On the cohomology of pseudoeffective line bundles

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in honor of Professor Yum-Tong Siu
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Goals

- Study sections and cohomology of holomorphic line bundles $L \to X$ on compact Kähler manifolds,
without assuming any strict positivity of the curvature
- Generalize the Nadel vanishing theorem
(and therefore Kawamata-Viehweg)
- Several known results already in this direction:
– Skoda division theorem (1972)
– Ohsawa-Takegoshi L^2 extension theorem (1987)
– more recent work of Yum-Tong Siu:
invariance of plurigenera (1998 → 2000),
analytic version of Shokurov's non vanishing theorem,
finiteness of the canonical ring (2007),
study of the abundance conjecture (2010) ...
– solution of MMP (BCHM 2006), D-Hacon-Păun (2010)

Basic concepts (1)

Let X = compact Kähler manifold, $L \to X$ holomorphic line bundle, h a hermitian metric on L.
Locally $L_{|U} \simeq U \times \mathbb{C}$ and for $\xi \in L_x \simeq \mathbb{C}$, $\|\xi\|_h^2 = |\xi|^2 e^{-\varphi(x)}$.
Writing $h = e^{-\varphi}$ locally, one defines the curvature form of L to be the real $(1,1)$-form

$$\Theta_{L,h} = \frac{i}{2\pi}\partial\overline{\partial}\varphi = -dd^c \log h,$$

$$c_1(L) = \{\Theta_{L,h}\} \in H^2(X,\mathbb{Z}).$$

Any subspace $V_m \subset H^0(X, L^{\otimes m})$ define a meromorphic map

$$\begin{aligned} \Phi_{mL} : X \smallsetminus Z_m &\longrightarrow \mathbb{P}(V_m) \quad \text{(hyperplanes of } V_m) \\ x &\longmapsto H_x = \{\sigma \in V_m\,;\ \sigma(x) = 0\} \end{aligned}$$

where Z_m = base locus $B(mL) = \bigcap \sigma^{-1}(0)$.

Basic concepts (2)

Given sections $\sigma_1, \ldots, \sigma_n \in H^0(X, L^{\otimes m})$, one gets a singular hermitian metric on L defined by

$$|\xi|_h^2 = \frac{|\xi|^2}{\big(\sum |\sigma_j(x)|^2\big)^{1/m}},$$

its weight is the plurisubharmonic (psh) function

$$\varphi(x) = \frac{1}{m}\log\Big(\sum |\sigma_j(x)|^2\Big)$$

and the curvature is $\Theta_{L,h} = \frac{1}{m} dd^c \log\varphi \geq 0$
in the sense of currents, with logarithmic poles along the base locus

$$B = \bigcap \sigma_j^{-1}(0) = \varphi^{-1}(-\infty).$$

One has

$$(\Theta_{L,h})_{|X \smallsetminus B} = \frac{1}{m}\Phi_{mL}^*\omega_{\mathrm{FS}} \quad \text{where} \quad \Phi_{mL} : X \smallsetminus B \to \mathbb{P}(V_m) \simeq \mathbb{P}^{N_m}.$$

Basic concepts (3)

Definition

- L is pseudoeffective (psef) if $\exists h = e^{-\varphi}$, $\varphi \in L^1_{\rm loc}$, (possibly singular) such that $\Theta_{L,h} = -dd^c \log h \geq 0$ on X, in the sense of currents.
- L is semipositive if $\exists h = e^{-\varphi}$ smooth such that $\Theta_{L,h} = -dd^c \log h \geq 0$ on X.
- L is positive if $\exists h = e^{-\varphi}$ smooth such that $\Theta_{L,h} = -dd^c \log h > 0$ on X.

The well-known Kodaira embedding theorem states that L is positive if and only if L is ample, namely:
$Z_m = B(mL) = \emptyset$ and

$$\Phi_{|mL|} : X \to \mathbb{P}(H^0(X, L^{\otimes m}))$$

is an embedding for $m \geq m_0$ large enough.

