Sheaf cohomology and the De Rham-Weil isomorphism theorem

We present here the most basic facts concerning sheaf cohomology, and restrict ourselves to paracompact spaces, i.e. Hausdorff topological spaces admitting locally finite partitions of unity for any open covering. The theory of sheaves is due to Jean Leray (between 1940 and 1945; he was then a war prisoner in Austria). Analytic sheaves were then developped by Oka, Cartan and Serre. For the applications to algebraic geometry involving Zariski topology (which is non Hausdorff), a more general approach of cohomology due to Grothendieck is needed, but the paracompact theory usually suffices for analytic geometry. Here, all sheaves are implicitly supposed to be at least sheaves of abelian groups, and correspondingly, sheaf morphisms are morphisms of sheaves of abelian groups.

§ 1. Čech Cohomology

§ 1.A. Definitions

Let X be a topological space, $\mathcal{A}$ a sheaf of abelian groups on X, and $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ an open covering of X. For the sake of simplicity, we set

$$U_{\alpha_0\alpha_1\ldots\alpha_q} = U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_q}.$$

The group $C^q(\mathcal{U}, \mathcal{A})$ of *Čech q-cochains* is the set of families

$$c = (c_{\alpha_0\alpha_1\ldots\alpha_q}) \in \prod_{(\alpha_0,\ldots,\alpha_q)\in I^{q+1}} \mathcal{A}(U_{\alpha_0\alpha_1\ldots\alpha_q}).$$

The group structure on $C^q(\mathcal{U}, \mathcal{A})$ is the obvious one deduced from the addition law on sections of $\mathcal{A}$. The *Čech differential* $\delta^q : C^q(\mathcal{U}, \mathcal{A}) \longrightarrow C^{q+1}(\mathcal{U}, \mathcal{A})$ is defined by the formula

$$(1.1)\qquad (\delta^q c)_{\alpha_0\ldots\alpha_{q+1}} = \sum_{0\leqslant j\leqslant q+1} (-1)^j\, c_{\alpha_0\ldots\widehat{\alpha_j}\ldots\alpha_{q+1}}{\restriction}_{U_{\alpha_0\ldots\alpha_{q+1}}},$$

and we set $C^q(\mathcal{U}, \mathcal{A}) = 0$, $\delta^q = 0$ for $q < 0$. In degrees 0 and 1, we get for example

$$(1.2)\qquad q = 0,\quad c = (c_\alpha),\quad (\delta^0 c)_{\alpha\beta} = c_\beta - c_\alpha{\restriction}_{U_{\alpha\beta}},$$

$$(1.2')\qquad q = 1,\quad c = (c_{\alpha\beta}),\quad (\delta^1 c)_{\alpha\beta\gamma} = c_{\beta\gamma} - c_{\alpha\gamma} + c_{\alpha\beta}{\restriction}_{U_{\alpha\beta\gamma}}.$$

Easy verifications left to the reader show that $\delta^{q+1} \circ \delta^q = 0$. We get therefore a cochain complex $\big(C^\bullet(\mathcal{U}, \mathcal{A}), \delta\big)$, called the *complex of Čech cochains* relative to the covering $\mathcal{U}$.

(1.3) Definition. *The Čech cohomology group of $\mathcal{A}$ relative to $\mathcal{U}$ is*

$$H^q(\mathcal{U}, \mathcal{A}) = H^q\big(C^\bullet(\mathcal{U}, \mathcal{A})\big).$$

Formula (1.2) shows that the set of Čech 0-cocycles is the set of families $(c_\alpha) \in \prod \mathcal{A}(U_\alpha)$ such that $c_\beta = c_\alpha$ on $U_\alpha \cap U_\beta$. Such a family defines in a unique way a global section $f \in \mathcal{A}(X)$ with $f_{\restriction U_\alpha} = c_\alpha$. Hence

$$H^0(\mathcal{U}, \mathcal{A}) = \mathcal{A}(X). \tag{1.4}$$

Now, let $\mathcal{V} = (V_\beta)_{\beta \in J}$ be another open covering of X that is finer than $\mathcal{U}$; this means that there exists a map $\rho : J \to I$ such that $V_\beta \subset U_{\rho(\beta)}$ for every $\beta \in J$. Then we can define a morphism $\rho^\bullet : C^\bullet(\mathcal{U}, \mathcal{A}) \longrightarrow C^\bullet(\mathcal{V}, \mathcal{A})$ by

$$(\rho^q c)_{\beta_0 \ldots \beta_q} = c_{\rho(\beta_0)\ldots\rho(\beta_q)} \restriction_{V_{\beta_0 \ldots \beta_q}} ; \tag{1.5}$$

