Exercises

Sylvain Courte Stéphane Guillermou

1 Hamilton's equations, differential calculus in $\mathbf{R}^n$

Exercise 1. For $0 < r < R$ and $n \geq 2$, find a volume preserving diffeomorphism of $\mathbf{R}^{2n}$ which maps $B^{2n}(R)$ into $B^2(r) \times \mathbf{R}^{2n-2}$.

Exercise 2. For $a_1, \dots, a_n > 0$ consider $H = \sum_j a_j(p_j^2 + q_j^2)$. Compute the associated Hamiltonian vector field and find the periodic solutions on a given level set $H^{-1}(c)$.

Exercise 3. Let g be a Riemannian metric on $\mathbf{R}^n$ given in coordinates $(q_1, \dots, q_n)$ by a symmetric matrix $(g_{ij}(q))_{ij}$. We define a Hamiltonian function H on $\mathbf{R}^{2n}$ with coordinates $(p_1, \dots p_n, q_1, \dots, q_n)$ by $H(p,q) = ||p||^2 = \sum_{ij} g_{ij}(q) p_i p_j$. Let $(p(t), q(t))$ be a flow line of X_H. Prove that

$$\ddot{q}_k = -\sum_{ij} \Gamma_{ij}^k \, \dot{q}_i \dot{q}_j,$$

where

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^{n} g^{kl} \left(\frac{\partial g_{lj}}{\partial q_i} + \frac{\partial g_{il}}{\partial q_j} - \frac{\partial g_{ij}}{\partial q_l} \right)$$

are the *Christoffel symbols* of the metric and $(g^{kl}(q))_{kl}$ denotes the inverse of the matrix $(g_{ij}(q))_{ij}$. (This equation gives the geodesics of the metric.)

Exercise 4. We use the same notations as in Exercice 3 and define another Hamiltonian function $H_1(p,q) = \|p\| = H(p,q)^{\frac{1}{2}}$ on the open subset $\{p \neq 0\}$. We introduce the hypersurface $S = \{||p|| = 1\}$ of $\mathbf{R}^{2n}$. We let $\mathbf{R}^*$ act on $\mathbf{R}^{2n}$ by multiplication on the variables p, that is, $\lambda \cdot (p,q) = (\lambda p_1, \dots \lambda p_n, q_1, \dots, q_n)$. Prove that $\varphi_{H_1}^t(\lambda \cdot (p,q)) = \lambda \cdot \varphi_{H_1}^t(p,q)$, for $\lambda > 0$, and that $\varphi_{H_1}^{2t}|_S = \varphi_H^t|_S$. Compute $\varphi_H^t(p,q)$ and $\varphi_{H_1}^t(p,q)$ in the case of the standard metric ($||p||^2 = \sum_i p_i^2$).

Exercise 5. Check that the exterior product $\wedge$ is associative and graded commutative.

Exercise 6. Check the formulas

$$\iota_v(\alpha \wedge \beta) = \iota_v \alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge \iota_v \beta,$$

$$\iota_v(a^* \alpha) = a^*(\iota_{a(v)} \alpha).$$

Exercise 7. Let E be a real vector space of dimension n. An element of $\bigwedge^k E^*$ is called *decomposable* if it is the exterior product of k elements of $\bigwedge^1 E^* = E^*$.

1. Show that all degree n or $n-1$ forms are decomposable.
2. For $\alpha \in E^*$ and $\omega \in \bigwedge^k E^*$, show that $\alpha \wedge \omega = 0$ if and only if $\omega = \alpha \wedge \beta$ for some $\beta \in \bigwedge^{k-1} E^*$.
3. If $\alpha, \beta, \gamma, \delta \in E^*$ are linearly independent, show that $\alpha \wedge \beta + \gamma \wedge \delta$ is not decomposable.

Exercise 8. Using the definition of the exterior derivative of a k-form α given by the formula:

$$(d\alpha)_p(v_0, \dots, v_k) = \lim_{t \to 0} \frac{1}{t} \sum_i (-1)^i (\alpha_{p+tv_i}(v_0, \dots, \widehat{v_i}, \dots, v_k) - \alpha_p(v_0, \dots, \widehat{v_i}, \dots, v_k)),$$

check the properties:

- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$,
- $d(\alpha + \beta) = d\alpha + d\beta$,
- $d^2 = 0$.

Exercise 9. For a smooth map $\varphi : U \to V$ and a k-form α on V, show that $d(\varphi^* \alpha) = \varphi^*(d\alpha)$.

Exercise 10. Let M be a manifold and $\Omega^*(M)$ the algebra of differential forms.

For a vector field X and a diffeomorphism $\psi : M \to M$, prove the following properties involving the Lie derivative $\mathcal{L}_X$, the contraction ι_X and the exterior derivative d:

1. $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X \alpha \wedge \beta + \alpha \wedge \mathcal{L}_X \beta$,
2. $\psi^* \circ \mathcal{L}_{\psi_* X} = \mathcal{L}_X \circ \psi^*$,
3. $\mathcal{L}_X \circ d = d \circ \mathcal{L}_X$,
4. $\mathcal{L}_X = d \circ \iota_X + \iota_X \circ d$ (Lie-Cartan's formula. Hint: use the unicity of an operator $\mathcal{L}_X$ satisfying the formulas (1), (3) and the additivity).

Exercise 11. Let U be an open subset of $\mathbf{R}^n$ which is starshaped about the origin and β a closed k-form on U with $k \in \mathbf{N}^*$. Prove that $d\alpha = \beta$ where

$$\alpha_x = \int_0^1 t^{k-1} \iota_x \beta_{tx} dt.$$

Exercise 12. Let U be an open set of $\mathbf{R}^n$, α a 1-form on U and X, Y vector fields on U. Show that

$$d\alpha(X, Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X, Y]).$$

2 Differential manifolds, symplectic linear algebra

Exercise 13. Show that the compatibility relation between atlases is an equivalence relation. Show the existence and uniqueness of a topology induced by an atlas.

Exercise 14. Show that a manifold structure on M is determined by the ring of smooth functions $M \to \mathbf{R}$.

Exercise 15. Give two different manifold structures on $\mathbf{R}$.

