ASYMPTOTIC INVARIANTS OF 3-DIMENSIONAL VECTOR FIELDS

PIERRE DEHORNOY

Abstract. In this survey article, we present several constructions of invariants for 3-dimensional volume-preserving vector fields under volume-preserving diffeomorphisms. After introducing helicity, we focus on invariants constructed using knot theory, following Arnol'd's strategy. Most invariants constructed in this way are actually very close to helicity, but we also present a few that are rather different. We conclude with some open questions.

These notes are dedicated to the following

Problem A. *Construct invariants of 3-dimensional volume-preserving vector fields up to volume-preserving diffeomorphisms.*

We did not specify what is the underlying manifold. For physical applications, it is natural to work on $\mathbb{R}^3$ or on a bounded domain of $\mathbb{R}^3$, often with the condition that the vector field is tangent to the boundary. For mathematical reasons, it is easier to work on compact manifolds, so the 3-sphere $\mathbb{S}^3 = \mathbb{R}^3 \cup \{\infty\}$ is a natural space. Actually most presented results hold both in $\mathbb{R}^3$ and $\mathbb{S}^3$, so we will alternate freely between those two manifolds, depending on what is more natural.

We also did not specify the regularity of the vector field and the regularity of the diffeomorphisms. Yet regularity is in general an important question in dynamical systems. For having a well-defined orbit flow and well-defined orbits, we need the vector field to be Lipschitz-continuous, but for simplicity we will generally assume C^∞. Likewise, many invariants are invariant under C^1 volume-preserving diffeomorphisms, but one can restrict to invariance under C^∞ volume-preserving diffeomorphisms for simplicity. It turns out that it is an open question for most constructions whether they are invariants under volume-preserving *homeomorphisms*.

Also comes the question of the volume. For physical applications, we are mostly interested in invariant measures given by the Euclidean volume, or a function times the Euclidean volume. But the richness of the mathematical approach is to deal with more general invariant measures, like for example the linear measures concentrated on periodic orbits, or the physical SRB measures (more on this in Section 1.c).

Problem A has roots in magnetohydrodynamics (MHD), a part of physics dedicated to the dynamics of magnetic fields, in particular in plasmas. Indeed the magnetic flow of an ideal plasma is time-dependent, but turns out to be transported by the velocity field of the plasma, so that the magnetic field at a given moment is the image of the magnetic field at another moment under a volume-preserving diffeomorphism. In order to understand the long-term behavior (as long as the ideal model is relevant), it is desirable to have invariants that help understanding how the magnetic flow may or may not evolve [AK98].

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Firstly, one can note that the number of fixed points and of periodic orbits is such an invariant. Secondly, periodic orbits form knots whose isotopy classes are a second class of invariants. However there exist C^1-volume-preserving vector fields on $\mathbb{S}^3$ without fixed points nor periodic orbits [KK96], so that the knot types are not sufficient to classify vector fields (there also exist analytic vector fields without periodic orbits, but they do not preserve any volume [Kup94]). Moreover these invariants are of rather local nature: knowing that a vector field contains a certain knot as periodic orbit does not necessarily says much about what happens away from this particular periodic orbit.

The quest for (global) invariants has been launched by Woltjer and Moreau with the discovery of helicity [Wol58, Mor61]. It was first defined (see Section 2) using differential forms, and invariance under diffeomorphism was first unnoticed. This invariance was proven by Moffatt [Mof69] who also remarked that helicity is actually related to knot theory, and more precisely to the linking number.

This connection was deepened by Arnol'd [Arn73] who showed that helicity is a sort of average linking number of pairs of orbits. More precisely a generic orbit of a vector field has in general no reason for being closed, but it is recurrent (meaning that it comes back close to its initial point). One then obtains a knot by connecting the two ends of any arc of orbit with a geodesic segment. Arnol'd proved that for almost every pair of points (p_1, p_2) the linking number of the two such arcs of orbits of length t_1, t_2 is asymptotic to a constant c_{p_1,p_2} times $t_1 t_2$. Moreover the function $(p_1, p_2) \mapsto c_{p_1,p_2}$ is integrable, and its integral equals... the helicity of the vector field!

As the world of knot and link invariants is large and rich, that it contains many different tractable objects, it is desirable to use Arnol'd's strategy in order to export these invariants to vector fields. Let us describe an ideal scheme which mimics Arnol'd's theorem:

- take a link invariant v, that is, a function that assigns to any link $k_1 \cup \cdots \cup k_i$ in $\mathbb{R}^3$ or $\mathbb{S}^3$ a real number and that is invariant under isotopy,
- for $\vec{X}$ a volume-preserving vector field and $p_1, \dots, p_i$ points in $\mathbb{R}^3$ or $\mathbb{S}^3$, consider the segments of orbits $\vec{X}$ of the form $\phi_{\vec{X}}^{[0,t_1]}(p_1), \dots, \phi_{\vec{X}}^{[0,t_i]}(p_i)$ where $(\phi_{\vec{X}}^t)_{t\in\mathbb{R}}$ denotes the flow of $\vec{X}$,
- close these segments using geodesic arcs to get knots $k_{\vec{X}}(p_1, t_1), \dots, k_{\vec{X}}(p_i, t_i)$,
- if for almost every $p_1, \dots, p_i$ the invariant $v(k_{\vec{X}}(p_1, t_1), \dots, k_{\vec{X}}(p_i, t_i))$ has an asymptotic behavior of the form $v_{\vec{X}}^{\infty}(p_1, \dots, p_i) \cdot t_1^{n_1} \dots t_i^{n_i}$ and the function $(p_1, \dots, p_i) \mapsto v_{\vec{X}}^{\infty}(p_1, \dots, p_i)$ is integrable with respect to the volume-measure, then the integral $\int_{(\mathbb{S}^3)^n} v_{\vec{X}}^{\infty}(p_1, \dots, p_i)\, d\mathrm{vol}$ is an invariant of $\vec{X}$ under volume-preserving diffeomorphism.

If v is a link invariant such that the above scheme works for every volume-preserving vector field $\vec{X}$, then v is an *asymptotic vector field invariant of order* $(n_1, \dots, n_i)$. Its value on $\vec{X}$ is defined as $v^{(n_1,\dots,n_i)}(\vec{X}) := \frac{1}{\mathrm{Vol}(\mathbb{S}^3)^n} \int_{(\mathbb{S}^3)^n} v_{\vec{X}}^{\infty}(p_1, \dots, p_i)\, d\mathrm{vol}$.

With this definition, Arnol'd's theorem [Arn73] states that the linking number is an asymptotic vector field invariant of order $(1, 1)$ whose value on a vector field equals helicity.

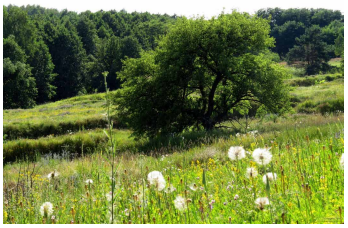
Recall that a vector field $\vec{X}$ is *ergodic* with respect to a probability measure μ if every $\vec{X}$-invariant measurable set has μ-measure 0 or 1. In this case every $\vec{X}$-invariant

function is almost-surely constant. In the previous setting, when $\vec{X}$ is ergodic, the function $(p_1, \dots, p_i) \mapsto v^\infty_{\vec{X}}(p_1, \dots, p_i)$ is almost surely constant, so that $v^{(n,\dots,n)}(\vec{X})$ can be computed using a single tuple of generic orbits.

Problem B (Arnol'd's question)**.** *Construct asymptotic vector field invariants whose value on ergodic vector fields is not a function of helicity.*

Sadly Problem B admits few answers yet. On the one hand there are some invariants (*e.g.*, crossing number [FH91]) for which Arnol'd's strategy is *likely* to work, but the invariant is not very tractable and the correspondence between the actual knot invariant and its vector field-counterpart is not proven yet.

On the other hand there are many invariants (*e.g.*, ω-signatures, Vassiliev invariants) for which Arnol'd's strategy is *known* to work, but the asympotitcs turn out to be a function the helicity when the vector field is ergodic (*i.e.*, $v(\vec{X})$ is a function of the helicity of $\vec{X}$). Let us however underline that this dependance is proven only for ergodic vector fields, so that these invariants may still say something on how the different ergodic components of a vector field are linked. Actually the family of invariants that fall in this second category is very large (Vassiliev invariants are even conjectured to be *total invariants*). So, if link invariants form a vast forest for which linking number is the first of many trees (left), asymptotic invariants for ergodic volume-preserving vector fields seem to form a forest in which there might be only one tree called helicity (right) !



A satisfactory explanation of this phenomenon has been given recently (actually between the time the course was given and these notes published!) by E. Kudrayvtseva [Kud14, Kud15] and A. Encisco, D. Peralta-Salas, and F. Torres de Lizaur [EPT16]. They show that if one looks for *very regular* invariants, then there is only one for ergodic vector fields, namely helicity. We will present their results.

