Holomorphic Morse inequalities, old and new

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Introduction and goals

Let X be a compact complex manifold, and $L \to X$ a holomorphic line bundle. Assume L equipped with a Hermitian metric h, written locally as $h = e^{-\varphi}$ in a trivialization. The curvature form of (L, h) is

$$\theta = \Theta_{L,h} = -\frac{i}{2\pi}\partial\overline{\partial}\log h = \frac{i}{2\pi}\partial\overline{\partial}\varphi.$$

Important problems in algebraic or analytic geometry

- Find upper and lower bounds for the dimensions of cohomology groups $h^q(X, L^{\otimes m} \otimes \mathcal{F})$ where $\mathcal{F}$ is a coherent sheaf, asymptotically as $m \to +\infty$, e.g. in terms of $\theta = \Theta_{L,h}$.
- (Harder question ?) In case $q = 0$ and $\mathcal{F}$ is invertible (say), try to analyze the base locus of $H^0(X, L^{\otimes m} \otimes \mathcal{F})$, i.e. the set of common zeroes of all holomorphic sections.

Holomorphic Morse inequalities (D-, 1985) provide workable answers in terms of the q-index sets of the curvature form.

Holomorphic Morse inequalities: main statement

The q-index set of a real $(1,1)$-form θ is defined to be

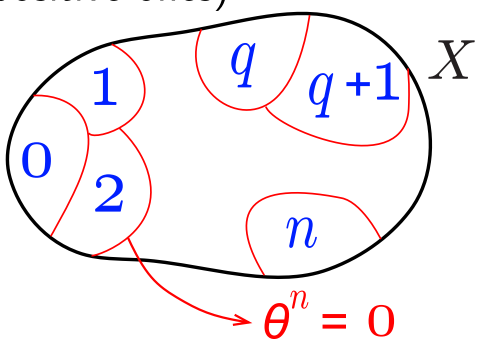
$$X(\theta, q) = \{x \in X \mid \theta(x) \text{ has signature } (n-q, q)\}$$

(exactly q negative eigenvalues and $n-q$ positive ones)

Set also $X(\theta, \leq q) = \bigcup_{0 \leq j \leq q} X(\theta, j)$.

$X(\theta, q)$ and $X(\theta, \leq q)$ are open sets.

$\mathrm{sign}(\theta^n) = (-1)^q$ on $X(\theta, q)$.



Theorem (D-, 1985)

Let $\theta = \Theta_{L,h}$ and $r = \operatorname{rank} \mathcal{F}$. Then, as $m \to +\infty$

$$\sum_{j=0}^{q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes \mathcal{F}) \leq r \frac{m^n}{n!} \int_{X(\theta, \leq q)} (-1)^q \theta^n + o(m^n).$$

Strategy of proof and consequences

The proof proceeds by considering the $\overline{\partial}$-complex and looking at the spectral theory of $\overline{\square} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ acting on sections of $L^{\otimes m} \otimes \mathcal{F}$.
The curvature form of $L^{\otimes m}$ is

$$m\theta = im \textstyle\sum_{j,k} \theta_{jk} dz_j \wedge d\overline{z}_k = i \textstyle\sum_{j,k} \theta_{jk} d\zeta_j \wedge d\overline{\zeta}_k$$

in rescaled coordinates $\zeta_j = \sqrt{m}\, z_j$. The "wavelength" of eigenfunctions is $\sim 1/\sqrt{m}$ and the estimates localize at this scale.

Various formulations of holomorphic Morse inequalities

- $h^q(X, L^{\otimes m} \otimes \mathcal{F}) \leq r \dfrac{m^n}{n!} \displaystyle\int_{X(\theta, q)} (-1)^q \theta^n + o(m^n)$.
- $h^q(X, L^{\otimes m} \otimes \mathcal{F}) \geq r \dfrac{m^n}{n!} \displaystyle\int_{\bigcup_{q-1 \leq j \leq q+1} X(\theta, j)} (-1)^q \theta^n - o(m^n)$.
- For $q = 0$, $h^0(X, L^{\otimes m} \otimes \mathcal{F}) \geq r \dfrac{m^n}{n!} \displaystyle\int_{X(\theta, \leq 1)} \theta^n - o(m^n)$.

