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# Variational approach for complex Monge-Ampère equations and geometric applications

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# Abstract and goals

• Recent work by Berman, Berndtsson, Boucksom, Eyssidieux, Guedj, Jonsson, Zeriahi (among others) leads to a new variational approach for the solution of Monge-Ampère equations on compact Kähler manifolds.

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• Applications include the existence and uniqueness of Kähler-Einstein metrics on $\mathbb{Q}$-Fano varieties with log terminal singularities, and a new proof by Berman-Boucksom-Jonsson of a uniform version of the Yau-Tian-Donaldson conjecture solved around 2013 by Chen-Donaldson-Sun.

# Kähler-Einstein metrics

To a Kähler metric on a compact complex $n$ fold $X$

$$\omega = i \sum_{1\le j,k\le n} \omega_{jk}(z)\, dz_j \wedge d\overline{z}_k, \quad d\omega = 0$$

one associates its Ricci curvature form

$$\mathrm{Ricci}(\omega) = \Theta_{\Lambda^n T_X, \Lambda^n \omega} = -dd^c \log\det(\omega_{jk})$$

where $d^c = \frac{1}{4i\pi}(\partial - \overline{\partial})$, $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$.

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This requires $\lambda\omega \in c_1(X)$, hence $(*)$ can be solved only when $c_1(X)$ is positive definite, negative definite or zero, and after rescaling $\omega$ by a constant, one can always assume that $\lambda \in \{0, 1, -1\}$.

# Kähler-Einstein $\Longleftrightarrow$ Monge-Ampère equation (1)

Fix a reference Kähler metric $\omega_0$ and put $\omega = \omega_0 + dd^c\varphi$. The KE condition $(*)$ is equivalent to

$$(**) \qquad (\omega_0 + dd^c\varphi)^n = e^{-\lambda\varphi + f}\omega_0^n.$$

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• When $\lambda = -1$ and $c_1(X) < 0$, i.e. $c_1(K_X) > 0$, Aubin has shown in 1978 that there is always a unique solution, hence a unique Kähler metric $\omega \in c_1(K_X)$ such that

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This is a very natural generalization of the existence of constant curvature metrics on complex algebraic curves, implied by Poincaré's uniformization theorem in dimension 1.

# Kähler-Einstein $\Longleftrightarrow$ Monge-Ampère equation (2)

• For $\lambda = 0$ and $c_1(X) = 0$, a celebrated result of Yau (solution of the Calabi conjecture, 1978) states that there exists a unique metric $\omega = \omega_0 + dd^c\varphi$ in the given cohomology class $\{\omega_0\}$ such that $\mathrm{Ricci}(\omega) = 0$.

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has a unique solution whenever $\int_X e^f \omega_0^n = \int_X \omega_0^n$.
Equivalently, the Ricci curvature form can be prescribed to be equal any given smooth closed $(1,1)$-form

$$\mathrm{Ricci}(\omega) = \rho,$$

provided that $\rho \in c_1(X)$.

# The case of Fano manifolds

For $\lambda = +1$, the equation to solve is

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This is possible only if $-K_X$ $(= \Lambda^n T_X)$ is ample. One then says that $X$ is a Fano manifold.

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When solutions exist, it is known by Bando Mabuchi (1987) that they are unique up to the action of the identity component $\mathrm{Aut}^0(X)$ in the complex Lie group of biholomorphisms of $X$.

**Berman-Boucksom-Jonsson 2015**

Let $X$ be a Fano manifold with finite automorphism group. Then $X$ admits a Kähler-Einstein metric if and only if it is uniformly K-stable.

Recently, Chen, Donaldson and Sun got this result under the more general assumption that $X$ is $K$-stable (Bourbaki/Ph. Eyssidieux, january 2015).

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$$\pi^*(K_X + \Delta) = K_{\tilde{X}} + E, \qquad E = \textstyle\sum_j a_j E_j$$

for some $\mathbb{Q}$-divisor $E$ whose push-forward to $X$ is $\Delta$ (since $X_{\rm sing}$ has codimension 2, the components $E_j$ that lie over $X_{\rm sing}$ yield $\pi_* E_j = 0$). The coefficient $-a_j \in \mathbb{Q}$ is known as the discrepancy of $(X, \Delta)$ along $E_j$.

# The klt condition ("Kawamata Log Terminal")

**Definition**

$(X, \Delta)$ is klt if $a_j < 1$ for all $j$.

