Introduction to symplectic and contact geometry

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Introduction: Hamilton's equations from mechanics

We consider the motion of a ponctual mass m at the position $q(t) \in \mathbf{R}^3$ at time t when the force is given by a potential $U(q)$. Newton's law gives $\ddot{q} = -(1/m)\nabla U$, where $\nabla U = (\frac{\partial U}{\partial q_1}, \ldots, \frac{\partial U}{\partial q_3})$. We define the total energy as $\mathcal{E}(t) = \frac{1}{2}m||\dot{q}||^2 + U(q(t))$. It is classical and easy to check that $\mathcal{E}$ is constant along any trajectory $q(t)$ satisfying the above equation.

It is usual to turn the above second order equation into a a first order one by adding variables p and setting $p(t) = m\dot{q}(t)$. We obtain the system in $\mathbf{R}^6$ with coordinates $(p_1, p_2, p_3, q_1, q_2, q_3)$:

$$\begin{cases} \dot{p}_i = -\dfrac{\partial U}{\partial q_i} \\ \dot{q}_i = \dfrac{p_i}{m}. \end{cases}$$

Setting $H(p,q) = \sum \frac{p_i^2}{2m} + U(q)$, this is rewritten

$$S(H) \qquad \begin{cases} \dot{p}_i = -\dfrac{\partial H}{\partial q_i} \\ \dot{q}_i = \dfrac{\partial H}{\partial p_i}. \end{cases}$$

We could consider a point on a line instead of $\mathbf{R}^3$, or several points and work in $\mathbf{R}^{3n}$. The same formalism applies. We will thus work in dimension n. We choose coordinates $(p,q) = (p_1, \ldots, p_n, q_1, \ldots, q_n) \in \mathbf{R}^{2n}$. For a given function $H(p,q,t)$ (maybe depending on time) defined on $\mathbf{R}^{2n+1}$ the above system of equations $S(H)$ is called the *Hamiltonian system* associated with H. It is called *autonomous* if H does not depend on t. The function H is called the *Hamiltonian* function of the system. We also introduce the Hamiltonian vector field of H

$$X_H(p,q,t) = \sum_i -\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i}.$$

Hence the trajectories which are solutions of $S(H)$ are the same things as the flow lines of X_H.

Lemma 0.1. *Let $(p(t), q(t))$ be a solution of $S(H)$. We set $H_1(t) = H(p(t), q(t), t)$. Then*

$$\frac{dH_1}{dt} = \frac{\partial H}{\partial t}.$$

In particular, if H is autonomous, then H is constant along the trajectories (the total energy is conserved).

Proof.
$$\frac{dH_1}{dt} = \frac{\partial H}{\partial t} + \sum_i \dot{p}_i \frac{\partial H}{\partial p_i} + \dot{q}_i \frac{\partial H}{\partial q_i} = \frac{\partial H}{\partial t} + \sum_i -\frac{\partial H}{\partial q_i}\frac{\partial H}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial H}{\partial q_i} = \frac{\partial H}{\partial t}$$

□

We denote $t \mapsto \varphi_H(p,q,t)$ the maximal solution of $S(H)$ which is equal to (p,q) at $t = 0$. We often also denote $\varphi_H^t(p,q) = \varphi_H(p,q,t)$ and $X_H^t(p,q) = X_H(p,q,t)$. Recall from differential equations that that the set of (p,q,t) for which $\varphi_H(p,q,t)$ is defined is an open set containing $\mathbf{R}^{2n} \times \{0\}$ and φ is a smooth map. Beware that $\varphi^{t+s}(p,q) \neq \varphi^t(\varphi^s(p,q))$ in general if H is not autonomous.

Definition 0.2. A *Hamiltonian diffeomorphism* of $\mathbf{R}^{2n}$ is a diffeomorphism which is of the form φ_H^T for some Hamiltonian H and some time T.

Lemma 0.3. *Hamiltonian diffeomorphisms of $\mathbf{R}^{2n}$ form a subgroup of all diffeomorphisms.*

Proof. For $H : \mathbf{R}^{2n+1} \to \mathbf{R}$, let U_H be the open set where φ_H is defined. If φ_H^T is defined for some $T > 0$, then U_H must contain $\mathbf{R}^{2n} \times [0,T]$. For any $\rho\colon \mathbf{R} \to \mathbf{R}$, setting $H_\rho(p,q,t) = \rho'(t)H(p,q,\rho(t))$, we have $X_{H_\rho}^t = \rho'(t)X_H^{\rho(t)}$ and, if $(p,q,\rho(t)) \in U_H$,

$$(d/dt)(\varphi_H^{\rho(t)}(p,q)) = \rho'(t)X_H^{\rho(t)}(\varphi_H^{\rho(t)}(p,q)) = X_{H_\rho}^t(\varphi_H^{\rho(t)}(p,q)).$$

Hence, if $\rho(0) = 0$, then $\varphi_H^{\rho(t)} = \varphi_{H_\rho}^t$ since they satisfy the same differential equation. Using a smooth function $\rho : \mathbf{R} \to [0,T]$ which vanishes for $t \le 0$ and equal to T for $t \ge T$, we obtain H_ρ for which φ_{H_ρ} is defined everywhere, $\varphi_{H_\rho}^t = \mathrm{id}$ for $t \le 0$ and $\varphi_{H_\rho}^t = \varphi_H^T$ for $t \ge T$.

Next consider two Hamiltonian diffeomorphisms $\varphi_{H_1}^{T_1}$ and $\varphi_{H_2}^{T_2}$. Assume H_1 and H_2 have been already modified as above. Let $H_2'(p,q,t) = H_2(p,q,t-T_1)$ so that $\varphi_{H_2'}^t = \varphi_{H_2}^{t-T_1}$. Now the function $H(p,q,t) = H_1(p,q,t)$ for $t \le T_1$ and $H(p,q,t) = H_2'(p,q,t)$ for $t \ge T_1$ is smooth and $\varphi_H^{T_1+T_2} = \varphi_{H_2}^{T_2} \circ \varphi_{H_1}^{T_1}$.

For a Hamiltonian diffeomorphism φ_H^T with H prepared as before, consider $H'(p,q,t) = -H(p,q,T-t)$. Then $\varphi_{H'}^t \circ \varphi_H^T = \varphi_H^{T-t}$ and in particular $(\varphi_H^T)^{-1} = \varphi_{H'}^T$ is hamiltonian. □

Hamiltonian diffeomorphisms are not arbitrary diffeomorphisms, a first restriction is the following.

Proposition 0.4 (Liouville-Gibbs)**.** *A Hamiltonian diffeomorphism of $\mathbf{R}^{2n}$ is volume (and orientation) preserving.*

Proof. Recall that the flow φ_t of a vector field $X_t = \sum_i X_t^i \frac{\partial}{\partial_{x_i}}$ on $\mathbf{R}^n$ preserves the volume (i.e., its jacobian $\det(d_x\varphi_t)$ equals 1 for each x,t) if and only if the divergence $\mathrm{div}X_t = \mathrm{tr}(dX_t) = \sum_i \frac{\partial X_t^i}{\partial_{x_i}}$ vanishes. Indeed $\det(d_x\varphi_0) = \det(\mathrm{id}) = 1$ and

$$\frac{d}{dt}\det(d_x\varphi_t) = \det(d_x\varphi_t)\mathrm{tr}((\frac{d}{dt}d_x\varphi_t)d_x\varphi_t^{-1}) = \det(d_x\varphi_t)\mathrm{tr}(d_x(X_t\circ\varphi_t)d_x\varphi_t^{-1}) = \det(d_x\varphi_t)\mathrm{div}(X_t)\circ\varphi_t.$$

Now, for a Hamiltonian vector field $X_H = \sum_i(-\frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i} + \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i})$, we compute $\mathrm{div} X_H = \sum_i(-\frac{\partial^2 H}{\partial p_i \partial q_i} + \frac{\partial^2 H}{\partial q_i \partial p_i}) = 0$. □

It turns out that Hamiltonian diffeomorphisms preserve a finer invariant than the volume and that conversely Hamiltonian diffeomorphisms are (locally) characterized by this property. Let ω be the skew-symmetric bilinear form on $\mathbf{R}^{2n}$ given by $\omega = \sum_i dp_i \wedge dq_i$.

Proposition 0.5. *(i) Any Hamiltonian diffeomorphism φ of $\mathbf{R}^{2n}$ preserves ω, that is, $\varphi^*\omega = \omega$.*

(ii) Let $(\varphi^t)_{t\in[0,1]}$ be an isotopy of diffeomorphisms of $\mathbf{R}^{2n}$ which preserve ω. Then it is a Hamiltonian isotopy: there exists a smooth function $H : \mathbf{R}^{2n} \times [0,1] \to \mathbf{R}$ such that $\varphi^t = \varphi^t_H$.

Proof. (i) Write $\varphi_t(p_i, q_i) = (p_i^t, q_i^t)$ so that $\varphi_t^*\omega = \sum dp_i^t \wedge dq_i^t$. Then

$$\begin{aligned}
\frac{d}{dt}\sum_i dp_i^t \wedge dq_i^t &= \sum_i d\dot{p}_i^t \wedge dq_i^t + dp_i^t \wedge d\dot{q}_i^t \\
&= \sum_i -d\frac{\partial H}{\partial q_i} \wedge dq_i^t + dp_i^t \wedge d\frac{\partial H}{\partial p_i} \\
&= \sum_{i,j} -\frac{\partial^2 H}{\partial q_j \partial q_i} dq_j^t \wedge dq_i^t - \frac{\partial^2 H}{\partial p_j \partial q_i} dp_j^t \wedge dq_i^t + \frac{\partial^2 H}{\partial q_j \partial p_i} dp_i^t \wedge dq_j^t + \frac{\partial^2 H}{\partial p_j \partial p_i}\, dp_i^t \wedge dp_j^t \\
&= 0
\end{aligned}$$

(ii) Since $d(\sum_i p_i dq_i) = \sum_i dp_i \wedge dq_i$, the 1-form $\sum_i p_i^t dq_i^t - \sum_i p_i dq_i$ is closed. Hence $\sum_i \dot{p}_i^t dq_i^t + p_i^t d\dot{q}_i^t$ is closed, and so is $\alpha_t = \sum_i \dot{p}_i^t dq_i^t - \dot{q}_i^t dp_i^t$ since it differs by the exact form $d(p_i^t \dot{q}_i^t)$. By Poincaré lemma, we may find a smooth function $K_t : \mathbf{R}^{2n} \to \mathbf{R}$ such that $\alpha_t = dK_t$ and writing $H_t = K_t \circ \varphi_t^{-1}$ (i.e., $H_t(p_t, q_t) = K_t(p, q)$), we get $\dot{p}_i^t = \frac{\partial H_t}{\partial q_i}$ and $\dot{q}_i^t = -\frac{\partial H_t}{\partial p_i}$. □

The 2-form ω is called the *standard symplectic form* on $\mathbf{R}^{2n}$. It is non-degenerate, which means that at any point $x \in \mathbf{R}^{2n}$, the "kernel" of ω_x, that is, $\{v \in T_x\mathbf{R}^{2n};\ \omega_x(v, w) = 0$ for all $w \in T_x\mathbf{R}^{2n}\}$ is $\{0\}$. This is equivalent to the condition that ω^n is a volume form. Here we have $\bigwedge^n \omega = n! dp_1 \wedge \ldots \wedge dp_n \wedge dq_1 \wedge \ldots \wedge dq_n$. Hence Proposition 0.4 actually follows from Proposition 0.5 since $\varphi_t^*\omega = \omega$ implies $\varphi_t^*\omega^n = \omega^n$. Another feature of ω is that it is closed: $d\omega = 0$. On more general manifolds, this is the definition of a symplectic form: a closed non-degenerate 2-form.

Thus, defining Hamiltonian diffeomorphisms as the diffeomorphisms of the "phase space" $\mathbf{R}^{2n}$ induced by the motion of a particle following mechanic's laws, we can characteristize these transforms by the property of preserving the standard symplectic form. Diffeomorphisms which preserve the symplectic form are called *symplectomorphisms.* Proposition 0.5 states that an isotopy of symplectomorphisms is locally Hamiltonian.

Though it is easy to find a diffeomorphism of $\mathbf{R}^{2n}$ which preserves the volume but not the symplectic form, it was unclear how much global constraint is added by this property. The following ground-breaking theorem of Gromov from 1985 is among the main and earliest results of *symplectic topology.*

Theorem 0.6 (Gromov, 1985)**.** *Let $0 < r < R$. There exists no Hamiltonian diffeomorphisms of $\mathbf{R}^{2n}$ which maps the ball $B^{2n}(R)$ into the cylinder $B^2(r) \times \mathbf{R}^{2n-2}$.*

This can be compared with Heisenberg's uncertainty principle in quantum mechanics: it is impossible to know precisely both the position and the speed of a particle (i.e., the coordinates (q_1, p_1)).

Our main goal for this course is to explain a proof of this result via the technique of *generating functions*. This proof was given by Viterbo in 1992 and is essentially different from the original proof by Gromov which is based on *holomorphic curves*.

Chapter 1

Differential manifolds

1.1 Differential calculus in $\mathbf{R}^n$

Exterior algebra Let E be a real vector space of dimension n. For $k \in \mathbf{N}$, we denote $\bigwedge^k E^*$ the vector space of all alternate multilinear forms $E^k \to \mathbf{R}$. By convention $\bigwedge^0 E^* = \mathbf{R}$. For $k > n$, $\bigwedge^k E^* = \{0\}$. We also consider $\bigwedge E^* = \oplus_{k\geq 0} \bigwedge^k E^*$ which carries a natural algebra structure: for $\alpha \in \bigwedge^k E^*$, $\beta \in \bigwedge^l E^*$ and $(v_1, \dots, v_{k+l}) \in E^{k+l}$, we define

$$\alpha \wedge \beta(v_1, \dots, v_{k+l}) = \sum_{\substack{\sigma \in \mathfrak{S}_{k+l} \\ \sigma(1) < \cdots < \sigma(k) \\ \sigma(k+1) < \cdots < \sigma(k+l)}} \epsilon(\sigma)\alpha(v_{\sigma(1)}, \dots, v_{\sigma(k)})\beta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+l)})$$

The *exterior product* $\wedge$ is then extended to $\bigwedge E^*$ by bilinearity. One then checks that it is associative and graded commutative (i.e. $\alpha \wedge \beta = (-1)^{kl}\beta \wedge \alpha$). For $\alpha_1, \dots, \alpha_k \in \bigwedge^1 E^*$ and $v_1, \dots, v_k \in E$,

$$\alpha_1 \wedge \cdots \wedge \alpha_k(v_1, \dots, v_k) = \det((\alpha_i(v_j))_{1\leq i,j\leq k}).$$

Given a basis $(e_1, \dots, e_n)$ of E, and the dual basis $(e_1^*, \dots, e_n^*)$, a basis of $\bigwedge^k E^*$ is given by $(e_{i_1} \wedge \cdots \wedge e_{i_k})$ where $i_1 < \cdots < i_k$. Hence $\bigwedge^k E^*$ has dimension $\binom{n}{k}$.

A linear map $a : E \to F$ induces a "pull-back" map $a^* : \bigwedge F^* \to \bigwedge E^*$ defined by $a^*\alpha(v_1, \dots, v_k) = \alpha(a(v_1), \dots, a(v_k))$ for $\alpha \in \bigwedge^k F^*$ and $v_1, \dots, v_k \in E$. This map preserves the graded algebra structure: $a^*(\alpha \wedge \beta) = (a^*\alpha) \wedge (a^*\beta)$.

Given $v \in E$ and $\alpha \in \bigwedge^k E^*$, we define the *interior product* by $\iota_v\alpha \in \bigwedge^{k-1} E^*$ (sometimes also denoted $v \lrcorner \alpha$)) by $\iota_v\alpha(v_1, \dots, v_{k-1}) = \alpha(v, v_1, \dots, v_{k-1})$ for all $v_1, \dots, v_{k-1} \in E$. The operator ι_v is a graded derivation:

$$\iota_v(\alpha \wedge \beta) = \iota_v\alpha \wedge \beta + (-1)^{\deg \alpha}\alpha \wedge \iota_v\beta.$$

Also with respect to pull-back, we have:

$$\iota_v(a^*\alpha) = a^*(\iota_{a(v)}\alpha).$$

Differential forms A *differential k-form* on an open set U of $\mathbf{R}^n$ is a smooth map $\alpha : U \to \bigwedge^k(\mathbf{R}^n)^*$. For $p \in U$, we usually write α_p instead of $\alpha(p)$. It can be uniquely written $\alpha = \sum_{i_1<\cdots<i_k} \alpha_{i_1,\ldots,i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$ for some smooth functions $\alpha_{i_1,\ldots,i_k} : U \to \mathbf{R}$. The differential df of a function $f : U \to \mathbf{R}$ is a differential 1-form. The exterior derivative $d\alpha$ of α is a differential $(k+1)$-form defined by

$$d\alpha = \sum_{i_1<\cdots<i_k} d\alpha_{i_1,\ldots,i_k} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

This formula is forced by the properties:

- $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{\deg \alpha} \alpha \wedge d\beta$,
- $d(\alpha + \beta) = d\alpha + d\beta$,
- $d^2 = 0$.

Alternatively, one can give a direct definition

$$(d\alpha)_p(v_0, \ldots, v_k) = \lim_{t \to 0} \frac{1}{t} \sum_i (-1)^i (\alpha_{p+tv_i}(v_0, \ldots, \widehat{v_i}, \ldots, v_k) - \alpha_p(v_0, \ldots, \widehat{v_i}, \ldots, v_k)),$$

and check the properties above to show that the definitions agree. The reason for $d^2 = 0$ is Schwarz's symmetry theorem for second derivatives.

Given open sets $U \subset \mathbf{R}^n$ and $V \subset \mathbf{R}^m$, and a smooth map $\varphi : U \to V$, the pullback operator φ^* is defined on functions by $\varphi^* f = f \circ \varphi$ and extended to k-forms uniquely provided we have:

- $\varphi^*(\alpha \wedge \beta) = \varphi^*\alpha \wedge \varphi^*\beta$,
- $\varphi^*(\alpha + \beta) = \varphi^*(\alpha + \beta)$,
- $\varphi^* d = d\varphi^*$.

The operator φ^* can also be defined directly by

$$(\varphi^*\alpha)_p(v_1, \ldots, v_k) = \alpha_{\varphi(p)}(d\varphi_p(v_1), \ldots, d\varphi_p(v_k)).$$

The reason for $\varphi^* d = d\varphi^*$ is the chain rule.

Vector fields A *vector field* on an open set U of $\mathbf{R}^n$ is a smooth map $X : U \to \mathbf{R}^n$. Given a diffeomorphism $\varphi : U \to V$, we define the push-forward vector field $\varphi_* X : V \to \mathbf{R}^n$ by the formula $\varphi_* X(x) = d\varphi_{\varphi^{-1}(x)} X(\varphi^{-1}(x))$. Beware that φ_* is not defined for arbitrary smooth maps. The pull-back φ^* is defined as $(\varphi^{-1})_*$.

The vector field X is determined by the operator $\mathcal{L}_X$ acting on functions by $\mathcal{L}_X(f) = df(X)$ (sometimes also denoted $X.f$). The operator $\mathcal{L}_X$ can then be uniquely extended to differential forms by the properties:

- $\mathcal{L}_X(\alpha \wedge \beta) = \mathcal{L}_X\alpha \wedge \beta + \alpha \wedge \mathcal{L}_X\beta$,
- $\mathcal{L}_X(\alpha + \beta) = \mathcal{L}_X\alpha + \mathcal{L}_X\beta$,

- $\mathcal{L}_X d = d\mathcal{L}_X$.

Recall from Cauchy-Lipschitz theorem that any vector field admits a local flow φ_X^t, i.e. $\varphi_X^0 = \mathrm{id}$ and $\frac{d}{dt}\varphi_X^t = X \circ \varphi_X^t$. A direct definition of $\mathcal{L}_X$ can be given as:

$$\mathcal{L}_X \alpha = \frac{d}{dt}|_{t=0} (\varphi_X^t)^* \alpha.$$

One may then check the properties above. The operator $\mathcal{L}_X$ also applies to vector fields with the same definition:

$$\mathcal{L}_X Y = \frac{d}{dt}|_{t=0} (\varphi_X^t)^* Y.$$

The *Lie bracket* of two vector fields X, Y can be defined as $[X,Y] = \mathcal{L}_X Y$. This vector field measures the infinitesimal defect of commutativity of the flows φ_t^X and φ_t^Y, namely we have:

$$[X,Y](x) = \frac{1}{2}\frac{d^2}{dt^2}|_{t=0} \varphi_Y^{-t} \circ \varphi_X^{-t} \circ \varphi_Y^t \circ \varphi_X^t(x).$$

Proposition 1.1. *Two vector fields X, Y locally commute, namely for all $x \in M$ and sufficiently small $s, t > 0$, $\varphi_X^s \circ \varphi_Y^t = \varphi_Y^t \circ \varphi_X^s$ if and only if $[X,Y] = 0$.*

We have the formulas

- $[\varphi_* X, \varphi_* Y] = \varphi_*[X,Y]$,
- $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}$,
- $[X,Y] = -[Y,X]$,
- $[X,[Y,Z]] + [Z,[X,Y]] + [Y,[Z,X]] = 0$,
- $[X, fY] = \mathcal{L}_X(f)Y + f\ [X,Y]$.

Another definition of the exterior derivative d can be given as follows: for a k-form α and vectors $v_0, \dots, v_k \in \mathbf{R}^n$ and $x \in U$, extend v_i to vector fields $X_0, \dots, X_k$ equal to $v_0, \dots, v_k$ at x and set

$$d\alpha(v_0,\dots,v_k) = \sum_i (-1)^i \mathcal{L}_{X_i}(\alpha(X_0,\dots,\widehat{X_i},\dots,X_k)) + \sum_{i<j} (-1)^{i+j} \alpha([X_i,X_j], X_0,\dots,\widehat{X_i},\dots,\widehat{X_j},\dots,X_k)$$

The result is independent of the choice of $X_0, \dots, X_k$ extending $v_0, \dots, v_k$. For 1-forms this gives:

$$d\alpha(v_0,v_1) = \mathcal{L}_{X_0}(\alpha(X_1)) - \mathcal{L}_{X_1}(\alpha(X_0)) - \alpha([X_0,X_1]).$$

Lie-Cartan formulas Given a k-form α, we can define the interior product $\iota_X\alpha$ pointwise: $(\iota_X\alpha)_p(v_1,\dots,v_k) = \alpha_p(X, v_1,\dots,v_k)$. We have the important *Lie-Cartan formula* which follows formally from previous formulas:

$$\mathcal{L}_X = d\circ\iota_X + \iota_X\circ d.$$

Indeed it holds on functions and both terms are derivations and commute with d.

Another useful formula is:

$$\frac{d}{dt}\,(\varphi_X^t)^*\alpha = (\varphi_X^t)^*(\mathcal{L}_X\alpha),$$

which follows from the direct definition of $\mathcal{L}_X$ and the property $\varphi_X^{t+s} = \varphi_X^t\circ\varphi_X^s$. We will also use the following elaboration on the previous formula:

Proposition 1.2. *If $(\varphi_t)_{t\in[0,1]}$ is an isotopy generated by a time-dependent vector field X_t (i.e., $\varphi_0 = \mathrm{id}$ and $\frac{d}{dt}\varphi_t = X_t\circ\varphi_t$) and $(\alpha_t)_{t\in[0,1]}$ is a smooth path of differential forms, then*

$$\frac{d}{dt}\varphi_t^*\alpha_t = \varphi_t^*(\mathcal{L}_{X_t}\alpha_t + \frac{d}{dt}\alpha_t).$$

Proof. Assume first that the formula holds at $t=0$. Set $\beta_h = \alpha_{t+h}$, $Y_h = X_{t+h}$ $\psi_h = \varphi_{t+h}\circ\varphi_t^{-1}$, and observe that ψ_h is an isotopy generated by Y_h and that

$$\frac{d}{dt}\varphi_t^*\alpha_t = \varphi_t^*(\frac{d}{dh}|_{h=0}\psi_h^*\beta_h) = \varphi_t^*(\mathcal{L}_{Y_0}\beta_0 + \frac{d}{dh}_{h=0}\beta_h) = \varphi_t^*(\mathcal{L}_{X_t}\alpha_t + \frac{d}{dt}\alpha_t).$$

Next we prove the formula at $t=0$. We have $\alpha_t = \alpha_0 + t\frac{d}{dt}|_{t=0}\alpha_t + o(t)$, $\varphi_t^*\alpha_t = \varphi_t^*\alpha_0 + t\varphi_t^*(\frac{d}{dt}|_{t=0}\alpha_t) + o(t)$ and hence

$$\frac{d}{dt}|_{t=0}(\varphi_t^*\alpha_t) = \frac{d}{dt}|_{t=0}(\varphi_t^*\alpha_0) + \frac{d}{dt}|_{t=0}\alpha_t.$$

It remains to prove that $\frac{d}{dt}|_{t=0}(\varphi_t^*\alpha_0) = \mathcal{L}_{X_0}\alpha_0$. Let ψ_t be the flow of X_0, we have $\psi_t = \varphi_t + o(t)$ and $d\psi_t = d\varphi_t + o(t)$, hence $\varphi_t^*\alpha_0 = \psi_t^*\alpha_0 + o(t)$. Finally, $\frac{d}{dt}|_{t=0}\psi_t^*\alpha_0 = \mathcal{L}_{X_0}\alpha_0$ is the definition of $\mathcal{L}_{X_0}$. ◻

1.2 Manifolds

We consider only *smooth* (i.e. of class C^∞) manifolds.

An *atlas* on a set M is a set $\mathcal{A}$ of couples (U,φ) such that

- for $(U,\varphi)\in\mathcal{A}$, U is a subset of M and φ is an injective map $U\to\mathbf{R}^n$ for some $n\in\mathbf{N}$ whose image is open,
- $M = \bigcup_{(U,\varphi)\in\mathcal{A}} U$,
- for $(U,\varphi),(V,\psi)\in\mathcal{A}$, $\varphi(U\cap V)$ is open and the map $\psi\circ\varphi^{-1} : \varphi(U\cap V)\to\psi(U\cap V)$ is smooth.

On the set of atlases for M, the relation "$\mathcal{A} \cup \mathcal{B}$ is an atlas" is an equivalence relation. An atlas induces a unique topology by requiring each U to be open and $\varphi : U \to \varphi(U)$ to be a homeomorphism. Equivalent atlases induce the same topology.

Definition 1.3. A *manifold* is a set M together with an equivalence class of atlases whose induced topology is Hausdorff and secound countable (the topology has a countable base).

The condition that M is secound countable is equivalent to the existence of a countable atlas. Note that the dimension of M (i.e., the integer n in the chart $\varphi : U \to \mathbf{R}^n$) is well-defined locally (since $\mathbf{R}^n$ and $\mathbf{R}^m$ are linearly isomorphic only for $n = m$) and locally constant, hence constant provided M is connected.

We can think of the definition of a manifold conversely as a process to build manifolds from open sets of $\mathbf{R}^n$. Namely given a set I, a collection of open sets $W_i \subset \mathbf{R}^n$, $i \in I$, open subsets $W_{ij} \subset W_i$ for $i, j \in I$, and diffeomorphisms $\varphi_{ji} : W_{ij} \to W_{ji}$ such that

- $W_{ii} = W_i$, $\varphi_{ii} = \mathrm{id}$,
- for $x \in W_{kj} \cap W_{ki}$, $\varphi_{jk}(x) \in W_{ji}$ and $\varphi_{ij} \circ \varphi_{jk}(x) = \varphi_{ik}(x)$ (Cocycle relation).

We can construct $M = \cup_{i \in I} W_i / \sim$ where $\sim$ is the equivalence relation generated by $y = \varphi_{ij}(x)$, the projection of W_i to M is a bijection onto its image U_i and hence its inverse is a chart (U_i, φ_i), whose collection forms an atlas on M. Moreover M is secound countable as soon as I is countable, and M is Hausdorff if one of the following condition holds:

1. for $x, y \in M$, there exists $i \in I$ such that $x, y \in U_i$.
2. for $i, j \in I$ and $K \subset W_i$ compact, $\varphi_{ij}(K \cap W_j)$ is closed in W_i.

Given a manifold M with an atlas $\mathcal{A}$ one can follow the process above with the open sets $\varphi(U)$ for all $(\varphi, U) \in \mathcal{A}$ and the resulting manifold is canonically isomorphic to M.

