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# On the Monge-Ampère volume of holomorphic vector bundles

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## Positive and ample vector bundles

Let $X$ be a projective $n$-dimensional manifold and $E \to X$ a holomorphic vector bundle of rank $r \geq 1$.

### Ample vector bundles

$E \to X$ is said to be ample in the sense of Hartshorne if the associated line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ on $\mathbb{P}(E)$ is ample.

By Kodaira, this is equivalent to the existence of a smooth hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ with positive curvature (equivalently, a negatively curved Finsler metric on $E^*$).

### Chern curvature tensor

This is $\Theta_{E,h} = i\nabla^2_{E,h} \in C^\infty(\Lambda^{1,1}T^*_X \otimes \mathrm{Hom}(E,E))$, which can be written

$$\Theta_{E,h} = i \sum_{1\leq j,k\leq n,\, 1\leq\lambda,\mu\leq r} c_{jk\lambda\mu} dz_j \wedge d\bar{z}_k \otimes e^*_\lambda \otimes e_\mu$$

in terms of an orthonormal frame $(e_\lambda)_{1\leq\lambda\leq r}$ of $E$.

## Positivity concepts for vector bundles

### Griffiths and (dual) Nakano positivity

One looks at the associated quadratic form on $S = T_X \otimes E$

$$\widetilde{\Theta}_{E,h}(\xi \otimes v) := \langle \Theta_{E,h}(\xi,\overline{\xi}) \cdot v, v\rangle_h = \sum_{1\le j,k\le n,\,1\le\lambda,\mu\le r} c_{jk\lambda\mu}\xi_j\overline{\xi}_k v_\lambda \overline{v}_\mu.$$

Then $E$ is said to be

- Griffiths positive (Griffiths 1969) if at any point $z \in X$
$$\widetilde{\Theta}_{E,h}(\xi \otimes v) > 0, \quad \forall \xi \in T_{X,z} \smallsetminus \{0\},\ \forall v \in E_z \smallsetminus \{0\}$$
- Nakano positive (Nakano 1955) if at any point $z \in X$
$$\widetilde{\Theta}_{E,h}(\tau) = \sum c_{jk\lambda\mu}\tau_{j,\lambda}\overline{\tau}_{k,\mu} > 0,\ \forall \underset{\neq 0}{\tau} = \sum \tau_{j,\lambda}\frac{\partial}{\partial z_j} \otimes e_\lambda \in T_{X,z} \otimes E_z.$$
- dual Nakano positive if at any point $z \in X$
$${}^T\widetilde{\Theta}_{E,h}(\tau) = \sum c_{jk\mu\lambda}\tau_{j,\lambda}\overline{\tau}_{k,\mu} > 0,\ \forall \underset{\neq 0}{\tau} = \sum \tau_{j,\lambda}\frac{\partial}{\partial z_j} \otimes e^*_\lambda \in T_{X,z} \otimes E^*_z.$$

$$\Theta_{E^*,h} = -{}^T\Theta_{E,h} = -\sum c_{jk\mu\lambda} dz_j \wedge d\overline{z}_k \otimes (e^*_\lambda)^* \otimes e^*_\mu.$$



## Relationships between these positivity concepts

### Easy and well known facts

$E$ (dual) Nakano positive $\Rightarrow$ $E$ Griffiths positive $\Rightarrow$ $E$ ample.

In fact $E$ Griffiths positive $\Rightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ positive:

$$\Theta_{\mathcal{O}_{\mathbb{P}(E)}(1)} = \omega_{\mathrm{FS}}([v]) + \sum c_{jk\lambda\mu}\frac{v_\lambda \overline{v}_\mu}{|v|^2} dz_j \wedge d\overline{z}_k, \quad z \in X,\ v \in E_z.$$

### Remark: dual Nakano positivity is somewhat better behaved

$E$ dual Nakano (semi)positive $\Rightarrow$ any quotient $Q = E/S$ is also dual Nakano (semi)positive.

$E$ generated by global sections $\Rightarrow$ $E$ dual Nakano semipositive.

