2}$ ,  $N = n + c$ .

In his PhD thesis, Y. Deng obtained the improved lower bound

$$d_{n,c} = 4\nu(2N - 1)^{2\nu+1} + 6N - 3 = O((2N)^{N+3}), \quad \nu = \lfloor \frac{N+1}{2} \rfloor.$$

The proof is obtained by selecting carefully certain special sections  $\sigma_j$  associated with "lacunary" polynomials of high degree.

## Category of directed manifolds

**Goal.** More generally, we are interested in curves  $f : \mathbb{C} \rightarrow X$  such that  $f'(\mathbb{C}) \subset V$  where  $V$  is a subbundle of  $T_X$ , or possibly a singular linear subspace, i.e. a closed irreducible analytic subspace such that  $\forall x \in X$ ,  $V_x := V \cap T_{X,x}$  is linear.

### Definition (Category of directed manifolds)

- **Objects** : pairs  $(X, V)$ ,  $X$  manifold/ $\mathbb{C}$  and  $V \subset T_X$
- **Arrows**  $\psi : (X, V) \rightarrow (Y, W)$  holomorphic s.t.  $\psi_* V \subset W$
- **"Absolute case"**  $(X, T_X)$ , i.e.  $V = T_X$
- **"Relative case"**  $(X, T_{X/S})$  where  $X \rightarrow S$
- **"Integrable case"** when  $[\mathcal{O}(V), \mathcal{O}(V)] \subset \mathcal{O}(V)$  (foliations)

### Canonical sheaf of a directed manifold $(X, V)$

When  $V$  is nonsingular, i.e. a subbundle, one simply sets

$$K_V = \det(V^*) \quad (\text{as a line bundle}).$$

## Canonical sheaf of a singular pair $(X, V)$

When  $V$  is singular, we first introduce the rank 1 sheaf  ${}^b\mathcal{K}_V$  of sections of  $\det V^*$  that are **locally bounded** with respect to a smooth ambient metric on  $T_X$ . One can show that  ${}^b\mathcal{K}_V$  is equal to the integral closure of the image of the natural morphism

$$\mathcal{O}(\Lambda^r T_X^*) \rightarrow \mathcal{O}(\Lambda^r V^*) \rightarrow \mathcal{L}_V := \text{invert. sheaf } \mathcal{O}(\Lambda^r V^*)^{**}$$

that is, if the image is  $\mathcal{L}_V \otimes \mathcal{I}_V$ ,  $\mathcal{I}_V \subset \mathcal{O}_X$ ,

$${}^b\mathcal{K}_V = \mathcal{L}_V \otimes \overline{\mathcal{I}}_V, \quad \overline{\mathcal{I}}_V = \text{integral closure of } \mathcal{I}_V.$$

### Consequence

If  $\mu : \tilde{X} \rightarrow X$  is a modification and  $\tilde{X}$  is equipped with the pull-back directed structure  $\tilde{V} = \overline{\tilde{\mu}^{-1}(V)}$ , then

$${}^b\mathcal{K}_V \subset \mu_*({}^b\mathcal{K}_{\tilde{V}}) \subset \mathcal{L}_V$$

and  $\mu_*({}^b\mathcal{K}_{\tilde{V}})$  increases with  $\mu$ .

## Canonical sheaf of a singular pair $(X, V)$ [cont.]

By Noetherianity, one can define a sequence of rank 1 sheaves

$$\mathcal{K}_V^{[m]} = \lim_{\mu} \uparrow \mu_*({}^b\mathcal{K}_{\tilde{V}})^{\otimes m}, \quad \mu_*({}^b\mathcal{K}_V)^{\otimes m} \subset \mathcal{K}_V^{[m]} \subset \mathcal{L}_V^{\otimes m}$$

which we call the **pluricanonical sheaf sequence** of  $(X, V)$ .

### Remark

The blow-up  $\mu$  for which the limit is attained may depend on  $m$ . We do not know if there is a  $\mu$  that works for all  $m$ .

This generalizes the concept of **reduced singularities** of foliations, which is known to work in that form only for surfaces.

