Compact Kähler manifolds with hermitian semipositive anticanonical bundle

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1. Main results

This short note is a continuation of our previous work [DPS93] on compact Kähler manifolds X with semipositive Ricci curvature. Our purpose is to state a splitting theorem describing the structure of such manifolds, and to raise some related questions. The foundational background will be found in papers by Lichnerowicz [Li67], [Li71], and Cheeger-Gromoll [CG71], [CG72]. Recall that a *Calabi-Yau manifold X* is a compact Kähler manifold with $c_1(X) = 0$ and finite fundamental group $\pi_1(X)$, such that the universal covering $\widetilde{X}$ satisfies $H^0(\widetilde{X}, \Omega^p_{\widetilde{X}}) = 0$ for all $1 \leq p \leq \dim X - 1$. A *symplectic manifold X* is a compact Kähler manifold admitting a holomorphic symplectic 2-form ω (of maximal rank everywhere); in particular $K_X = \mathcal{O}_X$. We denote here as usual

$$\Omega_X = T_X^\star, \qquad \Omega_X^p = \Lambda^p T_X^\star, \qquad K_X = \det(T_X^\star).$$

The following structure theorem generalizes the Bogomolov-Kobayashi-Beauville structure theorem for Ricci-flat manifolds ([Bo74a], [Bo74b], [Ko81], [Be83]) to the Ricci semipositive case.

Structure Theorem. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then*

i) *The universal covering $\widetilde{X}$ admits a holomorphic and isometric splitting*

$$\widetilde{X} \simeq \mathbb{C}^q \times \prod X_i$$

with X_i being either a Calabi-Yau manifold or a symplectic manifold or a manifold with $-K_{X_i}$ semipositive and $H^0(X_i, \Omega_{X_i}^{\otimes m}) = 0$ for all $m > 0$.

ii) *There exists a finite étale Galois covering $\widehat{X} \to X$ such that the Albanese variety $\mathrm{Alb}(\widehat{X})$ is a q-dimensional torus and the Albanese map $\alpha : \widehat{X} \to \mathrm{Alb}(\widehat{X})$ is a locally trivial holomorphic fibre bundle whose fibres are products $\prod X_i$ of the type described in i), all X_i being simply connected.*

iii) *We have $\pi_1(\widehat{X}) \simeq \mathbb{Z}^{2q}$ and $\pi_1(X)$ is an extension of a finite group Γ by the normal subgroup $\pi_1(\widehat{X})$. In particular there is an exact sequence*

$$0 \to \mathbb{Z}^{2q} \to \pi_1(X) \to \Gamma \to 0,$$

and the fundamental group $\pi_1(X)$ is almost abelian.

Recall that a line bundle L is said to be hermitian semipositive if it can be equipped with a smooth hermitian metric of semipositive curvature form. A sufficient condition for hermitian semipositivity is that some multiple of L is spanned by global sections; on the other hand, the hermitian semipositivity condition implies that L is numerically effective (nef) in the sense of [DPS94], which, for X projective algebraic, is equivalent to saying that $L \cdot C \geq 0$ for every curve C in X. Examples contained in [DPS94] show that all three conditions are different (even for X projective algebraic). By Yau's solution of the Calabi conjecture (see [Au76], [Yau78]), a compact Kähler manifold X has a hermitian semipositive anticanonical bundle $-K_X$ if and only if X admits a Kähler metric g with $\mathrm{Ricci}(g) \geq 0$. The isometric decomposition described in the theorem refers to such Kähler metrics.

In view of "standard conjectures" in minimal model theory it is expected that projective manifolds X with no nonzero global sections in $H^0(X, \Omega_X^{\otimes m})$, $m > 0$, are rationally connected. We hope that most of the above results will continue to hold under the weaker assumption that $-K_X$ is nef instead of hermitian semipositive. However, the technical tools needed to treat this case are still missing.

2. Bochner formula and holomorphic differential forms

Our starting point is the following well-known consequence of the Bochner formula.