the commutation property $\delta\rho^\bullet = \rho^\bullet\delta$ is immediate. If $\rho' : J \to I$ is another refinement map such that $V_\beta \subset U_{\rho'(\beta)}$ for all β, the morphisms $\rho^\bullet$, $\rho'^\bullet$ are homotopic. To see this, we define a map $h^q : C^q(\mathcal{U}, \mathcal{A}) \longrightarrow C^{q-1}(\mathcal{V}, \mathcal{A})$ by

$$(h^q c)_{\beta_0 \ldots \beta_{q-1}} = \sum_{0 \leqslant j \leqslant q-1} (-1)^j c_{\rho(\beta_0)\ldots\rho(\beta_j)\rho'(\beta_j)\ldots\rho'(\beta_{q-1})} \restriction_{V_{\beta_0 \ldots \beta_{q-1}}}.$$

The homotopy identity $\delta^{q-1} \circ h^q + h^{q+1} \circ \delta^q = \rho'^q - \rho^q$ is easy to verify. Hence $\rho^\bullet$ and $\rho'^\bullet$ induce a map depending only on $\mathcal{U}$, $\mathcal{V}$:

$$H^q(\rho^\bullet) = H^q(\rho'^\bullet) \ : \ H^q(\mathcal{U}, \mathcal{A}) \longrightarrow H^q(\mathcal{V}, \mathcal{A}). \tag{1.6}$$

Now, we want to define a *direct limit* $H^q(X, \mathcal{A})$ of the groups $H^q(\mathcal{U}, \mathcal{A})$ by means of the refinement mappings (1.6). In order to avoid set theoretic difficulties, the coverings used in this definition will be considered as subsets of the power set $\mathcal{P}(X)$, so that the collection of all coverings becomes actually a set.

(1.7) Definition. *The Čech cohomology group $H^q(X, \mathcal{A})$ is the direct limit*

$$H^q(X, \mathcal{A}) = \varinjlim_{\mathcal{U}} \ H^q(\mathcal{U}, \mathcal{A})$$

when $\mathcal{U}$ runs over the collection of all open coverings of X. Explicitly, this means that the elements of $H^q(X, \mathcal{A})$ are the equivalence classes in the disjoint union of the groups $\check{H}^q(\mathcal{U}, \mathcal{A})$, with an element in $H^q(\mathcal{U}, \mathcal{A})$ and another in $H^q(\mathcal{V}, \mathcal{A})$ identified if their images in $H^q(\mathcal{W}, \mathcal{A})$ coincide for some refinement $\mathcal{W}$ of the coverings $\mathcal{U}$ and $\mathcal{V}$.

If $\varphi : \mathcal{A} \to \mathcal{B}$ is a sheaf morphism, we have an obvious induced morphism $\varphi^\bullet : C^\bullet(\mathcal{U}, \mathcal{A}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{B})$, and therefore we find a morphism

$$H^q(\varphi^\bullet) : H^q(\mathcal{U}, \mathcal{A}) \longrightarrow H^q(\mathcal{U}, \mathcal{B}).$$

Let $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ be an exact sequence of sheaves. We have an exact sequence of groups

$$0 \longrightarrow C^q(\mathcal{U}, \mathcal{A}) \longrightarrow C^q(\mathcal{U}, \mathcal{B}) \longrightarrow C^q(\mathcal{U}, \mathcal{C}), \tag{1.8}$$

but in general the last map is not surjective, because every section in $\mathcal{C}(U_{\alpha_0,\ldots,\alpha_q})$ need not have a lifting in $\mathcal{B}(U_{\alpha_0,\ldots,\alpha_q})$. The image of $C^\bullet(\mathcal{U},\mathcal{B})$ in $C^\bullet(\mathcal{U},\mathcal{C})$ will be denoted $C^\bullet_{\mathcal{B}}(\mathcal{U},\mathcal{C})$ and called the complex of *liftable cochains* of $\mathcal{C}$ in $\mathcal{B}$. By construction, the sequence

$$0 \longrightarrow C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^q(\mathcal{U},\mathcal{B}) \longrightarrow C^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow 0 \tag{1.9}$$

is exact, thus we get a corresponding long exact sequence of cohomology

$$H^q(\mathcal{U},\mathcal{A}) \longrightarrow H^q(\mathcal{U},\mathcal{B}) \longrightarrow H^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow H^{q+1}(\mathcal{U},\mathcal{A}) \longrightarrow \cdots. \tag{1.10}$$

(1.11) Proposition. *Let $\mathcal{A}$ be a sheaf of modules over a sheaf of rings $\mathcal{R}$ on X. Assume that $\mathcal{R}$ is a soft sheaf (i.e., by definition, that $\mathcal{R}$ admits locally finite partitions of unity for every open covering of X). Then $H^q(\mathcal{U},\mathcal{A}) = 0$ for every $q \geqslant 1$ and every open covering $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ of X.*