Singular version of holomorphic Morse inequalities

We assume here that L is equipped with a possibly singular metric $h = e^{-\varphi}$ were φ is quasi-psh with analytic singularities, i.e. locally

$$\varphi(z) = c \log \sum_j |g_j(z)|^2 + u(z), \quad g_j \text{ holomorphic, } u \in C^\infty,\ c > 0.$$

Then L^2 estimates involve multiplier ideal sheaves $\mathcal{I}(m\varphi) \subset \mathcal{O}_X$

$$\mathcal{I}(m\varphi)_x = \Big\{f \in \mathcal{O}_{X,x}\,;\ \exists U \ni x \text{ s.t. } \int_U |f|^2 e^{-m\varphi} dV < +\infty\Big\}.$$

Theorem (L. Bonavero 1996 – proof based on blowing up)

The same estimates as above are still valid, when one considers instead the twisted cohomology groups

$$H^q(X, L^{\otimes m} \otimes \mathcal{I}(m\varphi) \otimes \mathcal{F})$$

and Morse integrals in the complement of $\Sigma = \varphi^{-1}(-\infty) =$ singular set of $\theta = \Theta_{L,h}$:

$$\int_{X(\theta,q) \smallsetminus \Sigma} (-1)^q \theta^n.$$

Algebraic versions of Morse inequalities

Assume here that X is projective algebraic / $\mathbb{C}$, and that $L = \mathcal{O}_X(A - B)$ where A and B are ample (or nef) $\mathbb{Q}$-divisors (such that $A - B$ is integral).

Observation (D-, 1996)

In the above situation, the holomorphic Morse inequalities hold after replacing the q-index Morse integral by the intersection number $\binom{n}{q} A^{n-q} \cdot B^q$, and in particular (S. Trapani, 1995)

$$h^0(X, L^{\otimes m} \otimes \mathcal{F}) \geq r\,\frac{m^n}{n!}(A^n - nA^{n-1} \cdot B) - o(m^n).$$

Proof. For $(1,1)$-forms $\alpha, \beta \geq 0$, elementary symmetric functions arguments yield

$$\mathbb{1}_{X(\alpha-\beta,\leq q)}(-1)^q(\alpha - \beta)^n \leq \sum_{j=0}^{q} (-1)^{q-j} \binom{n}{j} \alpha^{n-j} \wedge \beta^j.$$

Algebraic proof by F. Angelini (1996), via exact sequence arguments.

Algebraic Morse inequalities of Benoît Cadorel

Definition of adapted stratifications (projective case)

- An "adapted stratification" for L over X is a collection of non singular projective schemes $S = (S_j)$, $\dim S_j = j$, $S_n = X$, together with proper birational morphisms ψ_j of S_j onto the support $|D_j| = \psi_j(S_j)$ of a divisor D_j of S_{j+1}, such that, when putting $\Phi_j = \psi_{n-1} \circ \cdots \circ \psi_j : S_j \to X$, the pull-back $\Phi_j^* L$ satisfies $\Phi_j^* L \simeq \mathcal{O}_{S_j}(D_{j-1}) = \mathcal{O}_{S_j}(D_{j-1}^+ - D_{j-1}^-)$.
- The "truncated powers of the Chern class" $c_1(L,S)^k_{[q]}$ are codim k cycles supported on S_{n-k} ($= 0$ if $q \notin [0,k]$), defined inductively by $c_1(L,S)^0_{[0]} = [X]$, $c_1(L,S)^0_{[q]} = 0$ for $q \neq 0$, and $c_1(L,S)^k_{[q]} = \psi^*_{n-k}\big(c_1(L,S)^{k-1}_{[q]} \cdot D^+_{n-k} - c_1(L,S)^{k-1}_{[q-1]} \cdot D^-_{n-k}\big)$.