Lemma. *Let X be a compact n-dimensional Kähler manifold with $-K_X$ hermitian semipositive. Then every section of $\Omega_X^{\otimes m}$, $m \geq 1$ is parallel with respect to the given Kähler metric.*

Proof. The Lemma is an easy consequence of the Bochner formula

$$\Delta(\parallel u \parallel^2) = \parallel \nabla u \parallel^2 + Q(u),$$

where $u \in H^0(X, \Omega_X^{\otimes m})$ and $Q(u) \geq m\lambda_0 \parallel u \parallel^2$. Here λ_0 is the smallest eigenvalue of the Ricci curvature tensor. For details see for instance [Ko83]. Q.E.D.

The following definition of a modified Kodaira dimension $\kappa_+(X)$ is taken from Campana [Ca93]. As the usual Kodaira dimension $\kappa(X)$, this is a birational invariant of X. Other similar invariants have also been considered in [BR90] and [Ma93].

Definition. *Let Y be a compact complex manifold. We define*

 i) $\kappa_+(Y) = \max\{\kappa(\det \mathcal{F}) : \mathcal{F} \text{ is a subsheaf of } \Omega_Y^p \text{ for some } p > 0\}$,

 ii) $\kappa_{++}(Y) = \max\{\kappa(\det \mathcal{F}) : \mathcal{F} \text{ is a subsheaf of } \Omega_Y^{\otimes m} \text{ for some } m > 0\}$.

Here we let as usual $\det \mathcal{F} = (\Lambda^r \mathcal{F})^{**}$, where $r = \mathrm{rank}\mathcal{F}$ and κ is the usual Iitaka dimension of a line bundle.

Clearly, we have $-\infty \leq \kappa(Y) \leq \kappa_+(Y) \leq \kappa_{++}(Y)$ where $\kappa(Y) = \kappa(K_Y)$ is the usual Kodaira dimension. It would be interesting to know whether there are precise relations between $\kappa_+(Y)$ and $\kappa_{++}(Y)$, as well as with the weighted Kodaira dimensions defined by Manivel [Ma93]. The above lemma implies:

Proposition. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then $\kappa_{++}(X) \leq 0$.*

Proof. Assume $\kappa_{++}(X) > 0$. Then we can find $m > 0$ and a subsheaf $\mathcal{F} \subset \Omega_X^{\otimes m}$ with $\kappa(\det \mathcal{F}) > 0$. Hence there is some $\mu \in \mathbb{N}$ and $s \in H^0(X, (\det \mathcal{F})^\mu)$ with $s \neq 0$. Since $\kappa(\det \mathcal{F}) > 0$, s must have zeroes. Hence the induced section $\tilde{s} \in H^0(X, \Omega_X^{\otimes \mu r m})$ has zeroes too, r being the rank of $\mathcal{F}$. This contradicts the previous Lemma. Q.E.D.

Corollary. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Let $\phi : X \to Y$ be a surjective holomorphic map to a normal compact Kähler space. Then $\kappa(Y) \leq 0$. (Here $\kappa(Y) = \kappa(\hat{Y})$, where $\hat{Y}$ is an arbitrary desingularization of Y.)*

Proof. This follows from the inequalities $0 \geq \kappa_+(X) \geq \kappa_+(Y) \geq \kappa(Y)$. For the second inequality, which is easily checked by a pulling-back argument, see [Ca93]. Q.E.D.

3. Proof of the structure theorem

We suppose here that X is equipped with a Kähler metric g such that $\mathrm{Ricci}(g) \geq 0$, and we set $n = \dim_{\mathbb{C}} X$.