Proof. Let $(\psi_\alpha)_{\alpha\in I}$ be a partition of unity in $\mathcal{R}$ subordinate to $\mathcal{U}$, i.e. $\mathrm{Supp}(\psi_\alpha) \subset U_\alpha$ and $\sum_\alpha \psi_\alpha = 1_{\mathcal{R}}$, the sum being locally finite. We define $h^q : C^q(\mathcal{U},\mathcal{A}) \longrightarrow C^{q-1}(\mathcal{U},\mathcal{A})$ by

$$(h^q c)_{\alpha_0\ldots\alpha_{q-1}} = \sum_{\nu\in I} \psi_\nu\, c_{\nu\alpha_0\ldots\alpha_{q-1}} \tag{1.12}$$

where $\psi_\nu\, c_{\nu\alpha_0\ldots\alpha_{q-1}}$ is extended by 0 on $U_{\alpha_0\ldots\alpha_{q-1}} \cap \complement U_\nu$. It is clear that

$$(\delta^{q-1}h^q c)_{\alpha_0\ldots\alpha_q} = \sum_{\nu\in I} \psi_\nu \big(c_{\alpha_0\ldots\alpha_q} - (\delta^q c)_{\nu\alpha_0\ldots\alpha_q}\big),$$

i.e. $\delta^{q-1}h^q + h^{q+1}\delta^q = \mathrm{Id}$. Hence $\delta^q c = 0$ implies $\delta^{q-1}h^q c = c$ if $q \geqslant 1$. □

§ 1.B. Čech Cohomology on Paracompact Spaces

We prove here that Čech cohomology theory behaves well on paracompact spaces, namely, we get exact sequences of cohomology for any short exact sequence of sheaves.

(1.13) Proposition. *Assume that X is paracompact. If*

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$$

is a short exact sequence of sheaves, there is a "long" exact sequence

$$H^q(X,\mathcal{A}) \longrightarrow H^q(X,\mathcal{B}) \longrightarrow H^q(X,\mathcal{C}) \longrightarrow \check{H}^{q+1}(X,\mathcal{A}) \longrightarrow \cdots$$

which is the direct limit of the exact sequences (1.10) *over all coverings $\mathcal{U}$.*

Proof. We have to show that the natural map

$$\varinjlim\ H^q_{\mathcal{B}}(\mathcal{U},\mathcal{C}) \longrightarrow \varinjlim\ H^q(\mathcal{U},\mathcal{C})$$

is an isomorphism. This follows easily from the following lemma, which says essentially that every cochain in $\mathcal{C}$ becomes liftable in $\mathcal{B}$ after a refinement of the covering.

(1.14) Lifting lemma. *Let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be an open covering of X and $c \in C^q(\mathcal{U}, \mathcal{C})$. If X is paracompact, there exists a finer covering $\mathcal{V} = (V_\beta)_{\beta\in J}$ and a refinement map $\rho : J \to I$ such that $\rho^q c \in C^q_{\mathcal{B}}(\mathcal{V}, \mathcal{C})$.*

Proof. Since $\mathcal{U}$ admits a locally finite refinement, we may assume that $\mathcal{U}$ itself is locally finite. There exists an open covering $\mathcal{W} = (W_\alpha)_{\alpha\in I}$ of X such that $\overline{W}_\alpha \subset U_\alpha$. For every point $x \in X$, we can select an open neighborhood V_x of x with the following properties:

a) if $x \in W_\alpha$, then $V_x \subset W_\alpha$;

b) if $x \in U_\alpha$ or if $V_x \cap W_\alpha \neq \emptyset$, then $V_x \subset U_\alpha$;

c) if $x \in U_{\alpha_0\ldots\alpha_q}$, then $c_{\alpha_0\ldots\alpha_q} \in C^q(U_{\alpha_0\ldots\alpha_q}, \mathcal{C})$ admits a lifting in $\mathcal{B}(V_x)$.

Indeed, a) (resp. c)) can be achieved because x belongs to only finitely many sets W_α (resp. U_α), and so only finitely many sections of $\mathcal{C}$ have to be lifted in $\mathcal{B}$. b) can be achieved because x has a neighborhood V'_x that meets only finitely many sets U_α ; then we take

$$V_x \subset V'_x \cap \bigcap_{U_\alpha \ni x} U_\alpha \cap \bigcap_{U_\alpha \not\ni x} (V'_x \smallsetminus \overline{W}_\alpha).$$