Let $r$ be a positive integer such that $r(K_X + \Delta)$ is Cartier, and $\sigma$ a local generator of $\mathcal{O}(r(K_X + \Delta))$ on some open set $U \subset X$. Then the $(n, n)$ form

$$|\sigma|^{2/r} := i^{n^2}\,\sigma^{1/r} \wedge \overline{\sigma^{1/r}}$$

is a volume form with poles along $S = \mathrm{Supp}\,\Delta \cup X_{\mathrm{sing}}$.

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By the change of variable formula, the local integrability can be checked by pulling back $\sigma$ to $\tilde{X}$, in which case it is easily seen that the integrability occurs if and only if $a_j < 1$ for all $j$, i.e. when $(X, \Delta)$ is klt.

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In local coordinates

$$|\sigma|^{2/r} \sim \frac{\text{volume form}}{\prod |z_j|^{2a_j}}.$$

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Every form $\omega = \omega_0 + dd^c\psi \in \{\omega_0\}$ can be seen as the curvature form of a smooth hermitian metric $h$ on $\mathcal{O}(-(K_X + \Delta))$, whose weight is $\phi = u_0 + \psi$ where $u_0$ is a local potential of $\omega_0$, hence

$$\omega = \omega_0 + dd^c\psi = dd^c\phi$$

where $\phi$ is understood as the weight of a global metric formally denoted $h = e^{-\phi}$ on the $\mathbb{Q}$-line bundle $\mathcal{O}(-(K_X + \Delta))$.

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The inverse $e^{\phi}$ is a hermitian metric on $\mathcal{O}(K_X + \Delta)$. If $\sigma$ is a local generator of $\mathcal{O}(r(K_X + \Delta))$, the product $|\sigma|^{2/r}e^{\phi} = e^{\psi+u_0}$ is (locally) a smooth positive function whenever $\varphi$ is smooth, hence

$$e^{-\phi} = |\sigma|^{2/r}e^{-(\psi+u_0)}$$

is an integrable volume form on $X$ with poles along $S := \mathrm{Supp}\,\Delta \cup \{\text{singularities}\}$. The KE condition can be rewritten

$$(dd^c\phi)^n = c\,e^{-\phi} \quad \text{on } X \smallsetminus S \Leftrightarrow \mathrm{Ricci}(\omega) = \omega + [\Delta].$$

# The space of Kähler metrics

Let $A \in H^{1,1}_{\partial\overline{\partial}}(X, \mathbb{R})$ be a Kähler $\partial\overline{\partial}$-cohomology class, and let

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Here we are mostly interested in the Fano case $A = -K_X$ and the log Fano case $A = -(K_X + \Delta)$. Let $V = \int_X \omega_0^n = A^n$ be the volume of $\omega_0$.

## Definition

The space $\mathcal{K}_A$ of Kähler metrics (resp. $\mathcal{P}_A$ of Kähler potentials) is the set of Kähler metrics $\omega$ (resp. functions $\psi$) such that

$$\omega = \omega_0 + dd^c\psi > 0.$$

Here $\phi = u_0 + \psi$ is thought intrinsically as a hermitian metric $h = e^{-\phi}$ on $A$ with strictly plurisubharmonic (psh) weight $\phi$.

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Clearly $\mathcal{K}_A \simeq \mathcal{P}_A/\mathbb{R}$.

# The Riemannian structure on $\mathcal{P}_A$

The basic operator of interest on $\mathcal{P}_A$ is the Monge-Ampère operator

$$\mathcal{P}_A \to \mathcal{M}_+, \qquad \mathrm{MA}(\phi) = (dd^c\phi)^n = (\omega_0 + dd^c\psi)^n$$

According to Mabuchi the space $\mathcal{P}_A$ can be seen as some sort of infinite dimensional Riemannian manifold: a "tangent vector" to $\mathcal{P}_A$ is an infinitesimal variation $\delta\phi \in C^\infty(X, \mathbb{R})$ of $\phi$ (or $\psi$), and the infinitesimal Riemannian metric at a point $h = e^{-\phi}$ is given by

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X.X. Chen and his collaborators have studied the metric and geometric properties of the space $\mathcal{P}_A$, showing in particular that it is a path metric space (a non trivial assertion in this infinite dimensional setting) of nonpositive curvature in the sense of Alexandrov. A key step has been to produce almost $C^{1,1}$-geodesics which minimize the geodesic distance.