i) Let $(\widetilde{X}, g) \simeq \prod(X_i, g_i)$ be the De Rham decomposition of $(\widetilde{X}, g)$, induced by a decomposition of the holonomy representation in irreducible representations. Since the holonomy is contained in $\mathrm{U}(n)$, all factors (X_i, g_i) are Kähler manifolds with irreducible holonomy and holonomy group $H_i \subset \mathrm{U}(n_i)$, $n_i = \dim X_i$. By Cheeger-Gromoll [CG71], there is possibly a flat factor $X_0 = \mathbb{C}^q$ and the other factors X_i, $i \geq 1$, are compact. Also, the product structure shows that $-K_{X_i}$ is hermitian semipositive. It suffices to prove that $\kappa_{++}(X_i) = 0$ implies that X_i is a Calabi-Yau manifold or a symplectic manifold. In view of §2, the condition $\kappa_{++}(X_i) = 0$ means that there is a nonzero section $u \in H^0(X_i, \Omega_{X_i}^{\otimes m})$ for some $m > 0$. Since u is parallel by the lemma, it is invariant under the holonomy action, and therefore the holonomy group H_i is not the full unitary group $\mathrm{U}(n_i)$ (indeed, the trivial representation does not occur in the decomposition of $(\mathbb{C}^{n_i})^{\otimes m}$ in irreducible $\mathrm{U}(n_i)$-representations, all weights being of length m). By Berger's classification of holonomy groups [Bg55] there are only two remaining possibilities, namely $H_i = \mathrm{SU}(n_i)$ or $H_i = \mathrm{Sp}(n_i/2)$. The case $H_i = \mathrm{SU}(n_i)$ leads to X_i being a Calabi-Yau manifold. The remaining case $H_i = \mathrm{Sp}(n_i/2)$ implies that X_i is symplectic (see e.g. [Be83]).

ii) Set $X' = \prod_{i \geq 1} X_i$. The group of covering transformations acts on the product $\widetilde{X} = \mathbb{C}^q \times X'$ by holomorphic isometries of the form $x = (z, x') \mapsto (u(z), v(x'))$.

At this point, the argument is slightly more involved than in Beauville's paper [Be83], because the group G' of holomorphic isometries of X' need not be finite (X' may be for instance a projective space); instead, we imitate the proof of ([CG72], Theorem 9.2) and use the fact that X' and $G' = \mathrm{Isom}(X')$ are compact. Let $E_q = \mathbb{C}^q \rtimes U(q)$ be the group of unitary motions of $\mathbb{C}^q$. Then $\pi_1(X)$ can be seen as a discrete subgroup of $E_q \times G'$. As G' is compact, the kernel of the projection map $\pi_1(X) \to E_q$ is finite and the image of $\pi_1(X)$ in E_q is still discrete with compact quotient. This shows that there is a subgroup Γ of finite index in $\pi_1(X)$ which is isomorphic to a crystallographic subgroup of $\mathbb{C}^q$. By Bieberbach's theorem, the subgroup $\Gamma_0 \subset \Gamma$ of elements which are translations is a subgroup of finite index. Taking the intersection of all conjugates of Γ_0 in $\pi_1(X)$, we find a normal subgroup $\Gamma_1 \subset \pi_1(X)$ of finite index, acting by translations on $\mathbb{C}^q$. Then $\widehat{X} = \widetilde{X}/\Gamma_1$ is a fibre bundle over the torus $\mathbb{C}^q/\Gamma_1$ with X' as fibre and $\pi_1(X') = 1$. Therefore $\widehat{X}$ is the desired finite étale covering of X.

iii) is an immediate consequence of ii), using the homotopy exact sequence of a fibration. Q.E.D.

Corollary 1. *Let X be a compact Kähler manifold with $-K_X$ hermitian semi-positive. If $\widetilde{X}$ is indecomposable and $\kappa_+(X) = 0$, then X is Ricci-flat.*

Corollary 2. *Let X be a compact Kähler manifold with $-K_X$ hermitian semi-positive. Then, if $\widehat{X} \to X$ is an arbitrary finite étale covering,*

$$\kappa_+(X) = -\infty \iff \kappa_{++}(X) = -\infty$$
$$\iff \forall \widehat{X} \to X, \ \forall p \geq 1, \quad H^0(\widehat{X}, \Omega^p_{\widehat{X}}) = 0.$$

If $\kappa_+(X) = -\infty$, then $\chi(X, \mathcal{O}_X) = 1$ and X is simply connected.