Theorem (Cadorel, December 2019)

$$\sum_{0\le j\le q} (-1)^{q-j} h^j(X, L^{\otimes m} \otimes \mathcal{F}) \le \frac{(-1)^q r\, m^n}{n!} \deg c_1(L,S)^n_{[\le q]} + O(m^{n-1}).$$

Considerations and questions about base loci

Let (L,h) be a hermitian line bundle over X. If we assume that $\theta = \Theta_{L,h}$ satisfies $\int_{X(\theta,\le 1)} \theta^n > 0$, then we know that L is big, i.e. that $h^0(X, L^{\otimes m}) \ge c\, m^n$, for $m \ge m_0$ and $c > 0$, but this does not tell us anything about the base locus $\mathrm{Bs}(L) = \bigcap_{\sigma \in H^0(X, L^{\otimes m})} \sigma^{-1}(0)$.

Definition

The "iterated base locus" $\mathrm{IBs}(L)$ is obtained by picking inductively $Z_0 = X$ and $Z_k =$ zero divisor of a section σ_k of $L^{\otimes m_k}$ over the normalization of Z_{k-1}, and taking $\bigcap_{k, m_1, \ldots, m_k, \sigma_1, \ldots, \sigma_k} Z_k$.

Unsolved problem

Find a condition, e.g. in the form of Morse integrals (or analogs) for $\theta = \Theta_{L,h}$, ensuring for instance that $\mathrm{codim}\, \mathrm{IBs}(L) > p$.

We would need for instance to be able to check the positivity of Morse integrals $\int_{Z(\theta_{|Z}, \le 1)} \theta^{n-p}$ for Z irreducible, $\mathrm{codim}\, Z = p$.

Transcendental holomorphic Morse inequalities

Morse inequalities were initially found as a strengthening of Siu's solution of the Grauert-Riemenschneider conjecture characterizing Moishezon manifolds among compact complex manifolds.

In this general setting, we raised 25−30 years ago the following

Conjecture

Let X be a compact complex manifold and $\alpha \in H^{1,1}_{BC}(X,\mathbb{R})$ a Bott-Chern class, represented by closed real $(1,1)$-forms modulo $\partial\bar\partial$ exact forms. Assume α pseudoeffective, and set

$$\mathrm{Vol}(\alpha) = \sup_{T=\alpha+i\partial\bar\partial\varphi\geq 0} \int_X T_{ac}^n, \quad T \geq 0 \text{ current, } n = \dim X.$$

Then

$$\mathrm{Vol}(\alpha) \geq \sup_{\theta\in\{\alpha\},\ \theta\in C^\infty} \int_{X(\theta,\leq 1)} \theta^n$$

where

$$X(\theta,q) = q\text{-index set of } \theta = \big\{x\in X\,;\ \theta(x) \text{ has signature } (n-q,q)\big\}.$$

Conjecture on volumes of (1,1)-classes

Conjectural corollary (transcendental volume estimate)

Let X be compact Kähler, $\dim X = n$, and $\alpha,\beta \in H^{1,1}(X,\mathbb{R})$ be nef classes. Then $\mathrm{Vol}(\alpha-\beta) \geq \alpha^n - n\alpha^{n-1}\cdot\beta$.

By BDPP 2004, this conjecture yields a characterization of the dual of the pseudoeffective cone on arbitrary compact Kähler manifolds.

Observation (BDPP, 2004)

The volume estimate holds if X has deformation approximations by projective manifolds X_ν of maximal Picard number $\rho(X_\nu) = h^{1,1}$.