# Basic functionals (1)

Given $\phi_0, \phi \in \mathcal{P}_A$, one defines:

- The Monge-Ampère functional

$$E_{\phi_0}(\phi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X (\phi - \phi_0)(dd^c\phi)^j \wedge (dd^c\phi_0)^{n-j}$$

$$(***) \qquad = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int_X \psi(\omega_0 + dd^c\psi)^j \wedge \omega_0^{n-j}, \quad \psi = \phi - \phi_0.$$

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It is a primitive of the Monge-Ampère operator in the sense that $dE_{\phi_0}(\phi) = \frac{1}{V}\,\mathrm{MA}(\phi)$, i.e. for any path $[T, T'] \ni t \mapsto \phi_t$, one has

$$\frac{d}{dt} E_{\phi_0}(\phi_t) = \frac{1}{V} \int_X \dot{\phi}_t \,\mathrm{MA}(\phi_t) \quad \text{where } \dot{\phi}_t = \frac{d}{dt}\phi_t.$$

This is easily checked by a differentiation under the integral sign.

# Basic functionals (2)

As a consequence $E$ satisfies the cocycle relation

$$E_{\phi_0}(\phi_1) + E_{\phi_1}(\phi_2) = E_{\phi_0}(\phi_2),$$

so its dependence on $\phi_0$ is only up to a constant.

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Finally, if $\phi_t$ depends linearly on $t$, one has $\ddot{\phi}_t = \frac{d^2}{dt^2}\phi_t = 0$ and a further differentiation of $(***)$ yields

$$\frac{d^2}{dt^2}E_{\phi_0}(\phi_t) = \frac{n}{V}\int_X \dot{\phi}_t\, dd^c\dot{\phi}_t \wedge (dd^c\phi_t)^{n-1}$$
$$= -\frac{n}{V}\int_X d\dot{\phi}_t \wedge d^c\dot{\phi}_t \wedge (dd^c\phi_t)^{n-1} \le 0.$$

It follows that $E_{\phi_0}$ is concave on $\mathcal{P}_A$.

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# The $J$ and $J^*$ functionals

- The concavity of $E$ implies the nonnegativity of $J_{\phi_0}(\phi) := dE_{\phi_0}(\phi_0) \cdot (\phi - \phi_0) - E_{\phi_0}(\phi)$, This quantity is called the Aubin $J$-energy functional

$$J_{\phi_0}(\phi) = V^{-1} \int_X (\phi - \phi_0)(dd^c\phi_0)^n - E_{\phi_0}(\phi) \geq 0.$$

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- By exchanging the roles of $\phi$, $\phi_0$ and putting $J^*_{\phi_0}(\phi) = J_\phi(\phi_0) \geq 0$, the cocycle relation for $E$ yields $E_\phi(-\phi_0) = -E_{\phi_0}(\phi)$. The transposed $J$-energy functional is

$$\begin{aligned} J^*_{\phi_0}(\phi) :&= E_{\phi_0}(\phi) - V^{-1}\int_X (\phi-\phi_0)(dd^c\phi)^n \\ &= E_{\phi_0}(\phi) - V^{-1}\int_X \psi(\omega_0+dd^c\psi)^n \geq 0, \quad \psi = \phi-\phi_0. \end{aligned}$$

# The symmetric $I$ functional

- The *I-functional* is the symmetric functional defined by

$$I_{\phi_0}(\phi) = I_\phi(\phi_0) := -\frac{1}{V}\int_X (\phi-\phi_0)\big(\,\mathrm{MA}(\phi) - \mathrm{MA}(\phi_0)\big)$$

$$= \sum_{j=0}^{n-1} V^{-1}\int_X d(\phi-\phi_0)\wedge d^c(\phi-\phi_0)\wedge(dd^c\phi)^j\wedge(dd^c\phi_0)^{n-1-j} \geq 0.$$

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In fact $I_{\phi_0}(\phi) = J_{\phi_0}(\phi) + J^*_{\phi_0}(\phi)$, and one can also write

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$$I_{\phi_0}(\phi) = V^{-1}\left(\int_X \psi\,\omega_0^n - \int_X \psi(\omega_0+dd^c\psi)^n\right).$$

It satisfies the quasi-triangle inequality: $\exists c_n > 0$ s.t.

$$I_{\phi_0}(\phi) \leq c_n\big(I_{\phi_0}(\phi_1) + I_{\phi_1}(\phi)\big). \qquad \forall\ \phi_0,\ \phi_1,\ \phi \in \mathcal{P}_A.$$

# The Ding and Mabuchi functionals (1)

- In the Fano or log Fano setting, the Ding functional is defined by

$$D_{\phi_0} = L - L(\phi_0) - E_{\phi_0}, \quad \text{where} \quad L(\phi) = -\log \int_X e^{-\phi}.$$

# The Ding and Mabuchi functionals (1)

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Recall: $e^{-\phi}$ is integrable by the klt condition.