### Definition

We say that  $(X, V)$  is of **general type** if the **pluricanonical sheaf sequence**  $\mathcal{K}_V^{[\bullet]}$  is **big**, i.e.  $H^0(X, \mathcal{K}_V^{[m]})$  provides a generic embedding of  $X$  for a suitable  $m \gg 1$ .

## Definition of algebraic differential operators

Let  $(\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ ,  $t \mapsto f(t) = (f_1(t), \dots, f_n(t))$  be a curve written in some local holomorphic coordinates  $(z_1, \dots, z_n)$  on  $X$ . It has a local Taylor expansion

$$f(t) = x + t\xi_1 + \dots + t^k\xi_k + O(t^{k+1}), \quad \xi_s = \frac{1}{s!} \nabla^s f(0)$$

for some connection  $\nabla$  on  $V$ .

One considers the **Green-Griffiths bundle**  $E_{k,m}^{\text{GG}} V^*$  of polynomials of weighted degree  $m$  written locally in coordinate charts as

$$P(x; \xi_1, \dots, \xi_k) = \sum a_{\alpha_1 \alpha_2 \dots \alpha_k}(x) \xi_1^{\alpha_1} \dots \xi_k^{\alpha_k}, \quad \xi_s \in V,$$

also viewed as **algebraic differential operators**

$$\begin{aligned} P(f_{[k]}) &= P(f', f'', \dots, f^{(k)}) \\ &= \sum a_{\alpha_1 \alpha_2 \dots \alpha_k}(f(t)) f'(t)^{\alpha_1} f''(t)^{\alpha_2} \dots f^{(k)}(t)^{\alpha_k}. \end{aligned}$$

## Definition of algebraic differential operators [cont.]

Here  $t \mapsto z = f(t)$  is a curve,  $f_{[k]} = (f', f'', \dots, f^{(k)})$  its  **$k$ -jet**, and  $a_{\alpha_1 \alpha_2 \dots \alpha_k}(z)$  are supposed to holomorphic functions on  $X$ .

The reparametrization action :  $f \mapsto f \circ \varphi_\lambda$ ,  $\varphi_\lambda(t) = \lambda t$ ,  $\lambda \in \mathbb{C}^*$  yields  $(f \circ \varphi_\lambda)^{(k)}(t) = \lambda^k f^{(k)}(\lambda t)$ , whence a  $\mathbb{C}^*$ -action

$$\lambda \cdot (\xi_1, \xi_1, \dots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \dots, \lambda^k \xi_k).$$

$E_{k,m}^{\text{GG}}$  is precisely the set of polynomials of weighted degree  $m$ , corresponding to coefficients  $a_{\alpha_1 \dots \alpha_k}$  with  $m = |\alpha_1| + 2|\alpha_2| + \dots + k|\alpha_k|$ .

### Direct image formula

If  $J_k^{\text{nc}} V$  is the set of non constant  $k$ -jets, one defines the **Green-Griffiths bundle** to be  $X_k^{\text{GG}} = J_k^{\text{nc}} V / \mathbb{C}^*$  and  $\mathcal{O}_{X_k^{\text{GG}}}(1)$  to be the associated tautological rank 1 sheaf. Then we have

$$\pi_k : X_k^{\text{GG}} \rightarrow X, \quad E_{k,m}^{\text{GG}} V^* = (\pi_k)_* \mathcal{O}_{X_k^{\text{GG}}}(m)$$

# Generalized GGL conjecture

## Generalized GGL conjecture

If  $(X, V)$  is directed manifold of general type, i.e.  $\mathcal{K}_V^{[\bullet]}$  is big, then  $\exists Y \subsetneq X$  such that  $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ , one has  $f(\mathbb{C}) \subset Y$ .

**Remark.** Elementary by Ahlfors-Schwarz if  $r = \text{rk } V = 1$ .  
 $t \mapsto \log \|f'(t)\|_{V,h}$  is strictly subharmonic if  $r = 1$  and  $(V^*, h^*)$  big.