Theorem 1 (Xiao 2015, Popovici 2016)

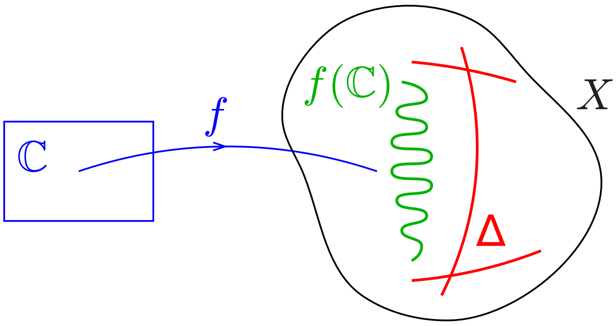
If $\alpha^n - n\alpha^{n-1}\cdot\beta > 0$, then $\alpha-\beta$ is a big class, i.e. $\mathrm{Vol}(\alpha-\beta) > 0$.

Theorem 2 (Witt-Nyström & Boucksom 2019)

The transcendental volume estimate holds if X is projective.

Entire curves in projective varieties and hyperbolicity

- The goal is to study (nonconstant) entire curves $f : \mathbb{C} \to X$ drawn in a projective variety$/\mathbb{C}$. The variety X is said to be Brody ($\Leftrightarrow$ Kobayashi) hyperbolic if there are no such curves.
- More generally, if $\Delta = \sum \Delta_j$ is a reduced normal crossing divisor in X, we want to study entire curves $f : \mathbb{C} \to X \setminus \Delta$ drawn in the complement of Δ.



If there are none, the log pair (X, Δ) is said Brody hyperbolic.

- The strategy is to show that under suitable conditions, such entire curves must satisfy algebraic differential equations.

k-jets of curves and k-jet bundles

Let X be a nonsingular n-dimensional projective variety over $\mathbb{C}$.

Definition of k-jets

For $k \in \mathbb{N}^*$, a k-jet of curve $f_{[k]} : (\mathbb{C}, 0)_k \to X$ is an equivalence class of germs of holomorphic curves $f : (\mathbb{C}, 0) \to X$, written $f = (f_1, \ldots, f_n)$ in local coordinates $(z_1, \ldots, z_n)$ on an open subset $U \subset X$, where two germs are declared to be equivalent if they have the same Taylor expansion of order k at 0 :

$$f(t) = x + t\xi_1 + t^2\xi_2 + \cdots + t^k\xi_k + O(t^{k+1}), \quad t \in D(0, \varepsilon) \subset \mathbb{C},$$

and $x = f(0) \in U$, $\xi_s \in \mathbb{C}^n$, $1 \le s \le k$.

Notation

Let J^kX be the bundle of k-jets of curves, and $\pi_k : J^kX \to X$ the natural projection, where the fiber $(J^kX)_x = \pi_k^{-1}(x)$ consists of k-jets of curves $f_{[k]}$ such that $f(0) = x$.

Algebraic differential operators

Let $t \mapsto z = f(t)$ be a germ of curve, $f_{[k]} = (f', f'', \ldots, f^{(k)})$ its k-jet at any point $t = 0$. Look at the $\mathbb{C}^*$-action induced by *dilations* $\lambda \cdot f(t) := f(\lambda t)$, $\lambda \in \mathbb{C}^*$, for $f_{[k]} \in J^k X$.

Taking a (local) connection ∇ on T_X and putting $\xi_s = f^{(s)}(0) = \nabla^s f(0)$, we get a trivialization $J^k X \simeq (T_X)^{\oplus k}$ and the $\mathbb{C}^*$ action is given by

$$(*) \qquad \lambda \cdot (\xi_1, \xi_2, \ldots, \xi_k) = (\lambda \xi_1, \lambda^2 \xi_2, \ldots, \lambda^k \xi_k).$$

We consider the Green-Griffiths sheaf $E_{k,m}(X)$ of homogeneous polynomials of weighted degree m on $J^k X$ defined by

$$P(x\,;\,\xi_1, \ldots, \xi_k) = \sum a_{\alpha_1 \alpha_2 \ldots \alpha_k}(x)\, \xi_1^{\alpha_1} \ldots \xi_k^{\alpha_k}, \quad \sum_{s=1}^k s|\alpha_s| = m.$$