The notion of smooth maps between manifolds can now be defined as maps which are smooth when written in charts. Similarly, one can define the *rank* of a map at a point as the rank of the differential in a chart, and the notions of *immersion*, *submersion*, *subimmersion* (i.e. locally constant rank maps), *embedding*, *diffeomorphism*.

Let M be a manifold of dimension n. A *submanifold* (of dimension k) of M is a subset N such that, for all $x \in N$, there exists a neighborhood U of x in M, a neighborhood V of 0 in $\mathbf{R}^n$ and a diffeomorphism $\varphi \colon U \to V$ such that $\varphi(U \cap M) = V \cap (\mathbf{R}^k \times \{0\})$.

Another important notion is the following: a smooth map between manifolds $f : E \to B$ is a *fibration* if for each point $x \in B$, there exists an open neighborhood U of x, a manifold F and a diffeomorphism $U \times F \to f^{-1}(U)$ respecting the projections to U. In particular, the *fibers* $f^{-1}(x)$ are submanifolds and are all diffeomorphic for $x \in U$ (hence also for all $x \in B$ if B is connected).

Vector Bundles We consider only smooth vector bundles over manifolds.

Definition 1.4. Let M be a manifold and E a set. A *vector bundle atlas* for a map $\pi : E \to M$ is a set $\mathcal{A}$ of couples (U, Φ), where U is an open subset of M, $\Phi : \pi^{-1}(U) \to \mathbf{R}^n \times U$ is a map respecting the projections to U, such that for any two charts (U,Φ), (V,Ψ) the map $\Psi\circ\Phi^{-1}$: $\mathbf{R}^n\times(U\cap V)\to\mathbf{R}^n\times(U\cap V)$ is smooth and for each $x\in U\cap V$, its restriction to $\{x\}\times\mathbf{R}^n$ is a linear map.

Two vector bundle atlases are equivalent if their union is still a vector bundle atlas. Each *fiber* $\pi^{-1}(x)$, for $x\in M$, inherits a vector space structure. Combined with an atlas for M, a vector bundle atlas gives an atlas for the *total space* E, hence E inherits a manifold structure for which π is smooth. The projection π is a fibration. As for manifolds, vector bundles can be constructed by gluing the open sets $U\times\mathbf{R}^n$ using the transition functions. A *section* of a vector bundle $\pi: E\to M$ is a smooth map $s: M\to E$ such that $\pi\circ s=\mathrm{id}$. Concretely, given a vector bundle atlas $\mathcal{A}$, it is a collection of smooth maps $s_U: U\to\mathbf{R}^n$ for all $(U,\Phi)\in\mathcal{A}$ subject to the appropriate gluing condition, namely $\Psi\circ\Phi^{-1}(s_U(x),x)=(s_V(x),x)$ for all charts (U,Φ), (V,Ψ).

Any natural operation which can be performed on vector spaces can be done also on vector bundles. For example, if E is a vector bundle we can define the dual E^* and the exterior powers $\bigwedge^k E^*$.

Let M be a manifold and $\mathcal{A}$ an atlas for M. The *tangent bundle* of M can be defined as follows. Consider the open sets $\mathbf{R}^n\times\varphi(U)$ and for two charts (U,φ), (V,ψ) the diffeomorphism $\mathbf{R}^n\times\varphi(U\cap V)\to\mathbf{R}^n\times\psi(U\cap V)$ given by $(u,x)\mapsto(d(\psi\circ\varphi^{-1})_x(u),\psi\circ\varphi^{-1}(x))$. This satisfies a cocycle relation and hence defines a vector bundle atlas on $TM=(\cup_{(U,\varphi)\in\mathcal{A}}\mathbf{R}^n\times\varphi(U))/\sim$ with the projection $\pi: TM\to M$. The tangent space T_xM at a point $x\in M$ is then simply the vector space $\pi^{-1}(x)$. Unraveling the definition, we see that an element of T_xM is an equivalence class of $(u,x)\in\mathbf{R}^n\times\varphi(U)$ where $(u,x)\sim(v,y)$ when $y=\psi\circ\varphi^{-1}(x)$ and $v=d(\psi\circ\varphi^{-1})_x(u)$. We define the cotangent bundle T^*M as the dual bundle of TM, and $\bigwedge^k T^*M$ the exterior powers.

Differential forms and vector fields on a manifold A differential k-form α on a manifold M is a smooth section of the bundle $\bigwedge^k T^*M$. Concretely, given an atlas $\mathcal{A}$ on M, α is the data of a differential k-form α_U on $\varphi(U)$ for each chart $(U,\varphi)\in\mathcal{A}$ such that $\alpha_U=(\psi\circ\varphi^{-1})^*\alpha_V$ for any two charts (U,φ), (V,ψ). The differential $d\alpha$ is well-defined by $d\alpha_U$ in each chart (U,φ) thanks to the relation $d(\psi\circ\varphi^{-1})^*=(\psi\circ\varphi^{-1})^*d$. A vector field is a smooth section X of TM, or concretely a vector field $X_U: U\to\mathbf{R}^n$ for each chart (U,φ) of an atlas $\mathcal{A}$, such that $(\psi\circ\varphi^{-1})_*X_U=X_V$ for any two charts $(U,\varphi),(V,\psi)$.

All the formulas that we have seen in $\mathbf{R}^n$ relating the operators φ^*, φ_*, $\wedge$, d, $\mathcal{L}_X$, i_X, $[.,.]$ are of local nature, and hence also hold on manifolds.

De Rham cohomology Let M be a manifold. We denote by $\Omega^i(M)$ the vector space of differential i-forms on M. The differential d gives a linear map $d\colon\Omega^i(M)\to\Omega^{i+1}(M)$. An i-form α is called *closed* if $d\alpha=0$ and *exact* if there exists $\beta\in\Omega^{i-1}(M)$ such that $\alpha=d\beta$. Since $d\circ d=0$ the exact forms are closed. The i^{th} group of de Rham cohomology is

$$H^i_{dR}(M)=\ker(d\colon\Omega^i(M)\to\Omega^{i+1}(M))/\mathrm{im}(d\colon\Omega^{i-1}(M)\to\Omega^i(M)).$$

Integration Assume that $\varphi\colon V\to U$ is a diffeomorphism of open sets in $\mathbf{R}^n$ and that $a\colon U\to\mathbf{R}$ is an integrable function. Then $\int_U a\,dx_1\cdots dx_n=\int_V(a\circ\varphi)|\det(d\varphi)|\,dy_1\cdots dy_n$. On the other hand, using the formula $(\varphi^*\alpha)_p(v_1,\dots,v_k)=\alpha_{\varphi(p)}(d\varphi_p(v_1),\dots,d\varphi_p(v_k))$ we see that $\varphi^*(a\,dx_1\wedge$

$\cdots \wedge dx_n) = (a\circ\varphi)(\det(d\varphi))\, dy_1 \wedge \cdots \wedge dy_n$. We say that φ is orientation preserving (resp. reversing) if $\det(d\varphi) > 0$ (resp. < 0). Any n-form ω on U is uniquely written $\omega = a\, dx_1 \wedge \cdots \wedge dx_n$ for some function a. We define $\int_U \omega = \int_U a\, dx_1 \cdots dx_n$. Our discussion implies that $\int_V \varphi^*\omega = \int_U \omega$ if φ is orientation preserving and $\int_V \varphi^*\omega = -\int_U \omega$ if φ is orientation reversing.

We say that a manifold M is orientable if there exists an atlas (U_i, φ_i), $i \in I$, such that the maps $\varphi_j \circ \varphi_i^{-1}$ preserve the orientation, for all $i, j \in I$. In this case an orientation of M is a choice of equivalence classes of such atlases (we say M is oriented if an orientation is chosen). If M is oriented, we say that an n-form ω is positive if it is written in local coordinates $\omega = a\, dx_1 \wedge \cdots \wedge dx_n$ with $a > 0$. If M is oriented, the cotangent space T_x^*M at any point x has a natural orientation given as follows: a basis $(e_1, \ldots, e_n)$ is positive (in the sense of a positive basis of an oriented vector space) if $e_1 \wedge \cdots \wedge e_n$ is positive in the above sense. The tangent space T_xM comes with the dual orientation.

If M is oriented of dimension n, we define the integral of an n-form ω on M, with compact support, as follows. We choose an atlas (U_i, φ_i), $i \in I$, and a partition of unity f_i, $i \in I$, subordinated to the covering U_i (that is, $\mathrm{supp}(f_i) \subset U_i$ and $\sum_i f_i = 1$). We then set

$$\int_M \omega = \sum_i \int_{U_i} (\varphi_i^{-1})^*(f_i\omega).$$

We can check that $\int_M \omega$ is independent of the choices of atlas and partition of unity. Moreover, if $\varphi\colon N \to M$ is an orientation preserving diffeomorphism, then $\int_N \varphi^*\omega = \int_M \omega$.

Stokes formula Let M be an oriented manifold of dimension n, $U \subset M$ a relatively compact open subset with a smooth boundary (that is, ∂U is a submanifold of M). For $x \in \partial U$ we say that $v \in T_xM$ is outward pointing if $\langle v, df_x\rangle > 0$ for a function $f\colon M \to \mathbf{R}$ such that $df_x \neq 0$ and $U = \{f < 0\}$ around x. The orientation of M induces an orientation of ∂U such that a basis $(e_1, \ldots, e_{n-1})$ of $T_x\partial U$ is positive if, for v outward pointing, $(v, e_1, \ldots, e_{n-1})$ is a positive basis of T_xM. With these orientations we have the Stokes formula: for any $(n-1)$-form α defined on a neighborhood of $\overline{U}$,

$$\int_U d\alpha = \int_{\partial U} \alpha.$$

In particular, if M is compact oriented, then, for any $\alpha \in \Omega^{n-1}(M)$, we have $\int_M d\alpha = 0$. Hence $\int_M\colon \Omega^n(M) \to \mathbf{R}$ factorizes through a natural map $\int_M\colon H^n_{dR}(M) \to \mathbf{R}$. We can show that this last map is an isomorphism when M is connected.

Homotopy groups A nice feature of fibrations is that they give rise to long exact sequences of homotopy groups.

For a topological space A, $x \in A$, and $k \in \mathbf{N}$, the *homotopy group* $\pi_k(A; x)$ is defined as the set of continuous maps $[0,1]^k \to A$ which maps the boundary of the cube to x, modulo homotopy among such maps. It has a natural group structure given by concatenating cubes. For $k = 1$, it is also called the fundamental (or Poincaré) group. For $k \geq 2$, $\pi_k(A; x)$ is abelian. Next for a pair of topological spaces (A, B) (i.e., A is a topological space and $B \subset A$ is endowed with the induced topology) and $x \in B$, we define the relative homotopy group $\pi_k(A, B; x)$ as the set of homotopy classes of continuous maps $[0,1]^k \to A$ which maps the boundary to B and a specific point, say $(1,0,\ldots,0)$, to x. There are obvious maps $\pi_k(B;x) \to \pi_k(A,x)$, $\pi_k(A;x) \to \pi_k(A,B;x)$ and also a boundary map $\pi_k(A,B;x) \to \pi_{k-1}(B;x)$ which form a long exact sequence:

$$\cdots \to \pi_{k+1}(A,B;x) \to \pi_k(B;x) \to \pi_k(A,x) \to \pi_k(A,B;x) \to \pi_{k-1}(B;x) \to \ldots.$$

Proposition 1.5. *Let $f : E \to B$ be a fibration, $x \in B$ and $F = f^{-1}(x)$. Then the map*

$$\pi_k(E,F;x) \to \pi_k(B;f(x))$$

induced by f is an isomorphism. In particular the long exact sequence of homotopy groups can be written:

$$\cdots \to \pi_{k+1}(B;f(x)) \to \pi_k(F;x) \to \pi_k(E,x) \to \pi_k(B;f(x)) \to \pi_{k-1}(F;x) \to \ldots.$$

Chapter 2

Symplectic geometry

2.1 Symplectic linear algebra

Symplectic vector spaces

Definition 2.1. A *symplectic vector space* (V, ω) is a finite-dimensional real vector space V together with a bilinear form ω which is non-degenerate (i.e., $\ker \omega = \{u \in V;\, \forall v \in V, \omega(u, v) = 0\} = \{0\}$) and skew-symmetric (i.e., $\forall u, v \in V, \omega(u, v) = -\omega(v, u)$).

Definition 2.2. The ω-orthogonal complement to a subspace F of V is denoted $F^{\perp_\omega} = \{u \in V, \forall v \in F, \omega(u, v) = 0\}$.

Lemma 2.3. $\dim F + \dim F^{\perp_\omega} = \dim V$.

Proof. The non-degeneracy condition says that the linear map $V \to V^*$ defined by $v \mapsto \iota_v \omega = \omega(v, .)$ has zero kernel and is thus an isomorphism since V is finite dimensional. Now $F^{\perp_\omega}$ is the kernel of the surjective map $V \to F^*$ obtained by composing with $V^* \to F^*$, hence $\dim V = \dim F^* + \dim F^{\perp_\omega}$. □

Definition 2.4. A subspace F is called

- *symplectic* if $F^{\perp_\omega} \cap F = \{0\}$ (or equivalently ω is non-degenerate on F),
- *isotropic* if $F \subset F^{\perp_\omega}$,
- *coisotropic* if $F^{\perp_\omega} \subset F$,
- *Lagrangian* if $F^{\perp_\omega} = F$,

Proposition 2.5. *Let (V, ω) be a symplectic vector space. Then there exists $n \in N$ such that $\dim V = 2n$ and a* symplectic basis *for V, namely a basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ such that $\omega = \sum_{i=1}^n e_i^* \wedge f_i^*$.*

Proof. We prove it by induction. If $\dim V = 0$, it is clear. Assume that, for some $k \geq 0$, the statement holds if $\dim V \leq k$ and take V of dimension $k+1$. Let $e_1 \in V \setminus \{0\}$. Since ω is non-degenerate there exists $f_1' \in V$ such that $\omega(e_1, f_1') \neq 0$. Take $f_1 = \frac{f_1'}{\omega(e_1, f_1')}$ so that $\omega(e_1, f_1) = 1$, and set $F = \langle e_1, f_1 \rangle$. Because ω is skew-symmetric, F necessarily has dimension 2 and it is a symplectic subspace: $F \oplus F^{\perp_\omega} = V$. By induction, $F^{\perp_\omega}$ has dimension $2n-2$ for some $n \geq 1$, and admits a symplectic basis $(e_2, \ldots, e_n, f_2, \ldots, f_n)$. □

In particular, any symplectic vector space is isomorphic to the standard symplectic space $\mathbf{R}^{2n}$ with symplectic form ω_0 represented by the matrix

$$\Omega_0 = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Example 2.6. *Let $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ be a symplectic basis of a symplectic vector space (V, ω).*

- *$F = \langle e_1, \ldots, e_k \rangle$ is isotropic,*
- *$F^{\perp_\omega} = \langle e_1, \ldots, e_k, e_{k+1}, \ldots, e_n, f_{k+1}, \ldots, f_n \rangle$ is coisotropic,*
- *for $k = n$, $F = F^{\perp_\omega}$ is Lagrangian.*

Lemma 2.7. *Let (V, ω) be a symplectic vector space and $F \subset V$ a subspace. If F is isotropic, then any basis $(e_1, \ldots, e_k)$ of F can be extended to a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of V.*

If F is coisotropic, then there exists a symplectic basis $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ of V such that $F = \langle e_1, \ldots, e_n, f_{k+1}, \ldots, f_n \rangle$.

Proof. (i) If $k = 1$, this follows from (the proof of) Proposition 2.5. We argue by induction and assume $k > 1$.

We set $G = \langle e_2, \ldots, e_k \rangle$. Then $G \subset F \subset F^{\perp_\omega} \subset G^{\perp_\omega}$ and $G^{\perp_\omega}$ has dimension one more than $F^{\perp_\omega}$. We choose $f_1 \in G^{\perp_\omega} \setminus F^{\perp_\omega}$. We must have $\omega(e_1, f_1) \neq 0$ and, multiplying f_1 by a scalar, we can assume $\omega(e_1, f_1) = 1$. We set $W = \langle e_1, f_1 \rangle$; it is a symplectic subspace. We have $G \subset W^{\perp_\omega}$ and we can apply the induction hypothesis to G. We obtain a basis $(e_2, \ldots, e_n, f_2, \ldots, f_n)$ of $W^{\perp_\omega}$. Adding (e_1, f_1) to this basis gives the result.

(ii) The second assertion follows from the first oen applied to $F^{\perp_\omega}$. □

Definition 2.8. The automorphisms φ of V which preserve ω, namely $\omega(\varphi(u), \varphi(v)) = \omega(u, v)$ for all $u, v \in V$, are called linear *symplectomorphisms* and form a subgroup denoted $\mathrm{Sp}(V, \omega)$ of $\mathrm{GL}(V)$. The corresponding subgroup of $\mathrm{GL}(2n, \mathbf{R})$ consisting of matrices A such that $A^T \Omega_0 A = \Omega_0$ is denoted $\mathrm{Sp}(2n, \mathbf{R})$.

Proposition 2.9. $\mathrm{Sp}(V, \omega) \subset \mathrm{SL}(V)$*, namely* $\det \varphi = 1$ *for all* $\varphi \in \mathrm{Sp}(V, \omega)$.

Proof. Recall that, if $\dim V = 2n$, $\Lambda^{2n} V^*$ is 1-dimensional and $\det \varphi$ can be defined as the endomorphism $\Lambda^n V^* \to \Lambda^n V^*$ induced by φ. Now since ω is non-degenerate, $\omega^n \neq 0 \in \Lambda^n V^*$. Indeed, in a symplectic basis, $\omega^n = (e_1^* \wedge f_1^* + \cdots + e_n^* \wedge f_n^*)^n = n! e_1^* \wedge f_1^* \wedge \cdots \wedge e_n^* \wedge f_n^*$. But $\varphi^*(\omega^n) = (\varphi^* \omega)^n = \omega^n$ and thus $\det \varphi = 1$. □

We now describe the process of *symplectic reduction*.

Proposition 2.10. *Let (V, ω) be a symplectic vector space and W a coisotropic subspace.*

1. *ω induces a symplectic form, say ω', on $W/W^{\perp_\omega}$,*

2. *For a subspace F of V we set $F^W = (F \cap W + W^{\perp_\omega})/W^{\perp_\omega}$, which is a subspace of $W/W^{\perp_\omega}$. Then $(F^{\perp_\omega})^W = (F^W)^{\perp_{\omega'}}$. In particular, if F is Lagrangian, then F^W is Lagrangian.*

Proof. 1. For $u_1, u_2 \in W$ and $v_1, v_2 \in W^{\perp_\omega}$, $\omega(u_1 + v_1, u_2 + v_2) = \omega(u_1, u_2)$, hence ω induces a form ω' on the quotient $W/W^{\perp_\omega}$. If $a_1 \in W/W^{\perp_\omega}$ is in the kernel of ω', then any lift $u_1 \in W$ is in $W^{\perp_\omega}$ and thus $a_1 = 0$.

2. We have

$$(F \cap W + W^{\perp_\omega})^{\perp_\omega} = (F \cap W)^{\perp_\omega} \cap W = (F^{\perp_\omega} + W^{\perp_\omega}) \cap W = F^{\perp_\omega} \cap W + W^{\perp_\omega},$$

where the last equality follows from $W^{\perp_\omega} \subset W$. Quotienting by $W^{\perp_\omega}$ and using the definition of ω' we obtain $(F^{\perp_\omega})^W = (F^W)^{\perp_{\omega'}}$. $\square$

If L is any vector space, the sum $L \oplus L^*$ has a natural symplectic structure, say ω_L, given by $\omega_L((v, v'), (w, w')) = \langle v, w' \rangle - \langle v', w \rangle$. The subspaces $L \oplus \{0\}$ and $\{0\} \oplus L^*$ are Lagrangian.

Proposition 2.11. *Let (V, ω) be a symplectic vector space and let L_1, $L_2 \subset V$ be two Lagrangian subspaces. We assume that $L_1 + L_2 = V$.*

1. *The map $u \colon L_2 \to L_1^*$, $v \mapsto (w \mapsto \omega(v, w))$, identifies L_2 with L_1^* and gives an isomorphism $(L_1 \oplus L_1^*, \omega_{L_1}) \mapsto (V, \omega)$, $(v, v^*) \mapsto v + u^{-1}(v^*)$.*

2. *Let $a \colon L_1 \to L_1^*$ be a symmetric linear map (that is, $a^* = a$). Then $L_a = \{v + u^{-1}(a(v));\ v \in L_1\}$ is a Lagrangian subspace of V which is transverse to L_2 ($L_a \oplus L_2 = V$).*

3. *Any Lagrangian subspace of V which is transverse to L_2 is of the type L_a for a unique symmetric map a.*

Proof. (1) follows from the non-degeneracy of ω and the definitions of u and ω_{L_1}.

(2) Using the isomorphism in (1), it is enough to see, for any $v, w \in L_1$, $\omega_{L_1}((v, a(v)), (w, a(w))) = \langle v, a(w) \rangle - \langle w, a(v) \rangle = 0$ which follows from the symmetry of a.

(3) Let $L \subset V$ be Lagrangian and transverse to L_2. Using the isomorphism in (1) we see L as a Lagrangian subspace of $L_1 \oplus L_1^*$. Then it is transverse to $\{0\} \oplus L_1^*$ and we can see it as the graph of a map $a \colon L_1 \to L_1^*$. Then for $v, w \in L_1$ we have $(v, a(v))$, $(w, a(w)) \in L$ and $\omega_{L_1}((v, a(v)), (w, a(w))) = \langle v, a(w) - a^*(w) \rangle$. Hence L is Lagrangian if and only if a is symmetric. $\square$

Hermitian vector spaces On a complex vector space, the multiplication by i defines an endomorphism J such that $J^2 = -\mathrm{id}$. Conversely, an endomorphism J with $J^2 = -\mathrm{id}$ on a real vector space determines a complex structure by $(a + ib)v = av + bJv$.

Definition 2.12. Let V be a finite-dimensional vector space. A symplectic form ω and a complex structure J are *compatible* if $g(u,v) = \omega(u,Jv)$ is a scalar product, namely $\omega(v,Jv) > 0$ for all $v \neq 0$ and $\omega(v,Jw) = \omega(w,Jv)$.

Given compatible ω and J, we define $\langle u,v\rangle = \omega(u,Jv) - i\omega(u,v)$ and check that this defines a Hermitian scalar product, namely

- $\langle u+v,w\rangle = \langle u,w\rangle + \langle v,w\rangle$,
- $\langle \lambda u,v\rangle = \lambda\langle u,v\rangle$,
- $\langle u,\lambda v\rangle = \overline{\lambda}\langle u,v\rangle$,
- $\langle u,v\rangle = \overline{\langle v,u\rangle}$,
- $\langle u,u\rangle \geq 0$ with equality if and only if $u = 0$.

On a hermitian vector space (V,ω,J,g) we have several subgroups of $\mathrm{GL}(V)$: $\mathrm{U}(V)$, $\mathrm{Sp}(V,\omega)$, $\mathrm{O}(V,g)$, $\mathrm{GL}(V,J)$ preserving respectively $\langle,\rangle$, ω, g and J.

Proposition 2.13. *Let (V,ω,J,g) be a hermitian vector space.*

$$\mathrm{U}(V) = \mathrm{O}(V,g) \cap \mathrm{Sp}(V,\omega) = \mathrm{O}(V,g) \cap \mathrm{GL}(V,J) = \mathrm{Sp}(V,\omega) \cap \mathrm{GL}(V,J).$$

Proof. Among the structures ω, g, J two out of three determine the third:

- $g(u,v) = \omega(u,Jv)$,
- $\omega(u,v) = -g(u,Jv)$,
- Given v, Jv is the inverse of $g(.,v) \in V^*$ by the map $V \to V^*$ induced by ω.

The result follows. □

Let (V,g) be a euclidean vector space, recall that an endomorphism $a \in \mathrm{End}(V)$ has an adjoint $a^* \in \mathrm{End}(V)$ defined by $g(au,v) = g(u,a^*v)$ for all $u,v \in V$. Then we define $\mathrm{Sym}(V,g) = \{a \in \mathrm{End}(V), a^* = a\}$ and the subsets $\mathrm{Sym}^+(V,g)$ (resp. $\mathrm{Sym}^{++}(V,g)$) of non-negative (resp. positive) symmetric endomorphisms.

Lemma 2.14. *The exponential map defines a diffeomorphism* $\exp : \mathrm{Sym}(V,g) \to \mathrm{Sym}^{++}(V,g)$*, its inverse is denoted* $\log$.

Proof. For all $a \in \mathrm{End}(V)$, we have $\exp(a)^* = \exp(a^*)$. Hence $\exp(\mathrm{Sym}(V,g)) \subset \mathrm{Sym}(V,g)$. For any $a \in \mathrm{End}(V)$ and $\lambda \in \mathbf{R}$, we have $\ker(a - \lambda\mathrm{id}) \subset \ker(\exp(a) - \exp(\lambda)\mathrm{id})$. Now any $a \in \mathrm{Sym}(V,g)$ can be diagonalized in an orthonormal basis, hence $V = \oplus_{\lambda\in\mathbf{R}} \ker(a - \lambda\mathrm{id}) = \oplus_{\mu\in\mathbf{R}} \ker(\exp(a) - \mu\mathrm{id})$ and thus $\ker(a - \lambda\mathrm{id}) = \ker(\exp(a) - \exp(\lambda)\mathrm{id})$ for all $\lambda \in \mathbf{R}$. This shows that $\exp(a)$ has positive eigenvalues, i.e. $\exp(a) \in \mathrm{Sym}^{++}(V,g)$, and that $\exp(a) = \exp(b)$ implies $a = b$ since a and b would then have the same eigenspaces. The inverse map log is well-defined by setting, for $s \in \mathrm{Sym}^{++}(V,g)$, $\log(s)$ to be the endomorphism with same eigenspaces as s and eigenvalues the logarithm of the eigenvalues of s.

Next we prove that it is a diffeomorphism. Since exp is defined by a normally convergent series, it is smooth. The differential of exp at $a \in \mathrm{End}(V)$ equals $e^a \frac{1-e^{-\mathrm{ad}_a}}{\mathrm{ad}_a}$ where $\mathrm{ad}_a(h) = ah - ha$. If a is diagonalizable with eigenvalues λ_i, then so does ad_a with eigenvalues $(\lambda_i - \lambda_j)$. Finally, since the function $\frac{1-e^{-x}}{x}$ nowhere vanishes on real numbers, we obtain that $d_a \exp$ is bijective. The result now follows from the inverse function theorem. □

Recall that an automorphism a of a euclidean vector space (V, g) admits a unique *polar decomposition*, namely $a = q \circ s$ where $q \in \mathrm{O}(V, g)$ and $s \in \mathrm{Sym}^{++}(V, g)$. Indeed, set $s = \exp(\frac{1}{2}\log(a^*a))$ and $q = as^{-1}$, and check $q^*q = s^{-1}a^*as^{-1} = s^{-1}s^2s^{-1} = \mathrm{id}$. This shows existence and uniqueness of the decomposition. This defines a map $P : \mathrm{GL}(V) \to \mathrm{O}(V, g) \times \mathrm{Sym}^{++}(V, g)$ which is a diffeomorphism.

Proposition 2.15. *Let* (V, ω, J, g) *be a hermitian vector space.*

1. *the logarithm maps* $\mathrm{Sym}^{++}(V, g) \cap \mathrm{Sp}(V, \omega)$ *on the subspace* $W = \{l \in \mathrm{Sym}(V), lJ + Jl = 0\}$.
2. $\mathrm{Sp}(V, \omega) = P^{-1}(\mathrm{U}(V) \times (\mathrm{Sym}^{++}(V, g) \cap \mathrm{Sp}(V, \omega))$.

Proof. 1. For $s \in \mathrm{Sym}^{++}(V, g)$, $s \in \mathrm{Sp}(V, \omega) \Leftrightarrow J^{-1}sJ = s^{-1}$. If $s = \exp(l)$, then $J^{-1}sJ = \exp(J^{-1}lJ)$ and $J^{-1}lJ = -l \Leftrightarrow J^{-1}sJ = \exp(-l) = s^{-1}$.

2. $s = \exp(\frac{1}{2}\log(a^*a))$ and $a^* \in \mathrm{Sp}(V, \omega)$, hence $\log(a^*a) \in W$ and $s \in \mathrm{Sp}(V, \omega)$. Finally $q = as^{-1} \in \mathrm{Sp}(V, \omega) \cap \mathrm{O}(V, g) = \mathrm{U}(V)$. □

Since W in the proposition above is a vector space, it follows that $\mathrm{Sp}(V, \omega)$ deformation retracts onto $\mathrm{U}(V)$. In particular, they have the same homotopy groups.