Positive cones

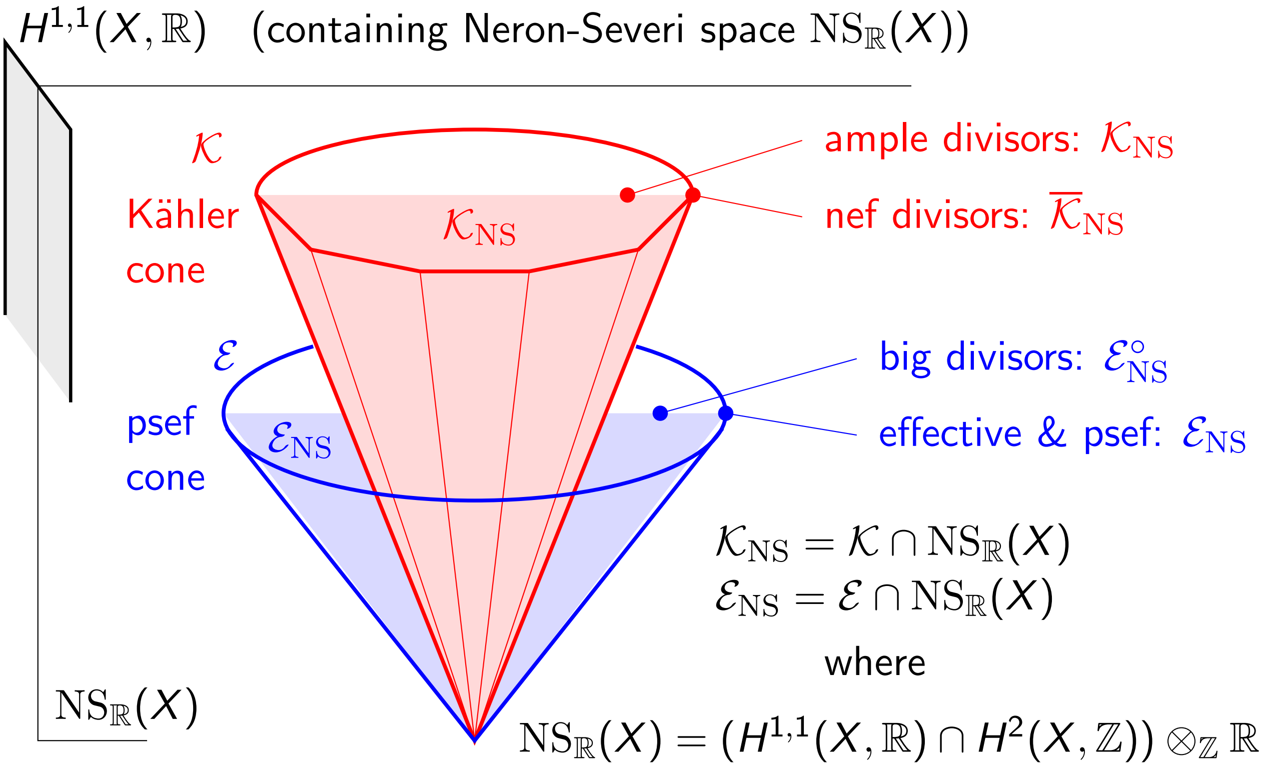
Definitions

Let X be a compact Kähler manifold.

- The Kähler cone is the (open) set $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{\omega\}$ of positive Kähler forms.
- The pseudoeffective cone is the set $\mathcal{E} \subset H^{1,1}(X, \mathbb{R})$ of cohomology classes $\{T\}$ of closed positive $(1,1)$ currents. This is a closed convex cone. (by weak compactness of bounded sets of currents).
- $\overline{\mathcal{K}}$ is the cone of "nef classes". One has $\overline{\mathcal{K}} \subset \mathcal{E}$.
- It may happen that $\overline{\mathcal{K}} \subsetneq \mathcal{E}$: if X is the surface obtained by blowing-up $\mathbb{P}^2$ in one point, then the exceptional divisor $E \simeq \mathbb{P}^1$ has a cohomology class $\{\alpha\}$ such that $\int_E \alpha = E^2 = -1$, hence $\{\alpha\} \notin \overline{\mathcal{K}}$, although $\{\alpha\} = \{[E]\} \in \mathcal{E}$.

Ample / nef / effective / big divisors

Positive cones can be visualized as follows :



Approximation of currents, Zariski decomposition

Definition

On X compact Kähler, a Kähler current T is a closed positive $(1,1)$-current T such that $T \geq \delta\omega$ for some smooth hermitian metric ω and a constant $\delta \ll 1$.