### Proposition

$E$ ample $\Rightarrow S^mE$ Nakano and dual Nakano $> 0$ for $m \gg 1$.

Berndtsson (2007): $E$ ample $\Rightarrow S^mE \otimes \det E$ Nakano $> 0$, $\forall m \ge 0$.

## Some counterexamples

### First (well known) observation

$E$ Griffiths positive $\not\Rightarrow$ $E$ Nakano positive.

For instance, $T_{\mathbb{P}^n}$ is easy shown to be ample and Griffiths positive for the Fubini-Study metric, but it is not Nakano positive. Otherwise the Nakano vanishing theorem would then yield

$$H^{n-1,n-1}(\mathbb{P}^n,\mathbb{C}) = H^{n-1}(\mathbb{P}^n,\Omega^{n-1}_{\mathbb{P}^n}) = H^{n-1}(\mathbb{P}^n, K_{\mathbb{P}^n}\otimes T_{\mathbb{P}^n}) = 0.$$

### Second observation (Liu, Sun, Yang, 2013)

$E$ Griffiths positive $\not\Rightarrow$ $E$ dual Nakano positive.

In fact, a variant of the Nakano vanishing theorem gives that
$E$ dual Nakano $> 0 \Rightarrow H^0(X,\Omega^p_X\otimes E^*) = 0$ for $p < n = \dim X$.

Take e.g. a smooth compact quotient $X = \mathbb{B}^n/\Gamma$ of the ball, $n \geq 2$.
Then $E = \Omega^1_X$ is Griffiths positive, but $\mathrm{Id} \in H^0(X,\Omega^1_X\otimes E^*) \neq 0$, so $E$ cannot be dual Nakano positive.



## Positivity thresholds

### Definition of a few thresholds

Let $E \to X$ be a holomorphic vector bundle such that $\det E = \Lambda^r E$ is ample.
One can introduce respectively the ample threshold $\tau_A(E)$, the Griffiths threshold $\tau_G(E)$, the Nakano threshold $\tau_N(E)$, the dual Nakano threshold $\tau_{N^*}(E)$ to be the infimum of $t \in \mathbb{Q}$ such that $E\otimes(\det E)^t$ is ample, i.e. $S^m(E\otimes(\det E)^t)$ is ample, resp. Griffiths, Nakano, dual Nakano positive.

Assume that $E$ is ample. One has $\tau_N(E) < 1$ (Berndtsson), $\tau_{N^*}(E) < 1$ (Liu-Sun-Yang), and the Griffiths conjecture
$E$ ample $\Rightarrow$ $E$ Griffiths $> 0$ is equivalent to asserting that $\tau_G(E) < 0$.

The previous counterexamples show that one may have
$\tau_N(E) \geq 0$ and $\tau_{N^*}(E) \geq 0$, but it could still wonder whether

$$E \text{ ample} \Rightarrow \tau_N(E) \leq 0,\ \tau_{N^*}(E) \leq 0 \ \ ?$$

## Determinantal functionals of the curvature tensor

If the Chern curvature tensor $\Theta_{E,h}$ is Nakano positive, one can introduce the $(n \times r)$-dimensional determinant of the Hermitian quadratic form on $T_X \otimes E$

$$\det{}_{T_X \otimes E}(\Theta_{E,h})^{1/r} := \det(c_{jk\lambda\mu})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

On the other hand, if $\Theta_{E,h}$ is dual Nakano positive, one can consider the $(n \times r)$-dimensional determinant of the "dual" Hermitian quadratic form on $T_X \otimes E^*$

$$\det{}_{T_X \otimes E^*}({}^T\Theta_{E,h})^{1/r} := \det(c_{jk\mu\lambda})^{1/r}_{(j,\lambda),(k,\mu)}\, idz_1 \wedge d\overline{z}_1 \wedge \ldots \wedge idz_n \wedge d\overline{z}_n.$$

These $(n,n)$-forms do not depend on the choice of coordinates $(z_j)$ on $X$, nor on the choice of the orthonormal frame $(e_\lambda)$ on $E$.