However, we can still look at less regular invariants. Indeed it turns out that there are (few) invariants (*e.g.*, Milnor's invariants, trunk) for which Problem B has a positive answer: $v^{(n_1,\dots,n_i)}(\vec{X})$ is not proportional to helicity. We will present some of them in the last section.

There already exist two excellent surveys on problem A [Gam06, Ghy07] and it seems hard to write better texts than these two. The goal of these notes is therefore to restrict our attention to Problem B and to present mostly results that have been proven in the last decade.

The plan is as follows: in Section 1 we present a short history of Problem A and connect it with hydro- and magnetodynamics; we also present examples of vector fields for which the study of knotted orbits and knot-theoretical invariants is easier. Such examples are useful for developing the intuition. In Section 2 we present the simplest vector field

invariant—helicity— and relate it with the simplest link invariant—linking number. In Section 3 we present other asymptotic invariants (signatures, Vassiliev invariants, quadratic linking numbers) which turn out to be proportional to helicity on ergodic vector fields. We also presents Encisco-Peralta-Salas-Torres de Lizaur's result about uniqueness of helicity. In Section 4 we explain how to derive vector field invariants that are *not* governed by helicity, and we finish with some questions in Section 5.

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1. Introduction: motivation and examples

We begin with a history section that reminds Helmholtz' laws for the motion of a perfect fluid [Hel1858]. These laws were at the origin of Thomson's theory of atoms [Tho1867] and motivated the foundation of knot theory by Tait [Tai1877]. We then give some examples of measure-preserving vector fields, so that the reader has some examples to test his intuition on.

1.a. Helmholtz' laws and connection with knot theory.

Helmholtz' laws simultaneously motivate Problem A and connect it with knot theory, as we explain below.

Euler's equations (1755) for the *velocity field* $\vec{u}_t(x)$ of an ideal (*i.e.,* inviscid, incompressible) fluid in $\mathbb{R}^3$ follow from Newton's laws of mechanics applied to infinitesimal volumes:

$$\vec{\nabla}\cdot\vec{u}_t = 0, \tag{1}$$

$$\frac{\partial\vec{u}_t}{\partial t} + (\vec{u}_t\cdot\vec{\nabla})\vec{u}_t + \vec{\nabla}p = \vec{0}. \tag{2}$$

Equation (1) transcribes the conservation of mass of the fluid (here $\vec{\nabla}\cdot\vec{u}_t$ denotes the divergence of $\vec{u}_t$), and Equation (2) transcribes the conservation of momentum (p stands for the pressure and $(\vec{u}_t\cdot\vec{\nabla})\vec{u}_t$ is the directional derivative of $\vec{u}_t$).

Helmholtz noted a remarkable property of these equations as follows. The local movement of the fluid around a particle is given by the differential $d\vec{u}_t$. It can be decomposed into a stretching part and a rotational part, given respectively by the symmetric and antisymmetric part of $d\vec{u}_t$. Given c_t an infinitesimal ellipse centered on the considered particle, the circulation of $\vec{u}_t$ along c given by $\oint_c \vec{u}_t\cdot\vec{dc}$ measures how $\vec{u}_t$ rotates on c. This quantity is bilinear in the two axes of the ellipse: it is hence a 2-form, which we denote by β_t. This can be thought of as a local plane and a local rotation in that plane. Given a volume form μ (for example the standard Euclidean volume), this rotation can be expressed by a vector whose direction is the axis of local rotation and whose length is the speed of local rotation. In coordinates one checks that the field $\vec{\omega}_t := \vec{\nabla}\times\vec{u}_t = \operatorname{curl}\vec{u}_t$ satisfies $\beta_t(\vec{y},\vec{z}) = \mu(\vec{\omega}_t,\vec{y},\vec{z})$ and that $\vec{\omega}_t$ is μ-preserving.

Now the key idea of Helmholtz is to compute the total derivative of the circulation on an arbitrary curve c_t with length element $d\vec{c}_t$ that is transported by the flow:

$$\begin{aligned}\frac{D}{Dt}\oint_{c_t}\vec{u}_t\cdot d\vec{c}_t &= \oint_{c_t}\left(\frac{D\vec{u}_t}{Dt}\cdot d\vec{c}_t+\vec{u}_t\cdot\frac{D(d\vec{c}_t)}{Dt}\right)\\ &= \oint_{c_t}\vec{\nabla}p_t\cdot d\vec{c}_t+\vec{u}_t\cdot d\vec{u}_t\\ &= \oint_{c_t}0+d\|\vec{u}_t\|^2=0.\end{aligned}$$

At the global level, this shows that the circulation on a curve is constant over time. At the infinitesimal level, this shows that the infinitesimal circulation —the form β_t— is transported by the flow: if we denote by ϕ^t the flow of $\vec{u}_t$ we have $\beta_t=(\phi^t)^*(\beta_0)$. Also for the curl we get $\vec{\omega}_t=(\phi^t)_*(\vec{\omega}_0)$: *for an ideal fluid, the vorticity field is transported by the velocity field*; one says that it is *frozen in*.

This theorem of Helmholtz has strong implications, in particular, since ϕ^t is a volume-preserving diffeomorphism for all t, every property of the vorticity field $\vec{\omega}_t$ that is preserved under volume-preserving diffeomorphism yields a time-independent invariant of the velocity field $\vec{u}_t$, hence of the original system. Among these properties, "$\vec{\omega}_t$ has a periodic orbit of a given knot type" is a remarkable one. Isolated periodic orbits are maybe not easy to detect, but a tubular neighborhood of a knot may also be invariant by the vector field, one then speak of a *knotted invariant tube*. "$\vec{\omega}_t$ has a knotted invariant tube of a given knot type" is then a property that is invariant under time-evolution. This is what led William Thomson (*a.k.a.* lord Kelvin) to imagine atoms as invariant vortex tubes in the ether fluid that was suppose to exist everywhere. The theory lasted several years before Thomson abandoned it himself, mostly because he could not find correspondances between the first knot tables he had and the known atoms that would reflect spectral properties of atoms (see the historical survey by D. Silver [Sil06]). However this hope led Peter G. Tait to found and develop knot theory, whose existence justifies *a posteriori* Thomson's attempt.

1.b. **Magnetohydrodynamics.** The connection between knot theory and fluid mechanics was freshened up one century after Helmholtz's discovery when Woltjer, an astrophysicist, remarked particularly stable patterns in the magnetic field of the crab nebula [Wol58].

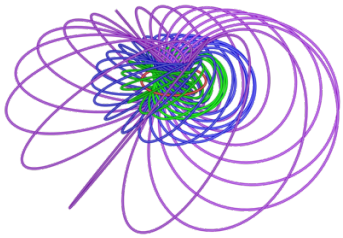
An ideal plasma is a perfectly conducting fluid. Its motion is directed by a velocity field $\vec{u}_t$ that describes the motion of particles, an electric field $\vec{E}_t$, and a magnetic field $\vec{B}_t$ that is volume-preserving. In the ideal model, the plasma is perfectly conducting. The magnetic field then satisfies $\frac{\partial\vec{B}_t}{\partial t}=\mathrm{curl}(\vec{u}\wedge\vec{B}_t)$. Working with a vector potential of $\vec{B}_t$ and using the incompressibility condition $\vec{\nabla}\cdot\vec{u}_t=0$, Woltjer derived the equation $\frac{\partial\vec{B}_t}{\partial t}+[\vec{u}_t,\vec{B}_t]=0$. This means exactly that the magnetic field $\vec{B}_t$ is frozen in the velocity field: magnetic lines can be distorted, but particles on the same magnetic line remain on the same magnetic line.

This ideal model is not accurate in general, but it is a good approximation of real phenomena in certain regimes. It fails for examples when magnetic lines are too "twisted" of "braided", in which case the magnetic lines reconnect, thus breaking Helmholtz' laws. Actually this reconnection phenomenon and the liberation of energy it induces are proposed

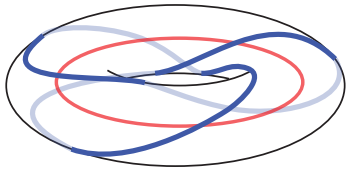
as an explanation of the huge temperature of the solar corona: while the temperature at the surface of the Sun is about 6000°K, the temperature in the corona 100 km above the surface is about 1.000.000°K, see for example [Pri99]. Note that if the invariants we are looking for in this text change under reconnection, they are likely to behave continuously, and therefore to be of interest even in this situation.

1.c. **Examples.** We now describe families of vector fields on subsets of $\mathbb{S}^3$ that are relevant and explain some of their properties. Remember that the flow of a vector field $\vec{X}$ is the one-parameter family of diffeomorphisms $(\phi^t_{\vec{X}})_{t\in\mathbb{R}}$ that describes the orbits of $\vec{X}$, namely $\frac{d}{dt}\phi^t_{\vec{X}}(p) = \vec{X}(\phi^t_{\vec{X}}(p))$. In some cases it is easier to describe the flow induced by the vector field, rather than the vector field itself.