- Given probability measures $\mu, \nu$ on a space $X$, the relative entropy $\mathrm{Entr}_\mu(\nu) \in [0, +\infty]$ of $\nu$ with respect to $\mu$ is defined as the integral

$$\mathrm{Entr}_\mu(\nu) := \int_X \log\left(\frac{d\nu}{d\mu}\right) d\nu,$$

if $\nu$ is absolutely continuous w.r.t. $\mu$; $\mathrm{Entr}_\mu(\nu) = +\infty$ otherwise.
Pinsker inequality: for all proba measures $\mu, \nu$ one has

$$\mathrm{Entr}_\mu(\nu) \geq \frac{1}{2}\|\mu - \nu\|^2 \geq 0.$$

In particular, $\mu = \nu \iff \mathrm{Entr}_\mu(\nu) = 0$.

# The Ding and Mabuchi functionals (2)

In the Fano or log Fano situation, the entropy functional $H_{\phi_0}(\phi)$ is defined to be the entropy of the probability measure $\frac{1}{V}(dd^c\phi)^n$ with respect to $e^{L(\phi_0)}e^{-\phi_0}$, namely

$$H_{\phi_0}(\phi) = \int_X \log\left(\frac{(dd^c\phi)^n/V}{e^{L(\phi_0)}e^{-\phi_0}}\right)\frac{(dd^c\phi)^n}{V} \geq 0.$$

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One gets the more explicit expression

$$M_{\phi_0}(\phi) = \int_X \log\left(\frac{e^{\phi}(dd^c\phi)^n}{V}\right)\frac{(dd^c\phi)^n}{V} - E_{\phi_0}(\phi) - L(\phi_0).$$

# Comparison properties

## Observation

If $c$ is a constant, then

$$E_{\phi_0}(\phi + c) = E_{\phi_0}(\phi) + c \quad \text{and} \quad L(\phi + c) = L(\phi) + c.$$

On the other hand, the functionals $I_{\phi_0}, J_{\phi_0}, J^*_{\phi_0}, D_{\phi_0}, H_{\phi_0}, M_{\phi_0}$ are invariant by $\phi \mapsto \phi + c$ and therefore descend to the quotient space $\mathcal{K}_A = \mathcal{P}_A/\mathbb{R}$ of Kähler metrics $\omega = dd^c\phi \in A$.

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## Comparison between $I$, $J$, $J^*$

The functionals $I$, $J$, $J^*$ are essentially growth equivalent:

$$\frac{1}{n} J_\phi(\phi_0) \leq J_{\phi_0}(\phi) \leq \frac{n+1}{n} J_{\phi_0}(\phi) \leq I_{\phi_0}(\phi) \leq (n+1) J_{\phi_0}(\phi).$$

# Comparison between Ding and Mabuchi functionals

## Proposition

Let $(X, \Delta)$ be a log Fano manifold. Then $M_{\phi_0}(\phi) \geq D_{\phi_0}(\phi)$ and, in case of equality, $\phi$ must be Kähler-Einstein.

*Proof.* From the definitions one gets

$$M - D = (H - J^*)(L - L(\phi_0) - E),$$

$$E_{\phi_0}(\phi) - J^*_{\phi_0}(\phi) = V^{-1} \int_X (\phi - \phi_0)(dd^c\phi)^n,$$

$$M_{\phi_0}(\phi) - D_{\phi_0}(\phi)$$

$$= \int_X \left( \log\left( \frac{(dd^c\phi)^n/V}{e^{L(\phi_0)}e^{-\phi_0}} \right) + (\phi - \phi_0) \right) \frac{(dd^c\phi)^n}{V} + L(\phi_0) - L(\phi)$$

$$= \int_X \log\left( \frac{(dd^c\phi)^n/V}{e^{L(\phi)}e^{-\phi}} \right) \frac{(dd^c\phi)^n}{V} \geq 0.$$

In case of equality, Pinsker implies KE condition: $\frac{(dd^c\phi)^n}{V} = e^{L(\phi)}e^{-\phi}$

# Non pluripolar products

- Bedford-Taylor Monge-Ampère products : for $u_j \in L^\infty_{\rm loc}$, one sets inductively

$$dd^c u_1 \wedge dd^c u_2 \wedge \ldots \wedge dd^c u_k := dd^c (u_1\, dd^c u_2 \wedge \ldots \wedge dd^c u_k)$$

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• Non pluripolar products (Guedj-Zeriahi)
Let $\mathcal{P}(X, \omega_0)$ be the set of $\omega_0$-psh potentials, i.e. $\phi = \phi_0 + \psi$ such that $dd^c\phi = \omega_0 + dd^c\psi \geq 0$.