In case $\Theta_{E,h}$ is Griffiths $> 0$, we have a functional

$$\mathrm{Grif}(\Theta_{E,h})(z) = \inf_{v \in E_z,\ |v|_h=1} \langle \Theta_{E,h}(z)v, v\rangle^n.$$



## Monge-Ampère volumes for vector bundles

If $E \to X$ is an ample vector bundle of rank $r$ that is Nakano positive (resp. dual Nakano positive), one can introduce its Monge-Ampère volume to be

$$\mathrm{MAVol}(E) = \sup_h \int_X \det{}_{T_X \otimes E}\left((2\pi)^{-1}\Theta_{E,h}\right)^{1/r},$$
$$\mathrm{MAVol}^*(E) = \sup_h \int_X \det{}_{T_X \otimes E^*}\left((2\pi)^{-1}\,{}^T\Theta_{E,h}\right)^{1/r},$$

where the supremum is taken over all smooth metrics $h$ on $E$ such that $\Theta_{E,h}$ is Nakano positive (resp. dual Nakano positive).

This supremum is always finite, and in fact

### Proposition

For any (dual) Nakano positive vector bundle $E$, one has

$$\mathrm{MAVol}(E) \le r^{-n} c_1(E)^n, \quad \mathrm{MAVol}^*(E) \le r^{-n} c_1(E)^n.$$

Equality occurs if and only if $E$ is projectively flat.

## Proof of the volume inequality

Assume e.g. $E$ nakano positive. Take $\omega_0 = \Theta_{\det E} > 0$ as a Kähler metric on $X$, and let $(\lambda_j)_{1\le j\le nr}$ be the eigenvalues of $\tilde\Theta_{E,h}$ as a hermitian form on $T_X \otimes E$, with respect to $\omega_0 \otimes h$. We have

$$\det{}_{T_X\otimes E}\left((2\pi)^{-1}\Theta_{E,h}\right)^{1/r} = \left(\prod_j \lambda_j\right)^{1/r} \omega_0^n$$

The inequality between geometric and arithmetic means $(\prod \lambda_j)^{1/nr} \le \frac{1}{nr}\sum \lambda_j$ implies, after raising to power $n$

$$\det{}_{T_X\otimes E}\left((2\pi)^{-1}\Theta_{E,h}\right)^{1/r} \le \left(\frac{1}{nr}\sum \lambda_j\right)^n \omega_0^n = \frac{\omega_0^n}{r^n} = \frac{1}{r^n}(\Theta_{\det E})^n.$$

Equality occurs iff all $\lambda_j$ are equal, i.e. $E$ projectively flat.

In case $E$ is Griffiths $> 0$, one can define

$$\mathrm{MAVol}_{\mathrm{Grif}}(E) = \sup_h \int_{z\in X} \inf_{v\in E_z,\ |v|_h=1} \Big((2\pi)^{-1}\langle \Theta_{E,h} v, v\rangle\Big)^n.$$

The Teissier-Hovanskii inequalities imply again $\mathrm{MAVol}_{\mathrm{Grif}}(E) \le \frac{1}{r^n} c_1(E)^n$ with equality iff $E$ is projectively flat.



## Further remarks

- In the split case $E = \bigoplus_{1\le j\le r} E_j$ and $h = \bigoplus_{1\le j\le r} h_j$, the inequality reads

$$\left(\prod_{1\le j\le r} c_1(E_j)^n\right)^{1/r} \le r^{-n} c_1(E)^n,$$

with equality iff $c_1(E_1) = \cdots = c_1(E_r)$.

- In the split case, it seems natural to conjecture that

$$\mathrm{MAVol}(E) = \left(\prod_{1\le j\le r} c_1(E_j)^n\right)^{1/r},$$

i.e. that the supremum is reached for split metrics $h = \bigoplus h_j$.

- The Euler-Lagrange equation for the maximizer is complicated (4th order!). It somehow extends the equation characterizing cscK metrics.

