Introduction to analytic geometry (course by Jean-Pierre Demailly)
Partial examination, 07/11/2019, 13:00 – 17:00

Any use of electronic devices is forbidden
Handwritten documents and lecture notes are allowed, printed documents are not (except exercise sheets)
Answers to questions can be developed in French or English
Justifications of more or less trivial questions need not be very long !
The four problems are independent.

1. (a) Let $\Omega \subset \mathbb{C}^n$ be an open set, let $E = \{w_k\}_{k\in\mathbb{N}} \subset \Omega$ be an infinite discrete (i.e. locally finite) subset of Ω. By looking at the sets $E_N = \{w_k\}_{k\geq N}$ and the associated ideals, show that the "global" ring $R = \mathcal{O}(\Omega)$ is never Noetherian.

(b) Let S be a complex analytic submanifold of Ω. If S is compact, prove that S is in fact of dimension 0 and consists of a finite set of points.
Hint. Apply the maximum principle to the coordinate functions.

(c) Show by an example that result (b) does not hold if one replaces "complex analytic submanifold" by real analytic (or even real algebraic) submanifold.

2. Let A be a germ of analytic set of pure dimension k in a neighborhood of 0 in $\mathbb{C}^n$. One assumes that coordinates $z = (z', z'')$, $z' = (z_1, \ldots, z_k)$, $z'' = (z_{k+1}, \ldots, z_n)$ have been chosen, and well as a small polydisk $D = D' \times D''$ in $\mathbb{C}^k \times \mathbb{C}^{n-k}$, such that the projection $p : A \cap D \to D'$, $z \mapsto z'$ defines a ramified covering, with ramification locus $\Sigma \subset D'$. One denotes by d the degree of the covering, and by $z'' = w_j(z')$, $1 \leq j \leq d$, the branches of A in a neighborhood of every point $z'_0 \in D' \smallsetminus \Sigma$.

(a) If $f \in \mathcal{O}(D)$ is a holomorphic function, show that

$$Q_f(z') = \prod_{(z',z'')\in A} f(z', z'') = \prod_{j=1}^{d} f(z', w_j(z'))$$

is holomorphic on $D' \smallsetminus \Sigma$ and actually extends holomorphically to D'.

(b) Let $B \subset A$ be an analytic subset defined as $B = \{z \in A / g_1(z) = \ldots = g_N(z) = 0\}$ with $g_j \in \mathcal{O}(D)$. The goal is to show that $p(B)$ is an analytic subset of D'. For $t = (t_1, \ldots, t_N) \in \mathbb{C}^N$, define

$$h(z', t) = Q_{t_1 g_1 + \cdots + t_N g_N}(z').$$

Show that h defines a holomorphic function on $D' \times \mathbb{C}^N$ that is a homogeneous degree d polynomial in $t = (t_1, \ldots, t_N)$. If $h(z', t) = \sum h_\alpha(z') t^\alpha$, show that $p(B)$ is the common zero set of the functions h_α.
Hint. For any finite set of non zero points in $\mathbb{C}^N$, one can find a linear form that does not vanish on any of them ...

(c) Let H be the hyperbola $z_1 z_2 = 1$ in $\mathbb{C}^2$ and $p : \mathbb{C}^2 \to \mathbb{C}$ be the first projection $p : (z_1, z_2) \mapsto z_1$. Is $p(H)$ analytic in $\mathbb{C}$? What happens ?

(d) Let $B \subset \mathbb{C}^n$ be a compact complex analytic set. The goal is to show that B is finite and of dimension 0 (thus generalizing the result of 1 b). One argues by induction on n, letting $p : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the projection to the first $n-1$ coordinates.
– case $n = 1$.
– for $n \geq 2$, show that the fibers of $p_{|B}$ must be finite.
– then show that $p(B)$ is analytic in $\mathbb{C}^{n-1}$ and conclude.
Hint. For any point $a' \in p(B)$ with $p^{-1}(a') \cap B = \{(a', a''_j) \in \mathbb{C}^{n-1} \times \mathbb{C} \,/\, 1 \leq j \leq m\}$, construct a neighborhood V' of a' and polynomials $P_{j,k}(z', z_n) \in \mathcal{O}(V')[z_n]$ so that $B_{V'} := B \cap p^{-1}(V')$ is the union of

$$B_{V',j} := B \cap (V' \times D(a''_j, \varepsilon)) = \{(z', z_n) \in V' \times D(a''_j, \varepsilon) \,/\, P_{j,k}(z', z_n) = 0,\ 1 \leq k \leq N_j\}, \quad 1 \leq j \leq m$$

and apply (b) with e.g. $B_{V'} \subset A := \{\prod_j P_{j,0}(z', z_n) = 0\}$ to infer that $p(B) \cap V'$ is analytic.

3. Let $\alpha \in]0, 1[$. One defines the *Hopf surface* X_α to be the quotient $(\mathbb{C}^2 \smallsetminus \{(0,0)\})/\Gamma$ by the discrete group $\Gamma \simeq \mathbb{Z}$ of homotheties h_α^k, $k \in \mathbb{Z}$, with $h_\alpha(z) = \alpha z$.