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The functions $\psi_\nu := \max\{\psi, -\nu\}$ are again $\omega_0$-psh and bounded for all $\nu \in \mathbb{N}$. The Monge-Ampère measures $(\omega_0 + dd^c\psi_\nu)^n$ are therefore well-defined in the sense of Bedford-Taylor, and one defines for any bidegree $(p, p)$ a positive current

$$T = \langle(\omega_1 + dd^c\psi_1) \wedge \ldots \wedge (\omega_p + dd^c\psi_p)\rangle = \lim_{\nu\to+\infty}$$

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Basic fact: $T$ is still closed [Proof uses ideas of Skoda & Sibony].

# Space of potentials of finite energy

One introduces for any $p \in [1, +\infty[$ the space

$$\mathcal{E}^p(X, \omega_0) := \Big\{ \phi = \phi_0 + \psi\,;\ \int_X |\psi|^p \,\mathrm{MA}(\omega_0 + dd^c\psi) < +\infty \Big\},$$

and $\int_X \mathrm{MA}(\omega_0 + dd^c\psi) = \int_X \omega_0^n$ ("full non pluripolar mass"). One says that functions $\psi \in \mathcal{E}^p(X, \omega_0)$ have *finite $\mathcal{E}^p$-energy*.

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$$\mathcal{T}^p(X, \omega_0) \subset \mathcal{T}^p_{\mathrm{full}}(X, \omega_0)$$

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$$\mathcal{T}^p(X, \omega_0) = \mathcal{E}^p(X, \omega_0)/\mathbb{R} \quad \text{via} \quad \phi \mapsto dd^c\phi = \omega_0 + dd^c\psi.$$

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It is important to note that $\mathcal{T}^p(X,\omega_0)$ is not a closed subset of $\mathcal{T}(X,\omega_0)$ for the weak topology.

# Finite energy extension of the functionals

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All functionals $E, L, I, J, J^*, D, H, M$ have a natural extension to arguments $\phi, \phi_0 \in \mathcal{E}^1(X, \omega_0)$, and $I, J, J^*, D, H, M$ descend to $\mathcal{T}^1(X, \omega_0) = \mathcal{E}^1(X, \omega_0)/\mathbb{R}$.

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## Theorem (BBGZ)

The map $T = \omega_0 + dd^c\psi \mapsto V^{-1}\langle T^n \rangle$ is a bijection between $\mathcal{T}^1(X, \omega_0)$ and the space of probability measures $\mathcal{M}^1(X, \omega_0)$ of finite energy.

Here one uses the Legendre-Fenchel transform

$$E_0^*(\mu) := \sup_{\phi=\phi_0+\psi\in\mathcal{E}^1(X,\omega_0)} \left( E_0(\psi) - \int_X \psi\, \mu \right) \in [0, +\infty]$$

where $E_0(\psi) = E_{\phi_0}(\phi_0 + \psi)$, and $\mu$ has finite energy if $E_0^*(\mu) < +\infty$.

# Sufficient conditions for existence of KE metrics

**Theorem (BBEGZ)**

For a current $\omega = dd^c\phi \in \mathcal{T}^1(X, A)$, the following conditions are equivalent.

(i) $\omega$ is a Kähler-Einstein metric for $(X, \Delta)$.

(ii) The Ding functional reaches its infimum at $\phi$ : $D_{\phi_0}(\phi) = \inf_{\mathcal{E}^1(X,A)/\mathbb{R}} D_{\phi_0}$.

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**Corollary (BBEGZ)**

Let $X$ be a $\mathbb{Q}$-Fano variety with log terminal singularities.

(i) The identity component $\mathrm{Aut}^0(X)$ of the automorphism group of $X$ acts transitively on the set of KE metrics on $X$,

(ii) If the Mabuchi functional of $X$ is proper, then $\mathrm{Aut}^0(X) = \{1\}$ and $X$ admits a unique Kähler-Einstein metric.