Here, we assume the coefficients $a_{\alpha_1 \alpha_2 \ldots \alpha_k}(x)$ to be holomorphic in x, and view P as a differential operator $P(f) = P(f\,;\, f', f'', \ldots, f^{(k)})$,

$$P(f)(t) = \sum a_{\alpha_1 \alpha_2 \ldots \alpha_k}(f(t))\, f'(t)^{\alpha_1} f''(t)^{\alpha_2} \ldots f^{(k)}(t)^{\alpha_k}.$$

Graded algebra of algebraic differential operators

In this way, we get a graded algebra $\bigoplus_m E_{k,m}(X)$ of differential operators. As sheaf of rings, in each coordinate chart $U \subset X$, it is a pure polynomial algebra isomorphic to

$$\mathcal{O}_X[f_j^{(s)}]_{1 \le j \le n,\, 1 \le s \le k} \quad \text{where} \quad \deg f_j^{(s)} = s.$$

If a change of coordinates $z \mapsto w = \psi(z)$ is performed on U, the curve $t \mapsto f(t)$ becomes $t \mapsto \psi \circ f(t)$ and we have inductively

$$(\psi \circ f)^{(s)} = (\psi' \circ f) \cdot f^{(s)} + Q_{\psi,s}(f', \ldots, f^{(s-1)})$$

where $Q_{\psi,s}$ is a polynomial of weighted degree s.

By filtering by the partial degree of $P(x; \xi_1, \ldots, \xi_k)$ successively in ξ_k, $\xi_{k-1}, \ldots, \xi_1$, one gets a multi-filtration on $E_{k,m}(X)$ such that the graded pieces are

$$G^\bullet E_{k,m}(X) = \bigoplus_{\ell_1 + 2\ell_2 + \cdots + k\ell_k = m} S^{\ell_1} T_X^* \otimes \cdots \otimes S^{\ell_k} T_X^*.$$

Logarithmic jet differentials

Take a logarithmic pair (X,Δ), $\Delta=\sum\Delta_j$ normal crossing divisor.

Fix a point $x\in X$ which belongs exactly to p components, say $\Delta_1,\ldots,\Delta_p$, and take coordinates $(z_1,\ldots,z_n)$ so that $\Delta_j=\{z_j=0\}$.

$\Longrightarrow$ log differential operators : polynomials in the derivatives

$$(\log f_j)^{(s)},\quad 1\le j\le p\quad\text{and}\quad f_j^{(s)},\quad p+1\le j\le n.$$

Alternatively, one gets an algebra of logarithmic jet differentials, denoted $\bigoplus_m E_{k,m}(X,\Delta)$, that can be expressed locally as

$$\mathcal{O}_X\big[(f_1)^{-1}f_1^{(s)},\ldots,(f_p)^{-1}f_p^{(s)},f_{p+1}^{(s)},\ldots,f_n^{(s)}\big]_{1\le s\le k}.$$

One gets a multi-filtration on $E_{k,m}(X,\Delta)$ with graded pieces

$$G^\bullet E_{k,m}(X,\Delta)=\bigoplus_{\ell_1+2\ell_2+\cdots+k\ell_k=m}S^{\ell_1}T_X^*\langle\Delta\rangle\otimes\cdots\otimes S^{\ell_k}T_X^*\langle\Delta\rangle$$

where $T_X^*\langle\Delta\rangle$ is the logarithmic tangent bundle, i.e., the locally free sheaf generated by $\frac{dz_1}{z_1},\ldots,\frac{dz_p}{z_p},dz_{p+1},\ldots,dz_n$.