Corollary 2.16. $\mathrm{Sp}(V, \omega)$ *is path-connected.*

Proof. It is enough to prove it for $\mathrm{U}(V)$ due to the previous proposition. $\mathrm{U}(V)$ is path-connected since any element of $\mathrm{U}(V)$ can be diagonalized in a orthonormal basis and its eigenvalues are complex numbers of moduli 1 which can be joined by continuous paths to 1 among such complex numbers. □

Corollary 2.17. *The fundamental group of* $\mathrm{Sp}(V, \omega)$ *is isomorphic to* $\mathbf{Z}$.

Proof. As in the previous proof, it is enough to prove it for $\mathrm{U}(V)$. Consider the (complex) determinant map $\det : \mathrm{U}(V) \to S^1$. It is a smooth fibration with fiber (over 1) the subgroup $\mathrm{SU}(V)$. The long exact sequence of homotopy groups associated to this fibration writes:

$$\pi_1 \mathrm{SU}(V) \to \pi_1 \mathrm{U}(V) \to \pi_1 S^1 \to \pi_0 \mathrm{SU}(V).$$

Since $\pi_1 S^1 = \mathbf{Z}$ it is enough to show that $\mathrm{SU}(V)$ is simply-connected. Elements of $\mathrm{SU}(V)$ are diagonalizable in a unitary basis, with eigenvalues λ_i, $i = 1, \ldots, n$, such that $\prod_i \lambda_i = 1$. One can then connect such an element to id by choosing path in S^1 joining $\lambda_2, \ldots, \lambda_n$ to 1 and setting $\lambda_1 = (\lambda_2 \times \cdots \times \lambda_n)^{-1}$. To prove that $\pi_1 \mathrm{SU}(V) = 0$, we proceed by induction. If $\dim_{\mathbf{C}} V = 1$, then $\mathrm{SU}(V) = \{1\}$. If $\dim_{\mathbf{C}} V \geq 1$, then consider the unit sphere $S(V)$, a vector $u \in \S(V)$ and the map $\mathrm{SU}(V) \to S(V)$ defined by $a \mapsto a(u)$. This is also a fibration whose fiber over u is naturally identified with $\mathrm{SU}(u^\perp)$ where $u^\perp$ denotes the complex hyperplane orthogonal to u. The long exact sequence associated with this fibration writes:

$$\pi_1\mathrm{SU}(u^\perp) \to \pi_1\mathrm{SU}(V) \to \pi_1 S(V).$$

By induction $\pi_1\mathrm{SU}(u^\perp) = 0$ and since $S(V)$ has dimension ≥ 2 (in fact ≥ 3), $\pi_1 S(V) = 0$. Hence $\pi_1\mathrm{SU}(V) = 0$. □

Recall that on a vector space V of dimension n, the *Grassmannian of k-planes* is the set of k-dimensional subspaces of V, denoted $\mathrm{Gr}_k(V)$. It has a natural structure of manifold of dimension $k(n-k)$, a smooth atlas being given as follows: for each $P \in \mathrm{Gr}_{n-k}(V)$ consider the subset $U_P \subset \mathrm{Gr}_k(V)$ of all L such that $L \oplus P = V$. Then U_P is naturally an affine space over $\mathrm{Hom}(V/P, P)$: given $L, L' \in U_P$, L' is the graph of a unique linear map $u : L = V/P \to P$. If $L \in U_P \cap U_{P'}$, the change of coordinates $\mathrm{Hom}(L,P) \to \mathrm{Hom}(L,P')$ is given by $u \mapsto \varphi \circ u \circ (\mathrm{id}_L + \pi' \circ u)^{-1}$ where $\pi' : P \to L$ is the projection parallel to P' and $\varphi : P \to P'$ is the (bijective) projection parallel to L. This change of coordinates is smooth and hence gives a smooth structure to $\mathrm{Gr}_k(V)$. Another point of view is to fix a scalar product on V and let $O(V)$ act on $\mathrm{Gr}_k(V)$. The action is transitive with stabilizer at a point L naturally identified with $O(L) \times \mathrm{O}(L^\perp)$. This shows furthermore that $\mathrm{Gr}_k(V)$ is compact. When $V = \mathbf{R}^n$, we also write $\mathrm{Gr}_k(V) = \mathrm{Gr}_{k,n}$ and get $\mathrm{Gr}_{k,n} \simeq \mathrm{O}(n)/\mathrm{O}(k) \times \mathrm{O}(n-k)$.

Given a symplectic vector space (V,ω), we denote by $\mathrm{Lag}(V,\omega)$ the *Lagrangian Grassmannian* the set of Lagrangian subspaces of V. If $\dim V = 2n$, then $\mathrm{Lag}(V,\omega)$ is a subset of $\mathrm{Gr}_n(V)$.

Proposition 2.18. *Let (V,ω) be a symplectic vector space of dimension $2n$. $\mathrm{Lag}(V,\omega)$ is a compact submanifold of dimension $\frac{n(n+1)}{2}$ of $\mathrm{Gr}_n(V)$.*

Proof. Let $L \in \mathrm{Lag}(V,\omega)$ and consider the chart U_L of $\mathrm{Gr}_n(V)$ made of all subspaces supplementary to L. The symplectic form ω induces an isomorphism $V/L \mapsto L^*$. Hence U_L is an affine space over $\mathrm{Hom}(L^*, L)$. For $u \in \mathrm{Hom}(L^*, L)$, the formal adjoint $u^* \in \mathrm{Hom}(L^*, (L^*)^*) = \mathrm{Hom}(L^*, L)$ lands in the same space, hence we may consider the subspace $\mathrm{Sym}(L^*, L)$ defined by the equation $u^* = u$. It turns out that $U_L \cap \mathrm{Lag}(V,\omega)$ is an affine subspace over $\mathrm{Sym}(L^*, L)$: if $P \in U_L \cap \mathrm{Lag}(V,\omega)$, $P' \in U_L$, then P' is the graph of $u : P \to L$ and P' is Lagrangian if and only if for all $(x,y) \in P$, $\omega(x + u(x), y + u(y)) = 0$. Since P and L are Lagrangian, this last condition is equivalent to $\omega(u(x), y) + \omega(x, u(y)) = 0$, or, under the isomorphism $P \simeq L^*$, $y(u(x)) = x(u(y))$, namely $u^* = u$. Finally, since $u^* = u$ is a closed condition and $\mathrm{Gr}_n(V)$ is compact, $\mathrm{Lag}(V,\omega)$ is also compact. □

For another description of $\mathrm{Lag}(V,\omega)$, fix a compatible complex structure J and let $\mathrm{U}(V)$ act on $\mathrm{Lag}(V,\omega)$. The action is transitive and the stabilizer is precisely $\mathrm{O}(L)$. Hence $\mathrm{Lag}(V,\omega)$ is $\mathrm{U}(V)/\mathrm{O}(L)$. If $V = \mathbf{R}^{2n}$, we also write $\mathrm{Lag}(n)$ and we have $\mathrm{Lag}(n) \simeq \mathrm{U}(n)/\mathrm{O}(n)$. One can check that the dimension is right: $\dim \mathrm{U}(n) - \dim \mathrm{O}(n) = n^2 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2}$.

Proposition 2.19. $\mathrm{Lag}(V,\omega)$ *is connected and* $\pi_1\mathrm{Lag}(V,\omega) \simeq \mathbf{Z}$.

Proof. Fix a Lagrangian subspace $L \in \mathrm{Lag}(V,\omega)$ and the induced identification $\mathrm{Lag}(V,\omega) \simeq \mathrm{U}(V)/\mathrm{O}(L)$. Since $\mathrm{U}(V)$ is connected, we get that $\mathrm{Lag}(V,\omega)$ is also connected. We have seen that the (complex) determinant $\mathrm{U}(V) \to S^1$ induces an isomorphism on π_1. Also, an element of $\mathrm{O}(L)$ seen in $\mathrm{U}(V)$ (in matrix language, the natural inclusion $\mathrm{O}(n) \subset \mathrm{U}(n)$) has determinant

± 1. Hence the map $\det^2 : \mathrm{U}(V)/\mathrm{O}(L) \to S^1$ is well-defined and we have the following diagram of fibrations:

$$\begin{array}{ccccc} \mathrm{O}(L) & \longrightarrow & \mathrm{U}(V) & \longrightarrow & \mathrm{U}(V)/\mathrm{O}(L) \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det^2} \\ \{\pm 1\} & \longrightarrow & S^1 & \xrightarrow{\cdot 2} & S^1 \end{array} \tag{2.1}$$

On homotopy groups, this induces a map of long exact sequences:

$$\begin{array}{ccccccccc} \pi_1\mathrm{O}(L) & \xrightarrow{0} & \pi_1\mathrm{U}(V) & \longrightarrow & \pi_1\mathrm{U}(V)/\mathrm{O}(L) & \xrightarrow{\delta} & \pi_0\mathrm{O}(L) & \longrightarrow & 0 \\ \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det} & & \downarrow{\scriptstyle \det^2} & & \downarrow{\scriptstyle \det} & & \downarrow \\ 0 & \longrightarrow & \mathbf{Z} & \xrightarrow{2} & \mathbf{Z} & \longrightarrow & \mathbf{Z}/2 & \longrightarrow & 0 \end{array} \tag{2.2}$$

Since $\pi_1\{\pm 1\} = 0$ and $\det : \pi_1\mathrm{U}(V) \to \mathbf{Z}$ is an isomorphism, we get that the map $\pi_1\mathrm{O}(L) \to \pi_1\mathrm{U}(V)$ is zero. Finally, since $\det : \pi_0\mathrm{O}(L) \to Z/2$ is an isomorphism, we obtain from the five lemma (simple diagram chasing) that $\det^2 : \pi_1\mathrm{U}(V)/\mathrm{O}(L) \to \mathbf{Z}$ is an isomorphism. $\square$

2.2 Symplectic manifolds

Definition 2.20. Let M be a manifold. A symplectic form on M is a 2-form $\omega \in \Omega^2(M)$ which is closed ($d\omega = 0$) and non-degenerate: for any $x \in M$, the form ω_x on the vector space T_xM is non-degenerate.

If (M, ω) is symplectic, all spaces (T_xM, ω_x) are symplectic by definition. In particular, as in the linear case, the non-degeneracy condition is equivalent to "ω^n is a volume form". In particular any symplectic manifold is oriented. It is usual to normalize ω^n and call $\nu = \frac{1}{n!}\omega^n$ the symplectic volume form.

The definitions given in the case of $\mathbf{R}^{2n}$ extend to the general case. A diffeomorphism φ from (X, ω) to (X', ω') is called symplectic (= *symplectomorphism*) if $\varphi^*\omega' = \omega$.

Since ω is non-degenerate, it gives an isomorphism between T_xM and T_x^*M at each $x \in M$. It induces an isomorphism between vector fields on M and 1-forms on M. More explicitly, to a vector field X we associate the 1-form $\iota_X(\omega)$. Conversely, for any 1-form α there exists a unique vector field X such that $\alpha = \iota_X(\omega)$.

For a (maybe time-dependent) function H on (X, ω) we denote by X_H (or X_H^t) the vector field defined by $\iota_{X_H}\omega = -dH$ (or $\iota_{X_H^t}\omega = -dH^t$ if H depends on time). We denote by φ_H^t the flow of X_H^t (if defined). They are called the Hamiltonian vector field and the Hamiltonian flow of H (which often called a Hamiltonian function).

Lemma 2.21. *Let X be a vector field on (M, ω) with a well-defined flow φ^t. Then $(\varphi^t)^*\omega = \omega$ if and only if $\iota_{X^t}(\omega)$ is closed (we say X is a symplectic vector field). Moreover X is the Hamiltonian vector field of some function if and only if $\iota_{X^t}(\omega)$ is exact (we say X is a Hamiltonian vector field).*

Proof. Since $\varphi_0^*\omega = \omega$, we have $\varphi_t^*\omega = \omega$ for all t if and only if $\frac{d}{dt}\varphi_t^*\omega = 0$ for all t. Then from the formula

$$\frac{d}{dt}\,\varphi_t^*\omega = \varphi_t^*(\mathcal{L}_{X_t}\omega) = \varphi_t^*(\iota_{X_t}d\omega + d(\iota_{X_t}\omega)) = \varphi_t^*(d(\iota_{X_t}\omega)),$$

this is equivalent to $\iota_{X_t}\omega$ being closed. Finally, $\iota_{X_t}\omega$ is exact: $\iota_{X_t}\omega = -dH_t$, by definition when φ_t is a Hamiltonian isotopy. □

Proposition 2.22. *The symplectomorphisms of (M,ω) which are of the form φ_H^T for some time-dependent Hamiltonian $H : M \times [0,1] \to \mathbf{R}$ form a normal subgroup, denoted* $\mathrm{Ham}(M,\omega)$ *of the group* $\mathrm{Diff}(M,\omega)$ *of all symplectomorphisms.*

Proof. The proof that $\mathrm{Ham}(M,\omega)$ is a subgroup is the same as in the case $M = \mathbf{R}^{2n}$. If $\psi \in \mathrm{Diff}(M,\omega)$ and φ_t is a Hamiltonian isotopy with Hamiltonian H_t, then $\psi \circ \varphi_t \circ \psi^{-1}$ is an isotopy generated by the vector field $\psi_*(X_{H_t})$. But $(\psi_*X_{H_t})\lrcorner\omega = \psi^*(X_{H_t}\lrcorner\psi^*\omega) = \psi^*(X_{H_t}\lrcorner\omega) = -d(H_t \circ \psi)$, and hence $\psi \circ \varphi_t \circ \psi$ is also a Hamiltonian isotopy generated by $H_t \circ \psi_t$. This shows that $\mathrm{Ham}(M,\omega)$ is a normal subgroup. □

Definition 2.23. If W is a submanifold of a symplectic manifold (M,ω) we say W is *isotropic* (resp. *coisotropic, Lagrangian*) if the same holds for $T_xW \subset T_xM$, for all $x \in W$.

Examples of symplectic manifolds and Lagrangian submanifolds The basic examples are in dimension 2. A symplectic structure on a surface is nothing but the datum of a volume form ω. Hence any orientable surface admits a symplectic structure. If two surfaces (S,ω) and (S',ω') are symplectomorphic, then they are in particular diffeomorphic and have the same volume. We will see soon using Moser's Lemma that the converse holds.

In a symplectic vector space of dimension 2, the Lagrangian subspaces are the lines. Hence the Lagrangian submanifolds of a surface (S,ω) are the curves.

Cotangent bundles

For any manifold M its cotangent bundle T^*M has a canonical symplectic form, ω_M. More precisely it even has a canonical 1-form λ_M called the *Liouville form* such that $\omega_M = d\lambda_M$ is symplectic.

When we take local coordinates on M, say on a local chart U, we often denote them by $q = (q_1,\dots,q_n)$. For such a choice of local chart, T^*U has natural coordinates associated with $(q_1,\dots,q_n)$, usually denoted $(p_1,\dots,p_n,q_1,\dots,q_n)$, such that $(p_1,\dots,p_n,q_1,\dots,q_n)$ are the coordinates of the point $\sum_{i=1}^n p_i dq_i(x)$ of T_x^*U where $x \in U$ has coordinates $(q_1,\dots,q_n)$.

In these local coordinates the Liouville form is defined by

$$\lambda_M = \sum_{i=1}^n p_i dq_i.$$

We could check that λ_M is indeed invariant by a change of coordinates in T^*M which is induced by a change of coordinates of M. But we can also give a coordinate-free definition of λ_M as follows. Let $\pi_M \colon T^*M \to M$ be the projection to the base (denoted π when there is no risk of confusion). Its differential gives $D\pi \colon TT^*M \to TM$. In particular, for $q \in M$ and $p \in T_q^*M$

defining $x = (p, q) \in T^*M$, we have $D\pi_x \colon T_xT^*M \to T_qM$. We define $(\lambda_M)_x \in T^*_xT^*M$, that is, $(\lambda_M)_x \colon T_xT^*M \to \mathbf{R}$ by

$$(\lambda_M)_x(v) = \langle D\pi_x(v), p\rangle \qquad \text{for all } v \in T_xT^*M.$$

As said above we define ω_M by $\omega_M = d\lambda_M$. It is exact by definition, hence closed. Using a local chart we obtain

$$\omega_M = \sum_{i=1}^n dp_i \wedge dq_i$$

which is the same formula as the symplectic form of $\mathbf{R}^{2n}$ already studied. In particular it is non-degenerate.

Proposition 2.24. *For $\theta \in \Omega^1(M)$, we write $s_\theta : M \to T^*M$ the corresponding section. There is a unique 1-form λ on T^*M such that for all $\theta \in \Omega^1(M)$, $s_\theta^*\lambda = \theta$.*

Proof. In local coordinates $(p_1, \ldots, p_n, q_1, \ldots, q_n)$ a 1-form θ writes $\theta = \sum_i f_i dq_i$ and the corresponding section writes $s_\theta(q_1, \ldots, q_n) = (f_1, \ldots, f_n, q_1, \ldots, q_n)$. Hence $s_\theta^*\lambda_M = \sum_i f_i dq_i = \theta$. This shows existence. For uniqueness, pick a vector $v \in T_x(T^*M)$ and a section s_θ whichis tangent to v at x, then $\lambda(v) = \lambda(ds_\theta \circ d\pi(v)) = (s_\theta^*\lambda)(d\pi(v)) = \theta(d\pi(v))$. Hence λ is determined by this property. □

The graph of a 1-form θ on M is Lagrangian if and only if $s_\theta^*(d\lambda) = 0$ if and only if $d\theta = 0$. This gives a first family of Lagrangian submanifolds of T^*M: the graphs of closed 1-forms. The zero-section is a particular case, and in fact all Lagrangian submanifolds sufficiently C^1-close to the zero-section are of this type. A second family is given by conormal bundles of submanifolds of M. Let $N \subset M$ be a submanifold. Then TN is a submanifold of TM and $T^*_NM = \{(p, q);\ q \in N,\ p \in (T_qN)^\perp\}$ is a submanifold of T^*M. We see that $\dim T^*_NM = \dim M$, for any N. By the definition $(\lambda_M)_{(p,q)}(v) = \langle D\pi_x(v), p\rangle$ we see that $\lambda_M|_{T^*_NM} = 0$. In particular $\omega_M|_{T^*_NM} = 0$ and T^*_NM is Lagrangian.

We remark that T^*_NM is also conic in the sense that $(p, q) \in T^*_NM$ implies $(ap, q) \in T^*_NM$, for all $a > 0$. Any conic Lagrangian submanifold is locally of this type. More generally let us set $\dot{T}^*M = T^*M \setminus 0_M$, where $0_M = \{(0, q);\ q \in\}$ is the zero section of T^*M. Let $L \subset \dot{T}^*M$ be a conic Lagrangian submanifold. Let $U \subset L$ be the open subset where the $\pi_M|_L \colon L \to M$ has maximal rank: $r = \max\{\operatorname{rank} D(\pi_M|_L)_x;\ x \in L\}$ and $U = \{x \in L;\ \operatorname{rank} D(\pi_M|_L)_x = r\}$. Then $N = \pi_M(U)$ is an immersed submanifold of M and $L = T^*_NM$ in a neighborhood of U. We remark that r is at most $\dim M - 1$; in this case N is some hypersurface and L is locally half of T^*_NM.

If M is also endowed with a Riemannian metric, the metric gives a Hamiltonian function on T^*M by $H(p, q) = ||p||^2$. In the case of a Euclidean metric in normal coordinates we have $H(p, q) = \sum_i p_i^2$ and $X_H = \sum_i 2p_i \frac{\partial}{\partial q_i}$. In general we can identify TM and T^*M through the metric. Then the sphere bundle of TM becomes the level set $H^{-1}(1)$. This is preserved by the flow φ_H of X_H and φ_H gets identified with the geodesic flow on the sphere bundle.

On $\dot{T}^*M$ we can also consider the Hamiltonian function $H(p, q) = ||p||$. Then φ_H sends a conic Lagrangian submanifold to a conic Lagrangian submanifold. For a compact submanifold N of M and for $t > 0$ small, $N_t = \{x \in M;\ d(x, N) < t\}$ is an open subset with a smooth boundary. Then $\varphi^t(T^*_NM)$ is half of $T^*_{\partial N_t}M$.

Complex projective space

We define $\mathbf{CP}^n$ as the set of complex lines in $\mathbf{C}^{n+1}$. If we let $\mathbf{C}^*$ act by multiplication on $\mathbf{C}^{n+1}$, we have $\mathbf{CP}^n = (\mathbf{C}^{n+1} \setminus \{0\})/\mathbf{C}^*$. Let $S^1 \subset \mathbf{C}^*$ be the unit circle and $S^{2n+1} \subset \mathbf{C}^{n+1}$ the unit sphere. We also have $\mathbf{CP}^n = S^{2n+1}/S^1$. The image of a point $(z_0, \dots, z_n)$ is denoted $[z_0 : \cdots : z_n]$. We have a natural structure of (complex) manifold on $\mathbf{CP}^n$ given by the $n+1$ charts $U_j = \{z_j \neq 0\}$ and $f_j \colon U_j \xrightarrow{\sim} \mathbf{C}^n$, $[z_0 : \cdots : z_n] \mapsto (\frac{z_0}{z_j}, \dots, \widehat{(\frac{z_j}{z_j})}, \dots, \frac{z_n}{z_j})$.

We identify $\mathbf{C}^{n+1}$ and $\mathbf{R}^{2n+2}$ via $z_j = x_j + iy_j$ and define the symplectic form $\omega_0 = \sum_j dx_j \wedge dy_j$. Then ω_0 is invariant by the action of S^1.

Lemma 2.25. *Let $j \colon S^{2n+1} \to \mathbf{C}^{n+1}$ be the inclusion and $\pi \colon S^{2n+1} \to \mathbf{CP}^n$ the projection. Then there exists a unique 2-form ω_{FS} on $\mathbf{CP}^n$ such that $j^*\omega_0 = \pi^*\omega_{FS}$. Moreover ω_{FS} is a symplectic form and, for any $x \in S^{2n+1}$, the tangent space $(T_{\pi(x)}\mathbf{CP}^n, \omega_{FS,\pi(x)})$ is the symplectic reduction of $(T_x\mathbf{C}^{n+1}, \omega_{0,x})$.*

The form ω_{FS} is called the *Fubini-Study* form.

Proof. (i) The existence of ω_{FS} comes from the general fact that $j^*\omega_0$ is S^1-invariant and its kernel contains the vertical vectors, that is, if $v \in T_xS^{2n+1}$ satisfies $D\pi_x(v) = 0$, then $\omega_{FS}(v,w) = 0$ for all $w \in T_xS^{2n+1}$ (indeed, the vector field $X = \sum y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j}$ is tangent to S^{2n+1} satisfies $X \lrcorner j^*\omega_0 = \sum x_j dx_j + y_j dy_j = 0$).

Indeed, for $y \in \mathbf{CP}^n$ and $v, w \in T_y\mathbf{CP}^n$ we choose $x \in S^{2n+1}$ and $v', w' \in T_xS^{2n+1}$ such that $\pi(x) = y$ and $d\pi_x(v') = v$, $d\pi_x(w') = w$. By the condition on the kernel the scalar $\omega_x^{tmp}(v,w) := \omega_0(v', w')$ is independent of v', w'. If x' is another point with $\pi(x') = y$, we can find $s \in S^1$ such that $s \cdot x = x'$. Let us denote $m_s \colon S^{2n+1} \to S^{2n+1}$ the multiplication by s. The invariance of ω_0 says $(\omega_0)_{x'}(Dm_s(v'), Dm_s(w')) = (\omega_0)_x(v', w')$. This proves $\omega_x^{tmp}(v,w) = \omega_{x'}^{tmp}(v,w)$. Hence we can define $(\omega_{FS})_y(v,w) := \omega_x^{tmp}(v,w)$. It is clear on this definition that $j^*\omega_0 = \pi^*\omega_{FS}$.

(ii) Since π is a submersion ($D\pi_x \colon T_xS^{2n+1} \to T_{\pi(x)}\mathbf{CP}^n$ is surjective), the condition $\pi^*(d\omega_{FS}) = d\pi^*\omega_{FS} = dj^*\omega_0 = 0$ implies $d\omega_{FS} = 0$. The uniqueness of ω_{FS} follows from the same argument.

(iii) At any $x \in S^{2n+1}$, the kernel of $\omega_0|_{T_xS^{2n+1}}$ is in fact exactly the set of vertical vectors. It follows that the kernel of $(\omega_{FS})_{\pi(x)}$ is $\{0\}$. This proves that $(T_{\pi(x)}\mathbf{CP}^n, \omega_{FS,\pi(x)})$ is the symplectic reduction of $(T_x\mathbf{C}^{n+1}, \omega_{0,x})$. Hence ω_{FS} is non-degenerate. □

There are two well-known Lagrangian submanifolds of $(\mathbf{CP}^n, \omega_{FS})$, the real projective space and the Clifford torus, defined by

$$\mathbf{RP}^n = \{[z_0 : \cdots : z_n] \in \mathbf{CP}^n;\ z_0, \dots, z_n \in \mathbf{R}\},$$
$$\mathbf{T}^n = \{[z_0 : \cdots : z_n] \in \mathbf{CP}^n;\ |z_0| = \cdots = |z_n|\}.$$

To prove this we use that $(T_{\pi(x)}\mathbf{CP}^n, \omega_{FS,\pi(x)})$ is the symplectic reduction of $(T_x\mathbf{C}^{n+1}, \omega_{0,x})$, for any $x \in S^{2n+1}$. To see that $\mathbf{RP}^n$ is Lagrangian, we write $\mathbf{RP}^n = \pi(\mathbf{R}^{n+1} \cap S^{2n+1})$. Then, for $x \in S^{2n+1}$, $T_{\pi(x)}\mathbf{RP}^n$ is the symplectic reduction of $T_x\mathbf{R}^{n+1}$, which is Lagrangian. For $\mathbf{T}^n$, we write $\mathbf{T}^n = \pi(C^{n+1})$, where $C \subset \mathbf{C}$ is the circle of radius $1/\sqrt{n+1}$. Since C is Lagrangian in $\mathbf{C}$, C^{n+1} is Lagrangian in $\mathbf{C}^{n+1}$ and we conclude as in the case of $\mathbf{RP}^n$.

2.3 Almost complex structures

Our manifold $M = \mathbf{CP}^n$ is actually a complex manifold in the sense that we can find an atlas (U_i, φ_i), $i \in I$, where φ_i maps U_i to an open subset of $\mathbf{C}^n$ and the maps $\varphi_j \circ \varphi_i^{-1}$ are holomorphic. The tangent spaces T_xM then have a complex structure. In this situation the linear operator "multiplication by i" is denoted by $J \colon TM \to TM$.

In general an *almost complex structure* on a (real) manifold M is an automorphism $J \colon TM \to TM$ of the tangent bundle such that $J^2 = -\mathrm{id}_{TM}$. (An almost complex structure does not necessarily come from a structure of complex manifold on M; if this is the case the almost complex structure is called *integrable*.)

When a manifold M has both a symplectic structure ω and an almost complex structure J we say that J and ω are *compatible* if

$$\begin{aligned} &\omega(Jv, Jw) = \omega(v, w) \qquad \text{for all } q \in M \text{ and } v,\, w \in T_qM, \\ &\omega(v, Jv) > 0 \qquad \text{for all } q \in M \text{ and } v \neq 0 \in T_qM. \end{aligned}$$

If J and ω are compatible, we obtain a Riemannian metric g on M by $g(v, w) = \omega(v, Jw)$. This metric is compatible with J in the sense that $g(Jv, Jw) = g(v, w)$. We have $\omega(v, w) = g(Jv, w)$. This gives a $1:1$ correspondence between non-degenerate 2-forms compatible with J and Riemannian metrics compatible with J.

An easy case is $M = \mathbf{C}^n$ with the above symplectic form $\omega_0 = \sum_j dx_j \wedge dy_j$ and J the natural complex structure. Then the associated metric is the usual Euclidean metric. We deduce the case of $M = \mathbf{CP}^n$.