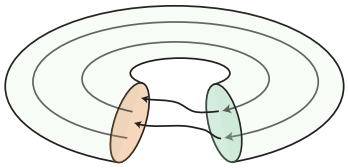
- The Hopf flow. Viewing $\mathbb{S}^3$ as the unit sphere in $\mathbb{C}^2$, the Hopf flow is defined by $\phi^t_{Hopf}(z_1, z_2) = (e^{it}z_1, e^{it}z_2)$. It preserves the volume given by the Haar measure on $\mathbb{S}^3$. All orbits are periodic of period 1. They form great circles that are pairwise linked once. The tori given by $|\frac{z_1}{z_2}| = \text{cst}$ are invariant, and the orbits are Villarceau circles on these tori. On the picture (drawn in $\mathbb{R}^3$ using stereographic projection from the point $(0, 1)$ in $\mathbb{C}^2 \simeq \mathbb{R}^4$), the circle $z_2 = 0$ corresponds to the red closed orbit. The circle $z_1 = 0$ is a vertical straight line going through the projection point.



- The Seifert flows generalize the previous example. For α, β two real parameters it is given by $\phi^t_{\alpha,\beta}(z_1, z_2) = (e^{i\alpha t}z_1, e^{i\beta t}z_2)$. It also preserves the Haar measure on $\mathbb{S}^3$. The tori $|\frac{z_1}{z_2}| = \text{cst}$ are still invariant, but the orbits now have slope β/α on each of them. When α, β are integers (or actually when their ratio is rational), the orbits are periodic of period $\text{lcm}(\alpha, \beta)$. They form (α, β)-torus knots, except the two orbits corresponding to $z_1 = 0$ and $z_2 = 0$ that are always trivial knots. The picture corresponds to the case $(\alpha, \beta) = (3, 2)$, in which orbits form trefoil knots. These examples are interestesting since torus knots are usually more easy to understand than general knots, and their invariants are more easily computed. Therefore when one wishes to understand the asymptotic behaviour of an invariant, it may help to first understand its behavior on periodic orbits of Seifert flows, that is, on torus knots.

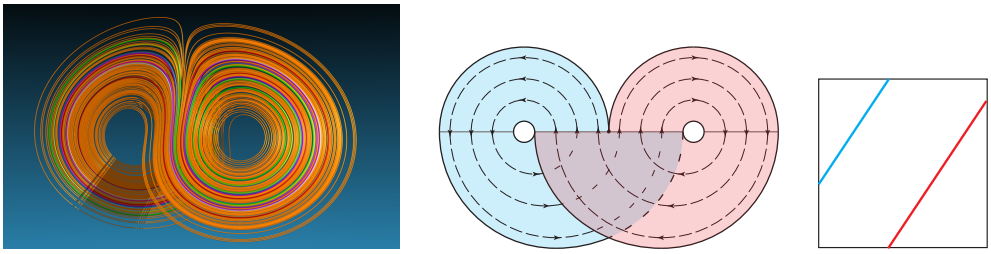


- Suspensions of automorphisms of the disc: for f an area-preserving diffeomorphism of the disc, one considers the suspension $\mathbb{D}^2 \times [0,1]/_{(p,1)\sim(f(p),0)}$ equipped with the vertical vector field. This is a topological torus and the vector field preserves the product volume. One can embed this torus into $\mathbb{S}^3$. The vector field thus obtained is not continuous at the boundary of the thorus, but one can easily extend it to a neighborhood of the embedded torus with a bump function and obtain a continuous, volume-preserving vector field.



If the torus is embedded in a knotted way, one obtains another vector field which is not the image of the previous one by an isotopy of the space. We call it a *knotted suspension*.

- The Lorenz flow [Lor63] is the flow on $\mathbb{R}^3$ that describes the solutions of the system $\dot{x} = -10x + 10y, \quad \dot{y} = 24x - y - xz, \quad \dot{z} = -8/3z + xy$. Its main feature is that it admits a strange attractor, that is, a branched surface on which orbits accumulate, keeping spiraling with a chaotic behavior (left).

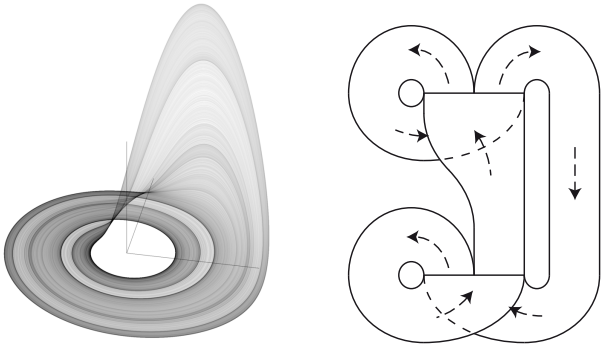


Understanding precisely the form and the dynamics of this attractor is difficult if one starts from the given equations. This is why a combinatorial model of the flow was introduced by Guckenheimer and Williams [Wil79] (center). The *geometric Lorenz attractor* is a branched surface supporting a semi-flow. It is obtained from two ribbons by gluing their extremities as shown on the picture. Identifying the gluing segment with $[0,1]$, the first return-map can be chosen of the form $x \mapsto a + b\,x \mod 1$, with two parameters $0 \leqslant a < 1 < b \leqslant 2$ (right). It is easily seen that such a map admits a dense set of periodic points. Lorenz geometric attractors hence contain infinitely many periodic orbits. These form non-trivial knots. Changing the parameters a, b changes the set of periodic orbits, yet the choice $a = 0, b = 2$ contains all the knots that appear for other parameters. These knots are called *Lorenz knots*. They are more complicated than torus knots, but still simpler than arbitrary knots (for example they are closures of positive braids, hence fibered knots), see [BW81, Ghy10, Deh11] for more on them. Hence they are good candidates for studying asymptotic behavior of knot invariants on orbits of vector fields, more complex than torus knots, but still rather well understood.

It has been proved [Tuc02] that the dynamics of the actual Lorenz equation is indeed (semi-)conjugated to the dynamics of some geometric attractor, so that

the geometric model reflects the behavior of the solutions of the Lorenz equations. The Lorenz flows (the original one or the geometric models) are dissipative and do not preserve any volume, so they are not directly eligible for our problem. However they admit invariant measures, like the Dirac linear measures whose mass is concentrated on a finite number of periodic orbits, or physical SRB-measures. The latters are obtained starting from the volume measure μ_B on an arbitrary ball B in $\mathbb{R}^3$, considering the image measure $(\phi^t)^*(\mu_B)$ obtained by pushing along the flow, and taking an accumulation point in the weak sense. Such a point is an invariant measure, called an SRB-measure. It can be thought of as "the invariant measures most compatible with volume when volume is not preserved" [You02].

Actually any differential system whose orbits do not all escape to infinity admits non-trivial invariant measures, so that the Lorenz flow is not an isolated example. Other similar examples include the Rössler flow [Ros76] (left) or the Ghrist flow [Ghr97] (right).



The latter is very interesting since it contains all knot types as periodic orbits. However when and how a given knot appears as periodic orbit of the Ghrist's attractor is still very badly understood, so that it is difficult to use this vector field for guiding the intuition. See the beautiful book [GHS97] for more examples.

What makes the Lorenz flow particularly interesting is the structure and abundance of its periodic orbits, plus the fact that their knot type is rather well understood.

In view of the previous list, one may ask: which flows are *not* eligible for our study? Almost none, since every flow on a compact manifold admits an invariant measure by the Krylov-Bogoliubov Theorem (see for example [KH95, Thm 4.1.1]). But we underlined the previous examples because the knot types of the periodic orbits are rather understood, and can serve as guiding lines.

2. Helicity and asymptotic linking number

For simplicity, we work in $\mathbb{S}^3$, although helicity can be defined for vector fields in arbitrary homology spheres, as well as on submanifolds of $\mathbb{S}^3$ with boundary, provided the vector field is tangent to the boundary.

2.a. **Woltjer-Moreau-Moffatt's helicity.** Given a volume form μ on $\mathbb{S}^3$ (for example the standard one), a vector field $\vec{u}$ induces a 2-form $\beta_{\vec{u}} = i_{\vec{u}}\mu$ according to the formula

$\beta_{\vec{u}}(\vec{y},\vec{z}) = \mu(\vec{u},\vec{y},\vec{z})$. Saying that $\vec{u}$ is μ-preserving amounts to the equation $L_{\vec{u}}\mu = 0$, where $L_{\vec{u}}$ is the Lie derivative along $\vec{u}$. By Cartan's formula $L_{\vec{u}} = i_{\vec{u}}d + di_{\vec{u}}$, we get $L_{\vec{u}}\mu = i_{\vec{u}}d\mu + di_{\vec{u}}\mu = d\beta_{\vec{u}}$. So the form $\beta_{\vec{u}}$ is closed. Since $\mathrm{H}^2(\mathbb{S}^3)$ is trivial, $\beta_{\vec{u}}$ is exact, so there exists a 1-form $\alpha_{\vec{u}}$ such that $\beta_{\vec{u}} = d\alpha_{\vec{u}}$. The 1-form $\alpha_{\vec{u}}$ is called a *form-potential* of $\vec{u}$. It is not unique, and the other form-potentials are obtained by adding a closed form.