Projectivized jets and direct image formula

Green Griffiths bundles

Consider $X_k:=J^kX/\mathbb{C}^*=\mathrm{Proj}\bigoplus_m E_{k,m}(X)$. This defines a bundle $\pi_k:X_k\to X$ of weighted projective spaces whose fibers are the quotients of $(\mathbb{C}^n)^k\smallsetminus\{0\}$ by the $\mathbb{C}^*$ action

$$\lambda\cdot(\xi_1,\ldots,\xi_k)=(\lambda\xi_1,\lambda^2\xi_2,\ldots,\lambda^k\xi_k).$$

Correspondingly, there is a tautological rank 1 sheaf $\mathcal{O}_{X_k}(m)$ [invertible only when $\mathrm{lcm}(1,\ldots,k)\,|\,m$], and a direct image formula

$$E_{k,m}(X)=(\pi_k)_*\mathcal{O}_{X_k}(m).$$

In the logarithmic case, we define similarly

$$X_k\langle\Delta\rangle:=\mathrm{Proj}\bigoplus_m E_{k,m}(X,\Delta)$$

and let $\mathcal{O}_{X_k\langle\Delta\rangle}(1)$ be the corresponding tautological sheaf, so that

$$E_{k,m}(X,\Delta)=(\pi_k)_*\mathcal{O}_{X_k\langle\Delta\rangle}(m).$$

Generalized Green-Griffiths-Lang conjecture

Generalized GGL conjecture

If (X, Δ) is a log pair of general type, in the sense that $K_X + \Delta$ is big, then there is a proper algebraic subvariety $Y \subsetneq X \smallsetminus \Delta$ containing all entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$.

One possible strategy is to show that such entire curves f must satisfy a lot of algebraic differential equations of the form $P(f; f', ..., f^{(k)}) = 0$ for $k \gg 1$. This is based on:

Fundamental vanishing theorem

[Green-Griffiths 1979], [D- 1995], [Siu-Yeung 1996], ...
Let A be an ample divisor on X. Then, for all global jet differential operators on (X, Δ) with coefficients vanishing on A, i.e. $P \in H^0(X, E_{k,m}(X, \Delta) \otimes \mathcal{O}(-A))$, and for all entire curves $f : \mathbb{C} \to X \smallsetminus \Delta$, one has $P(f_{[k]}) \equiv 0$.

Proof of the fundamental vanishing theorem

Simple case. First consider the compact case ($\Delta = 0$), and assume that f is a Brody curve, i.e. $\|f'\|_\omega$ bounded for some hermitian metric ω on X. By raising P to a power, we can assume A very ample, and view P as a $\mathbb{C}$ valued differential operator whose coefficients vanish on a very ample divisor A.

The Cauchy inequalities imply that all derivatives $f^{(s)}$ are bounded in any relatively compact coordinate chart. Hence $u_A(t) = P(f_{[k]})(t)$ is bounded, and must thus be constant by Liouville's theorem.

Since A is very ample, we can move $A \in |A|$ such that A hits $f(\mathbb{C}) \subset X$. But then u_A vanishes somewhere, and so $u_A \equiv 0$.

Logarithmic case. In the logarithmic case, one can use instead a "Poincaré type metric" ω. Removing the hypothesis f' bounded is more tricky. One possible way is to use the Ahlfors lemma and some representation theory.

Probabilistic cohomology estimate

Theorem (D-, PAMQ 2011 + recent work for logarithmic case)

Fix A ample line bundle on X, and hermitian structures $(T_X\langle\Delta\rangle, h)$, (A, h_A) with $\omega_A = \Theta_{A,h_A} > 0$. Let $\eta_\varepsilon = \Theta_{K_X+\Delta,\det h^*} - \varepsilon\omega_A$ and

$$L_{k,\varepsilon} = \mathcal{O}_{X_k\langle\Delta\rangle}(1) \otimes \pi_k^*\mathcal{O}_X\Big(-\frac{1}{kn}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)\varepsilon A\Big), \quad \varepsilon \in \mathbb{Q}_+.$$