Exercise 16. Let M be a manifold. We say that a subset $N \subset M$ is a (embedded) submanifold if, for any $x \in N$, we can find a chart U of M around x and coordinates $(x_1, \dots, x_n)$ such that $N \cap U = \{x_1 = \cdots = x_k = 0\}$.

Prove that a submanifold has a unique manifold structure such that the inclusion $i \colon N \to M$ is a morphism of manifolds.

Conversely, let $i \colon N \to M$ be a morphism of manifolds such that i is an immersion and an embedding of topological spaces (the topology of N is induced by that of M). Prove that $i(N)$ is a submanifold of M.

Exercise 17. Let $f \colon \mathbf{R}^n \to \mathbf{R}$ be a smooth function. We set $Z = f^{-1}(0)$. Assume that $df(x) \neq 0$ for all $x \in Z$. Prove that Z is a submanifold of $\mathbf{R}^n$.

More generally, if $f \colon M \to N$ is a morphism between manifolds and $df(x)$ is a surjective for all $x \in f^{-1}(0)$, then $f^{-1}(0)$ is a submanifold.

Exercise 18. Let $f \colon \mathbf{R} \to \mathbf{R}$ be a local diffeomorphism. Prove that the image of f is an open interval, I, and that f is a diffeomorphism from $\mathbf{R}$ to I.

Give a local diffeomorphism $f \colon \mathbf{R}^2 \to \mathbf{R}^2$ which is not a diffeomorphism onto its image.

Exercise 19. Let $\mathrm{O}(n) \subset \mathrm{Mat}_{n\times n} \simeq \mathbf{R}^{n^2}$ be the group of orthogonal matrices and $\mathrm{Sym}(n) \subset \mathrm{Mat}_{n\times n}$ the space of symmetric matrices. We have $\mathrm{Sym}(n) \simeq \mathbf{R}^{n(n-1)/2}$ and $\mathrm{O}(n) = f^{-1}(0)$, where $f \colon \mathrm{Mat}_{n\times n} \to \mathrm{Sym}(n)$, $A \mapsto AA^t$. Prove that $df_A(B) = BA^t + AB^t$ and deduce that $\mathrm{O}(n)$ is a submanifold of $\mathrm{Mat}_{n\times n}$.

Exercise 20. Let M be a manifold and let U, V be two open subsets such that $M = U \cup V$. We define the Mayer-Vietoris sequence

$$0 \to \Omega^k(M) \xrightarrow{\binom{r_U}{r_V}} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{(s_U, -s_V)} \Omega^k(U \cap V) \to 0,$$

where r_U, r_V, s_U, s_V are the obvious restriction maps. Prove that this is an exact sequence (that is, for any two consecutive maps f, g we have $g \circ f = 0$ and $\ker(g) = \mathrm{im}(f)$).

We recall the *snake lemma*: for a commutative diagram with two horizontal exact sequences

$$\begin{tikzcd}
 & A \arrow[r] \arrow[d, "u"] & A' \arrow[r] \arrow[d, "u'"] & A'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & B \arrow[r] & B' \arrow[r] & B'' &
\end{tikzcd}$$

we have an exact sequence $\ker(u) \to \ker(u') \to \ker(u'') \to \operatorname{coker}(u) \to \operatorname{coker}(u') \to \operatorname{coker}(u'')$.

Deduce the Mayer-Vietoris exact sequence in cohomology

$$\cdots \to H^k(M) \to H^k(U) \oplus H^k(V) \to H^k(U \cap V) \to H^{k+1}(M) \to \cdots .$$

Exercise 21. We can deduce from the Poincaré lemma that $H^k(M \times \mathbf{R}^i) \simeq H^k(M)$ for any manifold M and integers k, i. Using this and the previous exercice, prove by induction on n that $H^k_{dR}(T^n) \simeq \mathbf{R}^{\binom{n}{k}}$, where $T^n = (S^1)^n$ is the n-dimensional torus.

Exercise 22. Let $T^n = (S^1)^n$ be the n-dimensional torus. Let $0 \in S^1$ be a given point. For $I \subset \{1, \dots, n\}$ let $T_I \subset T^n$ be the subtorus of dimension $n - |I|$: $T_I = \{(\theta_1, \dots, \theta_n);\ \theta_i = 0$ if $i \notin I\}$. We define $a^k \colon \Omega^k(T^n) \to \mathbf{R}^{\binom{n}{k}}$, $\alpha \mapsto \prod_{|I|=k} \int_{T_I} (\alpha|_{T_I})$.

Check that a^k induces a map $b^k \colon H^k_{dR}(T^n) \to \mathbf{R}^{\binom{n}{k}}$. Prove that b^k is surjective, hence an isomorphism by the previous exercice.

Exercise 23. Let (V, ω) be a symplectic vector space and $L \subset V$ a Lagrangian subspace.

1. Show that there exists a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ such that $e_1, \dots, e_n \in L$.

2. Show that the group $\mathrm{Sp}(V, \omega)$ acts transitively on the set of Lagrangian subspaces.

Exercise 24. Show that a hyperplane is coisotropic.

Exercise 25. Show that $\mathrm{Sp}(4, \mathbf{R}) \neq \mathrm{SL}(4, \mathbf{R})$.

Exercise 26. Let (V, ω) be a symplectic vector space and φ an endomorphism of V. Show that φ is symplectic if and only if its graph $\Gamma_\varphi = \{(v, \varphi(v)), v \in V\}$ is Lagrangian in $(V \times V, \omega \oplus (-\omega))$.

Exercise 27. We say that two vector subspaces E, F of a vector space V are *transverse* if $E + F = V$. Let (V, ω) be a symplectic vector space, W a coisotropic subspace and L_1, L_2 two Lagrangian subspaces. We assume that L_1 is transverse to W and $L_2 \subset W$. Show that L_1 and L_2 are transverse if and only if their reductions with respect to W are transverse (in $W/W^{\perp_\omega}$).

Exercise 28. Let (V, ω) be a symplectic vector space and $q : V \to \mathbf{R}$ a positive definite quadratic form. Prove that there exist real numbers $0 < a_1 \leq \cdots \leq a_n$ and a symplectic basis $(e_1, \dots, e_n, f_1, \dots, f_n)$ such that

$$q = \sum_{i=1}^n a_i((e_i^*)^2 + (f_i^*)^2).$$

Prove that a tuple $(a_1, \dots, a_n)$ with the above property is unique (i.e. depends only on q).