Choose $\rho : X \to I$ such that $x \in W_{\rho(x)}$ for every x. Then a) implies $V_x \subset W_{\rho(x)}$, so $\mathcal{V} = (V_x)_{x\in X}$ is finer than $\mathcal{U}$, and ρ defines a refinement map. If $V_{x_0\ldots x_q} \neq \emptyset$, we have

$$V_{x_0} \cap W_{\rho(x_j)} \supset V_{x_0} \cap V_{x_j} \neq \emptyset \quad \text{for} \quad 0 \leqslant j \leqslant q,$$

thus $V_{x_0} \subset U_{\rho(x_0)\ldots\rho(x_q)}$ by b). Now, c) implies that the section $c_{\rho(x_0)\ldots\rho(x_q)}$ admits a lifting in $\mathcal{B}(V_{x_0})$, and in particular in $\mathcal{B}(V_{x_0\ldots x_q})$. Therefore $\rho^q c$ is liftable in $\mathcal{B}$. $\square$

(1.15) Corollary. *Let $0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{B} \longrightarrow \mathcal{C} \longrightarrow 0$ be an exact sequence of sheaves on X, and let U be a paracompact open subset. If $H^1(U, \mathcal{A}) = 0$, there is an exact sequence*

$$0 \longrightarrow \Gamma(U, \mathcal{A}) \longrightarrow \Gamma(U, \mathcal{B}) \longrightarrow \Gamma(U, \mathcal{C}) \longrightarrow 0.$$

§ 1.C. Leray's Theorem for acyclic coverings

We assume here that we are here in a topological space X such that every open subset U is paracompact (one can show that every metrizable space has this property). By definition of Čech cohomology, for every exact sequence of sheaves $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ there is a commutative diagram

$$(1.16)\qquad \begin{array}{ccccccccc} H^q(\mathcal{U}, \mathcal{A}) & \longrightarrow & H^q(\mathcal{U}, \mathcal{B}) & \longrightarrow & H^q_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) & \longrightarrow & H^{q+1}(\mathcal{U}, \mathcal{A}) & \longrightarrow & H^{q+1}(\mathcal{U}, \mathcal{B}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^q(X, \mathcal{A}) & \longrightarrow & H^q(X, \mathcal{B}) & \longrightarrow & H^q(X, \mathcal{C}) & \longrightarrow & H^{q+1}(X, \mathcal{A}) & \longrightarrow & H^{q+1}(X, \mathcal{B}). \end{array}$$

in which the vertical maps are the canonical arrows to the inductive limit.

(1.17) Theorem (Leray)**.** *Assume that $H^s(U_{\alpha_0\ldots\alpha_t}, \mathcal{A}) = 0$ for all indices $\alpha_0, \ldots, \alpha_t$ and all $s \geqslant 1$. Then $H^q(\mathcal{U}, \mathcal{A}) \simeq H^q(X, \mathcal{A})$ for every $q \geqslant 0$.*

We say that the covering $\mathcal{U}$ is *acyclic* (with respect to $\mathcal{A}$) if the hypothesis of Th. 1.17 is satisfied. Leray's theorem asserts that the cohomology groups of $\mathcal{A}$ on X can be computed by means of an arbitrary acyclic covering (if such a covering exists), without using the direct limit procedure.

Proof. By induction on q, the result being obvious for $q = 0$. Consider the exact sequence $0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$ where $\mathcal{B}$ is the sheaf of non necessarily continuous sections of $\widetilde{\mathcal{A}}$ and $\mathcal{C} = \mathcal{B}/\mathcal{A}$. As $\mathcal{B}$ is acyclic, the hypothesis on $\mathcal{A}$ and the long exact sequence of cohomology imply $H^s(U_{\alpha_0\ldots\alpha_t}, \mathcal{C}) = 0$ for $s \geqslant 1$, $t \geqslant 0$. Moreover $C^\bullet_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) = C^\bullet(\mathcal{U}, \mathcal{C})$ thanks to Cor. 1.15 applied on each open set $U_{\alpha_0\ldots\alpha_q}$. The induction hypothesis in degree q and diagram (1.16) give

$$\begin{array}{ccccccc}
H^q(\mathcal{U},\mathcal{B}) & \longrightarrow & H^q(\mathcal{U},\mathcal{C}) & \longrightarrow & H^{q+1}(\mathcal{U},\mathcal{A}) & \longrightarrow & 0 \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow & & \\
H^q(X,\mathcal{B}) & \longrightarrow & H^q(X,\mathcal{C}) & \longrightarrow & H^{q+1}(X,\mathcal{A}) & \longrightarrow & 0,
\end{array}$$

hence $H^{q+1}(\mathcal{U}, \mathcal{A}) \longrightarrow H^{q+1}(X, \mathcal{A})$ is also an isomorphism. □