Then for m sufficiently divisible, we have a lower bound

$$h^0(X_k, L_{k,\varepsilon}^{\otimes m}) = h^0\Big(X, E_{k,m}(X,\Delta)\otimes\mathcal{O}_X\Big(-\frac{m\varepsilon}{kn}\Big(1+\frac{1}{2}+\cdots+\frac{1}{k}\Big)A\Big)\Big)$$
$$\geq \frac{m^{n+kn-1}}{(n+kr-1)!}\,\frac{(1+\frac{1}{2}+\cdots+\frac{1}{k})^n}{n!\,(k!)^n}\Big(\int_{X(\eta,\leq 1)}\eta_\varepsilon^n - \frac{C}{\log k}\Big).$$

Corollary

If $K_X+\Delta$ is big and $\varepsilon>0$ is small, then η_ε can be taken >0, so $h^0(X_k, L_{k,\varepsilon}^{\otimes m}) \geq C_{n,k,\eta,\varepsilon}\, m^{n+kn-1}$ with $C_{n,k,\eta,\varepsilon}>0$, for $m \gg k \gg 1$.

Therefore, all $f:\mathbb{C}\to X\smallsetminus\Delta$ satisfy algebraic diff. equations.

A good Finsler metric on the k-jet bundle

Proof. Consider for simplicity the absolute (non logarithmic) case. Assume that T_X is equipped with a C^∞ connection ∇ and a hermitian metric h. One then defines a "weighted Finsler metric" on J^kX by taking $b=\mathrm{lcm}(1,2,...,k)$ and, $\forall f$, $f(0)=x\in X_k$,

$$\Psi_{h_k}(f_{[k]}) := \Big(\sum_{1\leq s\leq k}\|\varepsilon_s\nabla^s f(0)\|_{h(x)}^{2b/s}\Big)^{1/b}, \quad 1=\varepsilon_1\gg\varepsilon_2\gg\cdots\gg\varepsilon_k.$$

Letting $\xi_s=\varepsilon_s\nabla^s f(0)$, this can be viewed as a metric h_k on $L_k:=\mathcal{O}_{X_k}(1)$, and the curvature form of L_k is obtained by computing $\frac{i}{2\pi}\partial\overline{\partial}\log\Psi_{h_k}(f_{[k]})$ as a function of $(x,\xi_1,\ldots,\xi_k)$.

Modulo negligible error terms of the form $O(\varepsilon_{s+1}/\varepsilon_s)$, this gives

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi}\sum_{1\leq s\leq k}\frac{1}{s}\frac{|\xi_s|^{2b/s}}{\sum_t|\xi_t|^{2b/t}}\sum_{i,j,\alpha,\beta}c_{ij\alpha\beta}\frac{\xi_{s\alpha}\overline{\xi}_{s\beta}}{|\xi_s|^2}\,dz_i\wedge d\overline{z}_j$$

where $(c_{ij\alpha\beta})$ are the coefficients of the curvature tensor $-\Theta_{T_X,h}$ and $\omega_{\mathrm{FS},k}$ is the weighted Fubini-Study metric on the fibers of $X_k\to X$.

Evaluation of Morse integrals

The above expression can be simplified by using polar coordinates

$$x_s = |\xi_s|_h^{2b/s}, \qquad u_s = \xi_s/|\xi_s|_h = \nabla^s f(0)/|\nabla^s f(0)|.$$

Since the weighted projective space can be viewed as a circle quotient of the pseudosphere $\sum |\xi_s|^{2b/s} = 1$, we can take $\sum x_s = 1$, i.e. (x_s) in the $(k-1)$-dimensional simplex Δ^{k-1}, and we obtain

$$\Theta_{L_k,h_k} = \omega_{\mathrm{FS},k}(\xi) + \frac{i}{2\pi} \sum_{1\le s\le k} \frac{1}{s} x_s \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j$$

where $\omega_{\mathrm{FS},k}(\xi) = \frac{i}{2\pi b}\partial\overline{\partial}\log\sum_{1\le s\le k}|\xi_s|^{2b/s} > 0$ on fibers of $X_k \to X$.