## Strategy : fundamental vanishing theorem

[Green-Griffiths 1979], [Demailly 1995], [Siu-Yeung 1996]

$\forall P \in H^0(X, E_{k,m}^{\text{GG}} V^* \otimes \mathcal{O}(-A))$  : global diff. operator on  $X$  ( $A$  ample divisor),  $\forall f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ , one has  $P(f_{[k]}) \equiv 0$ .

## Theorem on existence of jet differentials (D-, 2010)

Let  $(X, V)$  be of general type, such that  ${}^b\mathcal{K}_V^{\otimes p}$  is a **big** rank 1 sheaf. Then  $\exists$  many global sections  $P, m \gg k \gg 1 \Rightarrow \exists$  alg. hypersurface  $Z \subsetneq X_k^{\text{GG}}$  s.t. all entire  $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$  satisfy  $f_{[k]}(\mathbb{C}) \subset Z$ .

## 1<sup>st</sup> step: take a Finsler metric on $k$ -jet bundles

Let  $J_k V$  be the bundle of  $k$ -jets of curves  $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$   
 Assuming that  $V$  is equipped with a hermitian metric  $h$ , one defines a "weighted Finsler metric" on  $J^k V$  by taking  $p = k!$  and

$$\Psi_{h_k}(f) := \left( \sum_{1 \leq s \leq k} \varepsilon_s \|\nabla^s f(0)\|_{h(x)}^{2p/s} \right)^{1/p}, \quad 1 = \varepsilon_1 \gg \varepsilon_2 \gg \dots \gg \varepsilon_k.$$

Letting  $\xi_s = \nabla^s f(0)$ , this can actually be viewed as a metric  $h_k$  on  $L_k := \mathcal{O}_{X_k^{\text{GG}}}(1)$ , with curvature form  $(x, \xi_1, \dots, \xi_k) \mapsto$

$$\Theta_{L_k, h_k} = \omega_{\text{FS}, k}(\xi) + \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{1}{s} \frac{|\xi_s|^{2p/s}}{\sum_t |\xi_t|^{2p/t}} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta} \frac{\xi_{s\alpha} \bar{\xi}_{s\beta}}{|\xi_s|^2} dz_i \wedge d\bar{z}_j$$

where  $(c_{ij\alpha\beta})$  are the coefficients of the curvature tensor  $\Theta_{V^*, h^*}$  and  $\omega_{\text{FS}, k}$  is the vertical Fubini-Study metric on the fibers of  $X_k^{\text{GG}} \rightarrow X$ .

The expression gets simpler by using polar coordinates

$$x_s = |\xi_s|_h^{2p/s}, \quad u_s = \xi_s / |\xi_s|_h = \nabla^s f(0) / |\nabla^s f(0)|.$$



## 2<sup>nd</sup> step: probabilistic interpretation of the curvature

In such polar coordinates, one gets the formula

$$\Theta_{L_k, h_k} = \omega_{\text{FS}, p, k}(\xi) + \frac{i}{2\pi} \sum_{1 \leq s \leq k} \frac{1}{s} x_s \sum_{i, j, \alpha, \beta} c_{ij\alpha\beta}(z) u_{s\alpha} \bar{u}_{s\beta} dz_i \wedge d\bar{z}_j$$

where  $\omega_{\text{FS}, k}(\xi)$  is positive definite in  $\xi$ . The other terms are a weighted average of the values of the curvature tensor  $\Theta_{V, h}$  on vectors  $u_s$  in the unit sphere bundle  $SV \subset V$ .

The weighted projective space can be viewed as a circle quotient of the pseudosphere  $\sum |\xi_s|^{2p/s} = 1$ , so we can take here  $x_s \geq 0$ ,  $\sum x_s = 1$ . This is essentially a sum of the form  $\sum \frac{1}{s} \gamma(u_s)$  where  $u_s$  are random points of the sphere, and so as  $k \rightarrow +\infty$  this can be estimated by a "Monte-Carlo" integral

$$\left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right) \int_{u \in SV} \gamma(u) du.$$

As  $\gamma$  is quadratic here,  $\int_{u \in SV} \gamma(u) du = \frac{1}{r} \text{Tr}(\gamma)$ .