Exercise 29. Let $\omega_0 = e_1^* \wedge f_1^* + e_2^* \wedge f_2^*$ be the standard symplectic form in $\mathbf{R}^4$.

(a) Prove that a non-zero alternate 2-form α on $\mathbf{R}^4$ is of rank 2 if and only if $\alpha \wedge \alpha = 0$.

(b) Deduce that the set of alternate 2-forms of rank 2, up to multiplication, is a quadric in the projective space $\mathbf{P}(\bigwedge^2 \mathbf{R}^{4*}) \simeq \mathbf{P}^5$ (that is, it is the vanishing locus of a homogeneous polynomial of degree 2).

(c) Let α be an alternate 2-form of rank 2 on $\mathbf{R}^4$. Prove that $\ker \alpha$ is Lagrangian if and only if $\alpha \wedge \omega_0 = 0$.

(d) Prove that the Grassmannian Lagrangian of $\mathbf{R}^4$ is a quadric in $\mathbf{P}(V) \simeq \mathbf{P}^4$, where $V = \{\alpha \in \bigwedge^2 \mathbf{R}^{4*};\ \alpha \wedge \omega_0 = 0\}$.

3 Symplectic linear algebra, symplectic manifolds

Exercise 30. Let (V, ω, g, J) be a Hermitian vector space. Show that a subspace L is Lagrangian if and only if JL is the g-orthogonal complement of L.

Exercise 31. Let (V, ω) be a symplectic vector space and $\mathcal{J}(V, \omega)$ the set of compatible complex structures on V.

1. Show that the group $\mathrm{Sp}(V, \omega)$ acts transitively on $\mathcal{J}(V, \omega)$ and that the stabilizer of J is $\mathrm{U}(V, \omega, J)$. Deduce from the polar decomposition that $\mathcal{J}(V, \omega)$ is diffeomorphic to a vector space.

2. Let $J_0 \in \mathcal{J}(V, \omega)$, W the space of anti-complex (i.e. $w \in \mathrm{End}(V)$, $J_0 \circ w + w \circ J_0 = 0$) symmetric endomorphisms and $B(W)$ the open unit ball in W for the operator norm $\|w\| = \sup_{x \neq 0} \frac{\|w(x)\|}{\|x\|}$.

   Show that the map $\Phi(w) = (\mathrm{id} + w) \circ J_0 \circ (\mathrm{id} + w)^{-1}$ defines a bijection from $B(W)$ to $\mathcal{J}(V, \omega)$.

Exercise 32. Let (V, ω) be a symplectic vector space and $u \in \mathrm{Sp}(V, \omega)$.

1. Prove that $\mathrm{im}(u^{-1} - \lambda \mathrm{id})^{\perp_\omega} = \ker(u - \lambda \mathrm{id})$.

2. Deduce that if λ is an eigenvalue of u, then so is $\frac{1}{\lambda}$ with the same multiplicity.

3. Show that 1 and -1 have even multiplicities as eigenvalues (possibly zero).

Exercise 33. Let (V, ω, g, J) be a Hermitian vector space, $A \in \mathrm{Sp}(V, \omega)$ and A^* its adjoint with respect to g.

1. Show that A^* is symplectic and the relation $g(A^* v, JA^* Jv) = -g(v, v)$ for $v \in V$.

2. Deduce that for $\|v\| = 1$, either $\|A^* v\| \geq 1$ or $\|A^* Jv\| \geq 1$.

3. Prove the linear non-squeezing theorem: when $V = \mathbf{C}^n$ if $A(B^{2n}(R)) \subset B^2(r) \times \mathbf{R}^{2n-2}$ then $R \leq r$.

Exercise 34. In $\mathbf{C}^n$ consider the loop of Lagrangian subspaces $L_t = A_t(\mathbf{R}^n)$ for $t \in \mathbf{R}/\mathbf{Z}$ where

$$A_t = \begin{pmatrix} e^{2i\pi kt} & 0 \\ 0 & iI_{n-1} \end{pmatrix}.$$

1. Compute the corresponding integer in $\pi_1(\mathrm{Lag}(n)) \simeq \mathbf{Z}$.

2. Let $\Sigma \subset \mathrm{Lag}(n)$ be the hypersurface consisting of Lagrangians L such that $\dim(L \cap \mathbf{R}^n) = 1$. Show that the loop L_t is transverse to Σ. How many times does it intersect Σ ?

Exercise 35. Let $(M_1, \omega_1), (M_2, \omega_2)$ be symplectic manifolds and $p_{1,2} : M_1 \times M_2 \to M_{1,2}$ the projections.

1. Check that $\omega = p_1^*\omega_1 - p_2^*\omega_2$ is a symplectic form on $M_1 \times M_2$.
2. Show that a map $\varphi : M_1 \to M_2$ is symplectic, i.e. $\varphi^*\omega_2 = \omega_1$ if and only if its graph is isotropic in $(M_1 \times M_2, \omega)$.

Exercise 36. Let M be a manifold. Prove that the canonical 1-form λ_M locally defined by $\sum p_i dq_i$ is invariant by coordinate changes induced from coordinate changes on M.

Exercise 37. 1. Let M be a Riemannian manifold. Compute the pull-back of the canonical 1-form λ_M under the isomorphism $TM \to T^*M$ induced by the inner product.

2. Let $j : M \to N$ be an isometric embedding. Prove that $(dj)^*\lambda_N = \lambda_M$ where $dj : TM \to TN$ is the differential of j.

Exercise 38. Is $(T^*S^1, dp \wedge dq)$ symplectomorphic to $(\mathbf{C}^*, \omega_0)$?

Exercise 39. Consider T^*S^1 with its standard symplectic structure.

1. Find a symplectic diffeomorphism of T^*S^1 which separates the zero section 0_{S^1} from itself.
2. Prove that if φ is a Hamiltonian diffeomorphism of T^*S^1, then $0_{S^1} \cap \varphi(0_{S^1})$ contains at least two points.

Exercise 40. A subset A of a symplectic manifold is *displaceable* if there exists a hamiltonian diffeomorphism φ such that $\varphi(A) \cap A = \emptyset$.