(1.18) Remark. The morphism $H^1(\mathcal{U}, \mathcal{A}) \longrightarrow H^1(X, \mathcal{A})$ is always injective. Indeed, (1.10) yields

$$\begin{array}{ccc}
H^0_{\mathcal{B}}(\mathcal{U},\mathcal{C})/\operatorname{Im} H^0(\mathcal{U},\mathcal{B}) & \xrightarrow{\simeq} & H^1(\mathcal{U},\mathcal{A}) \\
\downarrow & & \downarrow \\
H^0(X,\mathcal{C})/\operatorname{Im} H^0(X,\mathcal{B}) & \xrightarrow{\simeq} & H^1(X,\mathcal{A})
\end{array}$$

and $H^0(\mathcal{U}, \mathcal{B}) = H^0(X, \mathcal{B}) = \Gamma(X, \mathcal{B})$, while $H^0_{\mathcal{B}}(\mathcal{U}, \mathcal{C}) \longrightarrow H^0(X, \mathcal{C})$ is an injection. As a consequence, the refinement mappings $H^1(\mathcal{U}, \mathcal{A}) \to H^1(\mathcal{V}, \mathcal{A})$ are also injective. □

§ 2. The De Rham-Weil isomorphism theorem

Let $(\mathcal{L}^\bullet, d)$ be a resolution of a sheaf $\mathcal{A}$, that is a complex of sheaves such that we have an exact sequence

$$0 \longrightarrow \mathcal{A} \longrightarrow \mathcal{L}^0 \xrightarrow{d^0} \mathcal{L}^1 \xrightarrow{d^1} \mathcal{L}^2 \longrightarrow \cdots.$$

We assume in addition that all $\mathcal{L}^q$ are acyclic on X, i.e. $H^s(X, \mathcal{L}^q) = 0$ for all $q \geqslant 0$ and $s \geqslant 1$. Set $\mathcal{Z}^q = \ker d^q$. Then $\mathcal{Z}^0 = \mathcal{A}$ and for every $q \geqslant 1$ we get a short exact sequence

$$0 \longrightarrow \mathcal{Z}^{q-1} \longrightarrow \mathcal{L}^{q-1} \xrightarrow{d^{q-1}} \mathcal{Z}^q \longrightarrow 0.$$

Theorem 1.5 yields an exact sequence

$$(2.1) \qquad H^s(X, \mathcal{L}^{q-1}) \xrightarrow{d^{q-1}} H^s(X, \mathcal{Z}^q) \xrightarrow{\partial^{s,q}} H^{s+1}(X, \mathcal{Z}^{q-1}) \to H^{s+1}(X, \mathcal{L}^{q-1}) = 0.$$

If $s \geqslant 1$, the first group is also zero and we get an isomorphism

$$\partial^{s,q} : H^s(X, \mathcal{Z}^q) \xrightarrow{\simeq} H^{s+1}(X, \mathcal{Z}^{q-1}).$$

For $s=0$ we have $H^0(X,\mathscr{L}^{q-1})=\mathscr{L}^{q-1}(X)$ and $H^0(X,\mathscr{Z}^q)=\mathscr{Z}^q(X)$ is the q-cocycle group of $\mathscr{L}^\bullet(X)$, so the connecting map $\partial^{0,q}$ gives an isomorphism

$$H^q\big(\mathscr{L}^\bullet(X)\big)=\mathscr{Z}^q(X)/d^{q-1}\mathscr{L}^{q-1}(x)\xrightarrow{\widetilde{\partial}^{0,q}}H^1(X,\mathscr{Z}^{q-1}).$$

The composite map $\partial^{q-1,1}\circ\cdots\circ\partial^{1,q-1}\circ\widetilde{\partial}^{0,q}$ therefore defines an isomorphism

$$H^q\big(\mathscr{L}^\bullet(X)\big)\xrightarrow{\widetilde{\partial}^{0,q}}H^1(X,\mathscr{Z}^{q-1})\xrightarrow{\partial^{1,q-1}}\cdots\xrightarrow{\partial^{q-1,1}}H^q(X,\mathscr{Z}^0)=H^q(X,\mathscr{A}). \tag{2.2}$$

This isomorphism behaves functorially with respect to morphisms of resolutions. Our assertion means that for every sheaf morphism $\varphi:\mathscr{A}\to\mathscr{B}$ and every morphism of resolutions $\varphi^\bullet:\mathscr{L}^\bullet\longrightarrow\mathscr{M}^\bullet$, there is a commutative diagram

$$\begin{array}{ccc} H^s\big(\mathscr{L}^\bullet(X)\big) & \longrightarrow & H^s(X,\mathscr{A}) \\ \big\downarrow H^s(\varphi^\bullet) & & \big\downarrow H^s(\varphi) \\ H^s\big(\mathscr{M}^\bullet(X)\big) & \longrightarrow & H^s(X,\mathscr{B}). \end{array} \tag{2.3}$$