## On the Fulton Lazarsfeld inequalities (S. Finski)

A fundamental result due to Fulton-Lazarsfeld asserts that if $E \to X$ is an ample vector bundle, then all Schur polynomials $P(c_\bullet(E))$ in the Chern classes are numerically positive, i.e.

$$\int_Y P(c_\bullet(E)) > 0$$

for all irreducible cycles $Y$ of the appropriate dimension in $X$.

Recently, Siarhei Finski has shown

**Theorem (Finski 2020)**

If $(E, h)$ is a (dual) Nakano positive vector bundle, then all Schur polynomials $P(c_\bullet(E, h))$ in the Chern forms are pointwise positive $(k, k)$-forms (in the sense of the weak positivity of forms).

This is a compelling motivation to investigate the relationships between ampleness, Griffiths and Nakano positivity!



## Further recent results by Siarhei Finski

When $E \to X$ is an ample vector bundle, the symmetric powers $S^m E$ have enough sections to generate 1-jets for $m \geq m_0 \gg 1$, and one can immediately derive from there that

$$E \text{ ample} \Rightarrow S^m E \text{ dual-Nakano positive for } m \geq m_0 \gg 1.$$

Then it makes sense to wonder whether there is an asymptotic formula for the monge-Ampère volume $\mathrm{MAVol}(S^m E)$. S. Finski obtained more generally an asymptotic formula for the Monge-Ampère volume of direct images $E_m = \pi_*(L^m \otimes G)$ by any proper morphism $\pi : Y \to X$ of any line bundle $(L, h_L) > 0$ on $Y$.

**Theorem (S. Finski 2020)**

Given any volume form $d\nu$ on $X$, the direct images satisfy

$$\mathrm{MAVol}(E_m, h_{E_m}) \sim m^{\dim X} \int_X \exp\left(\frac{\int_Y \log\left(\omega_H^{\dim X}/\pi^*\nu\right)\omega^{\dim Y}}{\int_Y c_1(L)^{\dim Y}}\right) d\nu,$$

where $\omega = \Theta_{L,h_L} > 0$ on $Y$, and $\omega_H$ is its horizontal part.

## Matrix Monge-Ampère equations

### Basic idea

Assigning a "matrix Monge-Ampère equation"

$$\det_{T_X\otimes E}(\Theta_{E,h})^{1/r} = f > 0 \quad \text{or} \quad \mathrm{Grif}(\Theta_{E,h}) = f > 0$$

where $f$ is a positive $(n,n)$-form, may enforce the Nakano (resp. Griffiths) positivity of $\Theta_{E,h}$, especially if that assignment is combined with a continuity technique from an initial starting point where positivity is known.

Also, in order to compute thresholds, one could instead replace $E$ by $E\otimes(\det E)^t$ for a large value $t_0$ and try to decrease $t$ as much as possible.

In case $r = \mathrm{rank}\, E = 1$ and $h = h_0 e^{-\varphi}$, this is the same as solving a complex Monge-Ampère equation

$$(\Theta_{E,h})^n = (\omega_0 + i\partial\overline{\partial}\varphi)^n = f.$$



## Underdeterminacy of the equation

Assuming $E$ to be ample of rank $r > 1$, the equation

$$(**) \qquad \det_{T_X\otimes E}(\Theta_{E,h})^{1/r} = f > 0$$

becomes underdetermined, as the real rank of the space of hermitian matrices $h = (h_{\lambda\mu})$ on $E$ is equal to $r^2$, while $(**)$ provides only 1 scalar equation.

(Solutions might still exist, but lack uniqueness and a priori bounds.)

### Conclusion

In order to recover a well determined system of equations, one needs an additional "matrix equation" of rank $(r^2-1)$.