Lemma 2.26. *The complex structure J of $\mathbf{CP}^n$ is compatible with the Fubini-Study form ω_{FS}.*

Proof. For the atlas (U_j, f_j) given above, the maps $f_j \circ f_i^{-1}$ are holomorphic. Then the operator $J_{\mathbf{CP}^n}$ on $T\mathbf{CP}^n$ is obtained from $J_{\mathbf{C}^n}$ on $\mathbf{C}^n$ by pull-back through the f_i's. It can also be recovered as follows. Let $x \in S^{2n+1}$ be given and set $y = \pi(x) \in \mathbf{CP}^n$. The space $T_xS^{2n+1} \subset T_x\mathbf{C}^{n+1}$ contains a maximal complex subspace, namely $V_x = T_xS^{2n+1} \cap (i \cdot T_xS^{2n+1})$ and $d\pi_x \colon T_xS^{2n+1} \to T_y\mathbf{CP}^n$ identifies V_x and $T_y\mathbf{CP}^n$ as complex vector spaces, that is, $d\pi_x(J_{\mathbf{C}^{n+1}}v) = J_{\mathbf{CP}^n}(d\pi_x(v))$.

Recall that, for $v\ w \in V_x$, we have $(\omega_{FS})_y(d\pi_x(v), d\pi_x(w)) = \omega_0(v, w)$. Since the symplectic structure ω_0 and the complex structure of $\mathbf{C}^{n+1}$ are compatible, it follows that ω_{FS} and $J_{\mathbf{CP}^n}$ are also compatible. $\square$

There are many complex submanifolds of $\mathbf{CP}^n$; they are given by smooth algebraic subvarieties. For example we can consider "complete intersections": choose $k \leq n$ homogeneous polynomials $P_1, \ldots, P_k$ in $n+1$ variables and define $V_{P_1,\ldots,P_k} = \{[z_0 : \cdots : z_n];\ P_i(z_0, \ldots, z_n) = 0$, for all $i = 1, \ldots, k\}$. For a generic choice of P_i's $V_{P_1,\ldots,P_k}$ is a complex submanifold of $\mathbf{CP}^n$ of codimension k.

Now let $i \colon Z \hookrightarrow \mathbf{CP}^n$ be a complex submanifold. Then $i^*\omega_{FS}$ is a symplectic structure on Z. Indeed $i^*\omega_{FS}$ is closed since ω_{FS} is. The only non trivial fact is the non-degeneracy but in our situation it follows from the fact that ω_{FS} and $J_{\mathbf{CP}^n}$ are compatible. Indeed, since TZ is stable by $J_{\mathbf{CP}^n}$, it is enough to check that the symmetric form $i^*\omega_{FS}(\cdot, J\cdot)$ is non-degenerate and this follows from its positivity.

More generally a *Kähler manifold* is a complex manifold M, with complex structure J, endowed with a symplectic form ω such that ω and J are compatible. As in the case of $\mathbf{CP}^n$ we see that any complex submanifold of a Kähler manifold is also a symplectic submanifold (and in fact a Kähler manifold).

Examples of Kähler manifolds are complex tori $\mathbf{T}_L = \mathbf{C}^n/L$ where L is a lattice of $\mathbf{C}^n$, that is, a free abelian subgroup of rank $2n$, $L \simeq \mathbf{Z}^{2n}$. An element $z \in L$ acts on $\mathbf{C}^n$ by the translation $t_z\colon x \mapsto x+z$. Let $\pi\colon \mathbf{C}^n \to \mathbf{T}_L$ be the quotient map. Since t_z preserves the standard form ω_0 and $d\pi_x\colon T_x\mathbf{C}^n \xrightarrow{\sim} T_{\pi(x)}\mathbf{T}_L$ for any $x \in \mathbf{C}^n$, we deduce that ω_0 induces a unique 2-form $\omega_{\mathbf{T}_L}$ such that $\omega_0 = \pi^*(\omega_{\mathbf{T}_L})$. Then $\omega_{\mathbf{T}_L}$ is symplectic and compatible with the complex structure because the isomorphism $d\pi_x$ above identifies both the complex and symplectic structures of the tangent spaces $T_x\mathbf{C}^n$ and $T_{\pi(x)}\mathbf{T}_L$. Thus $\mathbf{T}_L$ is a Kähler manifold. However $\mathbf{T}_L$ is not a complex submanifold of some $\mathbf{CP}^N$ in general (the proof of this fact relies on the Kodaira embedding theorem – see for example [Griffiths-Harris, *Principles of algebraic geometry*, Chap. 2.6]).

All complex manifolds are not Kähler. An easy example is the Hopf surface defined by $M = (\mathbf{C}^2 \setminus \{0\})/\mathbf{Z}$, where $1 \in \mathbf{Z}$ acts on $\mathbf{C}^2 \setminus \{0\}$ by multiplication by 2. As in the case of $\mathbf{T}_L$, this multiplication respects the complex structure and induces a complex structure on M. However M is diffeomorphic to $S^3 \times S^1$ (the map $(z_1, z_2) \mapsto (\frac{(z_1,z_2)}{\|(z_1,z_2)\|}, \frac{\log(\|(z_1,z_2)\|)}{\log 2})$ induces a diffeomorphism $M \to S^3 \times \mathbf{R}/\mathbf{Z}$). We deduce that $H^2(M;\mathbf{R}) \simeq 0$. Hence M has no symplectic structure.

All symplectic manifolds are not Kähler. Here is an example by Kodaira and Thurston (see [McDuff-Salamon, *Introduction to symplectic topology*, Ex. 3.1.17]). Let Γ be $\mathbf{Z}^2 \times \mathbf{Z}^2$ endowed with the following multiplication: for $j = (j, j_1) \in \mathbf{Z}^2$ we set $A_j = \begin{pmatrix} 1 & j_1 \\ 0 & 1 \end{pmatrix}$ and we define $(j', k')\circ(j, k) = (j'+j, k'+A_{j'}k)$. Then Γ is a group and it acts on $\mathbf{R}^4$ by $(j,k)\cdot(x,y) = (j+x, k+A_j y)$. This action is free and preserves the form $\omega = dx_1 \wedge dx_2 + dy_1 \wedge dy_2$. Hence the quotient $M = \mathbf{R}^4/\Gamma$ is a symplectic manifold. We can see that M is compact (one fundamental domain of the action is contained in $[0,1]^4$). Its fundamental group is Γ and it follows that $H_1(M;\mathbf{Z}) \simeq \Gamma/[\Gamma,\Gamma]$, where $[\Gamma,\Gamma]$ is the subgroup of Γ generated by the commutators. We find $[\Gamma,\Gamma] \simeq 0 \oplus 0 \oplus \mathbf{Z} \oplus 0$, hence $H_1(M;\mathbf{Z}) \simeq \mathbf{Z}^3$. By Hodge theory, for a Kähler manifold N, the odd Betti numbers $\dim H_i(N;\mathbf{R})$ (i odd) are even. Hence our M cannot be Kähler.

2.4 Moser's lemma

We usually denote a path of k-forms α on a manifold M by $(\alpha_t)_{t\in[0,1]}$ and we set $\dot\alpha_t = \frac{d\alpha_t}{dt}$.

Proposition 2.27 (Moser's Lemma for symplectic forms)**.** *Let M be a manifold. Let $(\omega_t)_{t\in[0,1]}$ be a smooth path of symplectic forms on M. We assume to be given a smooth path of 1-forms $(\alpha_t)_{t\in[0,1]}$ such that $\dot\omega_t = d\alpha_t$ and α_t vanishes outside some compact set K. Then there exists an isotopy φ_t of M with support in K such that $\varphi_0 = \mathrm{id}$ and $\varphi_t^*\omega_t = \omega_0$.*

Proof. We are looking for a vector field X_t which vanishes outside K such that its flow φ_X^t satisfies $(\varphi_X^t)^*\omega_t = \omega_0$. By Proposition 1.2 we have $0 = \frac{d}{dt}((\varphi_X^t)^*\omega_t) = (\varphi_X^t)^*(\mathcal{L}_{X_t}\omega_t + \frac{d}{dt}\omega_t)$. Since ω_t is closed, we deduce, by the Lie-Cartan formula, $d(i_{X_t}\omega_t) = -\dot\omega_t = -d\alpha_t$. It is enough to solve the equation $i_{X_t}\omega_t = -\alpha_t$, which has a unique solution by the non-degeneracy of ω_t. Since $\mathrm{supp}(\alpha_t) \subset K$, we also have $\mathrm{supp}(X_t) \subset K$ and the flow of X_t (which is well-defined because K is compact) also has its support in K. By construction $(\varphi_X^t)^*\omega_t$ is constant, hence equal to ω_0, as required. □

Proposition 2.28 (Darboux's theorem for symplectic structures)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$. Then, for any point $x_0 \in M$, there exists a chart containing x_0 with coordinates $(p_1, \dots, p_n, q_1, \dots, q_n)$ such that $\omega = \sum_{i=1}^n dp_i \wedge dq_i$. (Such a chart is called a* Darboux chart*.)*

Proof. Since the statement is local we can assume $M = \mathbf{R}^{2n}$ and $x_0 = 0$. By a linear change of coordinates we can assume that $\omega(0) = (\sum_{i=1}^n dp_i \wedge dq_i)(0)$. We let ω_0 be the standard symplectic form of $\mathbf{R}^{2n}$ in these coordinates. By the Poincaré lemma there exists a 1-form α such that $\omega - \omega_0 = d\alpha$. We can assume that α vanishes at 0 to order 1. Hence there exists $C > 0$ such that $||\alpha(x)|| \le Cx^2$ and $||d\alpha(x)|| \le Cx$ for $||x|| \le 1$.

We choose a bump function $\rho\colon \mathbf{R}^{2n} \to [0, 1]$ such that $\rho(x) = 1$ for $||x|| \le 1$ and $\rho(x) = 0$ for $||x|| \ge 2$. For $\varepsilon \in (0, 1]$ and $t \in [0, 1]$ we set

$$\omega_{\varepsilon,t} = \omega_0 + td(\rho_\varepsilon \alpha), \qquad \text{where } \rho_\varepsilon(x) = \rho(\varepsilon^{-1}x).$$

We remark that $\mathrm{supp}(\rho_\varepsilon\alpha)$ is contained in the ball $B_{2\varepsilon}$ of radius 2ε. Hence $\omega_{\varepsilon,t}(x) = \omega_0(x)$ for $||x|| \ge 2\varepsilon$. For $||x|| \le 2\varepsilon$ we have

$$\begin{aligned}(\omega_{\varepsilon,t} - \omega_0)(x) &\le ||d\rho_\varepsilon||_\infty ||\alpha(x)|| + ||d\alpha(x)|| \\ &\le C\varepsilon^{-1}||d\rho||_\infty ||x||^2 + C||x|| \\ &\le (4C||d\rho||_\infty + 2C)\varepsilon.\end{aligned}$$

Since $\omega_0(x)$ is non-degenerate, we deduce that, for ε small enough, $\omega_{\varepsilon,t}(x)$ is also non-degenerate.

Hence the hypothesis of Moser's Lemma are satisfied and there exists an isotopy φ_t with support in $B_{2\varepsilon}$ such that $\varphi_t^*(\omega_{\varepsilon,t}) = \omega_{\varepsilon,0} = \omega_0$. By definition $\omega_{\varepsilon,1} = \omega$ inside B_ε and we obtain $\varphi_1^*(\omega) = \omega_0$ inside B_ε. Hence $(B_\varepsilon, \varphi_1)$ is a Darboux chart. □

The classification of symplectic forms up to isomorphism is difficult in general. But for closed surfaces we have the following corollary of Moser's lemma.

Proposition 2.29. *Let Σ be a connected closed oriented surface. Two symplectic forms ω, ω' are conjugate by an orientation-preserving diffeomorphism of Σ if and only if $\int_\Sigma \omega = \int_\Sigma \omega'$.*

Proof. If there exists an orientation-preserving diffeomorphism φ with $\varphi^*\omega = \omega'$, then $\int_\Sigma \omega' = \int_\Sigma \varphi^*\omega = \int_\Sigma \omega$. Conversely, if $\int_\Sigma \omega = \int_\Sigma \omega'$, then there exists a 1-form α on Σ such that $\omega = \omega' + d\alpha$ (since the integration morphism $H^2_{dR}(M) \to \mathbf{R}$ is an isomorphism). Set $\omega_t = \omega' + td\alpha$ for $t \in [0, 1]$, it is a smooth path of closed forms. Moreover ω_t is non-degenerate: since $\int_\Sigma \omega = \int_\Sigma \omega'$ and Σ is connected, ω and ω' induce the same orientation on Σ, namely for all basis of tangent vectors (u, v), we have $\omega(u, v)\omega'(u, v) > 0$, and hence $\omega_t(u, v) \ne 0$. According to Proposition 2.27, there exists an isotopy φ_t such that $\varphi_t^*\omega_t = \omega_0$. Hence φ_1 is an orientation-preserving diffeomorphism with $\varphi_1^*\omega = \omega'$. □

Note that any closed orientable surface admits an orientation-reversing diffeomorphism (this follows from the classification of closed orientable surfaces, it is generally false for higher dimensional manifolds), and hence a symplectic form ω and its opposite $-\omega$ are also conjugate by an *orientation-reversing* diffeomorphism.

Another application of Moser's lemma is the following tubular neighborhood theorem for Lagrangian submanifolds originally due to Weinstein.

Proposition 2.30 (Weinstein neighborhood theorem). *Let (M, ω) be a symplectic manifold and $i : L \to M$ a Lagrangian embedding. There exists a neighborhood U of the zero-section L of T^*L and an embedding $j : U \to M$ such that $j = i$ on L and $j^*\omega = \omega_L$.*

To prove this we need to recall two results. The first is the classical tubular neighborhood theorem.

Proposition 2.31 (Tubular neighborhood theorem). *Let M be a manifold, $i : L \to M$ an embedding, $\nu L = (i^*TM)/TL \to L$ the normal bundle of i and $\pi : TM|L \to \nu L$ the projection. Given any bundle morphism $\Phi : \nu L \to TM|L$ covering i such that $\pi \circ \Phi = \mathrm{id}$, there exist a neighborhood U of the zero-section in νL and an embedding $\varphi : U \to M$ such that $\varphi = i$ on L and $d\varphi = di \oplus \Phi$ along L (with respect to the decomposition $T(\nu L) = TL \oplus \nu L$ along L).*

Sketch of proof. Pick a Riemannian metric on M such that the decomposition $\Phi(\nu L) \oplus di(TL) = TM$ along L is orthogonal. Let $\psi_t : TM \to TM$ be the geodesic flow with respect to this Riemannian metric. Then for $x \in L$ and $v \in \nu L_x$, set $\varphi(x, v) = (\pi \circ \psi_1)(i(x), \Phi_x(v))$. We check that $\varphi = i$ and $d\varphi = di \oplus \Phi$ along L, and hence φ embeds a neighborhood of the zero-section of νL into M. □

Note that the morphism Φ in the above proposition is determined by the subbundle $\Phi(\nu L)$ which is supplementary to TL in $TM|L$. For each $x \in L$, the space of such $\Phi_x : \nu L_x \to TM_x$ is an affine space, hence it is always possible to find a global morphism Φ using a partition of unity.

Lemma 2.32 (Relative Poincaré lemma). *Let M be a manifold, $i : L \to M$ an embedding, and α a closed k-form with $k \geq 1$ such that $i^*\alpha = 0$. Then there exists a $(k-1)$-form β defined on an open neighborhood of $i(L)$ such that β vanishes to order 1 along $i(L)$ and $\alpha = d\beta$.*

Proof. When L is a point, we can pick a chart to reduce to the case M is a vector space and L is the origin, which is the usual Poincaré Lemma. Recall a simple proof in this case : consider the radial vector field $X(x) = x$, its flow $\varphi_t(x) = e^t x$ and set $\beta = \int_{-\infty}^0 \varphi_t^*(\iota_X \alpha) dt$ so that $d\beta = \int_{-\infty}^0 \varphi_t^*(d\iota_X \alpha) dt = \int_{-\infty}^0 \varphi_t^*(\mathcal{L}_X \alpha) dt = \int_{-\infty}^0 \frac{d}{dt}(\varphi_t^* \alpha) dt = \alpha$ and $\beta = O(|x|^2)$ (since $\alpha = O(|x|)$ and $X = O(|x|)$).

In the general case, the proof is similar but involves Proposition 2.31. Consider the fiberwise radial vector field Y on νL, namely Y is tangent to each fiber νL_x and coincides with the radial vector field on νL_x. Pick a tubular neighborhood of L, namely a star-shaped open neighborhood U of the zero-section of νL and an embedding $\varphi : U \to M$ such that $\varphi = i$ on L. Consider $X = \varphi_* Y$ and its flow φ_t, and set $\beta = \int_{-\infty}^0 \varphi_t^*(\iota_X \alpha) dt$. This is well-defined on $\varphi(U)$ and the same computation as above shows that $d\beta = \alpha - \lim_{t \to -\infty} \varphi_t^* \alpha$. But using the assumption $i^*\alpha = 0$ and the fact that $d\varphi_t$ converges to the projection to L when $t \to -\infty$, we obtain $d\beta = \alpha$. Finally in a coordinates chart (x, v) where $\{v = 0\} = L$, we have $\alpha = O(|v|)$, $X = O(|v|)$ and hence $\beta = O(|v|^2)$ and β vanishes to order 1 along L. $\square$

Proof of Proposition 2.30. Since L is Lagrangian there is an isomorphism $\nu L \to T^*L$ defined by $v \mapsto i_v\omega$. Recall that in a symplectic vector space the space of Lagrangian subspaces transverse to a given one is an affine space. Hence, using a partition of unity, we can construct a field L' of Lagrangian subspaces transverse to TL along L. Next consider the unique bundle morphism $\Phi : \nu L \to TM$ with image L' and with $\pi \circ \Phi = \mathrm{id}$ where $\pi : TM|L \to \nu L$ is the projection. Then the bundle morphism $di \oplus d\varphi : TL \oplus T^*L \to TM|L$ is symplectic provided $TL \oplus T^*L = T(T^*L)|L$ is equipped with the standard symplectic structure. Proposition 2.31 applied with Φ provides a neighborhood U of the zero-section in T^*L and an embedding $\varphi : U \to M$ such that $\varphi^*\omega = \omega_L$ along L. From Lemma 2.32, we get a 1-form α defined near L in T^*L such that $\varphi^*\omega = \omega_L + d\ \alpha$ and α vanishes to order 1 along L. The end of the proof is similar to the proof of Proposition 2.28: use a cutoff function and apply Moser's lemma. When L is non-compact, it is a bit more difficult since one needs ϵ to be a function on L. $\square$

Corollary 2.33. *Let (M, ω) be a closed symplectic manifold. Then a neighborhood of id_M in $\mathrm{Diff}(M, \omega)$ can be identified with a neighborhood of 0 in the vector space of closed 1-forms on M.*

Proof. We equipp $M \times M$ with the form $\omega_2 = \omega \oplus (-\omega)$. Then the diagonal Δ is a Lagrangian submanifold of $M \times M$. By Proposition 2.30 there exists a neighborhood U of the zero-section of $T^*\Delta$ and an embedding $j : U \to M$ such that $j^*\omega_2 = \omega_\Delta$.

If $\psi \in \mathrm{Diff}(M, \omega)$, then its graph Γ_ψ is also Lagrangian. If ψ is sufficiently C^1-close to id_M, then Γ_ψ is a subset of $j(U)$ and $L := j^{-1}(\Gamma_\psi)$ is C^1-close to the zero-section. Hence L can be written as the graph of 1-form: $L = \{(\alpha(q), q);\ q \in \Delta\}$ for $\alpha \in \Omega^1(L)$. Since L is Lagrangian, α is closed.

We can argue in the reverse direction and associated an element of $\mathrm{Diff}(M, \omega)$ with any closed 1-form close enough to 0. $\square$

Corollary 2.34. *Let (M, ω) be a closed symplectic manifold and let $\psi \in \mathrm{Ham}(M, \omega)$. If ψ is C^1-close enough to id_M, then ψ has at least two fixed points.*

Proof. We use the notations in the proof of the previous corollary. We assume $\psi = \psi^1$ for an isotopy ψ^t. Then Γ_ψ is the image of Δ by the isotopy $\mathrm{id}_M \times \psi^t$ of M^2. Hence L is the image of the zero-section by a Hamiltonian isotopy. This implies that the form α is exact. Hence $\alpha = df$ for some function f. The maximum and minimum of f give two intersection points of L with the zero-section. They correspond to fixed points of ψ. $\square$

Chapter 3

Contact geometry

3.1 Integrability of planes fields

Definition 3.1. Let M be an n-dimensional manifold. A *plane field* of codimension k is a (smooth) subbundle ξ of rank $(n-k)$ of the tangent bundle TM.

Such a plane field can be described locally as $\xi = \cap_{i=1}^k \ker \alpha_i$ for some linearly independent 1-forms α_i, or alternatively as $\xi = \langle X_1, \dots, X_{n-k}\rangle$ for some linearly independent vector fields X_j.

For $k = n-1$, it is a line field, and is locally spanned by a non-vanishing vector field: $\xi = \langle X \rangle$. The Cauchy-Lipschitz theorem about ordinary differential equations says that, at any point, there are smooth curves everywhere tangent to X.

This is no longer true for $k = n-2$: on $\mathbf{R}^3$ consider the plane field $\xi = \ker(dz - ydx)$ and assume that some surface S is everywhere tangent to ξ, then S is a graph $z = f(x, y)$ with $df = ydx$ which is a impossible since $d^2 f = 0$ and $d(ydx) = dy \wedge dx$.

Theorem 3.2 (Frobenius)**.** *Let ξ be a codimension-k plane field on a manifold M of dimension n. The following conditions are equivalent:*

1. *for each point $p \in M$, there exists a codimension k submanifold everywhere tangent to ξ and containing p,*

2. *for each $p \in M$ and for any local sections X, Y of ξ near p, we have $[X, Y] \in \xi$,*

3. *for each point $p \in M$ and any local vector fields X_i near p such that $\langle X_1, \dots, X_{n-k}\rangle = \xi$ there exists functions c_{ij}^l for $1 \le i, j, l \le n-k$ such that $[X_i, X_j] = \sum_l c_{ij}^l X_l$.*

4. *near each point $p \in M$, if α is a 1-form with $\xi \subset \ker \alpha$, then $d\alpha$ vanishes on ξ,*

5. *near each point $p \in M$, if α_i, $i = 1, \dots k$ are 1-forms such that $\xi = \cap_i \ker \alpha_i$ then there exists 1-forms β_{ij}, $1 \le i, j \le k$, such that $d\alpha_i = \sum_j \beta_{ij} \wedge \alpha_j$.*

Proof. (1) $\Rightarrow$ (2): let N be a codimension k submanifold tangent to ξ and containing p. Then X and Y are tangent to N, and hence $[X, Y] \in TN = \xi$.

(2) $\Rightarrow$ (1) : The statement is local, we may take $M = \mathbf{R}^n = \mathbf{R}^k \times \mathbf{R}^{n-k}$, $p = (0,0)$, $\xi_{(0,0)} = \{0\} \times \mathbf{R}^{n-k}$. Consider the projection $\pi : \mathbf{R}^n \to \mathbf{R}^{n-k}$. The vector fields $\frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n}$ of $\mathbf{R}^{n-k}$ uniquely lift to vector fields $X_{k+1}, \ldots, X_n$ under the projection π near $(0,0)$. On the one hand, $[X_i, X_j] \in \xi$ by hypothesis and on the other hand $d\pi([X_i, X_j]) = 0$ since $[\frac{\partial}{\partial_{x_i}}, \frac{\partial}{\partial_{x_j}}] = 0$. Hence $[X_i, X_j] = 0$ and the flows φ_j^t of X_j, $k+1 \leq j \leq n$, commute. This allows to define a map $f : \mathbf{R}^{n-k} \to \mathbf{R}^n$ in a neighborhood of 0 by $f(x_{k+1}, \ldots, x_n) = \varphi_{k+1}^{x_{k+1}} \circ \cdots \circ \varphi_n^{x_n}(0)$. We have $\frac{\partial f}{\partial x_{k+1}} = X_{k+1} \circ f$ and, using the commutativity of the flows, also $\frac{\partial f}{\partial x_j} = X_j \circ f$ for all $k+1 \leq j \leq n$. In particular, f is an immersion near 0 and it is everywhere tangent to ξ.

(2) $\Leftrightarrow$ (3): obvious.

(2) $\Leftrightarrow$ (4): it follows from the formula

$$d\alpha(X,Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X,Y]).$$

Indeed, assuming (2) if $\xi \subset \ker\alpha$ and $X, Y \in \xi$, extend X, Y to local sections of ξ. Hence $\alpha(X)$ and $\alpha(Y)$ vanish identically and we have $d\alpha(X,Y) = \alpha([X,Y])$ since $[X,Y] \in \xi \subset \ker\alpha$. Assuming (3), consider local 1-forms $\alpha_1, \ldots, \alpha_k$ with $\xi = \cap_i \ker\alpha_i$. If X, Y are local sections of ξ, then $\alpha_i[X,Y] = d\alpha_i(X,Y) = 0$ for all i and hence $[X,Y] \in \xi$.

(5) $\Rightarrow$ (4) is obvious. For (4) $\Rightarrow$ (5), pick local 1-forms $\alpha_{k+1}, \ldots, \alpha_n$ in order to form a local basis of T^*M, and a dual basis $(e_1, \ldots, e_n)$ of TM, i.e. $\alpha_i(e_j) = \delta_{ij}$. Then $(e_{k+1}, \ldots, e_n)$ form a basis of $\xi = \cap_{1\leq i\leq k} \ker\alpha_i$. Write $d\,\alpha_l = \sum_{1\leq i<j\leq n} c_{i,j}^l\ \alpha_i \wedge \alpha_j$ and since, for $l \leq k$, and $i, j \geq k+1$, $d\alpha_l(e_i, e_j) = c_{i,j}^l = 0$, we get the required formula. □

Definition 3.3. A hyperplane field satisfying one of the equivalent conditions of Theorem 3.2 is called *integrable.*

A small variation on the proof of theorem 3.2 shows that, an integrable plane field can be locally mapped to the horizontal plane field in a chart $\mathbf{R}^k \times \mathbf{R}^{n-k}$. Also the submanifolds everywhere tangent to ξ are uniquely determined by the plane field. Hence the manifold M is decomposed as a disjoint union of submanifolds which locally look like the product $\mathbf{R}^n \times \mathbf{R}^{n-k}$. Such a structure is also called a *foliation* of codimension k.

When ξ is a hyperplane field (i.e. in the case $k = 1$), the condition $d\alpha$ vanishes on ξ for some local form α with $\xi = \ker\alpha$ is equivalent to $\alpha \wedge d\alpha = 0$ and also to the existence of β such that $d\alpha = \beta \wedge \alpha$.

3.2 Contact manifolds

Definition 3.4. A *contact form* on a manifold M is a 1-form α such that $d\alpha$ is non-degenerate on $\ker\alpha$. A *contact structure* is a hyperplane field which is locally the kernel of a contact form. A *contact manifold* (M, ξ) is a manifold M endowed with a contact structure ξ.

The condition that $d\alpha$ is non-degenerate on $\ker\alpha$ implies that $\ker\alpha$ has even dimension $2n$, and thus M has dimension $2n+1 \geq 1$. Also the non-degeneracy condition is equivalent to $\alpha \wedge (d\alpha)^n \neq 0$ everywhere. If ξ is a hyperplane field and α, β are local 1-forms with $\xi = \ker\alpha = \ker\beta$, then there exists a non-vanishing function f with $\beta = f\alpha$ and thus $d\beta = df \wedge \alpha + f d\alpha$ implies $d\beta$ coincide with

$d\alpha$ on ξ up to a non-zero factor. Hence the condition that ξ is a contact structure is independent on the choice of defining 1-form, and thus any such 1-form is a contact form. In fact, the contact condition can be written without reference to a contact form as follows. Let ξ be a hyperplane field and consider the skew-symmetric bilinear form $\kappa_\xi : \xi \times \xi \to TM/\xi$ defined by $(X,Y) \mapsto -[X,Y] \mod \xi$ where the tangent vectors X and Y are extended arbitrarily to local sections of ξ. One checks that the expression $[X,Y] \mod \xi$ is C^∞-linear in X,Y and hence independent of the choice of local extensions. If $\xi = \ker\alpha$ for some local 1-form α and X,Y are local sections of ξ, then $d\alpha(X,Y) = \mathcal{L}_X(\alpha(Y)) - \mathcal{L}_Y(\alpha(X)) - \alpha([X,Y]) = \kappa(X,Y)$ under the isomorphism $\alpha : TM/\xi \to \mathbf{R}$. Hence ξ is a contact structure if and only if κ_ξ is non-degenerate.

The standard contact structure on $\mathbf{R}^{2n+1}$ with coordinates $(p_1,\dots,p_n,q_1,\dots,q_n,z)$ is $\xi = \ker\alpha$, where $\alpha = dz - \sum_i p_i dq_i$.