If $\mathscr{W}^q=\ker(d^q:\mathscr{M}^q\to\mathscr{M}^{q+1})$, the functoriality comes from the fact that we have commutative diagrams

$$\begin{array}{ccccccccc} 0\to & \mathscr{Z}^{q-1} & \to & \mathscr{L}^{q-1} & \to & \mathscr{Z}^q & \to 0\ , & H^s(X,\mathscr{Z}^q) \xrightarrow{\partial^{s,q}} & H^{s+1}(X,\mathscr{Z}^{q-1}) \\ & \big\downarrow\varphi^{q-1} & & \big\downarrow\varphi^{q-1} & & \big\downarrow\varphi^q & & \big\downarrow H^s(\varphi^q) & \big\downarrow H^{s+1}(\varphi^{q-1}) \\ 0\to & \mathscr{W}^{q-1} & \to & \mathscr{M}^{q-1} & \to & \mathscr{W}^q & \to 0\ , & H^s(X,\mathscr{W}^q) \xrightarrow{\partial^{s,q}} & H^{s+1}(X,\mathscr{W}^{q-1}). \end{array}$$

(2.4) De Rham-Weil isomorphism theorem. *If $(\mathscr{L}^\bullet,d)$ is a resolution of $\mathscr{A}$ by sheaves $\mathscr{L}^q$ which are acyclic on X, there is a functorial isomorphism*

$$H^q\big(\mathscr{L}^\bullet(X)\big)\longrightarrow H^q(X,\mathscr{A}).\qquad\square$$

(2.5) Example: De Rham cohomology. Let X be a n-dimensional paracompact differential manifold. Consider the resolution

$$0\to\mathbb{R}\to\mathscr{E}^0\xrightarrow{d}\mathscr{E}^1\to\cdots\to\mathscr{E}^q\xrightarrow{d}\mathscr{E}^{q+1}\to\cdots\to\mathscr{E}^n\to 0$$

given by the exterior derivative d acting on germs of $\mathscr{C}^\infty$ differential q-forms (cf. Example 2.2). The *De Rham cohomology groups* of X are precisely

$$H^q_{\mathrm{DR}}(X,\mathbb{R})=H^q\big(\mathscr{E}^\bullet(X)\big). \tag{2.6}$$

All sheaves $\mathscr{E}^q$ are $\mathscr{E}_X$-modules, so $\mathscr{E}^q$ is acyclic by Cor. 4.19. Therefore, we get an isomorphism

$$H^q_{\mathrm{DR}}(X,\mathbb{R})\xrightarrow{\simeq}H^q(X,\mathbb{R}) \tag{2.7}$$

from the De Rham cohomology onto the cohomology with values in the constant sheaf $\mathbb{R}$. Instead of using $\mathscr{C}^\infty$ differential forms, one can consider the resolution of $\mathbb{R}$ given by the exterior derivative d acting on currents:

$$0 \to \mathbb{R} \to \mathscr{D}'_n \xrightarrow{d} \mathscr{D}'_{n-1} \to \cdots \to \mathscr{D}'_{n-q} \xrightarrow{d} \mathscr{D}'_{n-q-1} \to \cdots \to \mathscr{D}'_0 \to 0.$$

The sheaves $\mathscr{D}'_q$ are also $\mathscr{E}_X$-modules, hence acyclic. Thanks to (2.3), the inclusion $\mathscr{E}^q \subset \mathscr{D}'_{n-q}$ induces an isomorphism

$$\text{(2.8)} \qquad H^q\big(\mathscr{E}^\bullet(X)\big) \simeq H^q\big(\mathscr{D}'_{n-\bullet}(X)\big),$$

both groups being isomorphic to $H^q(X,\mathbb{R})$. The isomorphism between cohomology of differential forms and singular cohomology (another topological invariant) was first established by [De Rham 1931]. The above proof follows essentially the method given by [Weil 1952], in a more abstract setting. As we will see, the isomorphism (2.7) can be put under a very explicit form in terms of Čech cohomology. We need a simple lemma.

(2.9) Lemma. *Let X be a paracompact differentiable manifold. There are arbitrarily fine open coverings $\mathscr{U} = (U_\alpha)$ such that all intersections $U_{\alpha_0\ldots\alpha_q}$ are diffeomorphic to convex sets.*