Proof. The equivalence of all three properties is a direct consequence of the structure theorem. Now, every étale covering $\widehat{X} \to X$ satisfies $\kappa_+(\widehat{X}) = \kappa_+(X) = -\infty$, hence $\chi(\widehat{X}, \mathcal{O}_{\widehat{X}}) = \chi(X, \mathcal{O}_X) = 1$ (by Hodge symmetry we have $h^p(X, \mathcal{O}_X) = 0$ for $p \geq 1$, whilst $h^0(X, \mathcal{O}_X) = 1$). However, if d is the covering degree, the Riemann-Roch formula implies $\chi(\widehat{X}, \mathcal{O}_{\widehat{X}}) = d\,\chi(X, \mathcal{O}_X)$, hence $d = 1$ and X must be simply connected. Q.E.D.

4. Related questions for the case $-K_X$ nef

In order to make the structure theorem more explicit, it would be necessary to characterize more precisely the manifolds for which $\kappa_+(X) = -\infty$. We expect these manifolds to be rationally connected, even when $-K_X$ is just supposed to be nef.

Conjecture. *Let X be a compact Kähler manifold such that $-K_X$ is nef and $\kappa_+(X) = -\infty$. Then X is rationally connected, i.e. any two points of X can be joined by a chain of rational curves.*

Campana even conjectures this to be true without assuming $-K_X$ to be nef.

Another hope we have is that a similar structure theorem might also hold in the case $-K_X$ nef. A small part of it would be to understand better the structure of the Albanese map. We proved in [DPS93] that the Albanese map is surjective when $\dim X \leq 3$, and if $\dim X \leq 2$ it is well-known that the Albanese map is a locally trivial fibration. It is thus natural to state the following

Problem. *Let X be a compact Kähler manifold with $-K_X$ nef. Is the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ a smooth locally trivial fibration?*

The following simple example shows, even in the case of a locally trivial fibration, that the structure group of transition automorphisms need not be a group of isometries, in contrast with the case $-K_X$ hermitian semipositive.

Example 1 (see [DPS94], Example 1.7). Let $C = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ be an elliptic curve, and let $E \to C$ be the flat rank 2 bundle associated to the representation $\pi_1(C) \to \mathrm{GL}_2(\mathbb{C})$ defined by the monodromy matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then the projectivized bundle $X = \mathbb{P}(E)$ is a ruled surface over C with $-K_X$ nef and not hermitian semipositive (cf. [DPS94]). In this case, the Albanese map $X \to C$ is a locally trivial $\mathbb{P}_1$-bundle, but the monodromy group is not relatively compact in $\mathrm{GL}_2(\mathbb{C})$, hence there is no invariant Kähler metric on the fibre.

Example 2. The following example shows that the picture is unclear even in the case of surfaces with $\kappa_+(X) = -\infty$. Let $\mathbf{p} = (p_1, \ldots, p_9)$ be a configuration of 9 points in $\mathbb{P}_2$ and let $\pi : X_{\mathbf{p}} \to \mathbb{P}_2$ be the blow-up of $\mathbb{P}_2$ with center $\mathbf{p}$. Here some of the points p_i may be infinitely near: as usual, this means that the blowing-up process is made inductively, each p_i being an arbitrary point in the blow-up of $\mathbb{P}_2$ at $(p_1, \ldots, p_{i-1})$. There is always a cubic curve C containing all 9 points (C is even unique if $\mathbf{p}$ is general enough). The only assumption we make is that C is nonsingular, and we let $C = \{Q(z_0, z_1, z_2) = 0\} \subset \mathbb{P}_2$, $\deg Q = 3$. Then C is an elliptic curve and $-K_{X_{\mathbf{p}}} = \pi^\star \mathcal{O}(3) - \sum E_i$ where $E_i = \pi^{-1}(p_i)$ are the exceptional divisors. Clearly Q defines a section of $-K_{X_{\mathbf{p}}}$, of divisor equal to the strict transform C' of C, hence $-K_{X_{\mathbf{p}}} \simeq \mathcal{O}(C')$, and $(-K_{X_{\mathbf{p}}})^2 = (C')^2 = C^2 - 9 = 0$. Therefore $-K_{X_{\mathbf{p}}}$ is always nef.