## 3<sup>rd</sup> step: getting the main cohomology estimates

$\Rightarrow$  the leading term only involves the trace of  $\Theta_{V^*, h^*}$ , i.e. the curvature of  $(\det V^*, \det h^*)$ , that can be taken  $> 0$  if  $\det V^*$  is big.

### Corollary of holomorphic Morse inequalities (D-, 2010)

Let  $(X, V)$  be a directed manifold,  $F \rightarrow X$  a  $\mathbb{Q}$ -line bundle,  $(V, h)$  and  $(F, h_F)$  hermitian. Define

$$L_k = \mathcal{O}_{X_k^{\text{GG}}}(1) \otimes \pi_k^* \mathcal{O}\left(\frac{1}{kr} \left(1 + \frac{1}{2} + \dots + \frac{1}{k}\right) F\right),$$

$$\eta = \Theta_{\det V^*, \det h^*} + \Theta_{F, h_F}.$$

Then for all  $q \geq 0$  and all  $m \gg k \gg 1$  such that  $m$  is sufficiently divisible, we have upper and lower bounds [ $q = 0$  most useful!]

$$h^q(X_k^{\text{GG}}, \mathcal{O}(L_k^{\otimes m})) \leq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n! (k!)^r} \left( \int_{X(\eta, q)} (-1)^q \eta^n + \frac{C}{\log k} \right)$$

$$h^q(X_k^{\text{GG}}, \mathcal{O}(L_k^{\otimes m})) \geq \frac{m^{n+kr-1}}{(n+kr-1)!} \frac{(\log k)^n}{n! (k!)^r} \left( \int_{X(\eta, q, q \pm 1)} (-1)^q \eta^n - \frac{C}{\log k} \right).$$



- **Fonctor “1-jet”** :  $(X, V) \mapsto (\tilde{X}, \tilde{V})$  where :
  - $\tilde{X} = P(V) =$  bundle of projective spaces of lines in  $V$
  - $\pi : \tilde{X} = P(V) \rightarrow X, \quad (x, [v]) \mapsto x, \quad v \in V_x$
  - $\tilde{V}_{(x,[v])} = \{\xi \in T_{\tilde{X},(x,[v])}; \pi_*\xi \in \mathbb{C}v \subset T_{X,x}\}$
- For every entire curve  $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$  tangent to  $V$   
 $f$  lifts as  $\begin{cases} f_{[1]}(t) := (f(t), [f'(t)]) \in P(V_{f(t)}) \subset \tilde{X} \\ f_{[1]} : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (\tilde{X}, \tilde{V}) \text{ (projectivized 1st-jet)} \end{cases}$
- **Definition.** *Semple jet bundles* :
  - $(X_k, V_k) = k$ -th iteration of fonctor  $(X, V) \mapsto (\tilde{X}, \tilde{V})$
  - $f_{[k]} : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X_k, V_k)$  is the **projectivized  $k$ -jet of  $f$** .
- **Basic exact sequences**

$$0 \rightarrow T_{X_k/X_{k-1}} \rightarrow V_k \xrightarrow{(\pi_k)^*} \mathcal{O}_{X_k}(-1) \rightarrow 0 \Rightarrow \text{rk } V_k = r$$

$$0 \rightarrow \mathcal{O}_{X_k} \rightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \rightarrow T_{X_k/X_{k-1}} \rightarrow 0 \text{ (Euler)}$$

## Direct image formula for Semple bundles

For  $n = \dim X$  and  $r = \text{rk } V$ , one gets a **tower of  $\mathbb{P}^{r-1}$ -bundles**

$$\pi_{k,0} : X_k \xrightarrow{\pi_k} X_{k-1} \rightarrow \cdots \rightarrow X_1 \xrightarrow{\pi_1} X_0 = X$$

with  $\dim X_k = n + k(r - 1)$ ,  $\text{rk } V_k = r$ ,

and **tautological line bundles  $\mathcal{O}_{X_k}(1)$  on  $X_k = P(V_{k-1})$** .