Let M be a manifold and $J^1M = \mathbf{R} \times T^*M$ the 1-jet space of M. Let z be the coordinate on $\mathbf{R}$ and λ_M the Liouville 1-form on T^*M. Then the 1-form $\alpha_M = dz - \lambda_M$ on J^1M is a contact form. A smooth function $f\colon M \to \mathbf{R}$ defines a section $s = J^1f$ of the fiber bundle $J^1M \to M$ by $J^1f(x) = (f(x), df(x))$. This section is tangent to $\ker\alpha_M$ (in other words $(J^1f)^*(\alpha_M) = 0$). Conversely let $s\colon M \to J^1M$, $s(x) = (f(x), \beta(x))$, be a section of J^1M; if s is tangent to $\ker\alpha_M$, then $s = J^1f$.

Let $M = S^{2n+1}$ be the unit sphere in $\mathbf{R}^{2n+2} = \mathbf{C}^{n+1}$. At each point $x \in M$, the complex part $\xi_x = T_xM \cap JT_xM$ of its tangent space has codimension 1. Then ξ is a contact structure on M which is called the standard contact structure on S^{2n+1}. A contact form is $\lambda_0 = \frac{1}{2}\sum_{j=0}^n y_j dx_j - x_j dy_j$.

Proposition 3.5. *Let (M,ξ) be a contact manifold of dimension $2n+1$. If L is a submanifold everywhere tangent to ξ, then $\dim L \leq n$.*

Proof. If $i : L \to M$ is the inclusion and α is a contact form for ξ, then $i^*\alpha = 0$, and thus $i^*d\alpha = 0$. Hence $di(TL)$ is an isotropic subspace of the symplectic vector space $(\xi, d\alpha)$, we find $\dim L \leq n$. □

Definition 3.6. A submanifold of a contact manifold which is everywhere tangent to the contact structure is called *isotropic*. An isotropic submanifold of maximal dimension is also called *Legendrian*.

Let (M,ξ) be a contact manifold and let α be a choice of contact form (the existence of α implies that ξ is co-oriented). There exists a unique vector field $Y = Y_\alpha$ on M such that $\alpha(Y) = 1$ and $\iota_Y d\alpha = 0$. Indeed the second condition says that Y is in $\ker d\alpha$ which is a line since $d\alpha$ is of rank $(\dim M - 1)$ and the first condition normalizes Y. This Y_α is called the *Reeb* vector field of α. We have $\mathcal{L}_{Y_\alpha}(\alpha) = 0$ and the flow of Y_α preserves α, hence ξ.

Let (M,ξ) be a contact manifold such that ξ is co-orientable. We choose a co-orientation. A *contactomorphism* of (M,ξ) is a diffeomorphism ψ of M which preserves ξ and its co-orientation. If α is a contact form for ξ, this is equivalent to $\psi^*(\alpha) = e^h\alpha$, where h is some function on M.

A *contact isotopy* is a smooth family of contactomorphisms ψ_t such that $\psi_t^*(\alpha) = e^{h_t}\alpha$, for some family of functions h_t, and $\psi_0 = \mathrm{id}_M$. When ψ_t is the flow of a vector field X_t, we obtain

$$\psi_t^*\mathcal{L}_{X_t}(\alpha) = \frac{d}{dt}\psi_t^*(\alpha) = \frac{d}{dt}(e^{h_t}\alpha) = \frac{dh_t}{dt}e^{h_t}\alpha = \psi_t^*(g_t\alpha),$$

where $g_t = \frac{dh_t}{dt} \circ \psi_t^{-1}$. Hence $\mathcal{L}_{X_t}(\alpha) = g_t\alpha$. Conversely, if $\mathcal{L}_{X_t}$ satisfies such a condition, it generates a contact isotopy. Hence we say that a vector field X is a *contact field* if $\mathcal{L}_X\alpha = g\alpha$ for some contact form α and some function g.

Lemma 3.7. *Let (M, ξ) be a contact manifold and α a contact form. Let Y_α be the Reeb vector field of α.*

(i) A vector field X is a contact vector field if and only if there exists a function $H\colon M \to \mathbf{R}$ such that

$$\iota_X\alpha = H, \qquad \iota_X d\alpha = (Y_\alpha(H))\alpha - dH. \tag{3.1}$$

(ii) For any function H on M there exists a unique (contact) vector field X satisfying (3.1).

Proof. (i) If (3.1) holds, then $\mathcal{L}_X(\alpha) = d(\iota_X\alpha) + \iota_X d\alpha = (Y_\alpha(H))\alpha$. Hence $\mathcal{L}_X\alpha = g\alpha$ with $g = Y_\alpha(H)$. Conversely, if $\mathcal{L}_X\alpha = g\alpha$, we set $H = \iota_X\alpha$ and we have

$$\iota_X d\alpha = \mathcal{L}_X\alpha - d(\iota_X\alpha) = g\alpha - dH.$$

Contracting with Y_α and using the definition of Y_α, we find $g = Y_\alpha(H)$.

(ii) For any $x \in M$ we have $T_xM = \langle Y_\alpha\rangle + \xi$. Let us write a vector field X as $X = fY_\alpha + Z$, with $Z \in \xi$. The second condition in (3.1) determines Z, since $d\alpha|_\xi$ is non-degenerate. The first condition gives f (and we find $f = H$). □

We have thus a correspondence between contact vector fields and functions on M. The Reeb vector field corresponds to $H = 1$.

Remark 3.8. There is another way to phrase Lemma 3.7 which does not involve a choice of contact form: the map which takes a contact vector field X to its corresponding section of TM/ξ is a bijection.

3.3 Moser's lemma

The application of Moser's lemma gives the following stability result for contact structures, originally due to Gray.

Theorem 3.9 (stability near a compact subset for contact structures)**.** *Let M be a manifold, K a compact subset of M and $(\xi_t)_{t\in[0,1]}$ a smooth family of contact structures on M such that $\xi_t = \xi_0$ on $TM_{|K}$. Then there exists an open neighborhood U of K and an isotopy of embeddings $(\varphi_t)_{t\in[0,1]} : U \to M$ such that $\varphi_t = \mathrm{id}$ on K, φ_0 is the inclusion and $\varphi_t^*\xi_t = \xi_0$.*

Proof. The isotopy φ_t will be generated by a time-dependent vector field X_t. Locally ξ_t is the kernel of a 1-form α_t. We define X_t by the conditions

$$X_t \in \xi_t \tag{3.2}$$

$$(X_t \lrcorner d\alpha_t + \dot\alpha_t) \wedge \alpha_t = 0 \tag{3.3}$$

Condition (3.3) is equivalent to $X_t \lrcorner d\,\alpha_t = -\dot\alpha_t$ when restricted to ξ_t and also to the existence of a function f_t such that $X_t \lrcorner d\alpha_t + \dot\alpha_t = f_t\alpha_t$. Since $d\alpha_t$ is non-degenerate on ξ_t, the vector field

X_t is well-defined by (3.2) and (3.3). Moreover, the vector field X_t is independent of the choice of contact form α_t: if $\beta = f_t\alpha_t$ with $f_t \neq 0$, we also have $(X_t \lrcorner d\beta_t + \dot{\beta}_t) \wedge \beta_t = 0$. So the definition of X_t makes sense globally on M. Recall from the theory of ODE's that the domain of definition of the flow φ_t of X_t is an open subset of $\mathbf{R} \times M$. Since $X_t = 0$ on K, this open subset contains $K \times [0,1]$, hence it contains $U \times [0,1]$ for some open neighborhood U of K. It remains to check that $\varphi_t^*\xi_t = \xi_0$:

$$\begin{aligned}\frac{d}{dt}(\varphi_t^*\alpha_t \wedge \alpha_0) &= \varphi_t^*(\mathcal{L}_{X_t}(\alpha_t) + \dot{\alpha}_t) \wedge \alpha_0\\ &= \varphi_t^*(f_t\alpha_t) \wedge \alpha_0\\ &= f_t \circ \varphi_t(\varphi_t^*\alpha_t \wedge \alpha_0),\end{aligned}$$

and $\varphi_0^*\alpha_0 \wedge \alpha_0 = 0$, hence $\varphi_t^*\alpha_t \wedge \alpha_0 = 0$ for all $t \in [0,1]$. □

Theorem 3.10 (Gray's theorem, closed manifold)**.** *Let M be a closed manifold and $(\xi_t)_{t\in[0,1]}$ a smooth family of contact structures on M. Then there exists an isotopy $(\varphi_t)_{t\in[0,1]}$ of M such that $\varphi_t^*\xi_t = \xi_0$.*

Proof. The proof is the same (actually, simpler) as for the previous statement: the vector field X_t defined by (3.2) and (3.3) integrates to an isotopy $(\varphi_t)_{t\in[0,1]}$ because M is closed. □

As in the symplectic case, this allows to prove the following local normal form theorem.

Theorem 3.11 (Darboux's theorem for contact structures)**.** *Let (M,ξ) be a contact manifold of dimension $2n+1$ and p a point of M. There exists an open set U of M containing p, an open set V of $\mathbf{R}^{2n+1}$ containing 0 and a diffeomorphism $\varphi : U \to V$ such that $\varphi(p) = 0$ and $\varphi_*\xi = \ker(dz - \sum y_i dx_i)$.*

Proof. Set $\alpha = dz - \sum y_i dx_i$ and $\zeta = \ker\alpha$. There is a chart $\psi : U \to V$ such that $\psi(p) = 0$, $(\psi_*\xi)_0 = \zeta_0$. Then ψ induces a isomorphism $TM_p/\xi_p \to \mathbf{R}^{2n+1}/\zeta_0 = \mathbf{R}$. By the classification of linear symplectic forms, we may modify ψ by postcomposition with a linear map of $\mathbf{R}^{2n+1}$ of the form $\begin{pmatrix} A & 0\\ 0 & 1\end{pmatrix}$ so that $d\psi_p$ conjugates the forms $(\kappa_\xi)_p$ and $(\kappa_\zeta)_0$. This means that if α is any contact form for ξ near p, then $(\psi_* d\beta)_0$ is a non-zero multiple of $d\alpha_0$ when restricted to ζ_0. Up to composing ψ with $(x_i, y_i, z) \mapsto (x_i, y_i, -z)$ we can assume that this multiple is positive. Then consider $\alpha_t = (1-t)\alpha + t\psi_*\beta$ and $\zeta_t = \ker\alpha_t$, and observe that $d\alpha_t$ is non-degenerate on ζ_t at 0 and thus in a neigborhood of 0. □

Theorem 3.12 (Darboux's theorem for contact forms)**.** *Let M be a manifold of dimension $2n+1$, α a contact form on M and p a point of M. There exists an open set U of M containing p, an open set V of $\mathbf{R}^{2n+1}$ containing 0 and a diffeomorphism $\varphi : U \to V$ such that $\varphi(p) = 0$ and $\alpha = \varphi^*(dz - \sum y_i dx_i)$.*

Proof. Using Darboux's theorem for contact structures, it is enough to prove that two contact forms α and β in a neighborhood of 0 in $\mathbf{R}^{2n+1}$ for the same contact structure are conjugate. We may assume that $\ker\alpha = \{z = 0\}$ at 0. We set $\alpha_t = (1-t)\alpha + t\beta$. We look for an isotopy φ_t

generated by $X_t = Y_t + H_t R_t$ where R_t is the Reeb vector field of α_t and $\alpha_t(Y_t) = 0$. Deriving the equation $\varphi_t^* \alpha_t = \alpha_0$ with respect to t gives the equations:

$$\dot{\alpha}_t(R_t) = -dH_t(R_t) \tag{3.4}$$

$$(Y_t \lrcorner d\alpha_t + \dot{\alpha}_t + dH_t) \wedge \alpha_t = 0 \tag{3.5}$$

We set $H_t = 0$ on $\mathbf{R}^{2n} \times \{0\}$ and then H_t is uniquely determined by (3.4) in a neighborhood of 0 since R_t is transverse to $\mathbf{R}^{2n} \times \{0\}$: explicitly, if ψ_t is the flow of R_t, then $H_t(\psi_t(x)) = -\int_0^t \dot{\alpha}_s(R_s \circ \psi_s) ds$. Then the vector field Y_t is uniquely determined by (3.5) and we observe that $Y_t = 0$ on $V \times \{0\} \times [0,1]$ (since $\dot{\alpha}_t \wedge \alpha_t = 0$). Hence the flow φ_t of X_t is well-defined for $t \in [0,1]$ on some neighborhood of 0 and we have $\varphi_t^* \alpha_t = \alpha_0$. □

There is also a neighborhood theorem for Legendrian submanifolds analogous to the one for Lagrangian submanifolds in symplectic geometry.

Theorem 3.13 (Weinstein's theorem for Legendrian submanifolds)**.** *Let (M, ξ) be a contact manifold and L a closed Legendrian submanifold of M. There exists a neighborhood U of the zero-section in $J^1 L$ and a contact embedding $U \to M$ which is the identity on L.*

Proof. Let $\xi_L = \ker(\alpha_L)$ be the contact structure on $J^1 L$. In the (conformal) symplectic bundle $\xi \to L$, there exists a Lagrangian subbundle which is supplementary to TL. This allows to define a bundle isomorphism $T^*L \oplus \mathbf{R} \simeq \nu_L J^1 L \to \nu_L M$ which sends ξ_L to ξ and also the curvature form κ_{ξ_L} to κ_ξ. The tubular neighborhood theorem allows to find a neighborhood of the zero-section U and an embeding $\varphi : U \to M$ which induces the above bundle isomorphism along L. Then we proceed as in the proof of Darboux's theorem: pick a contact form α for ξ and a linear path joining $\varphi_* \alpha_L$ and α. This gives a path of contact structures in a neighborhood of L and we can apply Theorem 3.9. □

Chapter 4

Morse cohomology

4.1 Morse functions and gradient vector fields

Let V be a manifold of dimension n. Morse theory provides links between the topology of V and real valued fonctions on V.

Definition 4.1. A function $f : V \to \mathbf{R}$ is called a *Morse function* if all its critical points are non-degenerate.

The *Hessian* of f at a critical point of p is well-defined as a quadratic form on T_qV. Its *index* (i.e., maximal dimension of a negative definite subspace) is denote $\mathrm{ind}(p)$. Observe that a function $f : V \to \mathbf{R}$ is Morse if and only if the section $df : V \to T^*V$ is transverse to the zero-section. In particular the critical points of a Morse function are isolated.

For example the function $(x, y, z) \mapsto z$ is a Morse function on $S^2 = \{x^2 + y^2 + z^2 = 1\}$. It has two critical points $(0, 0, -1)$ and $(0, 0, 1)$ of respective indices 0 and 2. They are non-degenerate since, in the local coordinates (x, y), we have $f = \pm\sqrt{1 - x^2 - y^2} = 1 - \frac{1}{2}(x^2 + y^2) + o(|x, y|^2)$. The function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^3$ is not Morse: 0 is a degenerate critical point. However $f(x) = x^3 - \epsilon x$ for $\epsilon > 0$ is Morse.

Lemma 4.2. *If $p \in V$ is a critical point of a function $f : V \to \mathbf{R}$, then there exists (so-called* Morse*) coordinates $(x_1, \dots, x_n)$ where $f = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$.*

Proof. According to the classification of quadratic forms on a real vector space, it is enough to prove that a Morse function $f : \mathbf{R}^n \to \mathbf{R}$ such that $f(0) = 0$ and $df(0) = 0$ is conjugate to its Hessian Q at 0 near 0. The functions $y_i = \frac{\partial f_t}{\partial x_i}$ are coordinates near the origin, hence Hadamard's lemma gives that each function g which vanishes at 0 can be written $g = \sum_i y_i g_i(y_1, \dots, y_n)$, and hence $g = \sum_i \frac{\partial f_t}{\partial x_i} g_i'(x_1, \dots, x_n)$ where $g_i'(x) = g_i(y)$. We set $f_t = (1 - t)f + tQ$ and look for an isotopy φ_t generated by a vector field X_t such that $\varphi_t^* f_t = f_0$. Deriving with respect to t gives $df_t(X_t) + \dot{f}_t = 0$. As explained above, there exists functions X_t^i such that $\dot{f}_t = -\sum_i X_t^i \frac{\partial f_t}{\partial x_i} = -df_t(X_t)$ where $X_t = \sum X_t^i \partial_{x_i}$. Moreover $X_t(0) = 0$ since $\dot{f}_t = Q - f = o(|x|^2)$. We obtain $Q = f \circ \varphi_1$ as desired. $\square$

Any manifold admits a Morse function and in fact a generic function is Morse. To prove this, we start with some basic things about transversality.

Definition 4.3. Let M, N be manifolds and P a submanifold of N. We say that a map $f : M \to N$ is *transverse* to P if for each $x \in M$ such that $f(x) \in P$, we have $df(T_xM) + T_{f(x)}P = T_{f(x)}N$.

An equivalent way to write the transversality condition is : the map $df : T_xM \to T_{f(x)}N/T_{f(x)}P$ is surjective. When $\dim M + \dim P < \dim N$, then f is transverse to P is equivalent to $f(M) \cap P = \emptyset$. When P is reduced to a point y, f is transverse to P is equivalent to y is a regular value of f.

Proposition 4.4. *Let $f : M \to N$ be a smooth map transverse to a submanifold P of N. Then $f^{-1}(P)$ is a submanifold of M.*

Proof. Let $x \in f^{-1}(P)$, there is an open set U containing $f(x)$ and a submersion $u \colon U \to \mathbf{R}^k$ such that $P \cap U = u^{-1}(0)$. Then $(u \circ f)^{-1}(0) = f^{-1}(P) \cap f^{-1}(U)$ and $f \circ u : f^{-1}(U) \to \mathbf{R}^k$ is a submersion. Indeed, $d_{f(x)} : T_xM \to T_{f(x)}N/T_{f(x)}P$ is surjective and $du : T_{f(x)}N/T_{f(x)}P \to \mathbf{R}^k$ is an isomorphism and $d_x(u \circ f)$ is the composition of these maps. □

The following result says that regular values are plentiful.

Theorem 4.5 (Sard)**.** *Let M, N be manifolds and $f : M \to N$ a smooth map. The set of critical values of f is of Lebesgue measure zero.*

There is no canonical measure on a manifold. However the notion of *subset of Lebesgue measure zero* (or *negligible* subset) makes sense: it means the image by any chart is of Lebesgue measure zero in $\mathbf{R}^n$. Since diffeomorphisms between open subsets of $\mathbf{R}^n$ map negligible subsets to negligible subsets, and since a manifold admits a *countable* atlas, the notion is well-defined. Using another terminology, Sard's theorem says that *almost all* values are regular (be careful that, for $y \in N$, if $y \notin f(M)$, y is also called a regular value (sic)).

We would like an analogous result for the more general concept of transversality. This will be achieved via the following lemma.

Lemma 4.6. *Let M, T, N be manifolds and P a submanifold of N. Let $F : M \times T \to N$ be a smooth map which is transverse to P, $\Sigma = F^{-1}(P)$. Let $\pi : \Sigma \to T$ be the projection. For $t \in T$, we denote by $f_t : M \to N$ the restriction of F to $M \times \{t\}$. Then f_t is transverse to P if and only if t is a regular value of π.*

Proof. Let $(x, t) \in \Sigma$. The equivalence is proved by inspecting the following diagram, where the vertical and horizontal lines are exact sequences.

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & T_xM \arrow[d] \arrow[dr, "df_t"] & & \\
0 \arrow[r] & T_{(x,t)}\Sigma \arrow[r] \arrow[dr, "d\pi"'] & T_{(x,t)}(M \times T) \arrow[r, "dF"] \arrow[d] & T_{f(x,t)}N/T_{f(x,t)}P \arrow[r] & 0 \\
 & & T_tT \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

□

Theorem 4.7 (Transversality theorem). *Let M, T, N be manifolds, P a submanifold of N and $F : M \times T \to N$ a smooth map which is transverse to P. The restriction $f_t : M \to N$ of F to $M \times \{t\}$ is transverse to P for almost all t.*

Proof. This follows directly from Theorem 4.5 and Lemma 4.6. □

Theorem 4.8. *For any manifold V there exist Morse functions on V.*

Proof. Recall that a function g is Morse if $dg\colon V \to T^*V$ is transverse to the zero-section $0_V \subset T^*V$.

(i) We first prove that, for any relatively compact open subset $W \subset V$, there exists $f\colon V \to \mathbf{R}$ such that $f|_W$ is Morse. We cover $\overline{W}$ by charts U_i, $i = 1 \ldots, N$, and find $U_i' \subset U_i$ compact, such that $\overline{W} \subset \bigcup_{i=1}^N U_i'$. We choose $\varphi_i\colon V \to \mathbf{R}$ with support in U_i such that $\varphi_i = 1$ on U_i'. We have coordinates $(x_1^i, \ldots, x_n^i)$ on U_i. We set $k = nN$ and define $f_1, \ldots, f_k\colon V \to \mathbf{R}$ by and set $f_{(n-1)i+j} = \varphi_i \cdot x_j^i$. Then the differentials $df_1(x), \ldots, df_k(x)$ generate T_x^*V, for each $x \in W$.

(ii) For $\underline{a} = (a_1, \ldots, a_k)$ we set $g_{\underline{a}} = \sum_i a_i f_i$. We also define $H\colon W \times \mathbf{R}^k \to T^*W$, by $H(x, \underline{a}) = \sum_i a_i df_i(x)$, $h_{\underline{a}} = H|_{W \times \{\underline{a}\}}$. Hence $dg_{\underline{a}} = h_{\underline{a}}$. We claim that H is transverse to $0_W \subset T^*W$. Indeed, for any $x \in W$, we have $T_{(0,x)}T^*W \simeq T_0(T_x^*W) \oplus T_x 0_W \simeq T_x^*W \oplus T_x 0_W$. It is then enough to check that $\operatorname{im}((dH_x)_{\underline{a}}) = T_x^*W$, for $(x, \underline{a})$ such that $H(x, \underline{a}) \in 0_W$, where H_x is the restriction $H_x\colon \{x\} \times \mathbf{R}^k \to T_x^*W$. We clearly have $(dH_x)_{\underline{a}}(\underline{b}) = \sum b_i df_i(x)$ and these sums generate T_x^*W by hypothesis. Hence H is transverse to $0_W \subset T^*W$. By the transversality theorem there exists $\underline{a}$ such that $dg_{\underline{a}} = h_{\underline{a}}$ is transverse to 0_W. Hence $g_{\underline{a}}$ is Morse.

(iii) In general we can find an increasing family W_n, $n \in \mathbf{N}$, of compact subsets such that $V = \bigcup_n W_n$ and W_n is contained in the interior of W_{n+1}. We can find $f_n\colon V \to \mathbf{R}$ which is Morse on some neighborhood of $W_n \setminus W_{n-1}$. We can assume that $\operatorname{supp} f_n \subset W_{n+1} \setminus W_{n-2}$. Hence $g_0 = \sum_n f_{2n}$ and $g_1 = \sum_n f_{2n+1}$ are well-defined. As in (ii) we can see that a generic combination $a_0 g_0 + a_1 g_1$ is Morse. □

Definition 4.9. A vector field X is called an *adapted gradient vector field* for a Morse function $f : V \to \mathbf{R}$ if $df(X) > 0$ away from the critical points of f and near each critical point, there exist Morse coordinates $(x_1, \dots, x_n)$ where $X = -x_1\partial_{x_1} - \dots - x_k\partial_{x_k} + x_{k+1}\partial_{x_{k+1}} + \dots + x_n\partial_{x_n}$. In this situation we call (f, X) a *Morse pair.*

Proposition 4.10. *Any Morse function $f : V \to \mathbf{R}$ admits an adapted gradient vector field.*

Proof. Near a point p which is not critical, a vector field which is not tangent to the level sets of f is an adapted gradient. Near a critical point, an adapted gradient can be constructed according to Lemma 4.2. Hence we can pick an open cover U_i consisting of one open set for each critical point and other open sets disjoint from the critical points, and adapted gradient vector field X_i on U_i. Pick a partition of unity ρ_i subordinated to the cover U_i and set $X = \sum \rho_i X_i$, it is an adapted gradient vector field. □

The above proposition can also be proved as follows: pick a Riemannian metric g on V which coincides with $\frac{1}{2}\sum dx_i^2$ in some Morse charts near the critical points (g can be similarly constructed using a partition of unity) and consider $X = \nabla_g f$. Observe however that if g is chosen arbitrarily then $\nabla_g(f)$ is not an adapted gradient since we may not find appropriate coordinates near the critical points (for instance the eigenvalues of $\nabla_g(f)$ at critical points could be different from ± 1). It is technically simpler to work with adapted gradient vector fields though it is not essential.

Lemma 4.11. *Let (f, X) be a Morse pair on a closed manifold V and p a critical point of f of index k. The subset $W^s(p) = \{q \in V; \varphi_X^t(q) \underset{t\to+\infty}{\longrightarrow} p\}$ is a submanifold diffeomorphic to $\mathbf{R}^k$. Similarly $W^u(p) = \{q \in V; \varphi_X^t(q) \underset{t\to-\infty}{\longrightarrow} p\}$ is diffeomorphic to $\mathbf{R}^{n-k}$.*

Proof. Near p we have coordinates $(x_1, \dots, x_n)$ where $X = -\sum_{i\le k} x_i\ \partial_{x_i} + \sum_{i>k} x_i\partial_{x_i}$. For a sufficiently small neighborhood U of 0, we have $W^s(p) \cap U = (\mathbf{R}^k \times \{0\}) \cap U$. Indeed, for any other point q, $f \circ \varphi_t(q)$ will get bigger than $f(p)$ and thus $\varphi_t(q)$ cannot converge to p. Hence $W^s(p)$ is a submanifold near p. Since φ_t preserves $W^s(p)$ and for any q, $\varphi_t(q) \in U$ for sufficiently large t, we get that $W^s(p)$ is a submanifold. Denote by $i : \mathbf{R}^k \to W^s(p)$ the identification above defined in a neighborhood of 0 and extend it by the formula $i(x) = \varphi_t(i(e^{-t}x))$ with t so large that $e^{-t}x$ enters the domain of definition of i. This is the required diffeomorphism. The result for $W^u(p)$ follows from the above applied to the vector field $-X$ which is an adapted gradient for $-f$. □

Proposition 4.12. *Let (f, X) be a Morse pair on a closed manifold V. Then V is the disjoint union of all $W^s(p)$ for p critical point of f.*

Proof. Let $q \in V$, by compactness of V there is an increasing sequence t_n converging to $+\infty$ such that $\varphi_X^{t_n}(q)$ converges to some point $p \in V$. If p were not critical point, then for all points r sufficiently close to p, we would have $f \circ \varphi_X^t(r) > f(p)$ for some $t > 0$. Hence for some large enough n we have t such that $f(\varphi_X^{t+t_n}(q)) > f(p)$ which contradicts the fact that $\varphi_X^{t_n}(q)$ converges to p. Hence p is a critical point, and for large enough n, $\varphi_X^{t_n}(q)$ enters a Morse chart where X has a standard form and we get that $\varphi_X^t(q)$ converges to p when $t \to +\infty$. □

According to the above proposition, the manifold V is covered by disjoint open sets each diffeomorphic to some $\mathbf{R}^k$. This decomposition of V was first observed R. Thom.

Here is an example of how a Morse function can help us understand the topology of a manifold.

Proposition 4.13. *Let V be a compact manifold with boundary and $f : V \to \mathbf{R}$ a Morse function, we assume that $\partial V = f^{-1}(-1) \cup f^{-1}(1)$ and that -1 and 1 are regular values of f, df is positive (resp. negative) on inward pointing vectors along $f^{-1}(-1)$ (resp along $f^{-1}(1)$). If f has no critical points, then V is diffeomorphic to $f^{-1}(-1) \times [-1, 1]$.*

Proof. Pick a gradient vector field X and set $Y = \frac{X}{df(X)}$. Then the map $f^{-1}(-1) \times [-1, 1] \to V$ defined by $(q, t) \mapsto \varphi_Y^t(q)$ is a diffeomorphism. Indeed arguing similarly as in Proposition 4.12 we see that, since M is compact and f has no critical points, any trajectory of Y goes from $f^{-1}(-1)$ to $f^{-1}(1)$. □

Since $V \times [-1, 1]$ is diffeomorphic to $V \times [-1, -1 + \varepsilon]$ for any $\varepsilon > 0$ through a diffeomorphism which is id near $V \times \{-1\}$ we can reformulate the proposition as follows: for any $a < b \in \mathbf{R}$ such that $[a, b]$ contains no critical values, the sublevel sets $\{f \leq b\}$ and $\{f \leq a\}$ are diffeomorphic.