Lemma 2.1. *The integral $\int_{\mathbb{S}^3} \alpha_{\vec{u}} \wedge d\alpha_{\vec{u}}$ is independent of the choice of the form-potential.*

Proof. For any closed-form θ, we have $(\alpha+\theta)\wedge d(\alpha+\theta) = \alpha\wedge d\alpha + \theta\wedge d\alpha = \alpha\wedge d\alpha + d(\theta\wedge\alpha)$, so that $\int_{\mathbb{S}^3}(\alpha+\theta)\wedge d(\alpha+\theta) - \alpha\wedge d\alpha = \int_{\mathbb{S}^3} d(\theta\wedge\alpha) = \int_{\partial\mathbb{S}^3}\theta\wedge\alpha$, by Stokes' formula. The latter integral is zero since $\mathbb{S}^3$ has no boundary. □

Definition 2.2. The *helicity* $\mathrm{Hel}(\vec{u},\mu)$ of $\vec{u}$ is the integral $\int_{\mathbb{S}^3}\alpha_{\vec{u}}\wedge d\alpha_{\vec{u}}$ for $\alpha_{\vec{u}}$ a form-potential of $\vec{u}$. By the previous result, it does not depend on the choice of the potential.

Note that helicity heavily depends on the choice of the invariant volume μ: different invariant volumes induce different helicities. Helicity can also be defined on domains with boundary, provided the vector field is tangent to the boundary.

Lemma 2.3. *The helicity of a μ-preserving vector field $\vec{u}$ is invariant under the action of μ-preserving diffeomorphisms.*

Proof. If f is a μ-preserving diffeomorphism, then we have $\beta_{f_*(\vec{u})} = f^*(\beta_{\vec{u}})$ and the 1-form $f^*(\alpha_{\vec{u}})$ satisfies $df^*(\alpha_{\vec{u}}) = f^*(d\alpha_{\vec{u}}) = \beta_{f_*(\vec{u})}$, so that $^*(\alpha_{\vec{u}})$ is a form-potential of $f_*(\vec{u})$, and $\int_{\mathbb{S}^3}\alpha_{f_*(\vec{u})}\wedge d\alpha_{f_*(\vec{u})} = \int_{\mathbb{S}^3}\alpha_{\vec{u}}\wedge d\alpha_{\vec{u}}$. □

Although being very concise the previous definition may look mysterious. Here is another interpretation of helicity that is important for us. It relies on the introduction of an auxiliary metric g (for example the standard metric on $\mathbb{S}^3$). The volume-preservation of $\vec{u}$ now reads $\operatorname{div}\vec{u} = 0$, that is, $\vec{\nabla}\cdot\vec{u} = 0$, and this equation implies the existence of a vector field $\vec{\omega}$ such that $\operatorname{curl}\vec{\omega} := \vec{\nabla}\times\vec{\omega} = \vec{u}$. Such a $\vec{\omega}$ is called a vector-potential of $\vec{u}$.

In this case the wedge product $\alpha_{\vec{u}}\wedge d\alpha_{\vec{u}}$ is equal to $\vec{\omega}\cdot\vec{u}$, so one gets

$$\mathrm{Hel}(\vec{u},\mathrm{vol}) = \int_{\mathbb{S}^3}\vec{\omega}\cdot\vec{u}\,d\mathrm{vol} = \int_{\mathbb{S}^3}\operatorname{curl}^{-1}(\vec{u})\cdot\vec{u}\,d\mathrm{vol}. \tag{3}$$

An important example of vector potential (on $\mathbb{R}^3$) is given by the Biot-Savard Formula: the vector field $\vec{w}$ defined by $\vec{w}(x) := \frac{1}{4\pi}\int_{\mathbb{R}^3\setminus\{x\}}\frac{\vec{u}(y)\times(x-y)}{\|x-y\|^3}dy$ satisfies $\operatorname{curl}\vec{w} = \vec{u}$. Using this potential, we then obtain a formula for the helicity of a vector field on $\mathbb{R}^3$: $\mathrm{Hel}(\vec{u},\mathrm{vol}) = \frac{1}{4\pi}\iint_{\mathbb{R}^3\times\mathbb{R}^3\setminus\mathrm{Diag}}\frac{\vec{u}(x)\cdot(\vec{u}(y)\times(x-y))}{\|x-y\|^3}dx\,dy$, where Diag denotes the set $\{(x,x)\,|\,x\in\mathbb{R}^3\}$. Since $\vec{x}\cdot(\vec{y}\times\vec{z}) = \det(\vec{x},\vec{y},\vec{z})$, we get

$$\mathrm{Hel}(\vec{u},\mathrm{vol}) = \frac{1}{4\pi}\iint_{\mathbb{R}^3\times\mathbb{R}^3\setminus\mathrm{Diag}}\frac{\det(\vec{u}(x),\vec{u}(y),x-y)}{\|x-y\|^3}dx\,dy \tag{4}$$

2.b. **Linking number.** Linking number is certainly the simplest invariant of 2-component links. For k_1, k_2 two disjoint knots in $\mathbb{R}^3$, their linking number $\mathrm{Lk}(k_1,k_2)$ admits several equivalent definitions [Rol76]:

- the number of signed crossings of the curves $\pi(k_1),\pi(k_2)$ for π a generic projection of $\mathbb{R}^3$ on a plane;

- the algebraic intersection number $\langle k_1, S_2\rangle$, where S_2 is any oriented surface whose oriented boundary coincides with k_2 (also called a *Seifert surface for* k_2);
- the degree of the Gauss map $G : \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{S}^2, (t_1, t_2) \mapsto \frac{\gamma_1(t_1)-\gamma_2(t_2)}{\|\gamma_1(t_1)-\gamma_2(t_2)\|}$, where γ_1, γ_2 are arbitrary parametrizations of the knots k_1, k_2;
- the Gauss Integral

$$\frac{1}{4\pi}\iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{\det(\dot\gamma_1(t_1), \dot\gamma_2(t_2), \gamma_2(t_2)-\gamma_1(t_1))}{\|\gamma_2(t_2)-\gamma_1(t_1)\|^3}\, dt_1\, dt_2. \tag{5}$$

The equivalence of the first, third and fourth definitions is not hard to check. Indeed the first one corresponds to counting the signed number of preimages of the north pole under the Gauss map. The fourth one amounts to compute the degree by integrating the pullback of the area form on $\mathbb{S}^2$. The equivalence with the second definition is harder to check. One option is to first check that two different surfaces induce the same intersection number, and then to prove that this number corresponds to the number of signed crossings using a particular surface (for example the one given by the Seifert Algorithm [Sei34]).

The connection of the Gauss Integral with magnetic fields goes back to Ampère. Indeed Biot-Savard Equation states that the magnetic field at x induced by a charged particle q moving with velocity $\vec u(y)$ is given (up to a multiplicative constant) by $\frac{q}{2\pi}\frac{\vec u(y)\wedge(x-y)}{\|x-y\|^3}$. Therefore the magnetic field generated at a point x by a closed loop γ_2 crossed by a constant current of intensity I is given by $\oint_{\gamma_2}\frac{I}{2\pi}\frac{\vec{dy}\wedge(x-y)}{\|x-y\|^3}$. So the circulation of the magnetic field along a closed loop γ_1 is given by $\oint_{\gamma_1}\oint_{\gamma_2}\frac{I}{2\pi}\frac{\vec{dy}\wedge(x-y)}{\|x-y\|^3}\cdot\vec{dx} = \oint_{\gamma_1}\oint_{\gamma_2}\frac{I}{2\pi}\frac{\det(\vec{dx},\vec{dy},x-y)}{\|x-y\|^3} = 2I\cdot\mathrm{Lk}(\gamma_1,\gamma_2)$, the last equality relying on Gauss Integral. This last equation generalizes Ampère's law (which corresponds to the case where γ_1 is a trivial knot).

Note that the previous definition can work for knots in $\mathbb{S}^3$: one first perturbs them so that they do not pass through the point ∞, and then considers the linking number of their stereographic projections in $\mathbb{R}^3$. One checks that an isotopy, even passing through ∞, leave the linking number invariant.

2.c. **Connection between helicity and linking number.** In his seminal paper [Mof69] Moffatt showed that for a field $\vec u$ localized on two infinitesimal tubes which are tubular neighbourhoods of two knots k_1, k_2 parametrized by $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \mathbb{R}^3$, Formula (4) for helicity takes the form $\mathrm{Hel}(\vec u) = \frac{1}{4\pi}\iint_{\mathbb{S}^1\times\mathbb{S}^1}\frac{\det(\vec u(\gamma_1(t_1)), \vec u(\gamma_2(t_2)), \gamma_2(t_2)-\gamma_1(t_1))}{\|\gamma_2(t_2)-\gamma_1(t_1)\|^3}\, dt_1\, dt_2$. Now since the knots k_1, k_2 are invariant, the vector field $\vec u$ at a given point $\gamma_i(t_i)$ is proportional to the tangent vector $\dot\gamma_i(t_i)$. Up to changing the parametrization, the proportionality factor is constant equal to the intensity I_i of the current in the corresponding knot. Comparing with Formula (5), Moffatt deduces $\mathrm{Hel}(\vec u) = \mathrm{Lk}(k_1, k_2) I_1 I_2$.