Proof. Select locally finite coverings $\Omega'_j \subset\subset \Omega_j$ of X by open sets diffeomorphic to concentric euclidean balls in $\mathbb{R}^n$. Let us denote by τ_{jk} the transition diffeomorphism from the coordinates in Ω_k to those in Ω_j. For any point $a \in \Omega'_j$, the function $x \mapsto |x-a|^2$ computed in terms of the coordinates of Ω_j becomes $|\tau_{jk}(x) - \tau_{jk}(a)|^2$ on any patch $\Omega_k \ni a$. It is clear that these functions are strictly convex at a, thus there is a euclidean ball $B(a,\varepsilon) \subset \Omega'_j$ such that all functions are strictly convex on $B(a,\varepsilon) \cap \Omega'_k \subset \Omega_k$ (only a finite number of indices k is involved). Now, choose $\mathscr{U}$ to be a (locally finite) covering of X by such balls $U_\alpha = B(a_\alpha, \varepsilon_\alpha)$ with $U_\alpha \subset \Omega'_{\rho(\alpha)}$. Then the intersection $U_{\alpha_0\ldots\alpha_q}$ is defined in Ω_k, $k = \rho(\alpha_0)$, by the equations

$$|\tau_{jk}(x) - \tau_{jk}(a_{\alpha_m})|^2 < \varepsilon^2_{\alpha_m}$$

where $j = \rho(\alpha_m)$, $0 \leqslant m \leqslant q$. Hence the intersection is convex in the open coordinate chart $\Omega_{\rho(\alpha_0)}$. □

Let Ω be an open subset of $\mathbb{R}^n$ which is starshaped with respect to the origin. Then the De Rham complex $\mathbb{R} \longrightarrow \mathscr{E}^\bullet(\Omega)$ is acyclic: indeed, the Poincaré lemma yields a homotopy operator $k^q : \mathscr{E}^q(\Omega) \longrightarrow \mathscr{E}^{q-1}(\Omega)$ such that

$$k^q f_x(\xi_1,\ldots,\xi_{q-1}) = \int_0^1 t^{q-1}\, f_{tx}(x,\xi_1,\ldots,\xi_{q-1})\,dt, \quad x \in \Omega, \quad \xi_j \in \mathbb{R}^n,$$
$$k^0 f = f(0) \in \mathbb{R} \quad \text{for} \quad f \in \mathscr{E}^0(\Omega).$$

Hence $H^q_{\mathrm{DR}}(\Omega,\mathbb{R}) = 0$ for $q \geqslant 1$. Now, consider the resolution $\mathscr{E}^\bullet$ of the constant sheaf $\mathbb{R}$ on X, and apply the proof of the De Rham-Weil isomorphism theorem to Čech cohomology groups over a covering $\mathscr{U}$ chosen as in Lemma 2.9. Since the intersections $U_{\alpha_0\ldots\alpha_s}$ are convex, all Čech cochains in $C^s(\mathscr{U},\mathscr{Z}^q)$ are liftable in $\mathscr{E}^{q-1}$ by means of k^q. Hence for all $s = 1, \ldots, q$ we have isomorphisms $\partial^{s,q-s} : H^s(\mathcal{U}, \mathcal{Z}^{q-s}) \longrightarrow H^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1})$ for $s \geqslant 1$ and we get a resulting isomorphism

$$\partial^{q-1,1} \circ \cdots \circ \partial^{1,q-1} \circ \widetilde{\partial}^{0,q} : H^q_{\mathrm{DR}}(X, \mathbb{R}) \xrightarrow{\simeq} H^q(\mathcal{U}, \mathbb{R})$$

We are going to compute the connecting homomorphisms $\partial^{s,q-s}$ and their inverses explicitly.

Let c in $C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ such that $\delta^s c = 0$. As $c_{\alpha_0\ldots\alpha_s}$ is d-closed, we can write $c = d(k^{q-s}c)$ where the cochain $k^{q-s}c \in C^s(\mathcal{U}, \mathcal{E}^{q-s-1})$ is defined as the family of sections $k^{q-s}c_{\alpha_0\ldots\alpha_s} \in \mathcal{E}^{q-s-1}(U_{\alpha_0\ldots\alpha_s})$. Then $d(\delta^s k^{q-s}c) = \delta^s(dk^{q-s}c) = \delta^s c = 0$ and

$$\partial^{s,q-s}\{c\} = \{\delta^s k^{q-s}c\} \in \check{H}^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1}).$$

The isomorphism $H^q_{\mathrm{DR}}(X, \mathbb{R}) \xrightarrow{\simeq} H^q(\mathcal{U}, \mathbb{R})$ is thus defined as follows: to the cohomology class $\{f\}$ of a closed q-form $f \in \mathcal{E}^q(X)$, we associate the cocycle $(c^0_\alpha) = (f_{\restriction U_\alpha}) \in C^0(\mathcal{U}, \mathcal{Z}^q)$, then the cocycle

$$c^1_{\alpha\beta} = k^q c^0_\beta - k^q c^0_\alpha \in C^1(\mathcal{U}, \mathcal{Z}^{q-1}),$$

and by induction cocycles $(c^s_{\alpha_0\ldots\alpha_s}) \in C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ given by

$$c^{s+1}_{\alpha_0\ldots\alpha_{s+1}} = \sum_{0\leqslant j\leqslant s+1} (-1)^j\, k^{q-s} c^s_{\alpha_0\ldots\widehat{\alpha_j}\ldots\alpha_{s+1}} \quad \text{on } U_{\alpha_0\ldots\alpha_{s+1}}. \tag{2.10}$$

The image of $\{f\}$ in $H^q(\mathcal{U}, \mathbb{R})$ is the class of the q-cocycle $(c^q_{\alpha_0\ldots\alpha_q})$ in $C^q(\mathcal{U}, \mathbb{R})$.