(a) Prove that X_α is C^∞ (or even C^ω)-diffeomorphic to the product of spheres $S^1\times S^3$ via the map

$$z\mapsto\Big(\exp(2\pi\mathrm{i}\log|z|/\log\alpha)\;,\;z/|z|\Big)\in S^1\times S^3.$$

(b) Show that X_α can be equipped with the structure of a complex analytic surface, and give explicitly an atlas consisting of 2 open sets in $\mathbb{C}^2$.

(c) Check that

$$\omega(z)=\frac{\mathrm{i}}{|z|^2}\,\partial\overline{\partial}\,|z|^2$$

defines a hermitian metric on X_α that is not a Kähler metric, but show however that $\partial\overline{\partial}\,\omega=0$.

(d) If $P(z),Q(z)$ are homogeneous polynomials of degree d on $\mathbb{C}^2$ without common zeroes, show that $\Phi(z)=(P(z),Q(z))$ defines a holomorphic morphism $\varphi:X_\alpha\to X_\beta$ for certain values of β (which ones ?).

(e) Observing that the universal cover of X_α is $\mathbb{C}^2\smallsetminus\{0\}$, conclude that any non constant holomorphic morphism $\varphi:X_\alpha\to X_\beta$ lifts to a holomorphic map $\Phi:\mathbb{C}^2\to\mathbb{C}^2$ such that $\Phi(\alpha z)=\beta^p\Phi(z)$ for some $p\in\mathbb{N}^*$. Infer that a necessary and sufficient condition for the existence of such morphisms is $\log\beta/\log\alpha\in\mathbb{Q}^*_+$, and that Φ must be homogeneous of some degree.

(f) Give an example of a pair of non homeomorphic *compact* complex surfaces, resp. of a pair of homeomorphic (and even diffeomorphic) but non biholomorphic ones.

4. Except for (g) below, let $X=\mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus, considered as a C^∞ manifold.

(a) Show that C^∞ differential forms of degree p on X can be interpreted as forms $u(x)=\sum_{|I|=p}u_I(x)dx_I$ on $\mathbb{R}^n$ (the summation is over length p increasing multi-indices), where each function u_I satisfies a certain periodicity condition.

(b) Infer from (a) that forms with constant coefficients give rise to a natural ring morphism $\varphi:\Lambda^p(\mathbb{R}^n)^*\to H^p_{\mathrm{DR}}(X,\mathbb{R})$.

(c) For $p+q=n$, one defines a bilinear map

$$H^p_{\mathrm{DR}}(X,\mathbb{R})\times H^q_{\mathrm{DR}}(X,\mathbb{R})\longrightarrow\mathbb{R},\quad(\{u\},\{v\})\mapsto\int_X u\wedge v$$

where X is given the usual orientation of $\mathbb{R}^n$, and $\{u\}$ denotes the cohomology class of a d-closed form u. Show that the above bilinear form is well defined.

(d) Derive from (c) that the cohomology class of a constant-coefficient form $u=\sum u_Idx_I$ is equal to 0 if and only if $u=0$, in other words that φ is injective.

Hint. Use $v=dx_{\complement I}$ where $\complement I$ means the complement of I in $\{1,2,\ldots,n\}$.

(e) Given $a\in\mathbb{R}^n$, one defines operators L_a,G_a and M acting on smooth p-forms u by $L_a(u)=v$ (resp. $G_au=w$, $Mu=\widetilde{u}$) where

$$v_I(x)=D_au_I(x),\quad w_I(x)=\int_0^1u_I(x+ta)\,dt,\quad\widetilde{u}_I(x)=\int_{a\in[0,1]^n}u_I(x+a)\,d\lambda(a),$$

where $d\lambda$ is the Lebesgue measure on $\mathbb{R}^n$ and D_a the derivative in direction a. Show that L_a,G_a and M commute with the exterior derivative d, and that $Mu=\widetilde{u}$ always has constant coefficients (i.e. independent of x). Compute explicitly $G_a\circ L_a$ and $d\circ M$.

(f) The "interior product" i_au of p-form u by a is defined to be the alternate $(p-1)$-form such that $i_au(x)(\xi_2,\ldots,\xi_p)=u(x)(a,\xi_2,\ldots,\xi_p)$. The well known "Lie derivative formula" (that can be admitted here) states that $d(i_au)+i_a(du)=L_au$. Infer from this that for the torus, the operator

$$h_a:C^\infty(X,\Lambda^pT^*_X)\to C^\infty(X,\Lambda^{p-1}T^*_X)$$

defined by $h_a=G_a\circ i_a$ satisfies the so called "homotopy formula" $d(h_a(u))+h_a(du)=v_a$ with $v_a(x)=u(x+a)-u(x)$, and that for every closed form u, v_a is cohomologous to zero. Finally, conclude from all the above results that $\widetilde{u}-u$ is cohomologous to zero and that φ is an isomorphism.

(g) if $X=\mathbb{C}^n/\Lambda$ is a compact complex torus (where Λ is a lattice in $\mathbb{C}^n\simeq\mathbb{R}^{2n}$), use a similar technique to show that there is an *injective morphism* given by constant-coefficient forms

$$\psi:\Lambda^{p,q}(\mathbb{C}^*)^n\to H^{p,q}_{\overline{\partial}}(X,\mathbb{C}).$$

Note. One can show that ψ is actually an isomorphism, but this is a bit harder than for De Rham cohomology (and the required technology has not yet been explained in the course!).