Noting that helicity is a quadratic form, Moffatt then remarks that if a field is localized on closed curves, helicity will be the sum of all pairwise linking numbers of orbits (multiplied by the corresponding intensities). This suggests that helicity is an average linking number.

In order to make Moffatt's idea more precise and to deal with the fact that orbits of flows are more likely to be open lines than closed curves, Arnol'd introduced [Arn73] a method to turn open segments of orbits into loops. The original definition was not precise enough to make the desired result true, but Vogel [Vog02] provided a correct refinement (which takes a simple form on a compact manifold with a metric, like $\mathbb{S}^3$).

Definition 2.4 (Arnol'd-Vogel)**.** Given a vector field $\vec{X}$ on $\mathbb{S}^3$, for p a point and t a positive time, the loop $k_{\vec{X}}(p,t)$ is defined as the concatenation of the segment of orbit of X starting at p and of length t with a geodesic arc connecting $\phi^t(p)$ to p. Such a loop is called an *almost periodic orbit* of $\vec{X}$.

If the short path connecting $\phi^t(p)$ to p intersects the segment of orbit or if the orbit starting at p is periodic of period less than t, then the loop $k_{\vec{X}}(p,t)$ is not embedded, but otherwise it is, and therefore defines a knot.

Theorem 2.5 (Arnol'd-Vogel)**.** *Assume that $\vec{X}$ is a vector field on $\mathbb{S}^3$ that preserves a measure μ not charging any periodic orbit. Then for μ-almost every pair of points p_1, p_2, the limit $\mathrm{Lk}^\infty_{\vec{X}}(p_1,p_2) := \lim_{t_1,t_2\to\infty} \frac{1}{t_1 t_2}\mathrm{Lk}(k_{\vec{X}}(p_1,t_1), k_{\vec{X}}(p_2,t_2))$ exists. Moreover, if $\vec{X}$ is μ-ergodic, then for almost every p_1, p_2 the limit equals $\frac{1}{\mu(\mathbb{S}^3)^2}\mathrm{Hel}(\vec{X},\mu)$.*

In other words, Lk is a $(1,1)$-asymptotic invariant which is proportional to the helicity on ergodic vector fields.

The proof is an application of the Birkhoff Ergodic Theorem. The flow $(\phi^t)_{t\in\mathbb{R}}$ of the vector field $\vec{X}$ induces a parametrization of its orbits, so that the tangent vector to an orbit at a given point coincides with the vector field $\vec{X}$ at that point. Then Gauss Integral for $\frac{1}{t_1 t_2}\mathrm{Lk}(k_{\vec{X}}(p_1,t_1), k_X(p_2,t_2))$ can be written as the sum of the integral $\frac{1}{t_1 t_2}\int_{[0,t_1]}\int_{[0,t_2]} \frac{\det(\vec{X}(\phi^{s_1}(p_1)), \vec{X}(\phi^{s_2}(p_2)), \phi^{s_2}(p_2)-\phi^{s_1}(p_1))}{\|\phi^{s_2}(p_2)-\phi^{s_1}(p_1)\|^3}\, ds_1\, ds_2$ and three other terms that depend on the geodesic arcs used to close the segments of orbits. Since $\mathbb{S}^3$ is compact, these additional terms are of the order of $|t_1| + |t_2|$, and in particular are negligible compared to $t_1 t_2$.

The above integral is a time-average. In order to apply the ergodic theorem, one needs to check that the function $(x,y) \mapsto \frac{\det(\vec{X}(x),\vec{X}(y),y-x)}{\|y-x\|^3}$ is integrable on $\mathbb{S}^3\times\mathbb{S}^3\setminus\mathrm{Diag}$. It is indeed the case (this is a non-trivial fact). Birkhoff's ergodic Theorem then implies that when t_1, t_2 tend to infinity, for almost every p_1, p_2 the time-average converges to an integrable function, and the space-average of this function equals $\int_{\mathbb{S}^3}\int_{\mathbb{S}^3} \frac{\det(\vec{X}(x),\vec{X}(y),y-x)}{\|y-x\|^3}\, dx\, dy$.

Example 2.6. Let us estimate the helicity of the flows given in Section 1.c.

For the Hopf flow, every orbit is periodic of period 1 and two orbits have linking +1. Therefore the function $\mathrm{Lk}^\infty_{\vec{X}_{\mathrm{Hopf}}}$ is constant equal to 1, so that $\mathrm{Hel}(\vec{X}_{\mathrm{Hopf}}) = \mu(\mathbb{S}^3) = 1$.

For the Seifert flow if α, β are two coprime integers, then the orbits of $\vec{X}_{\alpha,\beta}$ are periodic of period 1, and their linking number is also equal to $\alpha\beta$. Therefore the function $\mathrm{Lk}^{\infty}_{\vec{X}_{\alpha\beta}}$ is also constant equal to $\alpha\beta$, so that $\mathrm{Hel}(\vec{X}_{\alpha,\beta}) = \alpha\beta$. By continuity, the same holds for arbitrary α, β.

For the suspension of a diffeomorphism of the disc embedded in a trivial way in $\mathbb{S}^3$, the interpretation of helicity as average linking number implies that the helicity of a suspension equals the average rotation of the corresponding diffeomorphism of the disc. It was proven by Fathi [Fat80] that such an average rotation is given by the so-called Calabi invariant [Cal70].

For the Lorenz flow, it is not easy to get an exact value for helicity for arbitrary invariant measures. However every pair of orbits has negative linking number, so that for all invariant measures, the helicity is negative.

3. Asymptotic invariants proportional to helicity

We now give examples of knot invariants (ω-signatures, Vassiliev invariants) for which Arnol'd's scheme works, meaning that an asymptotic on long pieces of orbits of vector fields exists. These invariants are among the most common knot invariants and form a very rich family. For example the classical signature is among the simplest invariants that distinguish the left-handed trefoil, the right-handed trefoil and the figure-eight knot. Also Vassiliev invariants are conjecturally total invariants: any two knots are presumably distinguished by some Vassiliev invariant.

However we will see that this richness is not fully preserved when taking the asymptotics. Indeed all constructed vector field invariants turn out to be proportional for ergodic vector fields (remember that a vector field $\vec{X}$ is *ergodic* with respect to a probability measure μ if every $\vec{X}$-invariant set has measure 0 or 1). So these invariants do not give solutions to Problem B.

Still, let us underline that in Theorem 3.1, 3.2, 3.4, and 3.5, proportionality is known only when the measure is a volume-measure and the vector field is ergodic for this measure. For example if the considered invariant measure is supported on only one periodic orbit, then the asymptotic invariants we consider exist and are equal to their standard counterpart for the knot formed by the periodic orbit. It is an interesting question to understand what happens on SRB-measures (as for example those for the Lorenz flow): does proportionality to helicity also hold in this case?

3.a. Signatures, linear saddle invariants, and Gambaudo-Ghys' approach.

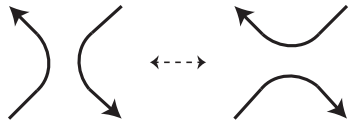
The signature σ is a classical knot invariant introduced by Trotter [Tro62]. It was later generalized by Tristram and Levine [Tri69, Lev69] into a one-parameter family σ_ω for $\omega \in \mathbb{S}^1$. These are among the simplest invariants to compute. Their definition relies on the introduction of a Seifert surface for the knot, but the invariants do not depend on the choice of this surface. Namely for S an orientable surface whose boundary coincides with a knot K, one can consider the *Seifert bilinear form* s on $\mathrm{H}_1(S, \mathbb{R})$ defined by $s([x], [y]) = \mathrm{Lk}(x, y^+)$, where x, y are arbitrary curves representing the respective homology classes and y^+ denotes the curve y pushed a bit off the surface in the positive normal direction. This linking number does not depend on the choice of the representative x, y, and one checks that the

form s is bilinear. The ω-signature $\sigma_\omega(K)$ is then defined as the signature of the hermitian form $(1-\omega)s + (1-\bar{\omega})^t s$. The classical signature corresponds to the case $\omega = -1$.

The ω-signature of a knot behaves rather nicely when one varies ω in the sense that it is a piecewise constant function that jumps only at the roots of the Alexander polynomial of the knot, and by a term at most twice the multiplicity. However the precise shape may be complicated and surprising. The result generalizing Arnol'd Theorem to signatures is due Gambaudo and Ghys (see the introduction for the notation $\sigma^{(2)}$).