It is easy to see that $-mK_{X_{\mathbf{p}}}$ may be generated or not by sections according to the choice of the 9 points p_i. In fact, if p_i' is the point of C' lying over p_i, we have

$$-K_{X_{\mathbf{p}}|C'} = \pi^\star(\mathcal{O}(3))_{|C'} \otimes \mathcal{O}(-\sum p_j') = \pi^\star\Big(\mathcal{O}(3)_{|C} \otimes \mathcal{O}(-\sum p_j)\Big).$$

Since $C' \simeq C$ is an elliptic curve and $-K_{X_{\mathbf{p}}|C'}$ has degree 0, there are nonzero sections in $H^0(C', -mK_{X_{\mathbf{p}}|C'})$ if and only if $L_{\mathbf{p}} = \mathcal{O}(3)_{|C} \otimes \mathcal{O}(-\sum p_j))$ is a torsion

point in $\mathrm{Pic}^0(C)$ of order dividing m. Such sections always extend to $X_{\mathbf{p}}$. Indeed, we may assume that m is exactly the order. Then $\mathcal{O}(-C') \otimes \mathcal{O}(-mK_{X_{\mathbf{p}}}) = \mathcal{O}((m-1)C')$ admits a filtration by its subsheaves $\mathcal{O}(kC')$, $0 \leq k \leq m-1$, and the H^1 groups of the graded pieces are $H^1(X_{\mathbf{p}}, \mathcal{O}_{X_{\mathbf{p}}}) = 0$ for $k = 0$ and

$$H^1(C', \mathcal{O}(kC')_{|C'}) = H^0(C', \mathcal{O}(-kC')) = 0 \ \text{ for } 0 < k < m.$$

Therefore $H^1(X_{\mathbf{p}}, \mathcal{O}(-C') \otimes \mathcal{O}(-mK_{X_{\mathbf{p}}})) = 0$, as desired. In particular, $-K_{X_{\mathbf{p}}}$ is hermitian semipositive as soon as $L_{\mathbf{p}}$ is a torsion point in $\mathrm{Pic}^0(C)$. In this case, there is a polynomial R_m of degree $3m$ vanishing of order m at all points p_i, such that the rational function R_m/Q^m defines an elliptic fibration $\varphi : X_{\mathbf{p}} \to \mathbb{P}_1$; in this fibration C is a multiple fibre of multiplicity m and we have $-mK_{X_{\mathbf{p}}} = \varphi^\star \mathcal{O}_{\mathbb{P}_1}(1)$. An interesting question is to understand what happens when $L_{\mathbf{p}}$ is no longer a torsion point in $\mathrm{Pic}^0(C)$ (this is precisely the situation considered by Ogus [Og76] in order to produce a counterexample to the formal principle for infinitesimal neighborhoods). In this situation, we may approximate $\mathbf{p}$ by a sequence of configurations $\mathbf{p}_m \subset C$ such that the corresponding line bundle $L_{\mathbf{p}_m}$ is a torsion point of order m (just move a little bit p_9 and take a suitable $p_{9,m} \in C$ close to p_9). The sequence of fibrations $X_{\mathbf{p}_m} \to \mathbb{P}_1$ does not yield a fibration $X_{\mathbf{p}} \to \mathbb{P}_1$ in the limit, but we believe that there might exist instead a holomorphic foliation on $X_{\mathbf{p}}$. In this foliation, C would be a closed leaf, and the generic leaf would be nonclosed and of conformal type $\mathbb{C}$ (or possibly $\mathbb{C}^\star$). If indeed the foliation exists and admits a smooth invariant transversal volume form, then $-K_{X_{\mathbf{p}}}$ would still be hermitian semipositive. We are thus led to the following question.

Question. *Let X be compact Kähler manifold with $-K_X$ nef and X rationally connected. Is then $-K_X$ automatically hermitian semipositive? In particular, is it always the case that $\mathbb{P}_2$ blown-up in 9 points of a nonsingular cubic curve has a semipositive anticanonical bundle?*

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