On the Monge-Ampère volume of holomorphic vector bundles

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Positive and ample vector bundles

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Chern curvature tensor

This is $\Theta_{E,h} = i\nabla^2_{E,h} \in C^\infty(\Lambda^{1,1}T^*_X \otimes \mathrm{Hom}(E,E))$, which can be written

$$\Theta_{E,h} = i \sum_{1\leq j,k\leq n,\, 1\leq\lambda,\mu\leq r} c_{jk\lambda\mu}\, dz_j \wedge d\overline{z}_k \otimes e^*_\lambda \otimes e_\mu$$

in terms of an orthonormal frame $(e_\lambda)_{1\leq\lambda\leq r}$ of E.

Positivity concepts for vector bundles

Griffiths and (dual) Nakano positivity

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi, \overline{\xi}) \cdot v, v \rangle_h = \sum_{1 \leq j,k \leq n,\, 1 \leq \lambda,\mu \leq r} c_{jk\lambda\mu} \xi_j \overline{\xi}_k v_\lambda \overline{v}_\mu .$$

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Then E is said to be

- Griffiths positive (Griffiths 1969) if at any point $z \in X$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall \xi \in T_{X,z} \smallsetminus \{0\},\ \forall v \in E_z \smallsetminus \{0\}$$

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$$\Theta_{E^*,h} = -{}^T\Theta_{E,h} = -\sum c_{jk\mu\lambda} dz_j \wedge d\overline{z}_k \otimes (e^*_\lambda)^* \otimes e^*_\mu.$$

Relationships between these positivity concepts

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Berndtsson (2007): E ample $\Rightarrow S^mE \otimes \det E$ Nakano > 0, $\forall m \geq 0$.

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$$H^{n-1,n-1}(\mathbb{P}^n,\mathbb{C}) = H^{n-1}(\mathbb{P}^n,\Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n}\otimes T_{\mathbb{P}^n}) = 0.$$

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E dual Nakano $> 0 \Rightarrow H^0(X,\Omega^p_X\otimes E^*) = 0$ for $p < n = \dim X$.

Take e.g. a smooth compact quotient $X = \mathbb{B}^n/\Gamma$ of the ball, $n \geq 2$. Then $E = \Omega^1_X$ is Griffiths positive, but $\mathrm{Id} \in H^0(X,\Omega^1_X\otimes E^*) \neq 0$, so E cannot be dual Nakano positive.

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Assume that E is ample. One has $\tau_N(E) < 1$ (Berndtsson), $\tau_{N^*}(E) < 1$ (Liu-Sun-Yang), and the Griffiths conjecture E ample $\Rightarrow$ E Griffiths > 0 is equivalent to asserting that $\tau_G(E) < 0$.

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The previous counterexamples show that one may have $\tau_N(E) \geq 0$ and $\tau_{N^*}(E) \geq 0$, but it could still wonder whether

$$E \text{ ample} \Rightarrow \tau_N(E) \leq 0,\ \tau_{N^*}(E) \leq 0\ \ ?$$

Determinantal functionals of the curvature tensor

If the Chern curvature tensor $\Theta_{E,h}$ is Nakano positive, one can introduce the $(n \times r)$-dimensional determinant of the Hermitian quadratic form on $T_X \otimes E$

$$\det{}_{T_X \otimes E}(\Theta_{E,h})^{1/r} := \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

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On the other hand, if $\Theta_{E,h}$ is dual Nakano positive, one can consider the $(n \times r)$-dimensional determinant of the "dual" Hermitian quadratic form on $T_X \otimes E^*$

$$\det{}_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r} := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

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In case $\Theta_{E,h}$ is Griffiths > 0, we have a functional

$$\mathrm{Grif}(\Theta_{E,h})(z) = \inf_{v\in E_z,\ |v|_h=1} \langle \Theta_{E,h}(z)v, v\rangle^n.$$