Theorem 3.1. [GG01] *Let $\vec{X}$ be a C^∞ volume-preserving vector field on $\mathbb{S}^3$. (a) For every $\omega = e^{2i\pi\theta}$, the ω-signature is an asymptotic vector field invariant of order (2). (b) If $\vec{X}$ is ergodic, we have $\sigma^{(2)}_\omega(\vec{X}) = 2\theta(1-\theta)\cdot \mathrm{Hel}(\vec{X})$.*

This theorem has then been generalized to other knot invariants [Baa07, Baa12] with little variation in the scheme of the proof. In order to explain this common scheme, we follow Baader and introduce the common feature of the considered invariants. A *saddle point move* is the local operation on links depicted below:



A real-valued link invariant τ is a *linear saddle invariant* if it is additive under disjoint union of links: $\tau(\ell_1 \cup \ell_2) = \tau(\ell_1) + \tau(\ell_2)$, and if for two oriented links ℓ_1, ℓ_2 that are related by a saddle point move one has $|\tau(\ell_1) - \tau(\ell_2)| \leqslant C$, where $C > 0$ is a constant not depending on ℓ_1, ℓ_2. The ω-signatures are examples of linear saddle invariants, but there are others, as for example Rasmussen s-invariant or certain concordance invariants.

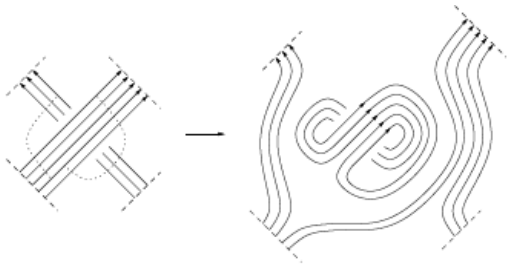
Theorem 3.2 ([Baa12])**.** *Let $\vec{X}$ be a C^∞ volume-preserving vector field on $\mathbb{S}^3$. (a) Every linear saddle invariant τ is an asymptotic vector field invariant of order (2). (b) If $\vec{X}$ is ergodic we have $\tau^{(2)}(\vec{X}) = C_\tau \cdot \mathrm{Hel}(\vec{X})$, where C_τ is an explicit constant independent of $\vec{X}$.*

The proof goes along two main steps. The first one is due to Gambaudo-Ghys and yields some *normal projections* for vector fields. The second in this context is due to Baader and consists in showing that linear saddle invariants behave well with respect to these normal projections. Let $\vec{X}$ be a vector field on $\mathbb{S}^3$ and τ a linear saddle invariant.

Firstly recall that a flow box for $\vec{X}$ is a submanifold of the form $D \times [0,1]$ for D a disc such that the vector field $\vec{X}$ is tangent to the direction given by the second coordinate. The general result proved by Gambaudo and Ghys [GG01] allows to decompose a large portion (say $1-\varepsilon_1$) of $\mathbb{S}^3$ into finitely many flow boxes $(B_i^{\vec{X}})_{i=1,\dots,n}$ that project well on a given plane in the sense that two boxes either do not overlap at all or they overlap transversally. Such projections are called *normal projections*. The combinatorics of the overlappings are recorded by a matrix $(e_{i,j})_{1\leqslant i,j\leqslant n}$ with $e_{i,j} = 0$ when the corresponding boxes do not overlap and $e_{i,j} = \pm 1$ when they do, the sign depending on whether the boxes overlap positively or negatively.

Secondly observe that, for $k_{\vec{X}}(p,t)$ an almost periodic orbit, the value $\tau(k_{\vec{X}}(p,t))$ depends mostly on its intersection with the boxes $B_1, \dots, B_n$. Indeed suppose that $k_{\vec{X}}(p,t)$ crosses m_i times B_i. Using at most $n(m_1 + \cdots + m_n)$ linear saddle point moves, one can

transform $k_{\vec{X}}(p,t)$ into the disjoint union of several torus links of type $T(m_i, e_{i,j}m_j)$, one for every pair of overlapping flow boxes (B_i, B_j), plus some remaining links depending on the portion of $\vec{X}$ that does not visit $B_1, \dots, B_n$.



Since a piece of orbit that visits a flow box stays in the box for a time that is bounded from above and from below, the number $m_1 + \cdots + m_n$ is linear in the length of $k_{\vec{X}}(p,t)$, and so the number of saddle point moves involved is also linear. Since we will prove that $\tau(k_{\vec{X}}(p,t))$ is quadratic in t, this linear number of saddle moves does not really count: up to another factor ε_2, the value $\tau(k_{\vec{X}}(p,t))$ is roughly equal to $\sum_{i,j} e_{i,j}\tau(T(m_i,m_j))$.

It remains to evaluate this last expression. Using the same properties of τ, for (p,q)-torus knots, the function $(p,q) \mapsto \tau(T(p,q))$ is almost-additive. Standard arguments imply that is equal to $C_\tau \cdot pq$, up to a factor ϵ_3, for some constant C_τ that only depends on τ.

Putting all of this together, we get that $\tau(k_{\vec{X}}(p,t))$ is equal to $C_\tau \sum_{i,j} e_{i,j}m_i m_j$ (up to a factor $\varepsilon_1+\varepsilon_2+\varepsilon_3$). Now the ergodic theorem implies that for almost every starting point p the number m_i of visits of $k_{\vec{X}}(p,t)$ in B_i is asymptotic to $q_i(p)\cdot t$, where $q_i(p)$ is an average frequency that depends of p, so that $\frac{1}{t^2}\tau(k_{\vec{X}}(p,t))$ is approximately $C_\tau \sum_{i,j} e_{i,j}q_i(p)q_j(p)$. This proves part (a) of the theorem.

For part (b), if $\vec{X}$ is ergodic, then the function q_i is almost surely constant. Since $\sum_{i,j} e_{i,j}q_iq_j$ actually computes the asymptotic linking number, we get $\tau^{(2)}(\vec{X}) = C_\tau \cdot \mathrm{Hel}(\vec{X})$.

3.b. **Vassiliev invariants and configuration integrals.** Vassiliev invariants are powerful invariants that conjecturally distinguish all knots (namely if k_1, k_2 are not isotopic, it is conjectured that there exists a Vassiliev invariant v such that $v(k_1) \neq v(k_2)$). A way to present them relies on chord diagrams [GPV00].

A *chord diagram* is a finite set of chords in a disc, each equipped with a sign and an orientation. A *Gauss diagram* for a knot k is of the same type: one starts from a planar projection $\pi(k)$ with d double points. For every double point of $\pi(k)$ we add to k a vertical arc that connects the two points of k that project to the double point. We orient this arc from top to bottom and we label it with a sign according to whether the crossing is positive or negative. This transforms k into a knotted graph, but we only keep its abstract structure and forget about the embedding, thus having a circle with d oriented and signed chords: the Gauss diagram of the projection. Of course, different projections yield different diagrams.

If D is a chord diagram and $\pi(k)$ a knot diagram of a knot k, the pairing $\langle D, \pi(k)\rangle$ is the signed number of appearances of D as a sub diagram of the Gauss diagram associated

to $\pi(k)$. In general $\langle D, \pi(k)\rangle$ depends on the diagram, so that $\langle D, \cdot\rangle$ is not a knot invariant. However by combining several diagrams one can obtain the invariance.

Theorem 3.3. [GPV00] *For every Vassiliev invariant v there exist chord diagrams $D_1, \dots, D_n$ and reals $c_1, \dots, c_n$ such that for every knot k and every diagram $\pi(k)$ of k one has $v(k) = \sum c_m \langle D_m, \pi(k)\rangle$.*

This approach allows Baader and Marché to prove that Vassiliev invariants also have an asymptotic behavior.

Theorem 3.4. [BM12] *Let $\vec{X}$ be a C^∞ volume-preserving vector field on $\mathbb{S}^3$. (a) Every Vassiliev knot invariant of order n is an asymptotic vector field invariant of order $(2n)$. (b) If $\vec{X}$ is ergodic we have $v^{(2n)}(\vec{X}) = C_v \cdot \mathrm{Hel}(\vec{X})^n$, where C_v is an explicit constant independent of $\vec{X}$.*

The original proof is once again a variation on Arnol'd-Gambaudo-Ghys' approach. It amounts to showing that for every diagram D and for a normal projection of $\vec{X}$ given by Gambaudo and Ghys [GG01], there is an asymptotic for $\langle D_i, \pi(k_{\vec{X}}(p,t))\rangle$. For this, given a chord diagram D_i, one divides the knot $k_{\vec{X}}(p,t)$ into $N >> n$ equal parts $s_1, \dots, s_N$. An apparition of the chord diagram D_i in $k_{\vec{X}}(p,t)$ corresponds to n crossing points, hence $2n$ times $t_1, \dots, t_{2n}$. Up to a small error, one can assume that these $2n$ points appear in $2n$ different segments of $s_1, \dots, s_N$, so that the corresponding times t_i are roughly independent. Once again, the probability of seeing a crossing is $\sum_{i,j} e_{i,j} q_i(p) q_j(p)$ for the same constants $q_i(p)$ as in the proof of Theorem 3.1. Therefore every term $\langle D_i, \pi(k_{\vec{X}}(p,t))\rangle$ is approximately $(\sum_{i,j} e_{i,j} q_i(p) q_j(p))^n$. By applying Birkhoff's ergodic theorem, this term is asymptotic to t^{2n} for almost every p, and the space average for ergodic vector fields is then $v^{(2n)}(\vec{X}) = (\sum c_m)(\sum_{i,j} e_{i,j} p_i p_j)^n$.