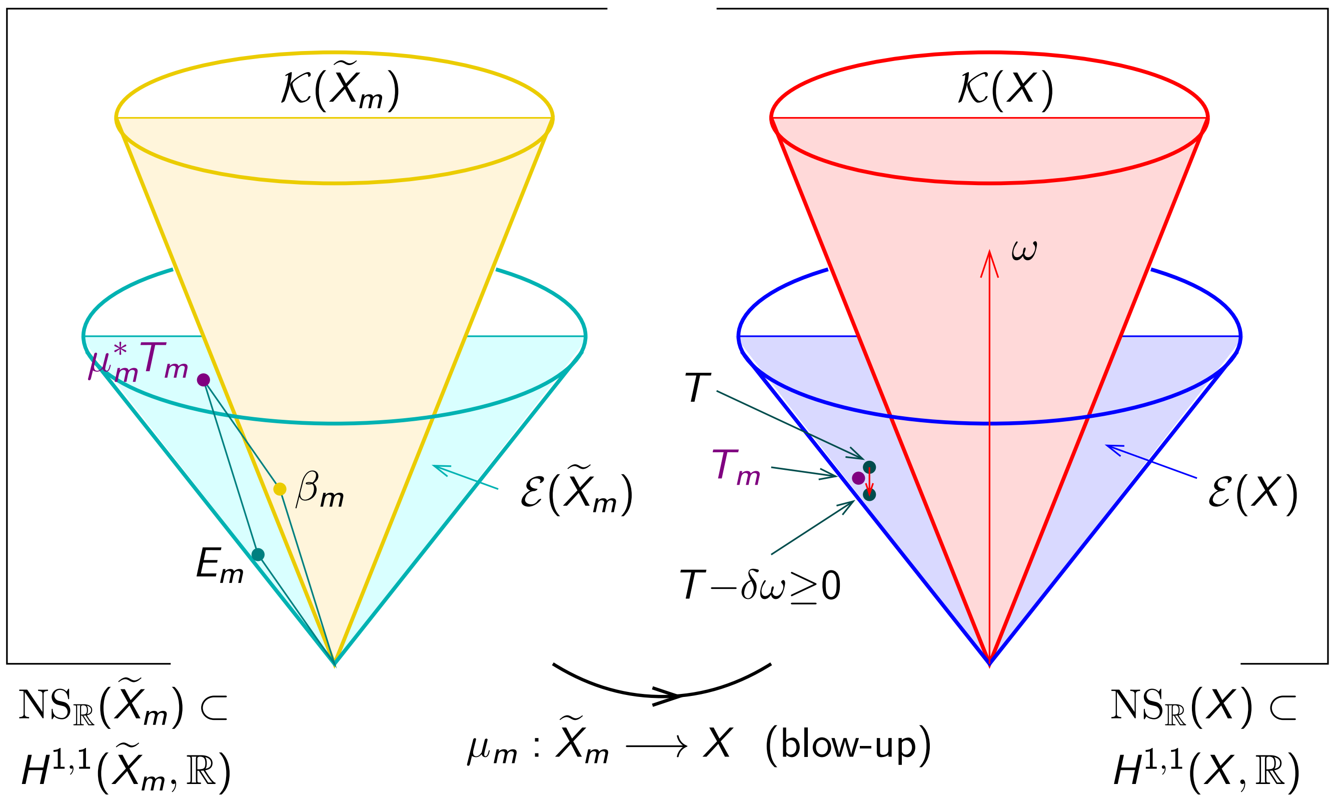
Easy observation

$\alpha \in \mathcal{E}^{\circ}$ (interior of $\mathcal{E}$) $\iff$ $\alpha = \{T\}$, $T =$ a Kähler current.
We say that $\mathcal{E}^{\circ}$ is the cone of big $(1,1)$-classes.

Theorem on approximate Zariski decomposition (D, '92)

Any Kähler current can be written $T = \lim T_m$ where $T_m \in \{T\}$ has analytic singularities & logarithmic poles,
i.e. $\exists$ modification $\mu_m : \widetilde{X}_m \to X$ such that $\mu_m^{\star} T_m = [E_m] + \beta_m$
where E_m is an effective $\mathbb{Q}$-divisor on $\widetilde{X}_m$ with coefficients in $\frac{1}{m}\mathbb{Z}$
and β_m is a Kähler form on $\widetilde{X}_m$.

Schematic picture of Zariski decomposition



Idea of proof of analytic Zariski decomposition

- Write locally

$$T = i\partial\overline{\partial}\varphi$$

for some strictly plurisubharmonic psh potential φ on X.