Monge-Ampère volumes for vector bundles

If $E \to X$ is an ample vector bundle of rank r that is Nakano positive (resp. dual Nakano positive), one can introduce its Monge-Ampère volume to be

$$\mathrm{MAVol}(E) = \sup_h \int_X \det{}_{T_X \otimes E}\left((2\pi)^{-1}\Theta_{E,h}\right)^{1/r},$$

$$\mathrm{MAVol}^*(E) = \sup_h \int_X \det{}_{T_X \otimes E^*}\left((2\pi)^{-1}\,{}^T\Theta_{E,h}\right)^{1/r},$$

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where the supremum is taken over all smooth metrics h on E such that $\Theta_{E,h}$ is Nakano positive (resp. dual Nakano positive).

This supremum is always finite, and in fact

Proposition

For any (dual) Nakano positive vector bundle E, one has

$$\mathrm{MAVol}(E) \le r^{-n} c_1(E)^n, \quad \mathrm{MAVol}^*(E) \le r^{-n} c_1(E)^n.$$

Equality occurs if and only if E is projectively flat.

Proof of the volume inequality

Assume e.g. E nakano positive. Take $\omega_0 = \Theta_{\det E} > 0$ as a Kähler metric on X, and let $(\lambda_j)_{1\le j\le nr}$ be the eigenvalues of $\tilde{\Theta}_{E,h}$ as a hermitian form on $T_X \otimes E$, with respect to $\omega_0 \otimes h$.

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$$\det{}_{T_X\otimes E}\left((2\pi)^{-1}\Theta_{E,h}\right)^{1/r} = \left(\prod_j \lambda_j\right)^{1/r}\omega_0^n$$

The inequality between geometric and arithmetic means $(\prod \lambda_j)^{1/nr} \le \frac{1}{nr}\sum \lambda_j$ implies, after raising to power n

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Equality occurs iff all λ_j are equal, i.e. E projectively flat.

In case E is Griffiths > 0, one can define

$$\mathrm{MAVol}_{\mathrm{Grif}}(E) = \sup_h \int_{z\in X} \inf_{v\in E_z,\ |v|_h=1} \left((2\pi)^{-1}\langle\Theta_{E,h}v, v\rangle\right)^n.$$

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The Teissier-Hovanskii inequalities imply again $\mathrm{MAVol}_{\mathrm{Grif}}(E) \le \frac{1}{r^n}c_1(E)^n$ with equality iff E is projectively flat.

Further remarks

- In the split case $E = \bigoplus_{1\le j\le r} E_j$ and $h = \bigoplus_{1\le j\le r} h_j$, the inequality reads

$$\Big(\prod_{1\le j\le r} c_1(E_j)^n\Big)^{1/r} \le r^{-n} c_1(E)^n,$$

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- The Euler-Lagrange equation for the maximizer is complicated (4th order!). It somehow extends the equation characterizing cscK metrics.

On the Fulton Lazarsfeld inequalities (S. Finski)

A fundamental result due to Fulton-Lazarsfeld asserts that if $E \to X$ is an ample vector bundle, then all Schur polynomials $P(c_\bullet(E))$ in the Chern classes are numerically positive, i.e.

$$\int_Y P(c_\bullet(E)) > 0$$

for all irreducible cycles Y of the appropriate dimension in X.

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A fundamental result due to Fulton-Lazarsfeld asserts that if $E \to X$ is an ample vector bundle, then all Schur polynomials $P(c_\bullet(E))$ in the Chern classes are numerically positive, i.e.

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This is a compelling motivation to investigate the relationships between ampleness, Griffiths and Nakano positivity!