The proofs that we sketched of Theorems 3.1, 3.2, and 3.4 may look rather technical and combinatorial compared to Arnol'd's proof of the asymptotic character of linking number. It is then natural to wonder whether there are more direct or more intuitive proofs of these results. This is indeed the case for Theorem 3.4 which has been given a proof relying on configuration space integrals by Komendarczyk and Volić [KV13]. This new proof does not rely on the decomposition into flow boxes that looked superfluous. However there are still many technical difficulties. The main point is that there exist integral formulas for Vassiliev invariants that generalize Gauss Integral, so that one can directly prove that every term $\langle D_m, \pi(k_{\vec{X}}(p,t))\rangle$ is asymptotic to a term of order $2n$, without using Gambaud-Ghys' normal projections.

This new proof has one advantage and one disadvantage. On the negative side, it is less explicit for the value of the proportionality factor C_v. On the positive side, it shows that if $v^{(2n)}(\vec{X}) = 0$, then there exists a lower-order asymptotic invariant, namely v is an order $2n-2$-asymptotic invariant. Also there is an induction: if $v^{(2n)}(\vec{X}) = v^{(2n-2)}(\vec{X}) = \cdots = v^{(2k+2)}(\vec{X}) = 0$, then v is an asymptotic vector field invariant of order $2k$. These lower order terms have no interpretation yet.

3.c. **Akhmetev's quadratic helicities.** In an attempt to define variations on the theme of helicity, one can use the intermediate step given by Arnol'd [Arn73]. Indeed Theorem 2.5

shows that for $\vec{X}$ a volume-preserving vector field, for almost-every pair of points (p_1, p_2), the limit $\mathrm{Lk}^\infty_{\vec{X}}(p_1,p_2) := \lim_{t_1,t_2\to\infty} \frac{1}{t_1 t_2}\mathrm{Lk}(k_{\vec{X}}(p_1,t_1), k_{\vec{X}}(p_2,t_2))$ exists. Helicity is defined as the integral of this function, and in case $\vec{X}$ is ergodic, $\mathrm{Lk}^\infty_{\vec{X}}$ is almost-surely constant.

Now if $\vec{X}$ is not ergodic, one can play with $\mathrm{Lk}^\infty_{\vec{X}}$ and wonder when we obtain other invariants. This was done by Akhmetev in at least two cases.

Theorem 3.5. [Akh12] *For $\vec{X}$ a volume-preserving vector field, the functions $(p_1,p_2)\mapsto \mathrm{Lk}^\infty_{\vec{X}}(p_1,p_2)^2$ and $(p_1,p_2,p_3)\mapsto \mathrm{Lk}^\infty_{\vec{X}}(p_1,p_2)\mathrm{Lk}^\infty_{\vec{X}}(p_1,p_3)$ are integrable and their integrals are invariant under volume-preserving diffeomorphism.*

The corresponding invariants are called *quadratic helicities* and denoted by $\mathrm{Hel}^{(2)}$ and $\mathrm{Hel}^{[2]}$ respectively. They are asymptotic invariants of order $(2,2)$ and $(2,1,1)$ respectively. Let us underline once again that if $\vec{X}$ is ergodic, then the function $(p_1,p_2)\mapsto \mathrm{Lk}^\infty_{\vec{X}}(p_1,p_2)$ is almost constant, so these quadratic helicities are just the square of the standard helicity. Their interest is then for non-ergodic flows. Of course it is easy to play with other combinations of higher degree, but it is then not obvious to decide when the obtained quantity is integrable and whether it is invariant under diffeomorphism.

3.d. **Helicity is the only C^1-invariant.** Looking at Theorems 3.1, 3.2, 3.4 and 3.5 one may wonder whether there exists any asymptotic invariant not proportional to helicity on ergodic vector fields. As we will see in the next section, these indeed exist. However we mention here a series of recent results that partly explains why it is not so easy to construct invariants different from helicity.

As shown by Lemma 2.1, the helicity of a field $\vec{X}$ may be defined by integrated the 3-form $\alpha_{\vec{X}}\wedge d\alpha_{\vec{X}}$ on the whole manifold. If $\vec{X}$ varies continuously, so do $\alpha_{\vec{X}}$ and $d\alpha_{\vec{X}}$, implying that the helicity varies continuously. The invariants we are looking for are functional on the space of volume-preserving vector fields. The natural notion of differentiability for such functionals is the *Fréchet derivative*. We then denote by $\mathcal{X}$ the set of C^1-volume preserving vector field on $\mathbb{S}^3$, with the natural C^1-topology. According with [Kud14, Kud15, EPT16], a function $I:\mathcal{X}\to\mathbb{R}$ is a *regular integral invariant* if it is invariant under volume-preserving diffeomorphisms and if the Fréchet derivative is obtained by integrating a continuous kernel K, namely $D_{\vec{X}}I(\vec{Y}) = \int_{\mathbb{S}^3} K(\vec{X})\cdot\vec{Y}$. Kudryavtseva on the one hand, and Encisco, Peralta-Salas and Torres de Lizaur on the other hand proved two local and global versions of the rough following statement

Theorem 3.6. [Kud15, EPT16] *Every regular integral invariant is a C^1-function of helicity.*

This result is remarkable and gives a satisfactory explanation why helicity appears that often. Let us however underline that it does not prevent the existence of invariant that are not functions of helicity. But it implies that such invariant cannot be too smooth. This is the case of the invariants we present in the next section.

4. Asymptotic invariants different from helicity

In this section we construct invariants that differ from helicity on ergodic vector fields. These constructions seem much more particular than those of the previous section in

the sense that we construct a few invariants, case by case, rather than obtaining infinite families as one could hope.

4.a. **Higher helicities.** The first way to generalize helicity was already suggested by Arnol'd and Khesin [AK98]. A first step generalization of linking number is Milnor's $\bar{\mu}$-invariant for 3-component links. For $k_1 \cup k_2 \cup k_3$ a link, $\bar{\mu}(k_1 \cup k_2 \cup k_3)$ is an element of $\mathbb{Z}/\gcd(\mathrm{Lk}(k_1,k_2),\mathrm{Lk}(k_2,k_3),\mathrm{Lk}(k_3,k_1))\mathbb{Z}$. In case all pairwise linking numbers are zero, we then obtain a well-defined integer. The similarity with linking number goes a bit further since there exist integral formulas for $\bar{\mu}$, called Massey products [Mas57]. We do not write them here since they are rather complicated, but their existence is important.

The corresponding scenario for vector fields would then require linking of arbitrary orbits to be zero. A particular case of this situation was studied by Komendarczyk:

Theorem 4.1 ([Kom09, Kom10])**.** *Let B_1, B_2, B_3 be three handlebodies in $\mathbb{S}^3$ or $\mathbb{R}^3$ supporting measure-preserving vector fields $\vec{X}_1, \vec{X}_2, \vec{X}_3$, such that the pairwise linking numbers of arbitrary pairs of orbits of $\vec{X}_i, \vec{X}_j$ with $i \neq j$ is zero. Then the integral over $B_1 \times B_2 \times B_3$ of the Massey product evaluated on $(\vec{X}_1, \vec{X}_2, \vec{X}_3)$ converges and is invariant under volume-preserving diffeomorphism.*

4.b. **Asymptotic crossing number.** Crossing number is one of the oldest knot invariants, but it is hard to compute in general. For k a knot in $\mathbb{R}^3$ and $\pi(k)$ a diagram of k (that is, a projection on a plane), $\mathrm{cr}(\pi(k))$ is defined as the number of double points of $\pi(k)$. The crossing number $\mathrm{Cr}(k)$ is then the minimum of $\mathrm{cr}(\pi(k))$ over all diagrams of k. In other words $\mathrm{cr}(\pi(k))$ is the number of preimages of the north pole under the Gauss map $G : \mathbb{S}^1 \times \mathbb{S}^1 \setminus \mathrm{Diag} \to \mathbb{S}^2, (t_1, t_2) \mapsto \frac{\gamma(t_1)-\gamma(t_2)}{\|\gamma(t_1)-\gamma(t_2)\|}$, where γ is an arbitrary parametrization of k that projects onto $\pi(k)$.

The difference with linking number is that $\mathbb{S}^1 \times \mathbb{S}^1 \setminus \mathrm{Diag}$ is not a closed surface, so that G does not have a well-defined degree. $\mathrm{Cr}(k)$ is then defined as the minimal number of preimages of the north pole under G over all projections of k.