Corollary 4.14 (Reeb)**.** *Let V be a closed manifold. If V admits a Morse function with only two critical points then V is homeomorphic to a sphere.*

Proof. (i) The critical points are necessarily a minimum and a maximum. After removing small open disks around each critical point in Morse coordinates, we obtain a manifold M subject to Proposition 4.13, hence M is diffeomorphic to $S^{n-1} \times [-1, 1]$ with $n = \dim V$. But V is obtained by gluing two disks D^n along $S^{n-1} \times \{\pm 1\}$. This implies that V is homeomorphic to S^n, as we see now.

(ii) Let $\varphi\colon S^{n-1} \xrightarrow{\sim} S^{n-1}$ be an homeomorphism and $X_\varphi = D^n \sqcup_\varphi D^n := (D^n \sqcup D^n)/\sim$, where $x \sim \varphi(x)$, for $x \in S^{n-1}$. We remark that there exists $\psi\colon D^n \to D^n$ such that $\psi|_{\partial D^n} = \varphi$. Indeed we can define ψ by $\psi(r, \theta) = r\varphi(\theta)$ in polar coordinates. Then the map $D^n \sqcup D^n \xrightarrow{\sim} D^n \sqcup D^n$ which is id on the first D^n and ψ on the second D^n induces $X_\varphi \xrightarrow{\sim} X_{\mathrm{id}} = S^n$. □

Let us remark that the stronger conclusion that V is diffeomorphic to a sphere is false: Milnor discovered in 1956 closed smooth manifolds of dimension 7 which admit Morse functions with two critical points but are nonetheless not diffeomorphic to S^7 (they are called *exotic spheres*).

Definition 4.15. A Morse pair (f, X) satisfies the *Morse-Smale condition* if for any critical points p, q, the submanifolds $W^u(p)$ and $W^s(q)$ intersect transversally. In this case we call (f, X) a *Morse-Smale pair.*

Theorem 4.16 (Smale)**.** *Let (f, X) be a Morse pair. There exists an adapted gradient vector field X' which coincides with X near the critical points and satisfies the Morse-Smale condition. Moreover X' can be chosen C^1-close to X.*

We admit this statement which can be proved by induction on the number of critical points. The induction step is essentially the following lemma.

Lemma 4.17. *Let V be a manifold and $X = (\partial/\partial s, X_s)$ be a vector field on $[0, 1] \times V$ projecting to $\partial/\partial s$. Let V_i^s, $i \in I$, V_j^u, $j \in J$, be finite families of submanifolds of $\{0\} \times V$ and $\{1\} \times V$ respectively. Then we can find a C^1-small deformation $X' = (\partial/\partial s, X'_s)$ of X near $\{\frac{1}{2}\} \times V$ such that $\Phi^1_{X''}(V_i^s)$ is transverse to V_j^u (inside $\{1\} \times V$), for all i, j.*

4.2 The Morse complex

Definition 4.18. Let (f, X) be a Morse pair. For p, q critical points of f, we set $\mathcal{M}(p,q) = W^u(p) \cap W^s(q) = \{x \in V; \varphi_X^t(x) \underset{t\to+\infty}{\longrightarrow} q \text{ and } \varphi_X^t(x) \underset{t\to-\infty}{\longrightarrow} p\}$.

Lemma 4.19. *If $p \neq q$, $\mathbf{R}$ acts freely on $\mathcal{M}(p,q)$ via the flow of X (that is, $\varphi_X^t(x) = x$ implies $t = 0$), and the quotient space is Hausdorff. More precisely, for any regular value $a \in]f(p), f(q)[$, the intersection $\mathcal{M}(p,q) \cap f^{-1}(a)$ is transverse (in particular a submanifold of $\mathcal{M}(p,q)$ and $f^{-1}(a)$) and the map $u_a \colon \mathcal{M}(p,q) \cap f^{-1}(a) \to \mathcal{M}(p,q)/\mathbf{R}$ is a homeomorphism. For another regular value $b \in]f(p), f(q)[$, the composition $u_b \circ u_a^{-1}$ is a diffeomorphism.*

Proof. (i) We remark that $\varphi_X^t(x) = x$ and $x \in W^u(p)$ already implies $x = p$. Since $p \notin W^s(q)$, it follows that $\mathcal{M}(p,q)$ has no fixed point under the flow action.

(ii) By the definition of a Morse pair, at any regular point x of f, the vector X_x and $T_x(f^{-1}f(x))$ generate T_xV. For $x \in \mathcal{M}(p,q)$ we have $X_x \in T_x\mathcal{M}(p,q)$ and the transversality follows.

Let us set $L_a = \mathcal{M}(p,q) \cap f^{-1}(a)$ and define $\psi \colon L_a \times \mathbf{R} \to \mathcal{M}(p,q)$, $(x,t) \mapsto \varphi_X^t(x)$. We have $d\psi_{(x,0)}(Y,1) = Y + X_x$; hence $d\psi_{(x,0)}$ is an isomorphism. Since $\psi(x,t) = \varphi_X^t\psi(x,0)$ it follows that $d\psi$ is a local diffeomorphism.

The function f is strictly increasing along any flow line. Hence any flow line in $\mathcal{M}(p,q)$ meets $f^{-1}(a)$ exactly once. This proves that ψ is also a bijection, hence a diffeomorphism. We obtain $L_a \times \mathbf{R} \simeq \mathcal{M}(p,q)$ (with compatible $\mathbf{R}$-actions) and $\mathcal{M}(p,q)/\mathbf{R} \simeq (L_a \times \mathbf{R})/\mathbf{R} \simeq L_a$. This proves that $\mathcal{M}(p,q)/\mathbf{R}$ is Hausdorff.

(iii) The composition $u_a \circ u_b^{-1} \colon L_b \to L_a$ coincides with $p \circ \psi^{-1}|_{L_b}$ where $p \colon L_a \times \mathbf{R} \to L_a$ is the projection. This proves that $u_a \circ u_b^{-1}$ is a C^∞ map. For the same reason its inverse is also a C^∞ map and they are diffeomorphisms. □

Definition 4.20. Let (f, X) be a Morse pair. We set $\mathcal{L}(p,q) = \mathcal{M}(p,q)/\mathbf{R}$. By the previous lemma this set has a natural structure of manifold.

For a sequence E_n, $n \in \mathbf{N}$, of subsets of V we set $\overline{\lim}_n E_n = \bigcap_{k=0}^\infty \overline{\bigcup_{n\geq k} E_n}$.

Lemma 4.21. *Let (f, X) be a Morse-Smale pair, p, q critical points. We assume that there exists a Morse chart U around p and a sequence of points $(p_n)_{n\in\mathbf{N}}$ in $U \cap W^s(q)$ converging to a point $p_\infty \in (W^u(p) \setminus \{p\})$. We let l_n be the flow line through p_n. Then we have either (1) or (2):*

(1) $p_\infty \in \mathcal{M}(p,q)$ and $\overline{\lim}_n l_n$ contains the flow line through p_∞; in particular $\mathrm{ind}(p) < \mathrm{ind}(q)$,

(2) there exists a critical point r with $\mathrm{ind}(p) < \mathrm{ind}(r) < \mathrm{ind}(q)$ *such that $\overline{\lim}_n l_n$ contains a flow line of $\mathcal{M}(p,r)$ and, up to taking a subsequence, there exist a Morse chart V around r, points $r_n \in V \cap l_n \subset V \cap W^s(q)$ converging to a point $r_\infty \in (W^u(r) \setminus \{r\})$.*

Proof. We set $a = f(p)$, $b = f(q)$. We choose a regular value $b' < b$ such that $[b', b[$ only contains regular values and $S = W^s(q) \cap f^{-1}(b')$ is a sphere of dimension $\mathrm{ind}(q) - 1$ contained in a Morse chart around q. For each n, the flow line l_n intersects S at one point q_n and we can write $q_n = \varphi_X^{t_n}(p_n)$.

(i-a) If the sequence t_n is bounded, we can take a subsequence and assume that t_n converges to some value T. Then $\varphi^T(p_\infty) \in S$ and we obtain $p_\infty \in \mathcal{M}(p,q)$. Since $\varphi_X^t(p_n)$ converges to $\varphi_X^t(p_\infty)$, for each $t \in \mathbf{R}$, the assertion on $\overline{\lim}_n l_n$ follows.

(i-b) If the sequence t_n is unbounded, up to taking a subsequence we can assume $t_n \geq n$. Since f is increasing along the flow lines we have, for $n \geq n_0$, $b' = f(\varphi_X^{t_n}(p_n)) \geq f(\varphi_X^{n_0}(p_n))$. Hence $b' \geq f(\varphi_X^{n_0}(p_\infty))$. Since this holds for all $n_0 \in \mathbf{N}$, Proposition 4.12 implies that $\varphi_X^t(p_\infty)$ converges to a critical point r with $f(r) \in]a, b'[$. Then $p_\infty \in \mathcal{M}(p,r)$ and this gives $\mathrm{ind}(p) < \mathrm{ind}(r)$ by the Morse-Smale property. The assertion on $\overline{\lim}_n l_n$ holds as in (i-a).

(ii) We consider a Morse chart V around r. We choose two values $c', c'' \in f(V)$ with $c' < f(r) < c''$ such that $S' = W^s(r) \cap f^{-1}(c')$ and $S'' = W^u(r) \cap f^{-1}(c'')$ are spheres. For n big enough, the flow line l_n meets $V \cap f^{-1}(c')$ at a point p'_n and $V \cap f^{-1}(c'')$ at a point r_n. The flow line through p_∞ also meets $V \cap f^{-1}(c')$ at a point p'_∞ and we have $p'_\infty = \lim_n p'_n$.

Since $\varphi_X^t(p_\infty)$ converges to r, we have in fact $p'_\infty \in S'$. This implies that the points r_n remain in some compact neighborhood of S''. Taking a subsequence we assume that r_n converges to some point r_∞. Then $r_\infty \in S''$: indeed, if $r_\infty \notin S''$, the flow induces a diffeomorphism between a neighborhood of r_∞ in $f^{-1}(c'') \setminus S''$ and a neighborhood of some point $p'_0 = \varphi_X^{-s}(r_\infty)$ in $f^{-1}(c') \setminus S'$. But then p'_n converges to p'_0; hence $p'_0 = p'_\infty \notin S'$.

(iii) We have proved the lemma except the bound $\mathrm{ind}(r) < \mathrm{ind}(q)$ in case (2). We can apply the lemma with $p^1 = r$ instead of p and with the sequence r_n instead of p_n. We have either $\mathrm{ind}(p^1) < \mathrm{ind}(q)$ (case (1)) or we find another critical point p^2 (case (2)) to which we can apply the lemma. As long as we are in case (2) we can apply the lemma and find a sequence of critical points p^1, p^2,... in $f^{-1}(]a,b[)$ with $\mathrm{ind}(p) < \mathrm{ind}(p_1) < \mathrm{ind}(p_2) < \dots$. This sequence must stop and the application of the lemma to the last point p_k must yields case (1). Hence $\mathrm{ind}(r) \leq \mathrm{ind}(p_k) < \mathrm{ind}(q)$. $\square$

Lemma 4.22. *Let (f,X) be a Morse-Smale pair, p,q critical points with $\mathrm{ind}(p)+1 = \mathrm{ind}(q)$. Then $\mathcal{L}(p,q)$ is finite.*

Proof. We assume that $\mathcal{M}(p,q)$ contains infinitely many flow lines l_n, $n \in \mathbf{N}$. Let us consider a Morse chart U around p. We choose $a > f(p)$ such that $S_p^u = U \cap W^u(p) \cap f^{-1}(a)$ is a sphere. We let p_n be the intersection point of l_n with S_p^u. Up to taking a subsequence we can assume that p_n converges to a point $p_\infty \in S_p^u$. We apply Lemma 4.21. We cannot be in case (2) of the lemma because $\mathrm{ind}(p) + 1 = \mathrm{ind}(q)$. Hence we are in case (1) and obtain $p_\infty \in \mathcal{M}(p,q)$.

But $\mathcal{M}(p,q)$ is a submanifold of dimension 1, hence we have a neighborhood Ω of p_∞ in V such that $\Omega \cap \mathcal{M}(p,q)$ is a segment of line. In particular $\Omega \cap \mathcal{M}(p,q)$ only contains points in the flow line of p_∞ which contradicts the convergence $p_n \to p_\infty$. $\square$

Definition 4.23. Let (f,X) be a Morse-Smale pair. Let $C^k(f,X)$ be the Z_2-vector space generated by the critical points of f and $d^k : C^k(f,X) \to C^{k+1}(f,X)$ the linear map defined by $dp = \sum_q \#\mathcal{L}(p,q)\, q$ where $\#$ denotes the cardinal modulo 2.

A "manifold with boundary" X is a closed subset of a manifold $\tilde{X}$ such that there exists an open subset $U \subset \tilde{X}$ with $\overline{U} = X$ and, near any point of ∂U, we can find coordinates $(x_1, \dots, x_n)$ such that $U = \{x_n > 0\}$. The compact manifolds with boundary of dimension 1 are the circle and the closed interval.

Lemma 4.24. *Let (f, X) be a Morse-Smale pair, p, q critical points with* $\mathrm{ind}(p) + 2 = \mathrm{ind}(q)$. *Then there is a compact manifold with boundary denoted $\bar{\mathcal{L}}(p, q)$ whose interior is diffeomorphic to $\mathcal{L}(p, q)$ and boundary diffeomorphic $\cup_{r,\mathrm{ind}(r)=\mathrm{ind}(p)+1}\mathcal{L}(p, r) \times \mathcal{L}(r, q)$.*

We recall that for a regular value $a \in \,]f(p), f(q)[$, $L_a = f^{-1}(a) \cap \mathcal{M}(p, q)$ is diffeomorphic with $\mathcal{L}(p, q)$. Then L_a is a submanifold of dimension 1 with finitely many connected components which are diffeomorphic to $\mathbf{R}$ or the circle. We can consider the closure $\overline{L_a}$ in V but it may be different from $\bar{\mathcal{L}}(p, q)$. In particular it can happen that L_a is $C \setminus \{x\}$ where C is a circle in $f^{-1}(a)$ and $x \in C$. Then $\overline{L_a} = C$ but $\bar{\mathcal{L}}(p, q)$ is a segment.

Proof. (i) We choose a regular value $a \in \,]f(p), f(q)[$. The statement is equivalent to the following:

(a) Let $i\colon [0, 1] \to V$ be a continuous map such that $i(]0, 1]) \subset L_a$ and $i(0) \notin L_a$. We let $M_i \subset \mathcal{M}(p, q)$ be the union of the flow lines through the points of $i(]0, 1])$. Then there exists another critical point r with $\mathrm{ind}(r) = \mathrm{ind}(p)+1$ and two flow lines $l' \subset \mathcal{M}(p, r)$, $l'' \subset \mathcal{M}(r, q)$ such that $\overline{M_i} \setminus \{p, q, r\} = M_i \cup l' \cup l''$,

(b) for any critical point r with $\mathrm{ind}(r) = \mathrm{ind}(p) + 1$ and any two flow lines $l' \subset \mathcal{M}(p, r)$, $l'' \subset \mathcal{M}(r, q)$, there exists a continuous map $i\colon [0, 1] \to V$ as in (a) such that $\overline{M_i} \setminus \{p, q, r\} = M_i \cup l' \cup l''$. Moreover, if i' is another such map, then we can find a neighborhood U of $i(0)$ such that $U \cap \mathrm{im}(i) = U \cap \mathrm{im}(i')$.

(ii) The assertion (a) follows from Lemma 4.21.

(iii) To prove (b) we first describe Morse charts. Let r be a critical point of index k. By definition there exist Morse coordinates $(x_1, \ldots, x_n)$ where $X = -x_1\partial_{x_1} - \cdots - x_k\partial_{x_k} + x_{k+1}\partial_{x_{k+1}} + \cdots + x_n\partial_{x_n}$. We denote $\underline{x} = (\underline{x}_-, \underline{x}_+)$, where $\underline{x}_- = (x_1, \ldots, x_k)$. We choose $\varepsilon, \eta > 0$ small enough and set

$$\begin{aligned} U &= \{\underline{x};\quad |f(\underline{x}) - f(r)| < \varepsilon,\quad ||x_-||^2 \cdot ||x_+||^2 < \eta(\eta + \varepsilon)\}, \\ \partial_\pm U &= \{\underline{x};\quad f(\underline{x}) - f(r) = \pm\varepsilon,\quad ||x_\mp||^2 \leq \eta\}, \\ \partial_0 U &= \{\underline{x};\quad |f(\underline{x}) - f(r)| \leq \varepsilon,\quad ||x_-||^2 \cdot ||x_+||^2 = \eta(\eta + \varepsilon)\}. \end{aligned}$$

By construction $\partial_\pm U$ is contained in the regular level set of $f(r) \pm \varepsilon$. Hence the flow lines meet $\partial_\pm U$ transversally. Moreover $\overline{U} \cap W^s(r) = \{(\underline{x}_-, 0);\ ||x_-||^2 \leq \varepsilon\}$, $\overline{U} \cap W^u(r) = \{(0, \underline{x}_+);\ ||x_+||^2 \leq \varepsilon\}$. The corresponding stable and unstable spheres are $S^s(r) = \{(\underline{x}_-, 0);\ ||x_-||^2 = \varepsilon\}$, $S^u(r) = \{(0, \underline{x}_+);\ ||x_+||^2 = \varepsilon\}$. Finally $\overline{U} \setminus (W^s(r) \cap W^u(r))$ is the union of the flow lines meeting $\partial_- U \setminus S^s(r)$ (or $\partial_+ U \setminus S^u(r)$); they enter $\overline{U}$ through $\partial_- U$ and exit through $\partial_+ U$. The flow induces a diffeomorphism

$$\Phi\colon \partial_- U \setminus S^s(r) \to \partial_+ U \setminus S^u(r), \qquad (\underline{x}_-, \underline{x}_+) \mapsto \left(\tfrac{||\underline{x}_+||}{||\underline{x}_-||}\,\underline{x}_-, \tfrac{||\underline{x}_-||}{||\underline{x}_+||}\,\underline{x}_+\right).$$

(iv) Now we assume to be given r and flow lines $l' \subset \mathcal{M}(p, r)$, $l'' \subset \mathcal{M}(r, q)$, as in (b). We set $k = \mathrm{ind}(r)$. We let $y' \in S^s(r)$, $y'' \in S^u(r)$ be the intersections of l', l'' with $S^s(r)$ and $S^u(r)$.

We also set $T = W^u(p) \cap \partial_- U$. The intersection is transverse. Since $\mathrm{ind}(p) = k - 1$, T is a submanifold of $\partial_- U$ of dimension $n - (k - 1) - 1 = n - k$. By the Morse-Smale property T intersects $S^s(r)$ transversally in a finite number of points. The point y' is one of them. We choose a neighborhood Ω of y' such that $\Omega \cap T \cap S^s(r) = \{y'\}$. Using the coordinates we consider the Morse chart U as embedded in $\mathbf{R}^n$ and we let $p''\colon U \to \mathbf{R}^{n-k}$ be the second projection. Since T intersects $S^s(r)$ transversally, $d(p''|_T)_{y'}$ is an isomorphism from $T_{y'}T$ to $T_0\mathbf{R}^{n-k}$. By the local inversion theorem, up to shrinking Ω we can write $\Omega \cap T = \{j(\underline{x}_+);\ \underline{x}_+ \in B^{n-k}(r)\}$, where $j(\underline{x}_+) = (f(\underline{x}_+), \underline{x}_+)$ for some function $f\colon \mathbf{R}^{n-k} \to \mathbf{R}^k$ defined in some ball $B^{n-k}(\rho)$ of radius ρ and center 0. We have $f(0) = y'$ and we can assume that f is nonvanishing.

Let $\mathbf{S}^{n-k-1}$ be the unit sphere in $\mathbf{R}^{n-k}$. We write in polar coordinates $\underline{x}_+ = (\theta, r) \in \mathbf{S}^{n-k-1} \times]0, \infty[$. We set $\Psi(\theta, r) = \Phi(j(\theta, r))$. We have $\Psi(\theta, r) = (\frac{rf(r\theta)}{||f(r\theta)||}, ||f(r\theta)||\,\theta)$. This expression for $\Psi(\theta, r)$ actually makes sense for $r \in]-\rho, \rho[$ and gives a C^∞ map $\Psi\colon \mathbf{S}^{n-k-1} \times]-\rho, \rho[\to \partial U_+$ (we use that f is nonvanishing and that $(\theta, r) \mapsto r\theta$ is C^∞). We can chek that Ψ is an embedding of $S^{n-k-1} \times]-\rho, \rho[$ in $S^u(r)$. Let T' be the image of this embedding; this is a submanifold of dimension $n-k$.

We remark that $\Psi(\theta, 0) = (0, \varepsilon\,\theta)$. Hence $\Psi(\mathbf{S}^{n-k-1} \times \{0\}) = S^u(r)$ is of codimension 1 in T'. We set $T'_+ = \Psi(\mathbf{S}^{n-k-1} \times]0, \rho[)$. Then T'_+ is a submanifold with boundary $S^u(r)$ in $\partial_+ U$ and Φ identifies T with T'_+.

(v) We thus have three submanifolds of $\partial_+ U$, namely $S^u(r)$, T' and $W^s(q) \cap \partial_+ U$, such that:

$W^s(q) \cap \partial_+ U$ meets $S^u(r)$ transversally at y'' by the Morse-Smale property,

$S^u(r)$ is a submanifold of T' of codimension 1.

Hence $W^s(q) \cap \partial_+ U$ also meets T' transversally at y''. Near y'' the intersection is a segment of line, say l_0, in T' intersecting $S^u(r)$ transversally at y''. Hence l_0 meets both sides, in particular T'_+, near y''. We set $l_0^+ = l_0 \cap T'_+$. Since $T'_+ \subset W^u(p)$ we get $l_0^+ \subset \mathcal{M}(p, q)$.

We set $a = f(r) + \varepsilon$ so that $\partial U_- \subset f^{-1}(a)$. We set $L_a = f^{-1}(a) \cap \mathcal{M}(p, q)$ as in (a). We can choose $i\colon [0, 1] \to V$ with $i(]0, 1]) \subset l_0^+$ and $i(0) = y''$. Then i satisfies the conditions in (b).

Conversely, if i' is another map satisfying the conditions in (b), $\overline{i'(]0, 1[)}$ contains y'' and thus $\mathrm{im}(i') \subset l_0$. But the closure of $\Phi^{-1}(i'(]0, 1[))$ must also meet l' and we deduce $i'(]0, 1[) \subset l_0^+$, which concludes the proof. $\square$

Theorem 4.25. $d^2 = 0$

Proof. We have

$$\langle d^2 p, r\rangle = \langle d \sum_q \#\mathcal{L}(p, q) q, r\rangle = \sum_q \#\mathcal{L}(p, q)\#\mathcal{L}(q, r) = \#(\bigcup_q \mathcal{L}(p, q) \times \mathcal{L}(q, r)) = 0,$$

since, by lemma 4.24, $\cup_q \mathcal{L}(p, q) \times \mathcal{L}(q, r)$ is the boundary of a one-dimensional compact manifold and thus consists in an even number of points. $\square$

Definition 4.26. The vector spaces $H^k(f, X) = \ker d^k/\mathrm{im} d^{k-1}$ are called the *Morse cohomology* groups (with $\mathbf{Z}/2$-coefficients).

The Morse cohomology groups a priori depend on the Morse-Smale pair (f, X). Observe first that the Morse complex depends only on the gradient field, namely if X serves as an adapted gradient for two Morse functions f and g, then $C(f, X) = C(g, X)$. This is the case for instance when g is equal to f up to a constant. In general, $C(f, X) \neq C(g, Y)$ but we have the following result.

Theorem 4.27. *Up to canonical isomorphism, the Morse cohomology groups are independent of the choice of the Morse-Smale pair (f, X). More precisely, there exist isomorphisms*

$$\varphi_{01} \colon H^*(f_1, X_1) \xrightarrow{\sim} H^*(f_0, X_0),$$

such that $\varphi_{00} = \mathrm{id}$ and $\varphi_{01} \circ \varphi_{12} = \varphi_{02}$.

Proof. **Preliminaries** Let us fix a Morse function $g : \mathbf{R} \to \mathbf{R}$ which has precisely two critical points: a maximum at 0 and a minimum et 1. Pick also a complete adapted gradient Y for g. Finally pick a non-increasing function $\rho : \mathbf{R} \to \mathbf{R}$ equal to 1 near $(-\infty, 0]$ and equal to 0 near $[1, +\infty)$.

Step 1 : Construction of a chain map φ_{01}.

For $(s, x) \in \mathbf{R} \times V$, set $f_{01}^s(x) = \rho(s) f_0(x) + (1 - \rho(s)) f_1(x)$, $X_{01}^s(x) = \rho(s) X_0(x) + (1 - \rho(s)) X_1(x)$ and $Z_{01}(s, x) = (Y(s), X_{01}^s(x))$. The zeroes of Z_{01} are the points $(0, p_0)$ and $(1, p_1)$ for p_0, p_1 critical points of f_0, f_1 respectively. We have $W^s(0, p_0) = \mathbf{R} \times W^s(p_0)$ and $W^u(0, p_0) = \{0\} \times W^u(p_0)$ near $\{0\} \times V$. Similarly, $W^s(1, p_1) = \{0\} \times W^s(p_1)$ and $W^u(1, p_1) = \mathbf{R} \times W^s(p_1)$ near $\{1\} \times V$. Since X_0 and X_1 are Morse-Smale vector fields, we deduce that $W^u(0, p_0)$ is transverse to $W^s(0, q_0)$, that $W^u(1, p_1)$ is transverse to $W^s(1, q_1)$, and that $W^u(0, p_0)$ is disjoint (hence transverse) from $W^s(1, p_1)$ for all critical points p_0, q_0 of f_0 and p_1, q_1 of f_1. We claim that after a suitable perturbation of X_{01} supported in $(0, 1) \times V$, we can also make the submanifolds $W^s(0, p_0)$ and $W^u(1, p_1)$ transverse to each other. This is proved similarly as Theorem 4.16 (see Lemma 4.17). We assume this perturbation has been done and keep the notation X_{01}. Next we set $h_{01}(s, x) = f_{01}^s(x) + g(s) + C\rho(s)$ and claim that, for $C > 0$ sufficiently large, h_{01} is Morse and Z_{01} is an adapted gradient for h_{01}. Indeed $Z_{01}(h_{01}) = (\rho'(s)(C + f_0 - f_1) + g'(s)) Y(s) + X_{01}^s(f_{01}^s)$ is positive in $(0, 1) \times V$ for sufficiently large C, near $\{s \le 0\}$ we have $(h_{01}, Z_{01}) = (f_0 + g + C, X_0 + Y)$ and near $\{s \ge 1\}$ we have $(h_{01}, X_{01}) = (f_1 + g, X_1 + Y)$. The critical points of h_{01} then coincide with the zeroes of Z_{01} and are non-degenerate. Though $\mathbf{R} \times V$ is not compact, the Morse complex of (h_{01}, Z_{01}) is well-defined. Indeed all trajectories between critical points are contained in $[0, 1] \times V$ so the proofs of Lemma 4.22 and Lemma 4.24 work verbatim. Since 0 is a maximum for g and 1 is a minimum, we have $C^k(h_{01}, Z_{01}) = C^{k-1}(f_0, X_0) \oplus C^k(f_1, X_1)$, and, since there are no trajectories from $(0, p_0)$ to $(1, p_1)$, the differential has the form $d = \begin{pmatrix} d_0 & \varphi_{01} \\ 0 & d_1 \end{pmatrix}$. The equation $d^2 = 0$ gives $d_0 \varphi_{01} + \varphi_{01} d_1 = 0$ and hence $\varphi_{01} \colon C(f_1, X_1) \to C(f_0, X_0)$ is a chain map.

In the case where $X_0 = X_1$, observe that $Z_{01} = (Y, X_0)$ is Morse-Smale and an adapted gradient for $h_{01}(s, x) = f_{01}^s(x) + g(s)$, and that $\varphi_{01} = \mathrm{id}$ under the obvious identification $C(f_0, X_0) = C(f_1, X_1)$.