### Observation 1 (from the Donaldson-Uhlenbeck-Yau theorem)

Take a Hermitian metric $\eta_0$ on $\det E$ so that $\omega_0 := \Theta_{\det E,\eta_0} > 0$. If $E$ is $\omega_0$-polystable, $\exists h$ Hermitian metric $h$ on $E$ such that

$$\omega_0^{n-1}\wedge\Theta_{E,h} = \tfrac{1}{r}\,\omega_0^n\otimes \mathrm{Id}_E \quad \text{(Hermite-Einstein equation, slope } \tfrac{1}{r}\text{)}.$$

## Resulting trace free condition

### Observation 2

The trace part of the above Hermite-Einstein equation is "automatic", hence the equation is equivalent to the trace free condition

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,h} = 0,$$

when decomposing any endomorphism $u \in \mathrm{Herm}(E,E)$ as

$$u = u^\circ + \tfrac{1}{r}\,\mathrm{Tr}(u)\,\mathrm{Id}_E \in \mathrm{Herm}^\circ(E,E) \oplus \mathbb{R}\,\mathrm{Id}_E, \quad \mathrm{tr}(u^\circ) = 0.$$

### Observation 3

The trace free condition is a matrix equation of rank $(r^2 - 1)$ !!!

### Remark

In case $\dim X = n = 1$, the trace free condition means that $E$ is projectively flat, and the Umemura proof of the Griffiths conjecture proceeds exactly in that way, using the fact that the graded pieces of the Harder-Narasimhan filtration are projectively flat.



## Towards a "cushioned" Hermite-Einstein equation

In general, one cannot expect $E$ to be $\omega_0$-polystable, but Uhlenbeck-Yau have shown that there always exists a smooth solution $q_\varepsilon$ to a certain "cushioned" Hermite-Einstein equation.

To make things more precise, let $\mathrm{Herm}(E)$ be the space of Hermitian (non necessarily positive) forms on $E$. Given a reference Hermitian metric $H_0 > 0$, let $\mathrm{Herm}_{H_0}(E,E)$ be the space of $H_0$-Hermitian endomorphisms $u \in \mathrm{Hom}(E,E)$; denote by

$$\mathrm{Herm}(E) \overset{\simeq}{\to} \mathrm{Herm}_{H_0}(E,E), \quad q \mapsto \widetilde{q} \ \text{ s.t. } \ q(v,w) = \langle \widetilde{q}(v), w\rangle_{H_0}$$

the natural isomorphism. Let also

$$\mathrm{Herm}^\circ_{H_0}(E,E) = \big\{q \in \mathrm{Herm}_{H_0}(E,E)\,;\ \mathrm{tr}(q) = 0\big\}$$

be the subspace of "trace free" Hermitian endomorphisms.

In the sequel, we fix $H_0$ on $E$ such that

$$\Theta_{\det E, \det H_0} = \omega_0 > 0.$$

# A basic result from Uhlenbeck and Yau

**Uhlenbeck-Yau 1986, Theorem 3.1**

For every $\varepsilon > 0$, there always exists a (unique) smooth Hermitian metric $q_\varepsilon$ on $E$ such that

$$\omega_0^{n-1} \wedge \Theta_{E,q_\varepsilon} = \omega_0^n \otimes \left(\frac{1}{r}\,\mathrm{Id}_E - \varepsilon \log \widetilde{q}_\varepsilon\right),$$

where $\widetilde{q}_\varepsilon$ is computed with respect to $H_0$, and $\log g$ denotes the logarithm of a positive Hermitian endomorphism $g$.

The reason is that the term $-\varepsilon \log \widetilde{q}_\varepsilon$ is a "friction term" that prevents the explosion of the a priori estimates, similarly what happens for Monge-Ampère equations $(\omega_0 + i\partial\overline{\partial}\varphi)^n = e^{\varepsilon\varphi + f}\omega_0^n$.