By holomorphic Morse inequalities, we need to evaluate an integral

$$\int_{X_k(\Theta_{L_h,h_k},\le 1)} \Theta_{L_k,h_k}^{N_k}, \qquad N_k = \dim X_k = n + (kn-1),$$

and we have to integrate over the parameters $z \in X$, $x_s \in \mathbb{R}_+$ and u_s in the unit sphere bundle $\mathbb{S}(T_X,1) \subset T_X$.

Probabilistic interpretation of the curvature

The signature of Θ_{L_k,h_k} depends only on the vertical terms, i.e.

$$\sum_{1\le s\le k} \frac{1}{s} x_s Q(u_s), \quad Q(u_s) := \frac{i}{2\pi} \sum_{i,j,\alpha,\beta} c_{ij\alpha\beta}(z)\, u_{s\alpha}\overline{u}_{s\beta}\, dz_i \wedge d\overline{z}_j.$$

After averaging over $(x_s) \in \Delta^{k-1}$ and computing the rational number $\int \omega_{\mathrm{FS},k}(\xi)^{nk-1} = \frac{1}{(k!)^n}$, what is left is to evaluate Morse integrals with respect to $(u_s) \in (\mathbb{S}(T_X,1))^k$ of "horizontal" $(1,1)$-forms given by sums $\sum \frac{1}{s} Q(u_s)$, where (u_s) is a sequence of "random points" on the unit sphere.

As $k \to +\infty$, this sum is asymptotically equivalent to a "Monte-Carlo" integral $(1 + \frac{1}{2} + \cdots + \frac{1}{k}) \int_{u\in\mathbb{S}(T_X,1)} Q(u)\, du$.

Now, $Q(u)$ quadratic form $\Rightarrow \int_{u\in\mathbb{S}(T_X,1)} Q(u)\, du = \frac{1}{n}\operatorname{Tr}(Q)$,

and we have $\operatorname{Tr}(Q) = \operatorname{Tr}(-\Theta_{T_X,h}) = \Theta_{\det T_X^*,\det h^*} = \Theta_{K_X,\det h^*}$.

The asserted Morse estimates follow.

A result on the base loci of jet differentials

Thorem (D-, 2021)

Let (X, Δ) be a pair of general type, i.e. such that $K_X + \Delta$ is big. Then there exists $k_0 \in \mathbb{N}$ and $\delta > 0$ with the following properties.
Let $Z \subset X_k$ be an irreducible algebraic subvariety that is a component of a complete intersection of irreducible hypersurfaces

$$\bigcap_{1 \le j \le \ell} \{k\text{-jets } f_{[k]} \in X_k \,;\, P_j(f) = 0\}, \quad P_j \in H^0(X, E_{s_j,m_j}(X,\Delta) \otimes L_j)$$

with $k \ge k_0$, $\mathrm{ord}(P_j) = s_j$, $1 \le s_1 < \cdots < s_\ell \le k$, $\sum_{1 \le j \le \ell} \frac{1}{s_j} \le \delta \log k$.

Then the Morse integrals $\int_{Z(L_{k,\varepsilon}, \le 1)} \Theta_{L_{k,\varepsilon}}^{\dim Z}$ of

$$L_{k,\varepsilon} = \mathcal{O}_{X_k\langle\Delta\rangle}(1) \otimes \pi_k^* \mathcal{O}_X\Big(-\frac{1}{kn}\Big(1 + \frac{1}{2} + \cdots + \frac{1}{k}\Big)\varepsilon A\Big)$$

are positive for $\varepsilon > 0$ small, hence $H^0(Z, L_{k,\varepsilon}^{\otimes m}) \ge c\, m^{\dim Z}$ for $m \gg 1$.

Unfortunately, seems insufficient to show that $\dim \mathrm{IBs}(L_{k,\varepsilon}) < n$.

The end

Thank you for your attention!