Step 2: Checking $\varphi_{01} \circ \varphi_{12} = \varphi_{02}$ on cohomology groups

Set $X_{02}^t = \rho(t) X_0 + (1 - \rho(t)) X_2$, $X_{12}^t = \rho(t) X_1 + (1 - \rho(t)) X_2$, $X_{012}^{s,t} = \rho(s) X_{02}^t + (1 - \rho(s)) X_{12}^t$ and $Z_{012}^{s,t} = (Y(s), Y(t), X_{012}^{s,t})$. Let us assume that X_{02}^s, X_{01}^s and X_{12}^s have been perturbed slightly as above in $(0, 1) \times V$ so that their sums with $Y(s)$ are Morse-Smale vector fields on $\mathbf{R} \times V$. This modifies the family $X_{012}^{s,t}$ in the regions $\{s \le 0, t \in (0, 1)\}$, $\{s \in (0, 1), t \le 0\}$ and $\{s \ge 1, t \in (0, 1)\}$ accordingly. The zeroes of Z_{012} are $(0, 0, p_0)$, $(1, 0, p_1)$, $(0, 1, p_2)$ or $(1, 1, p_2)$ for critical points p_0, p_1, p_2 of f_0, f_1, f_2 respectively. All stable manifolds and unstable manifolds of Z_{012} are transverse to each other except possibly $W^s(0, 0, p_0)$ and $W^u(1, 1, p_2)$. As before we claim that this can be ensured by a perturbation of $X_{012}^{s,t}$ supported in $(0,1)\times(0,1)\times V$ (see Theorem 4.16 and Lemma 4.17). Set $f_{012}^{s,t}=\rho(s)f_{02}^t+(1-\rho(s))f_{12}^t$ and $h_{012}(s,t,x)=f_{012}^{s,t}(x)+g(s)+g(t)+C(\rho(s)+\rho(t))$. As in the previous step, we claim that for $C>0$ sufficiently large, (h_{012},Z_{012}) is a Morse-Smale pair on $\mathbf{R}^2\times V$, and, due to the special form of Z_{012}, its Morse complex is well-defined and we have

$$C^k(f_{012},Z_{012})=C^{k-2}(f_0)\oplus C^{k-1}(f_1)\oplus C^{k-1}(f_2)\oplus C^k(f_2),$$

and the differential has the form

$$d=\begin{pmatrix} d_0 & \varphi_{01} & \varphi_{02} & \psi \\ 0 & d_1 & 0 & \varphi_{12} \\ 0 & 0 & d_2 & \mathrm{id} \\ 0 & 0 & 0 & d_2 \end{pmatrix}.$$

Then $d^2=0$ gives $d_0\circ\psi+\varphi_{01}\circ\varphi_{12}+\varphi_{02}+\psi\circ d_2=0$. Hence (with coefficients $\mathbf{Z}/2\mathbf{Z}$), $\varphi_{02}-\varphi_{01}\circ\varphi_{12}=d_0\circ\psi+\psi\circ d_2$ and it follows that $\varphi_{02}-\varphi_{01}\circ\varphi_{12}$ induces the zero map in cohomology, as required.

Step 3: Conclusion

In the case where $(f_2,X_2)=(f_1,X_1)$, if we choose different interpolations X_{01}^s and $\overline{X}_{01}^s$ (i.e., different perturbations ensuring the Morse-Smale condition), we obtain two chain maps $\varphi_{01},\overline{\varphi}_{01}\colon C(f_1,X_1)\to C(f_0,X_0)$, but by the previous step, $\varphi_{01}=\overline{\varphi}_{01}$ in cohomology. Hence the morphism φ_{01} induced in cohomology is independent of any choice (the choice of the constant C does not affect the chain map φ_{01}), and it is legitimate to write it φ_{01}.

In the case where $(f_2,X_2)=(f_0,X_0)$, we obtain $\varphi_{01}\circ\varphi_{10}=\mathrm{id}$ in cohomology and hence φ_{01} induces an isomorphism $H(f_1,X_1)\to H(f_0,X_0)$.

For the constant interpolation $X_{00}^s=X_0$, we have seen that $\varphi_{00}=\mathrm{id}$ at the chain level. Since φ_{00} is independent of any choice at the cohomology level, we obtain $\varphi_{00}=\mathrm{id}$ on cohomology.

We have thus constructed the required isomorphisms. □

Definition 4.28. We define $H^k(V)=\bigsqcup_{(f,X)}H^k(f,X)/\sim$ where (f,X) runs over the Morse-Smale pairs and, for $c_0\in H^k(f_0,X_0)$, $c_1\in H^k(f_1,X_1)$ we set $c_0\sim c_1$ if $c_0=\varphi_{01}(c_1)$. Then $H^k(V)$ is a vector space and, for a given Morse-Smale pair (f,X), the map $H^k(f,X)\to H^k(V)$ is an isomorphism. We call $H(V)$ the *Morse cohomology* of V. The numbers $b_k(V)=\dim H^k(V)$ are called the *Betti numbers* of V.

Proposition 4.29. *The number of critical points of a Morse function f on V is bounded below by $\sum_k b_k(V)$.*

Proof. Pick an adapted Morse-Smale gradient X. The number of critical points of index k of f is equal to $\dim C^k(f,X)$ and we have

$$\dim C^k(f,X)\geq\dim\ker(d:C^k(f,X)\to C^{k+1}(f,X))\geq\dim H^k(f,X)=b_k(V).$$

The result follows by summing over all k. □

The groups $H(V)$ satisfy the following functoriality property. If $\varphi:V\to W$ is a difffeomorphism then it induces a map $\varphi^*:H(W)\to H(V)$. Indeed, pick a Morse-Smale pair (f,X) on W and pull it back to V to get an isomorphism $H(V)\simeq H(\varphi^*f,\varphi^*X)=H(f,X)\simeq H(V)$. In particular the group of diffeomorphims of V naturally acts on $H(V)$.

Lemma 4.30. *Let V be a closed manifold and $\varphi : V \to V$ a diffeomorphism isotopic to the identity then the map $H(V) \to H(V)$ induced by φ is the identity.*

Proof. Let φ_t, $t \in [0,1]$, be an isotopy with $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$, and (f, X) a Morse-Smale pair on V. The path $(\varphi_t^* f, \varphi_t^* X)$ joins f and $\varphi^* f$ and we have isomorphisms $H(\varphi_t^* f, \varphi_t^* X) \simeq H(f, X)$ since the Morse complex is independent of t. The isomorphism $H(f, X) \to H(\varphi^* f, \varphi^* X)$ from Theorem 4.27 is then the identity (as it follows from its proof). Hence the induced isomorphism $\varphi^* : H(V) \to H(V)$ is the identity. □

4.3 Computations

If V is a closed manifold of dimension n, it is clear that $H^k(V) = 0$ if $k < 0$ or $k > n$.

The cohomology groups of a manifold satisfy an important symmetry property called the Poincaré duality.

Proposition 4.31. *Let V be a closed manifold of dimension n. Then for all $k \in \mathbf{Z}$, $H^{n-k}(V)$ is isomorphic to $H^k(V)$.*

Proof. Let (f, X) be a Morse-Smale pair on V and consider the Morse-Smale pair $(-f, -X)$. A critical point of index k for f is of index $n-k$ for $-f$, and hence $C^k(f, X) = C^{n-k}(-f, -X)$. Using the basis of $C^{n-k}(-f, -X)$ given by the critical points, we identify $C^{n-k}(-f, -X)$ with its dual and the adjoint of the differential d is a differential $\partial : C^{n-k}(-f, -X) \to C^{n-k-1}(-f, -X)$ (which is of degree -1). Moreover the homology groups of $(C^{n-k}(-f, -X), \partial)$ are isomorphic to the duals of the cohomology groups of $(C^{n-k}(-f, -X), d)$. Now observe that, under the identification $C^{n-k}(-f, -X) = C^k(f, X)$, the differential ∂ corresponds to d. The result follows since a finite dimensional vector space is isomorphic to its dual. □

The case $k = 0$ and $k = n$ are computed as follows.

Proposition 4.32. *Let V be a closed connected manifold of dimension n. We have $H^n(V) = \mathbf{Z}/2\mathbf{Z}$ and $H^0(V) = \mathbf{Z}/2/\mathbf{Z}$.*

Proof. Due to Proposition 4.31 it is enough to prove it for $H^0(V)$. □

The cohomology groups of a product of manifolds can be computed with the following so-called Künneth formula.

Proposition 4.33. *Let V, W be closed manifolds, we have*

$$H^k(V \times W) = \bigoplus_{i+j=k} H^i(V) \otimes H^j(W).$$

4.4 Filtered Morse complex

Let V be a closed manifold and (f, X) a Morse-Smale pair on V. Let $a, b \in [-\infty, +\infty]$ be regular values of f with $a < b$. We define a complex $C_{(a,b)}(f, X)$ (simply written $C_{(a,b)}$ when the choice of (f, X) is unambiguous) generated by the critical points of f with critical values between a and b, and with differential counting gradient trajectories of X between them as in Definition 4.23. The proof of $d^2 = 0$ (see Theorem 4.25) works verbatim for this situation. Note that $C_{(-\infty,+\infty)} = C$.

If $a, b, c \in [-\infty, +\infty]$ are regular values of f with $a < b < c$, then we have a short exact sequence of complexes:

$$0 \to C_{(b,c)} \to C_{(a,c)} \to C_{(a,b)} \to 0$$

which induces a long exact sequence of cohomology groups

$$\cdots \to H^{k-1}_{(a,b)} \to H^k_{(b,c)} \to H^k_{(a,c)} \to H^k_{(a,b)} \to H^{k+1}_{(b,c)} \to \dots \tag{4.1}$$

If $a, b, c, d \in [-\infty, +\infty]$ are regular values of f with $a < b < c < d$ then we have four short exact sequences and their associated long exact sequences of cohomology groups as (4.1) involving the triples (a, b, c), (a, b, d), (a, c, d), (b, c, d). Due to some commutativity relations such as the following

$$\begin{tikzcd}
 & C(a,c) \arrow[dr] & \\
C(a,d) \arrow[ur] \arrow[rr] & & C(a,b),
\end{tikzcd}$$

the four long exact sequences can be nicely arranged into the following commutative diagram (sometimes called an *exact braid*):

$$\begin{tikzcd}[column sep=small]
\dots \arrow[dr] \arrow[rr, bend left] & & H^{k-1}_{(a,b)} \arrow[dr] \arrow[rr, bend left] & & H^k_{(b,c)} \arrow[dr] \arrow[rr, bend left] & & H^{k+1}_{(c,d)} \arrow[dr] \arrow[rr, bend left] & & \dots \\
 & H^{k-1}_{(a,c)} \arrow[ur] \arrow[dr] & & H^k_{(b,d)} \arrow[ur] \arrow[dr] & & H^k_{(a,c)} \arrow[ur] \arrow[dr] & & H^{k+1}_{(b,d)} \arrow[ur] \arrow[dr] & \\
\dots \arrow[ur] \arrow[rr, bend right] & & H^k_{(c,d)} \arrow[ur] \arrow[rr, bend right] & & H^k_{(a,d)} \arrow[ur] \arrow[rr, bend right] & & H^k_{(a,b)} \arrow[ur] \arrow[rr, bend right] & & \dots
\end{tikzcd} \tag{4.2}$$

The following theorem is a refinement of Theorem 4.27, we leave its proof to the reader.

Theorem 4.34. *Let V be a closed manifold, (f_0, X_0) and (f_1, X_1) Morse-Smale pairs on V, a_0 and b_0 regular values of f_0 with $a_0 < b_0$ and a_1, b_1 regular values of f_1 with $a_1 < b_1$. If $\{f_0 < a_0\} = \{f_1 < a_1\}$ and $\{f_0 < b_0\} = \{f_1 < b_1\}$ then there exists an isomorphism $\varphi_{01} : H_{(a_1,b_1)}(f_1, X_1) \to H_{(a_0,b_0)}(f_0, X_0)$. Moreover these isomorphisms are compatible in the sense that $\varphi_{00} = \mathrm{id}$, $\varphi_{02} = \varphi_{01} \circ \varphi_{12}$ with obvious notations, and commute with all maps in* (4.2).

Definition 4.35. Let (f, X) be a Morse-Smale pair on a closed manifold V and a, b regular values of f with $a < b$. Set $V_a = \{f < a\}$ and $V_b = \{f < b\}$. Mimicking definition 4.28 we define $H^k(V_b, V_a)$ to be the vector space canonically isomorphic to $H^k_{(a,b)}(f, X)$. We call $H(V_b, V_a)$ the *relative Morse cohomology* of the pair (V_b, V_a).

When $a = -\infty$, we have $V_a = \emptyset$ and we write $H(V_b) = H(V_b, V_a)$. When $b = +\infty$, we have $V_b = V$, and $H(V_b, V_a) = H(V, V_a)$.

If $[a, b]$ is an interval of regular values, then $H(V_b, V_a) = 0$ and the restriction morphism $H(V_b) \to H(V_a)$ is an isomorphism.

4.5 Spectral invariants

Definition 4.36. Let V be a closed manifold, f a Morse function on V and $\alpha \in H(V)$ we define

$$c(f, \alpha) = \sup\{c \in \mathbf{R} \setminus \mathrm{vcrit}(f);\ r_{c,f}(\alpha) = 0\},$$

where $r_{c,f} \colon H(V) \to H(V_c)$ is the natural restriction morphism from (4.1) (associated to the triple $(-\infty, c, +\infty)$). The number $c(f, \alpha)$ is called the *spectral invariant* of f with respect to α.

Lemma 4.37. *If $\varphi : V \to V$ is a diffeomorphism of V, f a Morse function on V and $\alpha \in H(V)$, then $c(\varphi^* f, \varphi^* \alpha) = c(f, \alpha)$.*

Proof. Let (f, X) be a Morse-Smale pair, c a regular value of f and consider the long exact sequence (4.1) associated to the triple $(-\infty, c, +\infty)$. We have the following commutative diagram involving the isomorphisms φ^*:

$$\begin{array}{ccc} H(f, X) & \xrightarrow{\quad\varphi^*\quad} & H(\varphi^* f, \varphi^* X) \\ \Big\downarrow{\scriptstyle r_{c,f}} & & \Big\downarrow{\scriptstyle r_{c,\varphi^* f}} \\ H_{(-\infty,c)}(f, X) & \xrightarrow{\varphi^*} & H_{(-\infty,c)}(\varphi^* f, \varphi^* X). \end{array}$$

We deduce that $r_{c,f}(\alpha) = 0$ if and only if $r_{c,\varphi^* f}(\varphi^* \alpha) = 0$, and the result. □

The numbers $c(f, \alpha)$ are therefore invariants in the following sense: if there exists $\alpha \in H(V)$ such that $c(f, \alpha) \neq c(g, \alpha)$ then f and g are not conjugate by a diffeomorphism isotopic to the identity (see Lemma 4.30).

Lemma 4.38. *$c(f, \alpha)$ is a critical value of f.*

Proof. If a is a regular value, then for small enough $\varepsilon > 0$, f has no critical values in $[a - \varepsilon, a + \varepsilon]$. Hence $H(V_{a+\epsilon}, V_{a-\epsilon}) = 0$ and from the exact sequence (4.1) associated to the triple $(-\infty, a-\epsilon, a+\epsilon)$ we deduce that the restriction map $\varphi : H(f < a + \varepsilon) \to H(f < a - \epsilon)$ is an isomorphism.

If further $a = c(f, \alpha)$, then $r_{a+\epsilon,f}(\alpha) \neq 0$ and $r_{a-\epsilon,f}(\alpha) = 0$ which contradicts $\varphi \circ r_{a+\epsilon,f} = r_{a-\epsilon,f}$. □

For similar reasons, we also have

$$c(f, \alpha) = \inf\{c \in \mathbf{R} \setminus \mathrm{vcrit}(f);\ r_{c,f}(\alpha) \neq 0\}.$$

Definition 4.39. Let V be a closed connected manifold, 1 be the generator of $H^0(V)$ and μ the generator of $H^n(V)$. We define $c_-(f) = c(f, 1)$ and $c_+(f) = c(f, \mu)$. If V is reduced to a point, then $1 = \mu$ and we write $c(f) = c_+(f) = c_-(f)$.

Proposition 4.40. *Let V be a closed connected manifold and f a Morse function on V. We have $c_-(f) = \min f$ and $c_+(f) = \max f$.*

Proof. Let (f, X) be a Morse-Smale pair. As we saw in the proof of Proposition 4.32 the class 1 is represented in $H(f, X)$ by the sum of all minima of f. If a is a regular value with $a > \min f$ and $(\min f, a)$ contains no critical value, then the sum of all minima of f in V_a is a cocycle in $C(V_a)$ (since there are no critical points in $\{\min f < f < a\}$) and is non-zero since there is at least one minimum in $\{f = \min f\}$. Thus $r_{a,f}(1) \neq 0$ and $c(f, 1) \leq \min f$. If $a < \min f$, then $H(V_a) = 0$ and hence $c(f, 1) \geq \min f$.

The equality $c(f, \mu) = \max f$ follows from the previous one by Poincaré duality, see Proposition 4.31. □

Lemma 4.41. *For $\alpha \in H(V)$, the map $f \mapsto c(f, \alpha)$ is 1-Lipschitz for the C^0-norm , namely*

$$|c(f, \alpha) - c(g, \alpha)| \leq \sup_{x \in V} |f(x) - g(x)|.$$

Proof. Let $M = \sup_{x \in V} |f(x) - g(x)|$. For any $c \in \mathbf{R}$ and $\epsilon > 0$, we have $\{g \leq c - M - \epsilon\} \subset \{f < c\}$ and we can find a function h such that $\{h < c\} = \{f < c\}$ and $\{h < c\} = \{g < M - c - \epsilon\}$. If further c is a regular value of f and $c - M - \epsilon$ is a regular value of g we can ensure the same property for h and from the diagram (4.2) associated to h and the quadruple $(-\infty, c - M - \epsilon, c, +\infty)$, we extract the diagram

$$H(V) \to H(\{f < c\}) \to H(\{g < c - M - \epsilon\}).$$

If $c < c(f, \alpha)$, then α is sent to zero by the first map, hence also by the composition, and thus $c(g, \alpha) \geq c - M - \epsilon$. Hence (using Sard's lemma) $c(f, \alpha) - c(f, \alpha) \leq M$. The other inequality is proved similarly. □

4.6 Functions quadratic at infinity

Up to now we have considered the Morse complex only on *closed* manifolds. One should be careful when trying to extend it to open manifolds or manifolds with boundary. There are several issues. The first one is that the number of gradient trajectories between points of consecutive indices may not be finite, in which case one cannot even define the differential d. It could also be that d is well-defined but we do not have $d^2 = 0$. Finally it could be that d is defined and $d^2 = 0$ but the resulting cohomology group depends on choices, i.e. theorem 4.27 is wrong. On $\mathbf{R}$, the function $f(x) = x$ has no critical points, an hence $H(f, \partial_x) = 0$, while for $g(x) = x^2$, $H^0(g, x\partial_x) = \mathbf{Z}/2\mathbf{Z}$. One way to fix all these issues is to restrict ourselves to a class of functions and vector fields with prescribed behaviour at infinity. We will use the following setup.

Definition 4.42. A function $f : \mathbf{R}^k \times V \to \mathbf{R}$ is called quadratic at infinity, if there exists a quadratic form $Q : \mathbf{R}^k \to \mathbf{R}$ and a function $g : V \to \mathbf{R}$ such that $f(v, x) = Q(v) + g(x)$ outside of some compact set.

A non-degenerate quadratic form on $\mathbf{R}^k$ always admits an adapted linear gradient vector field. Simply take coordinates $(x_1, \ldots, x_k)$ such that $Q = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_k^2$ and set $X = -x_1\partial_{x_1} - \cdots - x_i\partial_{x_i} + x_{i+1}\partial_{x_{i+1}} + \cdots + x_k\partial_{x_k}$.

Definition 4.43. A Morse pair (f, X) is quadratic at infinity if there exists a quadratic form $Q : \mathbf{R}^k \to \mathbf{R}$, a linear vector field Y on $\mathbf{R}^k$, a function $g : V \to \mathbf{R}$ and an adapted gradient field at infinity and X is linear at infinity.

Given a Morse-Smale pair (f, X) quadratic at infinity on $\mathbf{R}^k \times V$, we can define its morse complex $C(f, X)$ (we still have finiteness of gradient trajectories between critical points of consecutive indices and $d^2 = 0$). A version of Theorems 4.27 and 4.34 holds in this context (with fixed quadratic form Q) so we can give the following definition.

Definition 4.44. Let V be a closed manifold and Q a non-degenerate quadratic form on $\mathbf{R}^n$. We define $H^k_Q(\mathbf{R}^k \times V)$ to be the vector space canonically isomorphic to $H^k(f, X)$ for any Morse-Smale pair (f, X) quadratic at infinity (with quadratic form Q). Similarly we define the filtered version $H_{Q,(a,b)}(f, X) = H_Q(\{f < b\}, \{f < a\})$ for a, b regular values of f, and the spectral invariants $c(f, \alpha)$ for $\alpha \in H_Q(\mathbf{R}^k \times V)$.

Proposition 4.45. *If Q is a non-degenerate quadratic form of index i on $\mathbf{R}^n$ then $H^j_Q(\mathbf{R}^k) = \mathbf{Z}/2\mathbf{Z}$ if $j = i$ and $H^j_Q(\mathbf{R}^k) = 0$ if $j \neq i$.*

Proof. Let Y be a linear adapted gradient for Q. Then the Morse complex of (Q, Y) is very simple: $C^j(Q, Y) = 0$ if $j \neq i$ and $C^i(Q, Y) = \mathbf{Z}/2\mathbf{Z}$, and $d = 0$. The result follows since $H_Q(\mathbf{R}^k)$ can be computed with any Morse function equal to Q outside of a compact set, in particular with Q itself. □

Let V be a closed manifold and Q a non-degenerate quadratic form of index i. In view of the previous proposition, the Künneth morphism $H^k(V) = H^k(V) \otimes H^i_Q(\mathbf{R}^k) \to H^{k+i}_Q(\mathbf{R}^k \times V)$ is an isomorphism, we denote it i_Q. If $\alpha \in H(V)$, we abusively write $c(f, \alpha) = c(f, i_Q(\alpha))$. In particular, $c_-(f) = c(f, i_Q(1))$ and $c_+(f) = c(f, \mu)$. Note that Proposition 4.40 no longer holds for functions quadratic at infinity since these are typically unbounded.

Proposition 4.46. *Let V be a closed connected manifold, f a function quadratic at infinity on $\mathbf{R}^k \times V$ and $x_0 \in V$. The function $f_{x_0}(v) = f(v, x_0)$ is quadratic at infinity on $\mathbf{R}^k$ and we have*

$$c_-(f) \leq c(f_{x_0}) \leq c_+(f).$$

Proof. We only prove the second inequality. The first one is proved by similar arguments. Let Q be the quadratic form on $\mathbf{R}^k$ associated to f and i the index of Q. Let n be the dimension of V.

(i) We claim that we can assume f_{x_0} is Morse. Indeed, one can find another function g with this additional property and arbitrarily C^0-close to f. From Lemma 4.41, we get

$$c(f_{x_0}) \leq c(g_{x_0}) + \varepsilon \leq c_+(g) + \varepsilon \leq c_+(f) + 2\varepsilon$$

where $\varepsilon = \sup |f - g|$.

(ii) Let $a \in \mathbf{R}$ and $\varepsilon > 0$ be given. We claim that there exists a function $g \colon V \to \mathbf{R}$ such that $g(x_0) = 0$ is the maximum of g and $\{f \leq a - \varepsilon\} \subset \{f_{x_0} + g \leq a\}$. Indeed the inclusion is implied by the relation $f(x_0, p) + g(x) \leq f(x, p) + \varepsilon$, for all $(x, p) \in V \times \mathbf{R}^n$. By hypothesis there exists a function $h \colon V \to \mathbf{R}$ such that $f(x, p) - f(x_0, p) = h(x)$ outside a compact set. Since V is compact, the quantity $f(x,p)-f(x_0,p)$ is bounded on $V\times\mathbf{R}^n$. In particular $x\mapsto \inf_{p\in\mathbf{R}^n}\{f(x,p)-f(x_0,p)+\varepsilon\}$ is also bounded (and equal ε at x_0). Hence we can find a smooth function g with a maximum at x_0 equal to 0 and satisfying the required relation.

(iii) Let X, Y be gradient like vector fields for f_{x_0} and g. Using Proposition 4.45 and Proposition 4.33, we obtain an isomorphism

$$H^*(V)\simeq H^*(g,Y)=H^*(g,Y)\otimes H^i(f_{x_0},X)\to H^{*+i}(f_{x_0}+g,X+Y)\simeq H_Q^{*+i}(\mathbf{R}^k\times V).$$

Through this isomorphism the class $i_Q(\mu)$ is represented by $z\otimes x_0$, where z is a cycle in $C^i(f_{x_0},X)$ representing the canonical class of $H^i(f_{x_0},X)$ (recall also that x_0 is the maximum of g, so that $dx_0=0$).

(iv) Observe that x_0 is a generator of $H^n(V)$. Indeed we have $dx_0=0$ and the morphism $e: C^n\to\mathbf{Z}/2$ which maps each generator to 1 satisfies $d\circ e=0$ because each critical point of index $n-1$ has precisely two trajectories to critical points of index n. Hence the map $p\mapsto(p,x_0)$ induces an isomorphism $H^*(f_{x_0},X)\to H^{n+*}(f_{x_0}+g)$. Since $g(x_0)=0$, we deduce the following commutative diagram, for $\varepsilon>0$ such that $a-\varepsilon\notin \mathrm{vcrit}(f)$,

$$\begin{tikzcd}
H^i(f_{x_0}) \arrow[r, "u", "\sim"'] \arrow[d, "r_a"] & H^{n+i}(f_{x_0}+g) \arrow[r, "v_a"] \arrow[d, "s_a"] & H^{n+i}(f<a-\varepsilon) \\
H^i(f_{x_0}<a) \arrow[r] & H^{n+i}(f_{x_0}+g<a) \arrow[ur] &
\end{tikzcd}$$

where the existence of the right hand triangle follows from the inclusion $\{f<a-\varepsilon\}\subset\{f_{x_0}+g<a\}$ given in (i). The arrow u is an isomorphism and $H^i(f_{x_0})\simeq\mathbf{Z}/2\mathbf{Z}$. By definition of $c(f_{x_0})$, the morphism r_a is zero as soon as $a<c(f_{x_0})$. It follows that s_a, and then v_a, are also zero.

Through the canonical isomorphism $H^{n+i}(f_{x_0}+g)\simeq H_Q^{*+i}(\mathbf{R}^k\times V)$ the morphism v_a corresponds to $r_{a,f}$. We deduce that $c(f_{x_0})-\varepsilon\le c_+(f)$, for all $\varepsilon>0$, and the result follows. $\square$

Chapter 5

Generating functions

5.1 Basics

Let V be a manifold, T^*V the cotangent bundle of V, $J^1V = J^1(V, \mathbf{R})$ the bundle of 1-jets of functions on V. Recall T^*V has a canonical 1-form λ_V whose differential ω_V is a symplectic form. Recall also that J^1V naturally splits as $\mathbf{R}\times T^*V$ and is endowed with the contact form $\alpha_V = dz - \lambda_V$ where z is the coordinate on the $\mathbf{R}$ factor. The projection of a Legendrian submanifold of J^1V to T^*V is an exact Lagrangian immersion. Given a function $f : V \to \mathbf{R}$, the image of $j^1f : V \to J^1V$ is a Legendrian submanifold of V: $(j^1f)^*\alpha_V = f^*dz - (df)^*\alpha_V = df - df = 0$. The projection of this Legendrian submanifold under $J^1V \to J^0V = \mathbf{R} \times V$ is the graph of the function f in the (almost) usual sense $\{(f(x), x); x \in V\}$. A Legendrian submanifold of J^1V which is transverse to the fibers of $J^1V \to V$ is necessarily of this type: given by the 1-jet of some function on V. We will present a tool to study more general Legendrian submanifolds of J^1V.

Lemma 5.1. *Let $p : E \to V$ be a surjective submersion and consider the subset H of T^*E consisting of covectors β which vanish on $\ker dp \subset TE$. Then H is a coisotropic submanifold and the projection p induces a symplectomorphism from the symplectic reduction of H to T^*V.*

Proof. Near a point of E, the submersion lemma provides a neighborhood of the form $U\times F$ where p is the first projection. In this neighborhood, T^*E splits as $T^*U \times T^*F$ with the product symplectic form and H corresponds to $T^*U \times 0_F$ which is a coisotropic submanifold. The characteristic foliation $\mathcal{F}$ of H (given by the symplectic orthogonal to TH at each point) is given by $\{0\} \times 0_F$, and hence the reduction $H/\mathcal{F}$ is identified with T^*U under the projection p. □

Definition 5.2. Let V be a manifold. A *generating function* over V is a couple (φ, f) where $\varphi : E \to V$ is a submersion and $f : E \to \mathbf{R}$ a function such that $df : E \to T^*E$ is transverse to the submanifold H of covectors vanishing on $\ker d\varphi$.