- Approximate T (again locally) as

$$T_m = i\partial\overline{\partial}\varphi_m, \qquad \varphi_m(z) = \frac{1}{2m}\log\sum_\ell |g_{\ell,m}(z)|^2$$

where $(g_{\ell,m})$ is a Hilbert basis of the space

$$\mathcal{H}(\Omega, m\varphi) = \{f \in \mathcal{O}(\Omega)\,;\ \int_\Omega |f|^2 e^{-2m\varphi} dV < +\infty\}.$$

- The Ohsawa-Takegoshi L^2 extension theorem (extending from a single isolated point) implies that there are enough such holomorphic functions, and thus $\varphi_m \geq \varphi - C/m$.
- Further, $\varphi = \lim_{m\to+\infty} \varphi_m$ by the mean value inequality.

"Movable" intersection of currents

Let $\mathcal{P}(X) =$ closed positive $(1,1)$-currents on X

$$H^{k,k}_{\geq 0}(X) = \big\{\{T\} \in H^{k,k}(X,\mathbb{R})\,;\ T \text{ closed} \geq 0\big\}.$$

Theorem (Boucksom PhD 2002, Junyan Cao PhD 2012)

$\forall k = 1, 2, \ldots, n$, $\exists$ canonical "movable intersection product"

$$\mathcal{P} \times \cdots \times \mathcal{P} \to H^{k,k}_{\geq 0}(X), \quad (T_1, \ldots, T_k) \mapsto \langle T_1 \cdot T_2 \cdots T_k\rangle$$

Method. $T_j = \lim_{\varepsilon\to 0} T_j + \varepsilon\omega$, can assume T_j Kähler.
Approximate each T_j by Kähler currents $T_{j,m}$ with logarithmic poles,take a simultaneous log-resolution $\mu_m : \widetilde{X}_m \to X$ such that

$$\mu_m^\star T_j = [E_{j,m}] + \beta_{j,m}.$$

and define

$$\langle T_1 \cdot T_2 \cdots T_k\rangle = \lim_{m\to+\infty} \uparrow \{(\mu_m)_\star(\beta_{1,m} \wedge \beta_{2,m} \wedge \ldots \wedge \beta_{k,m})\}.$$

Volume and numerical dimension of currents

Remark. The limit exists a weak limit of currents thanks to uniform boundedness in mass.
Uniqueness comes from monotonicity ($\beta_{j,m}$ "increases" with m)

Special case. The volume of a class $\alpha \in H^{1,1}(X,\mathbb{R})$ is

$$\mathrm{Vol}(\alpha) = \sup_{T\in\alpha} \langle T^n\rangle \quad \text{if } \alpha \in \mathcal{E}^\circ \text{ (big class)},$$

$$\mathrm{Vol}(\alpha) = 0 \qquad \text{if } \alpha \notin \mathcal{E}^\circ,$$

Numerical dimension of a current

$$\mathrm{nd}(T) = \max\big\{p \in \mathbb{N}\,;\ \langle T^p\rangle \neq 0 \quad \text{in } H^{p,p}_{\geq 0}(X)\big\}.$$

Numerical dimension of a hermitian line bundle (L, h)

$$\mathrm{nd}(L,h) = \mathrm{nd}(\Theta_{L,h}).$$

Generalized abundance conjecture

Numerical dimension of a class $\alpha \in H^{1,1}(X,\mathbb{R})$

If α is not pseudoeffective, set $\mathrm{nd}(\alpha) = -\infty$, otherwise

$$\mathrm{nd}(\alpha) = \max\big\{p\in\mathbb{N}\,;\ \exists T_\varepsilon\in\{\alpha+\varepsilon\omega\},\ \lim_{\varepsilon\to 0}\langle T_\varepsilon^p\rangle\wedge\omega^{n-p}\geq C>0\big\}.$$

Numerical dimension of a pseudo-effective line bundle

$$\mathrm{nd}(L) = \mathrm{nd}(c_1(L)).$$

L is said to be abundant if $\kappa(L) = \mathrm{nd}(L)$.

Subtlety ! Let E be the rank 2 v.b. = non trivial extension $0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0$ on C = elliptic curve, let $X = \mathbb{P}(E)$ (ruled surface over C) and $L = \mathcal{O}_{\mathbb{P}(E)}(1)$. Then $\mathrm{nd}(L) = 1$ but $\exists\,!$ positive current $T = [\sigma(C)] \in c_1(L)$ and $\mathrm{nd}(T) = 0$!!

Generalized abundance conjecture

For X compact Kähler, K_X is abundant, i.e. $\kappa(X) = \mathrm{nd}(K_X)$.