Further recent results by Siarhei Finski

When $E \to X$ is an ample vector bundle, the symmetric powers $S^m E$ have enough sections to generate 1-jets for $m \geq m_0 \gg 1$, and one can immediately derive from there that

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Theorem (S. Finski 2020)

Given any volume form $d\nu$ on X, the direct images satisfy

$$\mathrm{MAVol}(E_m, h_{E_m}) \sim m^{\dim X} \int_X \exp\left(\frac{\int_Y \log\left(\omega_H^{\dim X}/\pi^*\nu\right)\omega^{\dim Y}}{\int_Y c_1(L)^{\dim Y}}\right) d\nu,$$

where $\omega = \Theta_{L,h_L} > 0$ on Y, and ω_H is its horizontal part.

Matrix Monge-Ampère equations

Basic idea

Assigning a "matrix Monge-Ampère equation"

$$\det_{T_X \otimes E}(\Theta_{E,h})^{1/r} = f > 0 \quad \text{or} \quad \mathrm{Grif}(\Theta_{E,h}) = f > 0$$

where f is a positive (n, n)-form, may enforce the Nakano (resp. Griffiths) positivity of $\Theta_{E,h}$, especially if that assignment is combined with a continuity technique from an initial starting point where positivity is known.

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In case $r = \mathrm{rank}\, E = 1$ and $h = h_0 e^{-\varphi}$, this is the same as solving a complex Monge-Ampère equation

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Underdeterminacy of the equation

Assuming E to be ample of rank $r > 1$, the equation

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Conclusion

In order to recover a well determined system of equations, one needs an additional "matrix equation" of rank $(r^2 - 1)$.

Observation 1 (from the Donaldson-Uhlenbeck-Yau theorem)

Take a Hermitian metric η_0 on $\det E$ so that $\omega_0 := \Theta_{\det E, \eta_0} > 0$.
If E is ω_0-polystable, $\exists h$ Hermitian metric h on E such that

$$\omega_0^{n-1} \wedge \Theta_{E,h} = \frac{1}{r}\,\omega_0^n \otimes \mathrm{Id}_E \quad \text{(Hermite-Einstein equation, slope } \tfrac{1}{r}\text{)}.$$

Resulting trace free condition

Observation 2

The trace part of the above Hermite-Einstein equation is "automatic", hence the equation is equivalent to the trace free condition

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,h} = 0,$$

when decomposing any endomorphism $u \in \mathrm{Herm}(E, E)$ as

$$u = u^\circ + \frac{1}{r}\,\mathrm{Tr}(u)\,\mathrm{Id}_E \in \mathrm{Herm}^\circ(E, E) \oplus \mathbb{R}\,\mathrm{Id}_E, \quad \mathrm{tr}(u^\circ) = 0.$$

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Remark

In case $\dim X = n = 1$, the trace free condition means that E is projectively flat, and the Umemura proof of the Griffiths conjecture proceeds exactly in that way, using the fact that the graded pieces of the Harder-Narasimhan filtration are projectively flat.

Towards a "cushioned" Hermite-Einstein equation

In general, one cannot expect E to be ω_0-polystable, but Uhlenbeck-Yau have shown that there always exists a smooth solution q_ε to a certain "cushioned" Hermite-Einstein equation.

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To make things more precise, let $\mathrm{Herm}(E)$ be the space of Hermitian (non necessarily positive) forms on E. Given a reference Hermitian metric $H_0 > 0$, let $\mathrm{Herm}_{H_0}(E,E)$ be the space of H_0-Hermitian endomorphisms $u \in \mathrm{Hom}(E,E)$; denote by

$$\mathrm{Herm}(E) \xrightarrow{\simeq} \mathrm{Herm}_{H_0}(E,E), \quad q \mapsto \widetilde{q} \ \text{ s.t. } \ q(v,w) = \langle \widetilde{q}(v), w \rangle_{H_0}$$

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$$\mathrm{Herm}^\circ_{H_0}(E,E) = \big\{q \in \mathrm{Herm}_{H_0}(E,E)\,;\ \mathrm{tr}(q) = 0\big\}$$