### Theorem

$X_k$  is a smooth compactification of  $X_k^{\text{GG,reg}}/\mathbb{G}_k = J_k^{\text{GG,reg}}/\mathbb{G}_k$ , where  $\mathbb{G}_k$  is the group of  $k$ -jets of germs of biholomorphisms of  $(\mathbb{C}, 0)$ , acting on the right by reparametrization:  $(f, \varphi) \mapsto f \circ \varphi$ , and  $J_k^{\text{reg}}$  is the space of  $k$ -jets of regular curves.

### Direct image formula for invariant differential operators

$E_{k,m} V^* := (\pi_{k,0})_* \mathcal{O}_{X_k}(m) =$  sheaf of algebraic differential operators  $f \mapsto P(f_{[k]})$  acting on germs of curves  $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$  such that  $P((f \circ \varphi)_{[k]}) = \varphi'^m P(f_{[k]}) \circ \varphi$ .

## Induced directed structure on a subvariety

Let  $Z$  be an irreducible algebraic subset of some Semple  $k$ -jet bundle  $X_k$  over  $X$  ( $k$  arbitrary).

We define an induced directed structure  $(Z, W) \hookrightarrow (X_k, V_k)$  by taking the linear subspace  $W \subset T_Z \subset T_{X_k|Z}$  to be the closure of  $T_{Z'} \cap V_k$  taken on a suitable Zariski open set  $Z' \subset Z_{\text{reg}}$  where the intersection has constant rank and is a subbundle of  $T_{Z'}$ .

Alternatively, one could also take  $W$  to be the closure of  $T_{Z'} \cap V_k$  in the  $k$ -th stage  $(\mathcal{X}_k, \mathcal{A}_k)$  of the “absolute Semple tower” associated with  $(\mathcal{X}_0, \mathcal{A}_0) = (X, T_X)$  (so as to deal only with nonsingular ambient Semple bundles).

This produces an **induced directed subvariety**

$$(Z, W) \subset (X_k, V_k).$$

It is easy to show that  $\pi_{k,k-1}(Z) = X_{k-1} \Rightarrow \text{rk } W < \text{rk } V_k = \text{rk } V$ .

## Sufficient criterion for the GGL conjecture

### Definition

Let  $(X, V)$  be a directed pair where  $X$  is projective algebraic. We say that  $(X, V)$  is “**strongly of general type**” if it is of general type and for every irreducible alg. subvariety  $Z \subsetneq X_k$  that projects onto  $X$ ,  $X_k \not\subset D_k := P(T_{X_{k-1}/X_{k-2}})$ , the induced directed structure  $(Z, W) \subset (X_k, V_k)$  is of **general type modulo  $X_k \rightarrow X$** , i.e.  ${}^b\mathcal{K}_W \otimes \mathcal{O}_{X_k}(m)|_Z$  is **big** for some  $m \in \mathbb{Q}_+$ , after a suitable blow-up.

### Theorem (D-, 2014)

If  $(X, V)$  is **strongly of general type**, the **Green-Griffiths-Lang conjecture holds true** for  $(X, V)$ , namely there  $\exists Y \subsetneq X$  such that every non constant holomorphic curve  $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$  satisfies  $f(\mathbb{C}) \subset Y$ .

**Proof:** Induction on rank  $V$ , using existence of jet differentials.

## Definition

Fix an ample divisor  $A$  on  $X$ . For every irreducible subvariety  $Z \subset X_k$  that projects onto  $X_{k-1}$  for  $k \geq 1$ ,  $Z \not\subset D_k$ , and  $Z = X = X_0$  for  $k = 0$ , we define the **slope** of the corresponding directed variety  $(Z, W)$  to be  $\mu_A(Z, W) =$

$$\frac{\inf \{ \lambda \in \mathbb{Q}; \exists m \in \mathbb{Q}_+, {}^b\mathcal{K}_W \otimes (\mathcal{O}_{X_k}(m) \otimes \pi_{k,0}^* \mathcal{O}(\lambda A))|_Z \text{ big on } Z \}}{\text{rank } W}.$$

Notice that  $(X, V)$  is of general type iff  $\mu_A(X, V) < 0$ .