1. What are the closed Lagrangian submanifolds of S^2 ?
2. Which ones are displaceable ?

Exercise 41. Consider a square A of side 1 in $\mathbf{R}^2$ with the standard symplectic form $\omega_0 = dx \wedge dy$. Show that for any $\epsilon > 0$, there exists a compactly supported hamiltonian $H : \mathbf{R}^2 \times [0, 1] \to \mathbf{R}$ such that $\varphi_H^1(A) \cap A = \emptyset$ and for all t, $\sup H_t - \inf H_t < 1 + \epsilon$.

Exercise 42. Prove that a symplectomorphism of $\mathbf{R}^{2n}$ is isotopic through symplectomorphisms to a linear symplectomorphism. Are all symplectomorphisms of $\mathbf{R}^{2n}$ Hamiltonian ?

Exercise 43. Let (M, ω) be a symplectic manifold such that ω is exact, i.e. $\omega = d\lambda$.

1. For $\varphi \in \mathrm{Ham}(M, \omega)$ show that there exists a unique compactly supported function f such that $\varphi^*\lambda - \lambda = df_\varphi$.
2. Show that the formula $C(\varphi) = \int_M f_\varphi \omega^n$ defines a group homomorphism $\mathrm{Ham}(M, \omega) \to \mathbf{R}$.
3. Deduce that $\mathrm{Ham}(M, \omega)$ is not a simple group when ω is exact. (A famous result of A. Banyaga says that $\mathrm{Ham}(M, \omega)$ is simple when M is closed and connected).

4 Complex structures, Moser's lemma

Exercise 44. On the cylinder $\mathbf{R} \times \mathbf{R}/Z$ give two symplectic forms of infinite volume which are not conjugate by a diffeomorphism.

Exercise 45. The natural action of $\mathrm{U}(n+1)$ on $\mathbf{C}^{n+1}$ induces an action on $\mathbf{CP}^n$. Show that the Fubini-Study form ω_{FS} on $\mathbf{CP}^n$ is invariant by this action (that is, if we denote by $m_g \colon \mathbf{CP}^n \to \mathbf{CP}^n$ the action of an element $g \in \mathrm{U}(n+1)$, then $m_g^*(\omega_{FS}) = \omega_{FS}$).

Prove that a 2-form on $\mathbf{CP}^n$ which is invariant under the action of $\mathrm{U}(n+1)$ is a scalar multiple of ω_{FS}.

Exercise 46. We take homogeneous coordinates $[z_0 : \cdots : z_n]$ on $\mathbf{CP}^n$ and consider the chart $U_0 = \{z_0 \neq 0\}$ and $\varphi \colon \mathbf{C}^n \xrightarrow{\sim} U_0$, $(z_1, \dots, z_n) \mapsto [1 : z_1 : \cdots : z_n]$. Compute ω_{FS} in this chart.
Answer: we write $z_j = x_j + iy_j$, $dz_j = dx_j + idy_j$, $d\bar{z}_j = dx_j - idy_j$. Then $dx_j \wedge dy_j = \frac{i}{2} dz_j \wedge d\bar{z}_j$ and we find

$$\omega_{FS} = \frac{i}{2}\Big(\frac{\sum_{j=1}^n dz_j \wedge d\bar{z}_j}{1 + \sum_{j=1}^n z_j \bar{z}_j} - \frac{(\sum_{j=1}^n \bar{z}_j dz_j) \wedge (\sum_{j=1}^n z_j d\bar{z}_j)}{(1 + \sum_{j=1}^n z_j \bar{z}_j)^2} \Big).$$

For $n = 1$, $\omega_{FS} = \frac{i}{2} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = \frac{dx \wedge dy}{(1+x^2+y^2)^2}$.

Exercise 47. Let V be a vector space endowed with a symplectic form ω and a positive definite bilinear form q. Define $A \in \mathrm{GL}(V)$ by the identity $\omega(v, w) = q(Av, w)$, for all $v, w \in V$. Prove that there exists a unique self-adjoint and positive definite (with respect to q) $Q \in \mathrm{GL}(V)$ such that $Q^2 = -A^2$. Prove that Q commutes with A and that $J = Q^{-1}A$ is a complex structure on V which is compatible with ω.

Deduce that any symplectic manifold (M, ω) has an almost complex structure which is compatible with ω.

Exercise 48. Let (V, ω) be a symplectic vector space. Let $\mathcal{J}$ be the set of ω-compatible linear complex structures on V. Prove that $\mathcal{J}$ is a submanifold of $\mathrm{GL}(V)$ whose tangent space at $J \in \mathcal{J}$ is $T_J\mathcal{J} = \{A \in \mathrm{End}(V);\ AJ + JA = 0,\ \omega(A\cdot, \cdot) + \omega(\cdot, A\cdot) = 0\}$.

We define an almost complex structure I on $\mathcal{J}$ by $I_J(A) = -JA$, for $J \in \mathcal{J}$ and $A \in T_J\mathcal{J}$. Check that the 2-form Ω on $\mathcal{J}$ defined by $\Omega_J(A, B) = \frac{1}{2}\mathrm{tr}(AJB)$, for $J \in \mathcal{J}$ and $A, B \in T_J\mathcal{J}$, is a symplectic structure which is compatible with I.

Exercise 49. Let M be a manifold and $S \subset M$ a smooth compact connected orientable hypersurface. We let $i \colon S \to M$ be the embedding of S. Let ω_0, ω_1 be two symplectic forms on M. We assume that $i^*(\omega_0) = i^*(\omega_1)$. Prove that there exist a neighborhood U of S and $\varphi \colon U \to M$ such that $\varphi|_S = i$ and $\varphi^*(\omega_1) = \omega_0$.

Hint: Let $L \subset TS$ be the line bundle defined by $L_x = (T_xS)^{\perp_{\omega_0}} = (T_xS)^{\perp_{\omega_1}}$. Let $\nu \colon S \to \nu S$ be a section of the normal bundle and $\xi \colon S \to L$ a section of L. We can choose ν, ξ nowhere vanishing (why?). Let $\varepsilon = \pm 1$ be the sign of $\omega_0(\nu(x), \xi(x)) \cdot \omega_1(\nu(x), \xi(x))$ and choose $\psi \colon U \to U$ such that $\psi|_S = i$ and $d\psi_x(\nu(x)) = \varepsilon\nu(x)$. Prove that $\alpha := \psi^*\omega_1 - \omega_0$ is exact near S and that $\omega_t := \omega_0 + t\alpha$ is non-degenerate near S, for $t \in [0, 1]$.