A variant can be obtained by counting the average number of preimages of points (not only of the north pole). This number can be computed by integrating the pull-back of the area form on $\mathbb{S}^2$ by G, yielding

$$\mathrm{cr}_{\mathrm{av}}(\pi(k)) := \frac{1}{4\pi} \iint_{\mathbb{S}^1\times\mathbb{S}^1\setminus\mathrm{Diag}} \frac{|\det(\dot{\gamma}(t_1), \dot{\gamma}(t_2), \gamma(t_2)-\gamma(t_1))|}{\|\gamma(t_2)-\gamma(t_1)\|^3}\, dt_1\, dt_2.$$

This number is not necessarily an integer. One then defines $\mathrm{Cr}_{\mathrm{av}}(k)$ as the infimum of $\mathrm{cr}_{\mathrm{av}}(\pi(k))$ over all projections of k.

Now this definition can be copied for arbitrary volume-preserving vector fields

$$\mathrm{cr}^{\infty}_{\mathrm{av}}(\vec{X}) := \frac{1}{4\pi} \iint_{\mathbb{S}^3\times\mathbb{S}^3\setminus\mathrm{Diag}} \frac{|\det(\vec{X}(p_1), \vec{X}(p_2), p_2-p_1)|}{\|p_2-p_1\|^3}\, dp_1\, dp_2.$$

However it is not invariant under volume-preserving diffeomorphism and one has to minimize once again in order to get an invariant

$$\mathrm{Cr}^{\infty}_{\mathrm{av}}(\vec{X}) := \min_{\phi \text{ vol. pres.}} \mathrm{cr}^{\infty}_{\mathrm{av}}(\phi^*\vec{X}).$$

Taking the infimum over all volume-preserving diffeomorphisms may look like cheating in view of obtaining an invariant. However this invariant has some good properties. For $p \geqslant 1$, the L^p-energy of a vector field on a compact domain of $\mathbb{R}^3$ is the integral of its L^p-norm: $E_p(\vec{X}) := \int \|\vec{X}\|^p d\mathrm{vol}$. It is not invariant under volume-preserving diffeomorphism. In particular, in physical applications where the considered vector field is transported by volume-preserving diffeomorphisms, the energy is likely to decrease, but one wonders whether it can tend to 0. A property of helicity that we did not mention is that it yields a lower bound on the L^2-energy. Therefore a non-zero helicity implies that the energy cannot tend to 0 (see [AK98]). Asymptotic crossing number yields similar and better bounds:

Theorem 4.2 ([FH91])**.** *For $\vec{X}$ a volume-preserving vector field, its $3/2$-energy is bounded by $E_{3/2}(\vec{X}) \geqslant \left(\frac{16}{\pi}\right)^{1/4} \mathrm{Cr}^{\infty}_{\mathrm{av}}(\vec{X})^{3/4}$.*

4.c. **Asymptotic trunk.** Thin position for knots is a concept introduced by Gabai for solving the R-conjecture [Gab87], which states that if the 0-surgery manifold of a knot k in $\mathbb{S}^3$ is homeomorphic to $\mathbb{S}^1 \times \mathbb{S}^2$, then k is the unknot. Roughly, thin position corresponds to an embedding of a knot that is *as vertical as possible*. It yields several knot invariants— waist, width, trunk— that were formally defined and studied first by Ozawa [Oza10]. Trunk translates well to vector fields and its asymptotic character is not hard to prove. It is very close in spirit with braid index or bridge number (whose asymptotic character is still unknown).

A height function on $\mathbb{S}^3$ is a Morse function with only two critical points (one maximum and one minimum). A curve k is in Morse position with respect to a Morse function h if $h|_k$ is also a Morse function. In this case $h^{-1}(t) \cap k$ consists of finitely many points for every t, and the trunk of k with respect to h is defined as the maximum of this number over all levels of h: $\mathrm{tr}(k,h) := \max_t \sharp\{h^{-1}(t) \cap k\}$. Of course this maximum is not invariant under isotopy, but allowing h to change over height functions does, and one obtains the definition of the (knot-)trunk of k:

$$\mathrm{Tr}(k) := \min_{h \text{ height}} \max_t \sharp\{h^{-1}(t) \cap k\}.$$

Theorem 4.3 ([DR15])**.** *Trunk is an asymptotic invariant of order (1), and there is no function $f : \mathbb{R} \to \mathbb{R}$ such that for every ergodic μ-preserving vector field $\vec{X}$, the value $\mathrm{Tr}^{(1)}(\vec{X}, \mu)$ is given by $f(\mathrm{Hel}(\vec{X}, \mu))$.*

The point for proving this theorem is that one can actually define directly an analog of the trunk for vector fields, and check that is coincide with the asymptotic invariant. Namely in the definition of the knot-trunk we replace the number of intersection points of the knot with a surface by the absolute value of the *flux* of the vector field through the surface, thus defining $\vec{\mathrm{tr}}(\vec{X}, \mu, h) := \max_t \int_{h^{-1}(t)} |i_{\vec{X}}\mu|$, and

$$\vec{\mathrm{Tr}}(\vec{X}, \mu) := \inf_{h \text{ height}} \max_t \int_{h^{-1}(t)} |i_{\vec{X}}\mu|.$$

The key property for proving Theorem 4.3 is to check that $\vec{\mathrm{Tr}}(\vec{X}, \mu)$ is continuous with respect to both $\vec{X}$ and μ (but its not regular integral, for otherwise it would contradict

Theorem 3.6). This implies that if a long piece of orbit approximates μ, the knot-trunk of this orbit approximates the trunk of the vector field. It is enough to compute trunk and helicity of several examples to check independence. Actually, one checks that the trunk of the Seifert flow $\vec{X}_{\alpha,\beta}$ with respect to the standard volume on $\mathbb{S}^3$ is $\min(\alpha,\beta)$. Recall from Example 2.6 that the helicity of $\vec{X}_{\alpha,\beta}$ is $\alpha\beta$. The Seifert flow is not ergodic, but it can be approximated by ergodic flows. Since there is no function f such that $\min(\alpha,\beta) = f(\alpha\beta)$, the statement follows.

5. Questions

We finish with some speculations on how to construct new vector field invariants.

5.a. **Higher helicities.** This direction seems to be the most promising one. On one hand, Theorem 4.1 ensures that Milnor's invariant for 3-component links has a vector field analog in the restricted case where it is computed on a union of three domains that ensure that pairwise linking numbers are all zero. A less restricted case would correspond to a vector field defined on one single domain.

Question 5.1. Can one generalize Komendarczyk Theorem 4.1 to any vector field $\vec{X}$ such that $\mathrm{Lk}^{\infty}_{\vec{X}}(p_1,p_2) = 0$ for almost every p_1, p_2? In particular if $\vec{X}$ is ergodic and $\mathrm{Hel}(\vec{X}) = 0$?

Another direction is given by Komendarczyk-Volić's proof that Vassiliev invariants are asymptotic invariants. In particular they show that when helicity vanishes, then an order n Vassiliev invariant yields an order $2n-2$ asymptotic invariant.

Question 5.2. For $\vec{X}$ a vector field with $\mathrm{Hel}(\vec{X}) = 0$ and v an order n Vassiliev invariant, is there an interpretation for $v^{(2n-2)}(\vec{X})$? Are all these invariants proportional? Are they related to Milnor's invariant for 3-component links?

Of course there is no reason to restrict to triple linking numbers.

Question 5.3. Are there asymptotics for higher order Milnor's invariants? Are they related to lower order asymptotics of Vassiliev invariants when higher order asymptotics vanish?

5.b. **Order 2 invariants.** One of the easiest knot invariants to define is the genus (also called 3-genus). For K a knot, $g(K)$ is defined as the minimal genus over all Seifert surfaces for K. Unfortunately this invariant is rather difficult to compute. There are lower bounds given by inequalities $\sigma_\omega(K) \leqslant \deg(\Delta_K) \leqslant 2g(K)$ where Δ_K denotes the Alexander polynomial of K. Also there are upper bounds given by explicit constructions of Seifert surfaces (which may well not minimize genus), as for example the one given by applying Seifert's algorithm. In general these bounds do not match exactly, but for $\vec{X}$ a vector field in Gambaudo-Ghys' normal form, lower and upper bounds estimated on $k_{\vec{X}}(p,t)$ grow both quadratically with t (although not at the same rate).

Question 5.4. Is genus an order 2 asymptotic invariant? Is the degree of the Alexander polynomial an order 2 invariant?

Note that Baader showed that if one replaces 3-genus by slice-genus (a smaller 4-dimensional cousin), then the answer is yes, but $g^{(2)}_{\text{slice}}$ is then equal to $|\text{Hel}|$ on ergodic vector fields.

Also when there exists a Gambaudo-Ghys' normal projection that exhibits only positive crossings (as for example for Seifert flows or for the Lorenz flow), the lower and upper bounds are asymptotically the same for all invariant measures, so that Question 5.4 has two positive answers, but the obtained invariants once again equal helicity.

An interesting example is given by Ghrist' flow [GHS97], see the end of Section 1.c. It admits many invariant measures, and for many of them helicity vanishes. Some numerical computations done by the author suggest that $\deg(\Delta_{k_{\vec{X}_{\text{Ghrist}}}(p,t)})$ has a non-trivial quadratic asymptotic behavior in this case, suggesting a positive answer to Question 5.4.

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