We say that  $(X, V)$  is **A-jet-stable** (resp. **A-jet-semi-stable**) if  $\mu_A(Z, W) < \mu_A(X, V)$  (resp.  $\mu_A(Z, W) \leq \mu_A(X, V)$ ) for all  $Z \subsetneq X_k$  as above.

**Observation.** If  $(X, V)$  is of general type and A-jet-semi-stable, then  $(X, V)$  is strongly of general type.

# Criterion for the generalized Kobayashi conjecture

## Definition

Let  $(X, V)$  be a directed pair where  $X$  is projective algebraic. We say that  $(X, V)$  is "**algebraically jet-hyperbolic**" if for every irreducible alg. subvariety  $Z \subsetneq X_k$  s.t.  $X_k \not\subset D_k$ , the induced directed structure  $(Z, W) \subset (X_k, V_k)$  either has  $W = 0$  or is of **general type modulo  $X_k \rightarrow X$** .

## Theorem (D-, 2014)

If  $(X, V)$  is **algebraically jet-hyperbolic**, then  $(X, V)$  is **Kobayashi (or Brody) hyperbolic**, i.e. there are no entire curves  $f : (\mathbb{C}, T_{\mathbb{C}}) \rightarrow (X, V)$ .

Now, the hope is that a (very) generic complete intersection  $X = H_1 \cap \dots \cap H_c \subset \mathbb{P}^{n+c}$  of codimension  $c$  and degrees  $(d_1, \dots, d_c)$  s.t.  $\sum d_j \geq 2n + c$  yields  $(X, T_X)$  algebraically jet-hyperbolic.

# Invariance of “directed plurigenera” ?

One way to check the above property, at least with non optimal bounds, would be to show some sort of Zariski openness of the properties “strongly of general type” or “algebraically jet-hyperbolic”. One would need e.g. to know the answer to

## Question

Let  $(\mathcal{X}, \mathcal{V}) \rightarrow S$  be a proper family of directed varieties over a base  $S$ , such that  $\pi : \mathcal{X} \rightarrow S$  is a nonsingular deformation and the directed structure on  $X_t = \pi^{-1}(t)$  is  $V_t \subset T_{X_t}$ , possibly singular. Under which conditions is

$$t \mapsto h^0(X_t, \mathcal{K}_{V_t}^{[m]})$$

locally constant over  $S$  ?

This would be very useful since one can easily produce jet sections for hypersurfaces  $X \subset \mathbb{P}^{n+1}$  admitting meromorphic connections with low pole order (Siu, Nadel).

## Proof of the non optimal Kobayashi conjecture (Brotbek)

Let  $A \rightarrow Z$  be a very ample line bundle, and  $X_\sigma$  the hypersurface associated with  $\sigma \in H^0(Z, dA)$ ,  $d \gg 1$ . One looks at special sections

$$\sigma = \sum_{|I|=\delta} a_I \tau^{(p+k)I}, \quad a_I \in H^0(Z, \eta A), \quad \tau_j \in H^0(Z, A), \quad 1 \leq j \leq N$$

where the  $\tau_j$  are generic and  $d = \ell + (p+k)\delta$ ,  $p \gg 1$ . Let  $\mathcal{X} \rightarrow S$  be the corresponding family of hypersurfaces,  $\sigma \in S$ , and let  $\mathcal{X}_k \rightarrow S$  be the Semple construction relative to  $\mathcal{X} \rightarrow S$ .

A construction similar to Nadel’s meromorphic connections with low pole orders then produces certain Wronskian operators, and one shows that for generic  $\sigma$ , a certain functorial blow-up  $\mu_k : \widehat{\mathcal{X}}_k \rightarrow \mathcal{X}_k$  of the  $k$ -th stage carries an ample invertible sheaf

$$L_k = \mu_k^*(\mathcal{O}_{\mathcal{X}_k}(a_1, a_2, \dots, a_k) \otimes \mathcal{I}_k \otimes \pi_{k,0} A^{-1}) \text{ over } X_\sigma,$$

where  $\mathcal{I}_k$  is the ideal sheaf associated with a suitable family of **Wronskian operators**. This is enough to prove the conjecture.  $\square$

# Happy birthday Hajime !