Exercise 50. Let (M,ω) be a symplectic manifold and let Q be a coisotropic submanifold. We define the complementary distribution $TQ^{\perp_\omega} \subset TQ$ fiberwise (that is, for each $x \in Q$, $(TQ^{\perp_\omega})_x = (TQ)_x^{\perp_{\omega_x}}$). Show that $TQ^{\perp_\omega}$ is integrable and the leaves are isotropic.

Hint: Suppose that Q has codimension k, and choose functions $h_1,\ldots,h_k$ near $x \in Q$ which locally define Q (Q is the zero-locus of the h_i's and the dh_i's are independent). Show that the vector fields X_{h_i}, $i = 1,\ldots,k$, span the complementary distribution.

For two functions f, g we set $\{f,g\} = \omega(X_f, X_g)$. Check that $\{f,g\} = X_f(g) = -X_g(f)$ and $X_{\{f,g\}} = [X_f, X_g]$.

Prove that $\{h_i, h_j\}$ vanishes on Q for all i,j and conclude.

5 Contact geometry, Morse theory

Exercise 51. Let M be a manifold and ξ a hyperplane field on M.

If $\dim M = 3$, show that ξ is a contact structure if and only if, for any $x \in M$ and $v \in T_xM$, there exist vector fields X, Y around x which are tangent to ξ and satisfy $[X, Y]_x = v$.

Is it true if $\dim M > 3$?

Exercise 52. On $\mathbf{R}^3$ with coordinates (x, y, z) (or polar coordinates (r, θ, z)) we define $\alpha_0 = -xdy + dz$, $\alpha_1 = r^2 d\theta + dz$ and $\beta = (1 + r^2 + z^2)^{-2}\alpha_1$.

Prove that α_0 and α_1 are diffeomorphic (there exists a diffeomorphism ψ such that $\psi^*\alpha_1 = \alpha_0$).
Hint: Use $(x, y, z) \mapsto (ax, by, z + f(x, y))$.

Prove that α_1 and β are not diffeomorphic. **Hint:** The Reeb vector field of β has a periodic orbit (given by $r = 1$, $z = 0$).

Exercise 53. Let (M, ξ) be a contact manifold.

(i) Show that X is the Reeb field of some contact form which defines ξ if and only if X is a contact vector field transverse to ξ, i.e. $\alpha(X) \neq 0$, for any defining form α. In particular, not every contact vector field is the Reeb field of some contact form.

(ii) Let α be a contact form and Y the corresponding Reeb field. If β is any 1-form such that $\beta(Y) = 0$, prove that there is a unique vector field X which is tangent to $\ker\alpha$ and such that $\beta = \iota_X d\alpha$.

Exercise 54. Let ξ be the standard contact structure on S^{2n+1}. We prove that, for any point $p \in S^{2n+1}$, the contact manifold $(S^{2n+1} \setminus \{p\}, \xi)$ is contactomorphic with $\mathbf{R}^{2n+1}$ endowed with the standard structure.

(i) Define $f\colon \mathbf{C}^{n+1} \setminus (\mathbf{C}^n \times \{-1\}) \to \mathbf{C}^{n+1} \setminus (\mathbf{C}^n \times \{-i\})$ by $f(z, w) = (\frac{z}{w+1}, -i\frac{w-1}{w+1})$, for $z \in \mathbf{C}^n$, $w \in \mathbf{C}$. Show that f gives a biholomorphism from $B = \{(z, w);\ ||(z, w)|| < 1\}$ to $D = \{(z, w);\ \mathrm{Im}(w) > |z|^2\}$.

(ii) Set $Q = \partial D \simeq \mathbf{R}^{2n+1}$. At each point $x \in Q$, the complex part $\xi_x = T_xQ \cap JT_xQ$ of its tangent space has codimension 1. Prove that (Q, ξ) is contactomorphic with the standard contact structure on $\mathbf{R}^{2n+1}$.

(iii) Conclude.

Exercise 55. Let M be a manifold, λ_M the Liouville 1-form on T^*M. We put a Riemannian metric on M and set $h(p, q) = ||p||^2$. Let $S = h^{-1}(1) \subset T^*M$ be the unit sphere subbundle.

1. Show that $\alpha = \lambda_M|_S$ is a contact form on S. Show that the Reeb vector field Y_α is proportional to $X_h|_S$.

2. Show that if L is a conic Lagrangian submanifold of T^*M which is transverse to S and tangent to the vector field $p\partial_p$ then $S \cap L$ is a Legendrian submanifold of S.

Exercise 56. Let M be a manifold, ξ a cooriented hyperplane field on M and $\pi : T^*M \to M$ the projection. Consider $S_\xi = \{\beta \in T^*M |\ker\beta = \xi_{\pi(\beta)}$ as cooriented hyperplanes of $T_{\pi(\beta)}M\}$. Prove that ξ is a contact structure if and only if S_ξ is a symplectic submanifold of T^*M.

Exercise 57. Check that the Hessian at a critical point p of a function $f : V \to \mathbf{R}$ is well-defined as a quadratic form $T_pV \to \mathbf{R}$.

Exercise 58. Check that a critical point p of a function $f : V \to \mathbf{R}$ is non-degenerate if the section $df : V \to T^*V$ is transverse to the zero-section at p.

Exercise 59. Let $f(x, y, z) = y^2 + 2z^2$ considered as a function on $S^2 = \{x^2 + y^2 + z^2 = 1\}$. Find its critical points and show that it is a Morse function. Show that it induces a Morse function on $\mathbf{R}P^2 = S^2/\sim$ where $(x, y, z) \sim (-x, -y, -z)$.

Exercise 60 (Another proof of Morse's lemma)**.** Let $f : \mathbf{R}^n \to \mathbf{R}$ be a Morse function with a critical point at 0 and $f(0) = 0$.