Conversely, let (ψ_α) be a $\mathcal{C}^\infty$ partition of unity subordinate to $\mathcal{U}$. Any Čech cocycle $c \in C^{s+1}(\mathcal{U}, \mathcal{Z}^{q-s-1})$ can be written $c = \delta^s\gamma$ with $\gamma \in C^s(\mathcal{U}, \mathcal{E}^{q-s-1})$ given by

$$\gamma_{\alpha_0\ldots\alpha_s} = \sum_{\nu\in I} \psi_\nu\, c_{\nu\alpha_0\ldots\alpha_s},$$

(cf. Prop. 1.11)), thus $\{c'\} = (\partial^{s,q-s})^{-1}\{c\}$ can be represented by the cochain $c' = d\gamma \in C^s(\mathcal{U}, \mathcal{Z}^{q-s})$ such that

$$c'_{\alpha_0\ldots\alpha_s} = \sum_{\nu\in I} d\psi_\nu \wedge c_{\nu\alpha_0\ldots\alpha_s} = (-1)^{q-s-1} \sum_{\nu\in I} c_{\nu\alpha_0\ldots\alpha_s} \wedge d\psi_\nu.$$

For a reason that will become apparent later, we shall in fact modify the sign of our isomorphism $\partial^{s,q-s}$ by the factor $(-1)^{q-s-1}$. Starting from a class $\{c\} \in H^q(\mathcal{U}, \mathbb{R})$, we obtain inductively $\{b\} \in H^0(\mathcal{U}, \mathcal{Z}^q)$ such that

$$b_{\alpha_0} = \sum_{\nu_0,\ldots,\nu_{q-1}} c_{\nu_0\ldots\nu_{q-1}\alpha_0}\, d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{q-1}} \quad \text{on } U_{\alpha_0}, \tag{2.11}$$

corresponding to $\{f\} \in H^q_{\mathrm{DR}}(X, \mathbb{R})$ given by the explicit formula

$$f = \sum_{\nu_q} \psi_{\nu_q} b_{\nu_q} = \sum_{\nu_0,\ldots,\nu_q} c_{\nu_0\ldots\nu_q}\, \psi_{\nu_q} d\psi_{\nu_0} \wedge \ldots \wedge d\psi_{\nu_{q-1}}. \tag{2.12}$$

The choice of sign corresponds to (2.2) multiplied by $(-1)^{q(q-1)/2}$.

(2.13) Example: Dolbeault cohomology groups. Let X be a $\mathbb{C}$-analytic manifold of dimension n, and let $\mathscr{E}^{p,q}$ be the sheaf of germs of $\mathscr{C}^\infty$ differential forms of type (p,q) with complex values. For every $p = 0, 1, \ldots, n$, the Dolbeault-Grothendieck lemma shows that $(\mathscr{E}^{p,\bullet}, d'')$ is a resolution of the sheaf Ω^p_X of germs of holomorphic forms of degree p on X. The *Dolbeault cohomology groups* of X are defined by

$$H^{p,q}(X,\mathbb{C}) = H^q\big(\mathscr{E}^{p,\bullet}(X)\big). \tag{2.14}$$

Since the sheaves $\mathscr{E}^{p,q}$ are acyclic, we get the *Dolbeault isomorphism theorem*, originally proved by Dolbeault in 1953, which relates d''-cohomology and sheaf cohomology:

$$H^{p,q}(X,\mathbb{C}) \xrightarrow{\simeq} H^q(X,\Omega^p_X). \tag{2.15}$$

The case $p = 0$ is especially interesting:

$$H^{0,q}(X,\mathbb{C}) \simeq H^q(X,\mathscr{O}_X). \tag{2.16}$$

As in the case of De Rham cohomology, there is an inclusion $\mathscr{E}^{p,q} \subset \mathscr{D}'_{n-p,n-q}$ and the complex of currents $(\mathscr{D}'_{n-p,n-\bullet}, d'')$ defines also a resolution of Ω^p_X. Hence there is an isomorphism:

$$H^{p,q}(X,\mathbb{C}) = H^q\big(\mathscr{E}^{p,\bullet}(X)\big) \simeq H^q\big(\mathscr{D}'_{n-p,n-\bullet}(X)\big). \tag{2.17}$$