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In the sequel, we fix H_0 on E such that

$$\Theta_{\det E, \det H_0} = \omega_0 > 0.$$

A basic result from Uhlenbeck and Yau

Uhlenbeck-Yau 1986, Theorem 3.1

For every $\varepsilon > 0$, there always exists a (unique) smooth Hermitian metric q_ε on E such that

$$\omega_0^{n-1} \wedge \Theta_{E,q_\varepsilon} = \omega_0^n \otimes \left(\frac{1}{r} \, \mathrm{Id}_E - \varepsilon \log \widetilde{q}_\varepsilon \right),$$

where $\widetilde{q}_\varepsilon$ is computed with respect to H_0, and $\log g$ denotes the logarithm of a positive Hermitian endomorphism g.

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The reason is that the term $-\varepsilon \log \widetilde{q}_\varepsilon$ is a "friction term" that prevents the explosion of the a priori estimates, similarly what happens for Monge-Ampère equations $(\omega_0 + i\partial\overline{\partial}\varphi)^n = e^{\varepsilon\varphi+f}\omega_0^n$.

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The above matrix equation is equivalent to prescribing $\det q_\varepsilon = \det H_0$ and the trace free equation of rank (r^2-1)

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,q_\varepsilon} = -\varepsilon\, \omega_0^n \otimes \log \widetilde{q}_\varepsilon.$$

Search for an appropriate evolution equation

General setup

In this context, given $\alpha > 0$ large enough, it is natural to search for a time dependent family of metrics $h_t(z)$ on the fibers E_z of E, $t \in [0,1]$, satisfying a generalized Monge-Ampère equation

$$(D) \qquad \det_{T_X \otimes E}\left(\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_E\right)^{1/r} = f_t\,\omega_0^n, \quad f_t > 0,$$

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and trace free, rank $r^2 - 1$, Hermite-Einstein conditions

$$(T) \quad \omega_t^{n-1} \wedge \Theta^\circ_{E,h_t} = g_t$$

with smoothly varying families of functions $f_t \in C^\infty(X,\mathbb{R})$, Hermitian metrics $\omega_t > 0$ on X and sections

$$g_t \in C^\infty(X, \Lambda^{n,n}_{\mathbb{R}} T^*_X \otimes \mathrm{Herm}^\circ_{h_t}(E,E)), \quad t \in [0,1].$$

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Observe that this is a determined (not overdetermined!) system.

Choice of the initial state ($t = 0$)

We start with the Uhlenbeck-Yau solution $h_0 = q_\varepsilon$ of of the "cushioned" trace free Hermite-Einstein equation, so that $\det h_0 = \det H_0$, and take $\alpha > 0$ so large that

$$\Theta_{E,h_0} + \alpha\,\omega_0 \otimes \mathrm{Id}_E > 0 \text{ in the sense of Nakano.}$$

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If conditions (D) and (T) can be met for all $t \in [0, t_0]$, thus without any discontinuity or explosion of the solutions h_t, we infer from (D) that

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Question

Is the maximal existence time t_0 of the solution such that $(1 - t_0)\alpha = \tau_N(E)$ (Nakano threshold of E) ?

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One still has the freedom of adjusting f_t, ω_t and g_t in the general setup. There are in fact many possibilities:

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Proposition

Let (E, H_0) be a smooth Hermitian holomorphic vector bundle such that E is ample and $\omega_0 = \Theta_{\det E, \det H_0} > 0$. Then the system of determinantal and trace free equations

$$(D) \quad \det_{T_X \otimes E}\left(\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_E\right)^{1/r} = F(t, z, h_t, D_z h_t)$$

$$(T) \quad \omega_t^{n-1} \wedge \Theta^{\circ}_{E,h_t} = G(t, z, h_t, D_z h_t, D_z^2 h_t) \in \mathrm{Herm}^{\circ}(E, E)$$

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It is elliptic whenever the symbol η_h of the linearized operator $u \mapsto DG_{D^2 h}(t, z, h, Dh, D^2 h) \cdot D^2 u$ has an Hilbert-Schmidt norm

$$\sup_{\xi \in T_X^*,\, |\xi|_{\omega_t} = 1} \|\eta_{h_t}(\xi)\|_{h_t} \leq (r^2+1)^{-1/2}\, n^{-1}$$

for any metric h_t involved, e.g. if G does not depend on $D^2 h$.