1. Find functions $a_{ij}(x)$ such that $a_{ij} = a_{ji}$ and $f(x) = \sum a_{ij}(x)x_ix_j$ near 0.
2. Assume $a_{11} \neq 0$ and compute f after the change of coordinates $(x_1, \ldots, x_n) \mapsto (x'_1, x_2, \ldots, x_n)$ where
$$x'_1 = \sqrt{a_{11}(x)}(x_1 + \sum_{k>1} \frac{a_{k1}(x)}{2a_{11}(x)} x_k).$$
3. Prove Morse's lemma by induction on n.

Exercise 61. Let M be a closed submanifold of $\mathbf{R}^n$ and, for $a \in \mathbf{R}^n$, $f_a(x) = |x - x_a|^2$. Show that for almost all $a \in \mathbf{R}^n$, $f_a : M \to \mathbf{R}$ is a Morse function.

6 Morse complex

Exercise 62. Let V, W be closed manifolds. Let (f, X), (g, Y) be Morse pairs on V and W. Set $h = f + g$, $Z = X + Y$. Prove that (h, Z) is a Morse pair on $V \times W$. Prove that $C^k(h, Z) \simeq \bigoplus_{i+j=k} C^i(f, X) \otimes C^j(g, Y)$. Deduce $H^k(V \times W) \simeq \bigoplus_{i+j=k} H^i(V) \otimes H^j(W)$.

Exercise 63. Compute $H^i(S^n)$, $H^i(S^n \times S^m)$.

Exercise 64. Does there exists a Morse function on $S^2 \times S^2$ with exactly one minimum, one point of index 2, one maximum, and other critical points of index $1, 3$?

Exercise 65. Let V be a connected closed manifold.

1. Let $M \subset V$ be a submanifold and let $\gamma\colon [0,1] \to V$ be a path with $x_0 = \gamma(0)$ and $x_1 = \gamma(1)$ not in M. Prove that there exists an arbitrarily small C^∞ deformation of γ which is transverse to M. **Hint:** let $X_1, \dots, X_k$ be vector fields defined in a neighborhood U of $\mathrm{im}(\gamma)$ such that $T_xU = \langle X_i(x) \rangle$ for all $x \in U$; define $\tilde{\gamma}\colon [0,1] \times [-\varepsilon, \varepsilon]^k \to V$, $(t, t_1, \dots, t_k) \mapsto \Phi^{t_1}_{X_1} \circ \cdots \circ \Phi^{t_k}_{X_k}(\gamma(t))$ and prove that $\tilde{\gamma}$ is transverse to M.

   If $\operatorname{codim} V \geq 2$, deduce that $V \setminus M$ is connected.

2. Let (f, X) be a Morse pair on V. We assume now that f has no critical points of index 1.
   - Prove that f has only one minimum, say p_0.
   - Let $\gamma\colon [0,1] \to V$ be a loop based at p_0. Prove that there exists a small deformation of γ which does not meet the unstable sets $W^u(p)$ for $p \neq p_0$.
   - Deduce that $\pi_1(V) = 0$. (Example $V = \mathbf{CP}^n$.)

Exercise 66. Recall that $f(x, y, z) = y^2 + 2z^2$, considered as a function on $S^2 = \{x^2 + y^2 + z^2 = 1\}$, induces a Morse function on $\mathbf{RP}^2 = S^2/\sim$ where $(x, y, z) \sim (-x, -y, -z)$. Compute $H^*(\mathbf{RP}^2)$.

Exercise 67. Let V be a closed orientable surface. Let f be a Morse function on V with exactly one minimum and one maximum. Prove that the number c of critical points of index 1 is independent of such an f. Compute $H^*(V)$ in function of c.

Exercise 68. Poincaré duality. Let V be a closed manifold of dimension n. Prove that for all $k \in \mathbf{Z}$, $H^{n-k}(V)$ is isomorphic to $(H^k(V))^*$. **Hint:** Let (f, X) be a Morse-Smale pair on V and consider the Morse-Smale pair $(-f, -X)$. A critical point of index k for f is of index $n-k$ for $-f$, and hence $C^k(f, X) \simeq C^{n-k}(-f, -X)$. Using dual basis we can identify these spaces with their duals. Check that the differential of one complex gets identified with the adjoint differential of the other.

Exercise 69. The snake lemma: for a commutative diagram with two horizontal exact sequences

$$\begin{tikzcd}
& A \arrow[r] \arrow[d, "u"] & A' \arrow[r] \arrow[d, "u'"] & A'' \arrow[r] \arrow[d, "u''"] & 0 \\
0 \arrow[r] & B \arrow[r] & B' \arrow[r] & B'' &
\end{tikzcd}$$

we have an exact sequence $\ker(u) \to \ker(u') \to \ker(u'') \to \operatorname{coker}(u) \to \operatorname{coker}(u') \to \operatorname{coker}(u'')$.

Exercise 70. Let V be a connected closed manifold. We want to prove that $H^0(V) \simeq \mathbf{Z}/2\mathbf{Z}$. Let (f, X) be a Morse pair and let E^k be the set of points of index k. We set $\delta = \sum_{p \in E^0} p$ and $\Gamma = \bigcup_{(p,q) \in E^0 \times E^1} \overline{\mathcal{M}(p,q)}$. For $E \subset E^0$ we set $\Gamma_E = \bigcup_{(p,q) \in E \times E^1} \overline{\mathcal{M}(p,q)}$.

1. Prove that $d\delta = 0$.

2. Let $E \subset E^0$ be such that $d(\sum_{p \in E} p) = 0$. Prove that $\Gamma_E \cap \Gamma_{E \setminus E^0} = \emptyset$.

3. For a critical point q and $E \subset E^0$ we set $W^s_E(q) = \bigcup_{p \in E} \mathcal{M}(p,q)$. For $i \geq 0$ we also set $W^{s,i}(q) = \bigcup_{p \in E^i} \mathcal{M}(p,q)$ (we recall that $W^{s,i}(q)$ is a submanifold of codimension i of $W^s(q)$). Let $E^0 = E' \sqcup E''$ be a partition of E^0. Prove that $W'(q) = W^s_{E'}(q) \sqcup W^s_{E''}(q) \sqcup W^{s,1}(q)$ is open in $W^s(q)$ and connected.