The above matrix equation is equivalent to prescribing $\det q_\varepsilon = \det H_0$ and the trace free equation of rank $(r^2 - 1)$

$$\omega_0^{n-1} \wedge \Theta^\circ_{E,q_\varepsilon} = -\varepsilon\,\omega_0^n \otimes \log \widetilde{q}_\varepsilon.$$



# Search for an appropriate evolution equation

**General setup**

In this context, given $\alpha > 0$ large enough, it is natural to search for a time dependent family of metrics $h_t(z)$ on the fibers $E_z$ of $E$, $t \in [0,1]$, satisfying a generalized Monge-Ampère equation

$$(D) \qquad \det{}_{T_X \otimes E}\big(\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_E\big)^{1/r} = f_t\,\omega_0^n, \quad f_t > 0,$$

and trace free, rank $r^2 - 1$, Hermite-Einstein conditions

$$(T) \quad \omega_t^{n-1} \wedge \Theta^\circ_{E,h_t} = g_t$$

with smoothly varying families of functions $f_t \in C^\infty(X, \mathbb{R})$, Hermitian metrics $\omega_t > 0$ on $X$ and sections

$$g_t \in C^\infty(X, \Lambda^{n,n}_{\mathbb{R}} T^*_X \otimes \mathrm{Herm}^\circ_{h_t}(E,E)), \quad t \in [0,1].$$

Observe that this is a determined (not overdetermined!) system.

## Choice of the initial state ($t = 0$)

We start with the Uhlenbeck-Yau solution $h_0 = q_\varepsilon$ of of the "cushioned" trace free Hermite-Einstein equation, so that $\det h_0 = \det H_0$, and take $\alpha > 0$ so large that

$$\Theta_{E,h_0} + \alpha\,\omega_0 \otimes \mathrm{Id}_E > 0 \text{ in the sense of Nakano.}$$

If conditions $(D)$ and $(T)$ can be met for all $t \in [0, t_0]$, thus without any discontinuity or explosion of the solutions $h_t$, we infer from $(D)$ that

$$\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_E > 0 \quad \text{in the sense of Nakano}$$

for all $t \in [0, t_0]$.

**Question**

Is the maximal existence time $t_0$ of the solution such that $(1 - t_0)\alpha = \tau_N(E)$ (Nakano threshold of $E$) ?



## Possible choices of the right hand side

One still has the freedom of adjusting $f_t$, $\omega_t$ and $g_t$ in the general setup. There are in fact many possibilities:

**Proposition**

Let $(E, H_0)$ be a smooth Hermitian holomorphic vector bundle such that $E$ is ample and $\omega_0 = \Theta_{\det E, \det H_0} > 0$. Then the system of determinantal and trace free equations

$$(D)\quad \det{}_{T_X \otimes E}\big(\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_E\big)^{1/r} = F(t, z, h_t, D_z h_t)$$

$$(T)\quad \omega_t^{n-1} \wedge \Theta^\circ_{E,h_t} = G(t, z, h_t, D_z h_t, D_z^2 h_t) \in \mathrm{Herm}^\circ(E, E)$$

(where $F > 0$), is a well determined system of PDEs.

It is elliptic whenever the symbol $\eta_h$ of the linearized operator $u \mapsto DG_{D^2h}(t, z, h, Dh, D^2h) \cdot D^2 u$ has an Hilbert-Schmidt norm

$$\sup_{\xi \in T_X^*,\, |\xi|_{\omega_t} = 1} \|\eta_{h_t}(\xi)\|_{h_t} \le (r^2 + 1)^{-1/2}\, n^{-1}$$

for any metric $h_t$ involved, e.g. if $G$ does not depend on $D^2 h$.

## Proof of the ellipticity

The (long, computational) proof consists of analyzing the linearized system of equations, starting from the curvature tensor formula

$$\Theta_{E,h} = i\overline{\partial}(h^{-1}\partial h) = i\overline{\partial}(\widetilde{h}^{-1}\partial_{H_0}\widetilde{h}),$$

where $\partial_{H_0} s = H_0^{-1}\partial(H_0 s)$ is the $(1,0)$-component of the Chern connection on $\mathrm{Hom}(E,E)$ associated with $H_0$ on $E$.

Let us recall that the ellipticity of an operator

$$P : C^\infty(V) \to C^\infty(W), \quad f \mapsto P(f) = \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha f(x)$$

means the invertibility of the principal symbol

$$\sigma_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\,\xi^\alpha \in \mathrm{Hom}(V,W)$$

whenever $0 \neq \xi \in T^*_{X,x}$.