Proof of the ellipticity

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\overline{\partial}(h^{-1}\partial h) = i\overline{\partial}(\widetilde{h}^{-1}\partial_{H_0}\widetilde{h}),$$

where $\partial_{H_0} s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection on $\mathrm{Hom}(E,E)$ associated with H_0 on E.

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Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \to C^\infty(W), \quad f \mapsto P(f) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\,\xi^\alpha \in \mathrm{Hom}(V,W)$$

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For instance, on the torus $\mathbb{R}^n/\mathbb{Z}^n$, $f \mapsto P_\lambda(f) = -\Delta f + \lambda f$ has an invertible symbol $\sigma_{P_\lambda}(x,\xi) = -|\xi|^2$, but P_λ is invertible only for $\lambda > 0$.

A more specific choice of the right hand side

Theorem

The elliptic differential system defined by

$$\det_{T_X\otimes E}\left(\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes \mathrm{Id}_E\right)^{1/r}=\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\lambda} a_0(z),$$

$$\omega_t^{n-1}\wedge\Theta_{E^\circ,h_t}=-\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\mu}(\log\widetilde{h}_t^\circ)\,\omega_0^n \quad \text{w.r.t. Kähler metric}$$

$$\omega_t=\frac{1}{r\alpha+1}\mathrm{tr}\left(\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes\mathrm{Id}_E\right)>0,$$

possesses an invertible elliptic linearization for $\varepsilon\geq\varepsilon_0(h_t)$ and $\lambda\geq\lambda_0(h_t)(1+\mu^2)$, with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough.

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$$\det_{T_X\otimes E}\big(\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes \mathrm{Id}_E\big)^{1/r}=\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\lambda} a_0(z),$$

$$\omega_t^{n-1}\wedge\Theta_{E^\circ,h_t}=-\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\mu}(\log\widetilde{h}_t^\circ)\,\omega_0^n\ \text{ w.r.t. Kähler metric}$$

$$\omega_t=\frac{1}{r\alpha+1}\mathrm{tr}\big(\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes \mathrm{Id}_E\big)>0,$$

possesses an invertible elliptic linearization for $\varepsilon\geq\varepsilon_0(h_t)$ and $\lambda\geq\lambda_0(h_t)(1+\mu^2)$, with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough.

Corollary

Under the above conditions, starting from the Uhlenbeck-Yau solution h_0 such that $\det h_0=\det H_0$ at $t=0$, the PDE system still has a solution for $t\in[0,t_0]$ and $t_0>0$ small.

A more specific choice of the right hand side

Theorem

The elliptic differential system defined by

$$\det_{T_X\otimes E}\left(\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes \mathrm{Id}_E\right)^{1/r}=\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\lambda} a_0(z),$$

$$\omega_t^{n-1}\wedge\Theta_{E^\circ,h_t}=-\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^{\mu}(\log\widetilde{h}_t^\circ)\,\omega_0^n \quad \text{w.r.t. Kähler metric}$$

$$\omega_t=\frac{1}{r\alpha+1}\mathrm{tr}\left(\Theta_{E,h_t}+(1-t)\alpha\,\omega_0\otimes\mathrm{Id}_E\right)>0,$$

possesses an invertible elliptic linearization for $\varepsilon\geq\varepsilon_0(h_t)$ and $\lambda\geq\lambda_0(h_t)(1+\mu^2)$, with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough.

Corollary

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Proof. Compute total symbol of linearized system + linear algebra.

Joyeuse et active retraite, Ahmed !

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