4. Deduce that, if $W^s_{E'}(q)$ and $W^s_{E''}(q)$ are non empty, then there exist $p_0 \in E'$, $p_1 \in E''$, $r \in E^1$, $z \in \mathcal{M}(r,q)$ and sequences (x_n), (y_n) in $\mathcal{M}(p_0,q)$ and $\mathcal{M}(p_1,q)$ converging to z. Deduce that $\mathcal{M}(p_0,r)$ and $\mathcal{M}(p_1,r)$ are non empty.

5. Now we prove that Γ is connected. We assume by contradiction that we can write $E^0 = E' \sqcup E''$ such that $\Gamma_{E'} \cap \Gamma_{E''} = \emptyset$. Let $q \in E^k$. Prove that either $W^s_{E''}(q) = \emptyset$ or $W^s_{E'}(q) = \emptyset$ (do first $k = 1$, then the general case). Equivalently we have either $W^s(q) = \overline{W^s_{E'}(q)}$ or $W^s(q) = \overline{W^s_{E''}(q)}$. We say that q is respectively of type E' or E''.

6. Prove by induction on $\operatorname{ind}(r)$ that, if $\mathcal{M}(q,r) \neq \emptyset$, then q and r are of the same type. We define $V' = \{x \in V;\ \lim_{t \to -\infty} \Phi^t_X(x) \text{ is of type } E'\}$ and V'' in the same way. Prove that V' and V'' are closed.

7. Conclude $H^0(V) \simeq \mathbf{Z}/2\mathbf{Z}$.

7 spectral invariants, generating functions

Exercise 71. Let V be a closed manifold, (f, X) a Morse-Smale pair on V and $a < b < c$ regular values of f with the following properties:

- $\mathrm{crit}_k(f) = \mathrm{crit}_k(f) \cap \{a < f < b\} = \mathrm{crit}(f) \cap \{a < f < b\}$,
- $\mathrm{crit}_{k+1}(f) = \mathrm{crit}_{k+1}(f) \cap \{b < f < c\} = \mathrm{crit}(f) \cap \{b < f < c\}$.

Show that the operator $H^k_{(a,b)}(f, X) \to H^{k+1}_{(b,c)}(f, X)$ from the long exact sequence of the triple (a, b, c) corresponds to the differential $d : C^k(f, X) \to C^{k+1}(f, X)$ under some natural identifications.

Exercise 72. Let V be a closed manifold, (f, X) a Morse-Smale pair on V such that there is at most one critical point in each level set.

1. Let $\alpha \in H^k(V)$ and p a critical point such that $f(p) = c(f, \alpha)$. Prove that p is of index k. Is p necessarily a cocycle ?
2. Show that any $\alpha \in H^k(V)$ can be represented as a sum of critical points with value $\geq c(f, \alpha)$.
3. Let $\alpha_1, \dots, \alpha_l \in H^k(V)$. Show that if $c(\alpha_i)$ are distinct, then (α_i) are free.
4. Show that $\{c(f, \alpha); \alpha \in H^k(V)\}$ is of cardinal $b_k(V)$.

Exercise 73. Let V be a closed manifold and (f, X) a Morse-Smale pair quadratic at infinity on $\mathbf{R}^k \times V$. For p and q critical points of f, show that the union of gradient trajectories from p to q is contained in a compact set of $\mathbf{R}^k \times V$. Deduce that the Morse complex of (f, X) is well-defined.

Exercise 74. Let $E \to V$ be a vector bundle and $Q : E \to \mathbf{R}$ a fiberwise quadratic form. Show that there exists another vector bundle $E' \to V$, a fiberwise quadratic form $Q' : E' \to \mathbf{R}$, an integer i and a vector bundle isomorphism $u : \mathbf{R}^k \times V \to E \oplus E'$ such that $(Q \oplus Q') \circ u = -v_1^2 - \dots - v_i^2 + v_{i+1}^2 + \dots + v_k^2$.

Exercise 75. Let $\varphi : \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^n$ be the second projection and $f : \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}$ a function. Check that (φ, f) is a generating function if and only if the matrix

$$\begin{pmatrix} \frac{\partial^2 f}{\partial v^2}(v, q) & \frac{\partial^2 f}{\partial q \partial v}(v, q) \end{pmatrix}$$

is of rank k at each point (v, q) such that $\frac{\partial f}{\partial v}(v, q) = 0$.

Exercise 76. Let $f : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ be defined as $f(v, q_1, q_2) = v^4 + q_1 v^2 + q_2 v$. Determine for each (q_1, q_2) the number of solutions v of the equation $\frac{\partial f}{\partial v}(v, q_1, q_2) = 0$. Show that f is a generating function and sketch a drawing of the front of the Legendrian surface generated by f.

Exercise 77. Let $\varphi : S^3 \to \mathbf{C}P^1 = S^2$ be the hopf fibration $\varphi(z_1, z_2) = [z_1 : z_2]$, and $f : S^3 \to \mathbf{R}$ the function $f(z_1, z_2) = \mathrm{Im}(z_2)$. Show that (φ, f) is a generating function and picture the front of the corresponding Legendrian surface of $J^1 S^2$.

Exercise 78. Let Γ be an ellipse in the plane $\mathbf{R}^2$, U the open set bounded by Γ, $f : \Gamma \times U \to \mathbf{R}$ the function defined by $f(v, q) = \|v - q\|$ and $\varphi : \Gamma \times U \to U$ the second projection.

1. In the case where Γ is a circle, show that (φ, f) is a generating function and draw the front of the corresponding Legendrian surface in J^1U.

2. Study what happens when Γ is close to being a circle.

8 Generating functions 2

Exercise 79. Let $f(x,q)\colon \mathbf{R}^k\times V\to\mathbf{R}$ be a generating function for a Legendrian submanifold L of $J^1(V)$.

1. Prove that $f_x\colon V\to\mathbf{R}$ is a Morse function for almost all x.

2. We assume that L is transverse to $\mathbf{R}\times 0_V$, where 0_V is the zero section of T^*V. Prove that f is a Morse function.