For instance, on the torus $\mathbb{R}^n/\mathbb{Z}^n$, $f \mapsto P_\lambda(f) = -\Delta f + \lambda f$ has an invertible symbol $\sigma_{P_\lambda}(x,\xi) = -|\xi|^2$, but $P_\lambda$ is invertible only for $\lambda > 0$.



## A more specific choice of the right hand side

**Theorem**

The elliptic differential system defined by

$$\det{}_{T_X\otimes E}\left(\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_E\right)^{1/r} = \left(\frac{\det H_0(z)}{\det h_t(z)}\right)^\lambda a_0(z),$$

$$\omega_t^{n-1} \wedge \Theta_{E^\circ,h_t} = -\varepsilon\left(\frac{\det H_0(z)}{\det h_t(z)}\right)^\mu (\log \widetilde{h}_t^\circ)\,\omega_0^n \quad \text{w.r.t. Kähler metric}$$

$$\omega_t = \frac{1}{r\alpha+1}\mathrm{tr}\left(\Theta_{E,h_t} + (1-t)\alpha\,\omega_0 \otimes \mathrm{Id}_E\right) > 0,$$

possesses an invertible elliptic linearization for $\varepsilon \ge \varepsilon_0(h_t)$ and $\lambda \ge \lambda_0(h_t)(1+\mu^2)$, with $\varepsilon_0(h_t)$ and $\lambda_0(h_t)$ large enough.

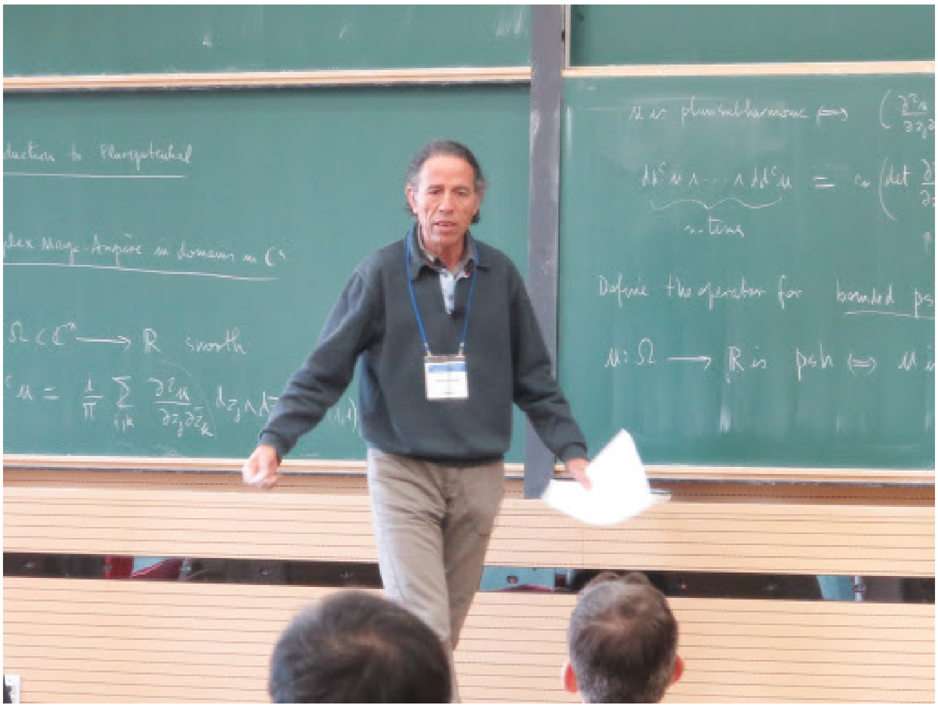
**Corollary**

Under the above conditions, starting from the Uhlenbeck-Yau solution $h_0$ such that $\det h_0 = \det H_0$ at $t=0$, the PDE system still has a solution for $t \in [0, t_0]$ and $t_0 > 0$ small.

**Proof.** Compute total symbol of linearized system + linear algebra.

# The end

**Joyeuse et active retraite, Ahmed !**





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