Hard Lefschetz theorem with pseudoeffective coefficients

Let (L,h) be a pseudo-effective line bundle on a compact Kähler manifold (X,ω) of dimension n, and for $h = e^{-\varphi}$, let $\mathcal{I}(h) = \mathcal{I}(\varphi)$ be the multiplier ideal sheaf:

$$\mathcal{I}(\varphi)_x := \big\{f\in\mathcal{O}_{X,x}\,;\ \exists V\ni x,\ \int_V |f|^2 e^{-\varphi}dV_\omega < +\infty\big\}.$$

The Nadel vanishing theorem claims that

$$\Theta_{L,h}\geq\varepsilon\omega \implies H^q(X,K_X\otimes L\otimes\mathcal{I}(h)) = 0 \ \text{ for } q\geq 1.$$

Hard Lefschetz theorem (D-Peternell-Schneider 2001)

Assume merely $\Theta_{L,h}\geq 0$. Then, the Lefschetz map : $u\mapsto\omega^q\wedge u$ induces a surjective morphism :

$$\Phi^q_{\omega,h} : H^0(X,\Omega_X^{n-q}\otimes L\otimes\mathcal{I}(h)) \longrightarrow H^q(X,\Omega_X^n\otimes L\otimes\mathcal{I}(h)).$$

Idea of proof of Hard Lefschetz theorem

Main tool. "Equisingular approximation theorem":

$$\varphi = \lim \downarrow \varphi_\nu \quad \Rightarrow \quad h = \lim h_\nu$$

with:

- $\varphi_\nu \in C^\infty(X \smallsetminus Z_\nu)$, where Z_ν is an increasing sequence of analytic sets,
- $\mathcal{I}(h_\nu) = \mathcal{I}(h), \quad \forall \nu$,
- $\Theta_{L,h_\nu} \geq -\varepsilon_\nu \omega$.

(Again, the proof uses in several ways the Ohsawa-Takegoshi theorem).

Then, use the fact that $X \smallsetminus Z_\nu$ is Kähler complete, so one can apply (non compact) harmonic form theory on $X \smallsetminus Z_\nu$, and pass to the limit to get rid of the errors ε_ν.

Generalized Nadel vanishing theorem

Theorem (Junyan Cao, PhD 2012)

Let X be compact Kähler, and let (L, h) be pseudoeffective on X. Then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0 \quad \text{for } q \geq n - \mathrm{nd}(L, h) + 1,$$

where

$$\mathcal{I}_+(h) = \lim_{\varepsilon \to 0} \mathcal{I}(h^{1+\varepsilon}) = \lim_{\varepsilon \to 0} \mathcal{I}((1+\varepsilon)\varphi)$$

is the "upper semicontinuous regularization" of $\mathcal{I}(h)$.

Remark 1. Conjecturally $\mathcal{I}_+(h) = \mathcal{I}(h)$. This might follow from recent work by Bo Berndtsson on the openness conjecture.

Remark 2. In the projective case, one can use a hyperplane section argument, provided one first shows that $\mathrm{nd}(L, h)$ coincides with H. Tsuji's algebraic definition $(\dim Y = p)$:

$$\mathrm{nd}(L, h) = \max \left\{ p \in \mathbb{N} \,;\, \exists Y^p \subset X,\ h^0(Y, (L^{\otimes m} \otimes \mathcal{I}(h^m))_{|Y}) \geq c m^p \right\}.$$

Proof of generalized Nadel vanishing (projective case)

Hyperplane section argument (projective case). Take $A =$ very ample divisor, $\omega = \Theta_{A,h_A} > 0$, and $Y = A_1 \cap \ldots \cap A_{n-p}$, $A_j \in |A|$. Then

$$\langle \Theta^p_{L,h} \rangle \cdot Y = \int_X \langle \Theta^p_{L,h} \rangle \cdot Y = \int_X \langle \Theta^p_{L,h} \rangle \wedge \omega^{n-p} > 0.$$

From this one concludes that $(\Theta_{L,h})_{|Y}$ is big.

Lemma (J. Cao)

When (L,h) is big, i.e. $\langle \Theta^n_{L,h} \rangle > 0$, there exists a metric $\widetilde{h}$ such that $\mathcal{I}(\widetilde{h}) = \mathcal{I}_+(h)$ with $\Theta_{L,\widetilde{h}} \geq \varepsilon\omega$ [Riemann-Roch].