# Test configurations

## Definition

A test configuration $(\mathcal{X}, \mathcal{A})$ for a ($\mathbb{Q}$-)polarized projective variety $(X, A)$ consists of the following data :

(i) a flat and proper morphism $\pi : \mathcal{X} \to \mathbb{C}$ of algebraic varieties; one denotes by $X_t = \pi^{-1}(t)$ the fiber over $t \in \mathbb{C}$.

(ii) a $\mathbb{C}^*$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{C}$;

(iii) an isomorphism $X_1 \simeq X$.

(iv) a $\mathbb{C}^*$-linearized ample line bundle $\mathcal{A}$ on $\mathcal{X}$ ; one puts $A_t = \mathcal{A}_{|X_t}$.

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$K$ stability (and uniform $K$-stability) is defined in termes of certain numerical invariants attached to arbitrary test configurations.

# Donaldson-Futaki invariants

## Donaldson-Futaki invariant

Let $N_m = h^0(X, mA)$ and $w_m \in \mathbb{Z}$ be the weight of the $\mathbb{C}^*$-action on the determinant $\det H^0(\mathcal{X}_0, m\mathcal{A}_0)$. Then there is an asymptotic expansion

$$\frac{w_m}{mN_m} = F_0 + m^{-1}F_1 + m^{-2}F_2 + \ldots \; .$$

and one defines $\mathrm{DF}(\mathcal{X}, \mathcal{A}) := -2F_1$.

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The polarized variety $(X, A)$ is said K-stable if $\mathrm{DF}(\mathcal{X}, \mathcal{A}) \geq 0$ for all normal test configurations, with equality iff $(\mathcal{X}, \mathcal{A})$ is trivial.

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## Generalized Yau-Tian-Donaldson conjecture

Let $(X, A)$ be a polarized variety. Then $X$ admits a cscK metric (short hand for Kähler metric with constant scalar curvature) $\omega \in c_1(A)$ if and only if $(X, A)$ is K-stable.

# Uniform K-stability

The Duistermaat-Heckman measure $\mathrm{DH}_{(X,A)}$ is the proba distribution measure of the $\mathbb{C}^*$-action weights:

$$\mathrm{DH}_{(X,A)} = \lim_{m\to\infty} \sum_{\lambda\in\mathbb{Z}} \frac{\dim H^0(X,mA)_\lambda}{\dim H^0(X,mA)}\,\delta_{\lambda/m}, \quad \delta_p := \text{Dirac at } p,$$

where $H^0(X,mA) = \bigoplus_{\lambda\in\mathbb{Z}} H^0(X,mA)_\lambda$ is the weight space decomposition. For each $p\in[1,\infty]$, the $L^p$-norm $\|(\mathcal{X},\mathcal{A})\|_p$ of an ample test configuration $(\mathcal{X},\mathcal{A})$ is defined as the $L^p$ norm

$$\|(\mathcal{X},\mathcal{A})\|_p = \left(\int_{\mathbb{R}} |\lambda - b(\mu)|^p\,d\mu(\lambda)\right)^{1/p}, \quad b(\mu) = \int_{\mathbb{R}} \lambda\,d\mu(\lambda).$$

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**Definition (Székelyhidi)**

The polarized variety $(X,A)$ is said to be $L^p$-uniformly K-stable if there exists $\delta>0$ such that $\mathrm{DF}(\mathcal{X},\mathcal{A}) \ge \delta\,\|(\mathcal{X},\mathcal{A})\|_p$ for all normal test configurations. [Note: only possible if $p < \frac{n}{n-1)}$.]

# Sufficiency of uniform K-stability

**Berman-Boucksom-Jonsson 2015**

Let $X$ be a Fano manifold with finite automorphism group. Then $X$ admits a Kähler-Einstein metric if and only if it is uniformly K-stable (in a related and simpler "non archimedean" sense).

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Let $A = -K_X$. A ray $(\phi_t)_{t\geq 0}$ in $\mathcal{P}_A$ corresponds to an $S^1$-invariant metric $\Phi$ on the pull-back of $-K_X$ to the product of $X$ with the punctured unit disc $\mathbb{D}^*$. The ray is called subgeodesic when $\Phi$ is plurisubharmonic (psh for short). Denoting by $F$ any of the functionals $M, D$ or $J$, there is a limit

$$\lim_{t\to+\infty} \frac{F(\phi_t)}{t} = F^{\mathrm{NA}}(\mathcal{X}, \mathcal{A})$$

Here $F^{\mathrm{NA}}$ can be seen as the corresponding "non-Archimedean" functional.