Exercise 80. Let $f\colon \mathbf{R}^n\to\mathbf{R}$ be a function and let $J^0(f)=\{(z,q)\in J^0\mathbf{R}^n;\ z=f(q)\}$ be its graph and $J^1(f)=\{(z,p,q)\in J^1\mathbf{R}^n;\ z=f(q),\ p=df(q)\}$. We see $J^0(f)$ as the enveloppe of its tangent planes: for $x\in\mathbf{R}^n$ we let $g_x(q)=g(x)+\langle q-x, df(x)\rangle$ be the affine function giving the tangent plane at $(f(q),q)$. We set $G(x,q)=g_x(q)$. Is G a generating function ? What is the link between the Legendrian set L_G defined by G and $J^1(f)$?

Exercise 81. Let (v,q) be the coordinates on $\mathbf{R}^k\times\mathbf{R}^n$. Let $F\colon \mathbf{R}^k\times\mathbf{R}^n\to\mathbf{R}$ be a generating function for some immersed Legendrian submanifold $L_F\subset J^1(\mathbf{R}^n)$. We define $G\colon \mathbf{R}^{2n+k}\times\mathbf{R}^n\to\mathbf{R}$ by $G(x,y,v,q)=F(v,y)+\langle x,q-y\rangle$. Prove that G is another generating function for L_F.

Exercise 82. Let Γ be a Legendrian curve in $J^1(\mathbf{R})$ given by $\gamma(x)=(z(x),p(x),q(x))$. We assume that p' never vanishes. Let $\pi\colon J^1\to J^0$ be the projection. Let $F(x,q)$ be the function such that $q\mapsto F(x,q)$ is the line tangent to $\pi(\Gamma)$ at $\pi(\gamma(x))$. Check that F is a generating function for Γ.

Example: give a generating function for the curve Γ such that $\pi(\Gamma)=\{z^2=q^3\}$.

Exercise 83. Let $h=h(p)$ be a function on $J^1(\mathbf{R})$ (with coordinates (z,p,q)) only depending on p. Its contact vector field is $X_h=(h(p)-ph'(p))\frac{\partial}{\partial z}-h'(p)\frac{\partial}{\partial q}$.

1. Let $g(q)=aq+b$. Check that the image of $J^1(g)$ by $\varphi^t=\varphi^t_{X_h}$ is a one jet $J^1(g^t)$ for an affine function g^t.

2. Let $C\subset\mathbf{R}^2=J^0(\mathbf{R})$ (coordinates (z,q)) be a curve with no tangent lines with direction $\frac{\partial}{\partial q}$. Check that there exists a unique Legendrian curve $\tilde C\subset J^1(\mathbf{R})$ with $\pi(\tilde C)=C$, $\pi\colon J^1\to J^0$. Let D be another such curve tangent with C at $A=(z_0,q_0)$. Let $\tilde A\in\tilde C\cap\tilde D$ be above A. Let Φ be a contact diffeomorphism of $J^1(\mathbf{R})$. We assume that $\pi(\Phi(\tilde C))$ is a smooth curve around $A'=p(\Phi(\tilde A))$. Check that $\pi(\Phi(\tilde C))$ is tangent to $\pi(\Phi(\tilde D))$ at A'.

3. Let Γ and $F(x,q)$ be as in Ex. 82. Give a generating function for $\varphi^t(\Gamma)$.

4. Example: let Γ be such that $\pi(\Gamma)$ is the parabola $\{z=ax^2\}$ and let $h(p)=\sqrt{p^2+1}$ be the Hamiltonian of the geodesic flow. Give a generating function for $\varphi^t(\Gamma)$. What is the first value of t such that $\pi(\varphi^t(\Gamma))$ is not smooth ?

9 Generating functions 3

Exercise 84. Let V be a manifold, W a submanifold of V, $\varphi : E \to V$ a submersion and $f : E \to \mathbf{R}$ a function. Set $F = \varphi^{-1}(W)$, $\psi : F \to W$ the restriction of φ, $g : F \to \mathbf{R}$ the restriction of f and $C_W = J^1V \times_V W$ (i.e. C_W consists of all 1-jets of functions on V at a point of W). Show that the following conditions are equivalent:

1. (f, φ) is a generating function over a neighborhood of W and $L_{(f,\varphi)}$ is tranverse to C_W,
2. (g, ψ) is a generating function over W.

Exercise 85. Let V be a manifold, $(\varphi_t)_{t\in[0,1]}$ a contact isotopy of J^1V, and L a Legendrian submanifold of J^1V. Construct a Legendrian submanifold $\mathcal{L}$ of $J^1(V \times [0, 1])$ whose reduction to $J^1(V \times \{t\})$ is $\varphi_t(L)$ for all $t \in [0, 1]$.

Exercise 86. Let V be a manifold, W a submanifold of V, $C_W = J^1V \times_V W$, $\rho : C_W \to J^1W$ the reduction, $H_t : J^1V \to \mathbf{R}$ and $h_t : J^1W \to \mathbf{R}$ such that $H_t = h_t \circ \rho$ on C_W. Prove that X_{H_t} is tangent to C_W and lifts X_{h_t} under ρ.

Exercise 87. Let V be a manifold, $\pi : J^1V \to \mathbf{R}\times V$ the projection, and L a connected Legendrian submanifold of J^1V such that $\pi : L \to \mathbf{R} \times V$ is an embedding and the projection $L \to V$ is proper. Prove that L is the 1-jet graph of a function $f : U \to \mathbf{R}$ for some open set U of V.

Exercise 88. Let V be a closed submanifold of $\mathbf{R}^n$. Pick a norm $\|.\|$ on the vector space $\mathbf{R}^n$ and endow $C^\infty(V, V)$ with the distance

$$d_1(\varphi, \psi) = \sup_{x\in V} \|\varphi(x) - \psi(x)\| + \sup_{x\in V} \|d_x\varphi - d_x\psi\|,$$

and the induced topology (called the *C^1-topology*). Prove that the group of diffeomorphisms of V is an open subset of $C^\infty(V, V)$.

Exercise 89. Let V be a closed manifold and $(f_t)_{t\in[0,1]} : V \to \mathbf{R}$ a smooth family of Morse functions such that the set critical values of f_t is independent of t. Prove that there exists an isotopy $(\varphi_t)_{t\in[0,1]}$ of V such that $f_t \circ \varphi_t = f_0$.