Then Nadel $\Rightarrow H^q(X, K_X \otimes L \otimes \mathcal{I}_+(h)) = 0$ for $q \geq 1$.

Conclude by induction on $\dim X$ and the exact cohomology sequence for the restriction to a hyperplane section.

Proof of generalized Nadel vanishing (Kähler case)

Kähler case. Assume $c_1(L)$ nef for simplicity. Then $c_1(L) + \varepsilon\omega$ Kähler. By Yau's theorem, solve Monge-Ampère equation:

$$\exists h_\varepsilon \text{ on } L, \quad (\Theta_{L,h_\varepsilon} + \varepsilon\omega)^n = C_\varepsilon \omega^n.$$

Here $C_\varepsilon \geq \binom{n}{p} \langle \Theta^p_{L,h} \rangle \cdot (\varepsilon\omega)^{n-p} \sim C\varepsilon^{n-p}$, $p = \mathrm{nd}(L,h)$.

Ch. Mourougane argument (PhD 1996). Let $\lambda_1 \leq \ldots \leq \lambda_n$ be the eigenvalues of $\Theta_{L,h} + \varepsilon\omega$ w.r.to ω. Then

$$\lambda_1 \ldots \lambda_n = C_\varepsilon \geq \text{Const } \varepsilon^{n-p}$$

and

$$\int_X \lambda_{q+1} \ldots \lambda_n \, \omega^n = \int_X \Theta^{n-q}_{L,h} \wedge \omega^q \leq \text{Const}, \quad \forall q \geq 1,$$

so $\lambda_{q+1} \ldots \lambda_n \leq C$ on a large open set $U \subset X$ and

$$\lambda_q^q \geq \lambda_1 \ldots \lambda_q \geq c\varepsilon^{n-p} \quad \Rightarrow \quad \lambda_q \geq c\varepsilon^{(n-p)/q} \text{ on } U,$$

$$\sum_{j=1}^q (\lambda_j - \varepsilon) \geq \lambda_q - q\varepsilon \geq c\varepsilon^{(n-p)/q} - q\varepsilon > 0 \text{ for } q > n-p.$$

Final step: use Bochner-Kodaira formula

λ_j = eigenvalues of $(\Theta_{L,h_\varepsilon}+\varepsilon\omega) \Rightarrow$ (eigenvalues of Θ_{L,h_ε}) $= \lambda_j - \varepsilon$.
Bochner-Kodaira formula yields

$$\|\partial u\|_\varepsilon^2 + \|\partial^* u\|_\varepsilon^2 \geq \int_X \Big(\sum_{j=1}^q (\lambda_j - \varepsilon)\Big)|u|^2 e^{-\varphi_\varepsilon} dV_\omega.$$

Then one has to show that one can take the limit by assuming integrability with $e^{-(1+\delta)\varphi}$, thus introducing $\mathcal{I}_+(h)$.

Application to Kähler geometry

Definition (Campana)

A compact Kähler manifold is said to be simple if there are no positive dimensional analytic sets $A_x \subset X$ through a very generic point $x \in X$.

Well-known fact

A complex torus $X = \mathbb{C}^n/\Lambda$ defined by a sufficiently generic lattice $\Lambda \subset \mathbb{C}^n$ is simple, and in fact has no positive dimensional analytic subset $A \subsetneq X$ at all.

In fact $[A]$ would define a non zero (p,p)-cohomology class with integral periods, and there are no such classes in general.

It is expected that simple compact Kähler manifolds are either generic complex tori, generic hyperkähler manifolds and their finite quotients, up to modification.

On simple Kähler 3-folds

Theorem (Campana - D - Verbitsky, 2013)

Let X be a compact Kähler 3-fold without any positive dimensional analytic subset $A \subsetneq X$. Then
X is a complex 3-dimensional torus.

Sketch of proof

- Every pseudoeffective class is nef, i.e. $\overline{\mathcal{K}} = \mathcal{E}$ (D, '90)
- K_X is pseudoeffective: otherwise X would be covered by rational curves (Brunella 2008), hence in fact nef.
- All multiplier ideal sheaves $\mathcal{I}(h)$ are trivial
- $H^0(X, \Omega_X^{n-q} \otimes K_X^{\otimes m-1}) \to H^q(X, K_X^{\otimes m})$ is surjective
- Hilbert polynomial $P(m) = \chi(X, K_X^{\otimes m})$ is bounded, hence $\chi(X, \mathcal{O}_X) = 0$.
- Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a biholomorphism.

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