A generating function over V induces an immersed Legendrian submanifold of J^1V in the following way. Consider first the Lagrangian submanifold which is the image of $df : E \to J^1E$, it is transverse to H, and hence its reduction gives an exact Lagrangian submanifold of T^*V in view of Proposition 2.10 and Lemma 5.1. Together with the value of the function f, this is lifted to an immersed Legendrian submanifold.

Concretely, near a point $e \in E$ we pick local coordinates (v, q) such that $\varphi(v, q) = q$. Then $df(v, q) \in H$ if and only if $\frac{\partial f}{\partial v} = 0$. The transversality assumption implies that $\Sigma = df^{-1}(H)$ is a submanifold of E. The Legendrian immersion associated with f is the map $i_f : \Sigma \to J^1V$ defined locally by $(v, q) \mapsto (f(v, q), \frac{\partial f}{\partial q}(v, q), q)$. We can check locally that i_f is isotropic:

$$i_f^* \alpha_V = df - \frac{\partial f}{\partial q} dq = \frac{\partial f}{\partial v} dv = 0$$

since $\frac{\partial f}{\partial v}$ vanishes on Σ. Moreover, the transversality condition means that the matrix $\left(\frac{\partial^2 f}{\partial v^2} \, \frac{\partial^2 f}{\partial q \partial v} \right)$ has maximal rank at each point of Σ. The differential of i_f writes $\begin{pmatrix} 0 & \frac{\partial f}{\partial q} \\ \frac{\partial^2 f}{\partial q \partial v} & \frac{\partial^2 f}{\partial q^2} \\ 0 & 1 \end{pmatrix}$. If $(\delta v, \delta_x) \ker di_f$, then $\delta_x = 0$, $\frac{\partial^2 f}{\partial x \partial v} \delta_v = 0$ and $\frac{\partial^2 f}{\partial v^2} \delta_v = 0$, but the matrix $\begin{pmatrix} \frac{\partial^2 f}{\partial v^2} \\ \frac{\partial^2 f}{\partial q \partial v} \end{pmatrix}$ is injective and thus $\delta_v = 0$. Hence i_f is an immersion. If we think of f as a family of functions parametrized by V, then Σ can be viewed as the set of fiberwise critical points. Moreover a point $(v, q) \in \Sigma$ corresponds to a non-degenerate critical point if and only if $d\varphi : T\Sigma \to TV$ is an isomorphism at (v, q).

The function $f : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ defined by $f(v, q) = v^3 - (1 - q^2)v$ is a generating function. The function $g : \mathbf{R} \times \mathbf{R}^2 \to \mathbf{R}$ defined by $f(v, q_1, q_2) = v^4 + q_1 v^2 + q_2 v = 0$. The corresponding Legendrian surface has a cusp edge along the curve $8q_1^3 + 27q_2^2 = 0$.

Not every Legendrian submanifold of J^1V can be described by a generating function. However the obstructions to this are necessarily global due to the following result.

Proposition 5.3. *Let V be a manifold, $\pi : J^1V \to V$ the natural projection, $L \subset J^1V$ a Legendrian submanifold and $x \in L$. There is a generating function for L near x.*

Proof. Let n be the dimension of V and k the dimension of the intersection of T_xL with the tangent space to the fibers of $J^1V \to J^0V = \mathbf{R} \times V$. Consider local functions $q_1, \dots, q_k$ on V defined near $\pi(x)$ such that $q_i(x) = 0$ for all i and $(dq_1(x), \dots, dq_k(x))$ span a basis of the above intersection. Then pick local functions $q_{k+1}, \dots, q_n$ vanishing at $\pi(p)$ in such a way that $q_1, \dots, q_n$ form a coordinate system on V centered at $\pi(x)$. These coordinates induce a local trivialization $J^1V = \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^n$ and we denote $p_1, \dots, p_n$ the dual coordinates to $(q_1, \dots, q_n)$ and z the coordinate in $\mathbf{R}$. We claim that $(p_1, \dots, p_k, q_{k+1}, \dots, q_n)$ form a coordinate system on L near p, due to the Legendrian condition. Indeed, let $v \in T_xL$. At x, for $i \leq k$, we have $dq_i(v) = \omega(\partial_{p_i}, v) = 0$ since $\partial_{p_i} \in T_xL$ and L is Legendrian. Now if we have $\sum_{i=1}^k \lambda_i dp_i + \sum_{i=k+1}^n \mu_i dq_i = 0$ on T_xL, then plugging ∂_{p_i} for $i = 1 \dots, k$, gives $\lambda_i = 0$ and then $\mu_i = 0$ since the projection $T_xL \to T_{\pi(x)}V$ has rank $n-k$ by assumption. Consider $\psi : \mathbf{R}^k \times \mathbf{R}^{n-k} \to L$ be the corresponding local diffeomorphism $\psi(p_1, \dots, p_k, q_{k+1}, \dots, q_n)$, $\varphi : \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}^n$ the submersion given by $\varphi(p_1, \dots, p_k, q_1, \dots, q_n) = (q_1, \dots, q_n)$ and $f : \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R}$ the function defined by

$$f = z \circ \psi + \sum_{i=1}^k p_i(q_i - q_i \circ \psi).$$

We claim that (f,φ) is a generating function for L near x. Indeed,

$$\begin{aligned}
df &= \psi^* dz + \sum_{i=1}^k p_i(dq_i - \psi^* dq_i) + \sum_{i=1}^k (q_i - q_i \circ \psi) dp_i \\
&= \sum_{i=1}^n \psi^*(p_i dq_i) + \sum_{i=1}^k p_i(dq_i - \psi^* dq_i) + \sum_{i=1}^k (q_i - q_i \circ \psi) dp_i \\
&= \sum_{i=1}^k p_i \psi^* dq_i + \sum_{i=k+1}^n \psi^* p_i dq_i + \sum_{i=1}^k p_i(dq_i - \psi^* dq_i) + \sum_{i=1}^k (q_i - q_i \circ \psi) dp_i \\
&= \sum_{i=k+1}^n \psi^* p_i dq_i + \sum_{i=1}^k p_i dq_i + \sum_{i=1}^k (q_i - q_i \circ \psi) dp_i
\end{aligned}$$

Hence $\Sigma = \{\frac{\partial f}{\partial p_i} = 0, i = 1, \dots, k\} = \{q_i = q_i \circ \psi, i = 1, \dots, k\}$ and $i_f(\Sigma) = \{(z \circ \psi, p_1, \dots, p_k, p_{k+1} \circ \psi, \dots, p_n \circ \psi, q_1 \circ \psi, \dots, q_k \circ \psi, q_{k+1}, \dots, q_n)\} = L$. $\square$

Definition 5.4. A generating function over V is called *quadratic at infinity* if it is of the form $E = \mathbf{R}^k \times V$, $\varphi = pr_2$ and $f : E \to \mathbf{R}$ is quadratic at infinity.

5.2 Chekanov-Sikorav's theorem

Theorem 5.5. *Let V be a manifold and L a Legendrian submanifold of J^1V admitting a generating function $f \colon \mathbf{R}^k \times V \to \mathbf{R}$. Let $\varphi_t \colon J^1V \to J^1V$, $t \in [0,1]$, be a compactly supported contact isotopy of J^1V. Then $\varphi_1(L)$ also admits a generating function $g \colon \mathbf{R}^l \times V \to \mathbf{R}$ for some integer l.*

Moreover if f is quadratic at infinity, then we can choose g quadratic at infinity as well.

Remark 5.6. It is not essential that L is a Legendrian submanifold in the above statement. In fact we could even remove the transversality assumption in the definition of a generating function. The proof given below would work verbatim.

On J^1V we have a contact form $\alpha_V = dz - \sum_i p_i dq_i$. The Reeb vector field is $Y = \frac{\partial}{\partial z}$. Recall that a function $h \colon J^1V \to \mathbf{R}$ induces a contact vector field X_h given by $X_h = h\frac{\partial}{\partial z} + X_h'$ where $X_h' \in \xi_V = \ker \alpha_V$ is defined by $X_h' \lrcorner d\alpha_V = -dh|_{\xi_V}$. We find

$$X_h = \Big(h - \sum_i p_i \frac{\partial h}{\partial p_i}\Big)\frac{\partial}{\partial z} + \sum_i \Big(\frac{\partial h}{\partial q_i} + p_i \frac{\partial h}{\partial z}\Big)\frac{\partial}{\partial p_i} - \sum_i \frac{\partial h}{\partial p_i}\frac{\partial}{\partial q_i}.$$

Corollary 5.7 (A conjecture of Arnol'd)**.** *Let V be a closed manifold and let $\varphi_t \colon T^*V \to T^*V$, $t \in [0,1]$ be a compactly supported Hamiltonian isotopy. Let $0_V \subset T^*V$ be the zero section. Let $t_0 \in [0,1]$ be such that $\varphi_{t_0}(0_V)$ intersects 0_V transversally. Then the intersection consists of at least $\sum_i \dim H^i(V)$ points.*

Proof. Let $h_t(p,q)$ be the Hamiltonian function of φ_t. Then $h'_t(z,p,q) = h_t(p,q)$ defines a contact isotopy ψ_t such that $\pi \circ \psi_t = \varphi_t \circ \pi$, where $\pi\colon J^1V \to T^*V$ is the projection. We set $L = \{0\} \times 0_V$ and $S = \bigcup_{t\in[0,1]} \psi_t(L)$. Multiplying h'_t by a bump function which is 1 on S, we obtain another isotopy ψ'_t with compact support and such that $\psi'_t(L) = \psi_t(L)$ for all $t \in [0,1]$.

The Legendrian L has a generating function quadratic at infinity (the zero function) and, by the previous theorem, the same holds for $\psi_t(L)$. Let f_t be a generating function for $\psi_t(L)$ which is quadratic at infinity. Then $\varphi_t(0_V) \cap 0_V$ is in bijection with $\psi_t(L) \cap (\mathbf{R} \times 0_V)$, which is also in bijection with the critical points of f_t. If the intersection is transverse, then f_t is Morse and the number of critical points must be greater than the sum of Betti numbers. $\square$

Idea of the proof of Theorem 5.5. A generating function $f(v,q)$ on $\mathbf{R}^k \times V$ can be seen as a family of functions $f(-,q)$ on $\mathbf{R}^k$ parametrized by V which are generically Morse. It can also be seen in the other way as a family of functions $f_v = f(v,-)$ on V parametrized by $\mathbf{R}^k$. In this case φ_t moves $J^1(f_v) \subset J^1V$ and, for a given v and small t (say $t < t_0(v)$), $\varphi_t(J^1(f_v))$ remains the 1-jet of some function, say g_v^t. We can set $f^t(v,q) = g_v^t(q)$ and it gives a generating function for $\varphi_t(L)$. The problem is that we cannot bound $t_0(v)$ from below in general.

The solution given here gives such a bound for the particular case of affine functions f_v on $V = \mathbf{R}^n$. Then we reduce to this case by embedding V into some $\mathbf{R}^n$ and adding auxiliary variables.

Lemma 5.8. *Let $\Phi\colon J^1V \to J^1V$ be a contact diffeomorphism and let $F^t(v,q)\colon \mathbf{R}^k \times V \to V$, $t = 0,1$ be generating functions for $L_0, L_1 \subset J^1(V)$. We set $F_v^t(q) = F^t(v,q)$ and we assume that $\Phi(J^1(F_v^0)) = J^1(F_v^1)$ for all v. Then $\Phi(L_0) = L_1$.*

Proof. We take coordinates (z,p,q) on $J^1(V)$ and (z,w,p,v,q) on $J^1(\mathbf{R}^k \times V)$. We define $M \subset J^1(\mathbf{R}^k \times V)$ as $M = \{w = 0\}$. The Legendrian L_0 is obtained from $J^1(F^0) \subset J^1(\mathbf{R}^k \times V)$ by $L_0 = p_V(M \cap J^1(F^0))$, where p_V is the projection to $J^1(V)$.

Since Φ is contact there exists a function μ on $J^1(V)$ such that $\Phi^*(\alpha) = \mu\alpha$. Now Φ induces a contact diffeomorphism Ψ on $J^1(\mathbf{R}^k \times V)$ such that $\Phi \circ p_V = p_V \circ \Psi$. Writing $(z',p',q') = \Phi(z,p,q)$, it is given by $\Psi(z,w,p,v,q) = (z',w',p',v,q')$ with $w'_i = \mu(p,q,t)w_i$ for all i. In particular $\Psi(M) = M$.

We have $J^1(F^0) = \{(F^0(v,q), \frac{\partial F^0}{\partial v}, \frac{\partial F^0}{\partial q}, v, q)\}$. Our hypothesis is that $\Psi(J^1(F^0)) = \{(F^1(v,q'), h(v,q'), \frac{\partial F^1}{\partial q'}, v, q')\}$ for some function $h = (h_1,\dots,h_k)$. Writing that $\Psi(J^1(F^0))$ is Legendrian we find $h_i = \frac{\partial F^1}{\partial v_i}$. Hence $\Psi(J^1(F^0)) = J^1(F^1)$ and the lemma follows. $\square$

Proof of Theorem 5.5. **(i) Embedding into $\mathbf{R}^n$**

By Whitney's theorem we can find an embedding $j : V \to \mathbf{R}^n$ for some n. In the sequel we consider j as an inclusion map and omit to write it. The function $f : \mathbf{R}^k \times V \to \mathbf{R}$ is extended to $\mathbf{R}^k \times \mathbf{R}^n$ arbitrarily.

(ii) Fragmentation

For $N \in \mathbf{N}^*$ and $j \in 1,\dots,N$, we set $\varphi_{j,N} = \varphi_{\frac{j}{N}} \circ \varphi^{-1}_{\frac{j-1}{N}}$ and get $\varphi_1 = \varphi_{N,N} \circ \cdots \circ \varphi_{1,N}$. Since V is embedded in $\mathbf{R}^n$ we can measure the C^1-distance concretely as follows : for maps $\varphi, \psi : J^1V \to J^1V$ define $d_1(\varphi,\psi) = \sup_{x\in V} \|\varphi(x) - \psi(x)\| + \sup_{x\in V} \ \|d\varphi(x) - d\psi(x)\|$ for some norm $\|.\|$ on the vector space $\mathbf{R}^{1+n+n} = J^1\mathbf{R}^n$ and the induced operator norm. We claim then that $\max_j d_1(\varphi_{j,N}, \mathrm{id})$ converges to 0 when N goes to $+\infty$. This follows from the fact that φ_t has compact support.

(iii) Action on affine jets

For $a \in \mathbf{R}^n$, $b \in \mathbf{R}$, consider the function $z_{a,b} : V \to \mathbf{R}$ defined by $z_{a,b}(q) = \langle a, q\rangle + b$ and its 1-jet extension $J^1 z_{a,b} : V \to J^1V$ given by $J^1 z_{a,b}(q) = (\langle a, q\rangle + b, a, q)$. Consider the projection $\pi : J^1V \to V$ and the map $Q_{a,b} : V \to V$ given by $Q_{a,b} = \pi \circ \varphi \circ J^1 z_{a,b}$. Set $K = \pi(\mathrm{supp}(\varphi))$. Outside of K, we have $Q_{a,b}(q) = q$. Thanks to the previous step, we can assume that φ is C^1-close to the identity, and thus $Q_{a,b}$ is uniformly C^1-close to q as a function of a, b and q. In particular, we may assume that $Q_{a,b}$ is a diffeomorphism of V for all a, b. Then we set $Z_{a,b}(q) = \pi' \circ \varphi \circ J^1 z_{a,b} \circ Q_{a,b}^{-1}(q)$ where π' is the projection $J^1V \to \mathbf{R}$. These definitions are made so that graph in J^1V of the 1-jet of $z_{a,b}$ is mapped by φ to that of $Z_{a,b}$.

(iv) Linearization

We define a new function $f_1 : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^k \times V \to \mathbf{R}$ by the formula

$$f_1(x, y, v, q) = f(v, y) + \langle x, q - y\rangle.$$

We check easily that $L_{f_1} = L_f$. Moreover with the notations above we have $f_1(x, y, v, q) = z_{x,f(v,y)-\langle x,y\rangle}(q)$.

Next we define $f_2 : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^k \times V \to \mathbf{R}$ by

$$f_2(x, y, v, q) = Z_{x,f(v,y)-\langle x,y\rangle}(q).$$

Lemma 5.8 ensures that $L_{f_2} = \varphi(L_{f_1})$. This proves our result: $g = f_2$ is the required generating function for $\varphi(L_f)$.

(v) Interpolation at infinity

It remains to prove that if f is quadratic at infinity we can ensure the same property for g. By quadratic at infinity we mean the following: for (q, v) outside of some compact set of $\mathbf{R}^k \times V$, we have $f(q, v) = Q(v)$ for some non-degenerate quadratic form Q.

In the first step above, we may assume that the embedding j is proper (this is automatic if V is compact). Also when extending f to $\mathbf{R}^k \times \mathbf{R}^n$ we ensure that: $f(q, v) = Q(v)$ outside of some compact set of $\mathbf{R}^k \times \mathbf{R}^n$.

Then we define $f_3 : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^k \times \mathbf{R} \to \mathbf{R}$ by the formula

$$f_3(x, Y, v, q) = f_2(x, q - Y, v, q)$$

and check easily that $L_{f_2} = L_{f_3}$ (f_3 is obtained from f_2 by composing with a fibered diffeomorphism).

Next, we pick a compactly supported function $\chi : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^k \to [0, 1]$ and set

$$f_4(x, Y, v, q) = \chi(x, Y, v) f_3(x, Y, v, q) + (1 - \chi(x, Y, v))(Q(v) + \langle x, Y\rangle).$$

For (x, Y, v) large enough and $q \in V$, we have $f_4(x, Y, v, q) = Q(v) + \langle x, Y\rangle$ which is a non-degenerate quadratic form on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^k$. For q outside of K, we have $f_2(x, y, v, q) = f_1(x, y, v, q)$ and thus

$$f_4(x, Y, v, q) = \chi(x, Y, v) f(v, q - Y) + (1 - \chi(x, Y, v))Q(v) + \langle x, Y\rangle.$$

If (x, Y, v) is bounded and q large (here we use use the properness of $V \to \mathbf{R}^n$), then $q - Y$ is large and we have $f(v, q-Y) = Q(v)$, hence $f_4(x, Y, v, q) = Q(v) + \langle x, Y\rangle$. We conclude that the function f_4 is quadratic at infinity. However we have to choose χ suitably to ensure $L_{f_4} = L_{f_3}$.

We first argue that for (x, Y, v) large enough say, $|(x, Y, v)| \geq C > 0$, and $q \in V$ the function f_3 has no fiberwise critical point. This follows from the proof of Lemma 5.8 or a computation (simpler than what we do below). Hence if we choose χ so that $\chi = 1$ on the $|(x, y, v)| \leq C$, then we only have to check that f_4 has no fiberwise critical point in the region of interpolation (the support of $d\chi$).

For $q \in V \setminus K$, the above expression for f_4 gives $\frac{\partial f_4}{\partial x} = Y$, $\frac{\partial f_4}{\partial Y} = x$ and $\frac{\partial f_4}{\partial v} = \frac{\partial Q}{\partial v}$ up to bounded fonctions on $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^k \times (V \setminus K)$. Hence for (x, Y, v) large enough, there is no fiberwise critical point for f_4. For $q \in K$, we need to compute a little more. With the notations above and $a = x$, $b = f(v, q-Y) + \langle x, Y\rangle - \langle x, q\rangle$, we have:

$$f_4 = \chi Z_{a,b} + (1-\chi) z_{a,b} + (1-\chi)(Q(v) - f(v, q-Y)),$$

and the fiber derivatives write:

$$\frac{\partial f_4}{\partial x} = Y + \frac{\partial \chi}{\partial x}(Z_{a,b} - z_{a,b}) + \chi(\frac{\partial Z_{a,b}}{\partial a} - \frac{\partial z_{a,b}}{\partial a}) + \chi(\frac{\partial Z_{a,b}}{\partial b} - \frac{\partial z_{a,b}}{\partial b})(Y-q) - \frac{\partial \chi}{\partial x}(Q(v) - f(v, q-Y)),$$

$$\frac{\partial f_4}{\partial Y} = x - \frac{\partial f}{\partial q}(v, q-Y) + \frac{\partial \chi}{\partial Y}(Z_{a,b} - z_{a,b}) + \chi(\frac{\partial Z_{a,b}}{\partial b} - \frac{\partial z_{a,b}}{\partial b})(x - \frac{\partial f}{\partial q}(v, q-Y)) - \frac{\partial \chi}{\partial Y}(Q(v) - f(v, q-Y)),$$

$$\frac{\partial f_4}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial \chi}{\partial v}(Z_{a,b} - z_{a,b}) + \chi(\frac{\partial Z_{a,b}}{\partial b} - \frac{\partial z_{a,b}}{\partial b})\frac{\partial f}{\partial v} - \frac{\partial \chi}{\partial v}(Q(v) - f(v, q-Y)).$$

The terms $|Z_{a,b} - z_{a,b}|$, $|\frac{\partial Z_{a,b}}{\partial a} - \frac{\partial z_{a,b}}{\partial a}|$, $|Q(v) - f(v, q-Y)|$ are bounded on $\mathbf{R}^k \times \mathbf{R}^n \times \mathbf{R}^n \times V$. Let us assume that φ is sufficiently C^1-close to id so that $|\frac{\partial Z_{a,b}}{\partial b} - \frac{\partial z_{a,b}}{\partial b}| < 1$. We conclude that the triple $(\frac{\partial f_4}{\partial x}, \frac{\partial f_4}{\partial Y}, \frac{\partial f_4}{\partial v})$ is arbitrary large for (x, Y, v) large and uniformly for $q \in K$. Hence if the region $\chi = 1$ is chosen large enough and say $\|d\chi\| \leq 1$, then f_4 has no fiberwise critical points in the interpolation region. This finishes the proof. □

We can deduce from this theorem the following 1-parametric version.

Theorem 5.9. *Let V be a manifold, $f : \mathbf{R}^k \times V \to \mathbf{R}$ a generating function quadratic at infinity, $(\varphi_t)_{t\in[0,1]}$ a compactly supported contact isotopy of J^1V. Then there exists a generating function quadratic at infinity $G : \mathbf{R}^l \times V \times [0,1] \to \mathbf{R}$ over $V \times [0,1]$ such that G_0 is equivalent to f and $L_{G_t} = \varphi_t(L_f)$ for all $t \in [0,1]$.*

Proof. The contact isotopy φ_t lifts to a contact isotopy of $J^1(V \times [0,1]) = J^1V \times T^*[0,1]$ as follows. We have $\varphi_t^*\alpha_V = k_t\alpha_V$ for some positive functions k_t, and $\alpha_V(\frac{d}{dt}\varphi_t) = h_t \circ \varphi_t$. We set $\Phi_s(z, p, q, \tau, t) = (\varphi_{st}(z, p, q), k_{st}\tau + sh_{st} \circ \varphi_{st}(z, p, q), t)$ and we have

$$\Phi_s^*(\alpha_V - \tau dt) = k_{st}\alpha_V + \alpha(\frac{d}{dt}\ \varphi_{st})dt - k_{st}\tau dt - sh_{st} \circ \varphi_{st} dt = k_{st}(\alpha_V - \tau dt).$$

Also we may extend Φ_s to a compactly suppoted contact isotopy of $J^1(V \times \mathbf{R})$.

Set $F : \mathbf{R}^k \times V \times \mathbf{R} \to \mathbf{R}$ defined by $F(v,q,t) = f(v,q)$, it is a generating function for $L_f \times \{\tau = 0\} \subset J^1(V \times \mathbf{R})$. We apply Theorem 5.5 to F and Φ_s and obtain a generating function quadratic at infinity G for $\Phi_1(L_F)$. By construction, for all $t \in [0,1]$, $\Phi_1(L_{F_t}) = \varphi_t(L_f)$ and thus $L_{G_t} = \varphi_t(L_f)$.

It remains to show that G_0 is equivalent to f. A look at the proof of Theorem 5.5 shows that when φ_t is the identity the process transforms a generating function $f(v,q)$ into $f'(x,Y,v,q) = f(v,q-Y) + \langle x, Y\rangle$. Moser's method can be used to prove that f' differs from $f(v,q) + \langle x, Y\ \rangle$ by a fibered diffeomorphism. The result follows. □

5.3 Viterbo's uniqueness theorem

Our goal here is to prove the following result.

Theorem 5.10. *Let V be a closed connected manifold and $f_i : \mathbf{R}^{k_i} \times V \to \mathbf{R}$, $i = 1,2$, be generating functions quadratic at infinity for a Legendrian submanifold which is contact isotopic to the zero-section. Then there exist fiberwise quadratic forms $Q'_i : \mathbf{R}^{k'_i} \times V \ \to \mathbf{R}$, $i = 1,2$, and a diffeomorphism $\varphi : R^{k_1+k'_1} \times V \to \mathbf{R}^{k_2+k'_2} \times V$ preserving the projection to V such that $f_1 + Q'_1 = (f_2 + Q'_2) \circ \varphi$.*

We collect first a few lemmas to be used in the proof of this theorem.

Lemma 5.11. *Let M be a closed manifold of dimension m and N a manifold of dimension n. Assume $2(m+1) \leq n$. If two embeddings $f, g : M \to N$ are homotopic then they are isotopic.*

Proof. Let $F : M \times [0,1] \to N$ be a homotopy with $F_0 = f$, $F_1 = g$. After a small perturbation of F supported in $M \times (0,1)$, we can assume that F is an immersion with double points if $2(m+1) = n$ or an embedding if $2(m+1) < n$. In the first case, by another small perturbation we can assume that the double points are of the type $f(x,t) = f(x',t')$ with $t \neq t'$ hence in both cases F_t is an embedding for all t. □

Lemma 5.12. *Two maps $S^k \to S^k$ are homotopic if and only of they have the same degree, i.e. induce the same map $H_k(S^k) \to H_k(S^k)$.*

Lemma 5.13. *Let V be a manifold, and $(f_t)_{t\in[0,1]} \colon \mathbf{R}^k \times V$ a family of generating functions quadratic at infinity which transversally generate the same Legendrian immersion L. Then there exists a family of fibered diffeomorphisms $(\varphi_t)_{t\in[0,1]} \colon \mathbf{R}^k \times V \to \mathbf{R}^k \times V$ such that $f_t \circ \varphi_t = f_0$ and $\varphi_0 = \mathrm{id}$.*

Proof. Moser's method. □

Proof of Theorem 5.10. We first prove the result for the case of the zero-section and then show how to reduce to this case.

Proof for the zero-section

Let $f : \mathbf{R}^k \times V \to \mathbf{R}$ be a generating function for the zero-section. For $q \in V$, we write $f_q : V \to \mathbf{R}$ the corresponding function on $V = V \times \{q\}$.

(1) By assumption, for each $q \in V$, f_q has a unique critical point $v_0(q)$. Up to replacing f by $f \circ \varphi$ where $\varphi(v, q) = (v + v_0(q), q)$, we may assume that $v_0(q) = 0$.

(2) Let Q_q be the Hessian of f_q at 0, it is a non-degenerate quadratic form. A parametric version of Morse's lemma provides a smooth family of compactly supported diffeomorphisms $\varphi_q \colon \mathbf{R}^k \to \mathbf{R}^k$ such that $f_q \circ \varphi_q = Q_q$ near 0. We may thus assume that $f_q = Q_q$ near 0 for all q. Since V is connected, the index of Q_q is independent of q, we denote it i.

(3) Pick a linear adapted gradient vector field X_q for Q_q and a complete gradient vector field Y_q for f_q which coincides with X_q near 0. Then after a linear change of coordinates on $\mathbf{R}^k$, we have $Q_q = -x_1^2 - \cdots - x_i^2 + x_{i+1}^2 + \cdots + x_k^2$ and $X_q = -x_1\partial_{x_1} - \cdots - x_i\partial_{x_i} + x_{i+1}\partial_{x_{i+1}} + \cdots + x_k\partial_{x_k}$. For brevity we write $x_- = (x_1, \ldots, x_i)$ and $x_+ = (x_{i+1}, \ldots, x_k)$. The linear automorphism which preserve Q_q and X_q correspond to the group $O(i) \times O(k-i)$. For $\epsilon > 0$, the subset $M_\epsilon = \{(-\epsilon \leq Q \leq \epsilon, |x_-||x_+| \leq \epsilon\}$ is therefore well-defined by Q_q and X_q (i.e. independent of the choice of linear coordinates $(x_1, \ldots, x_n)$) and in particular well-defined globally on V. We fix $\epsilon > 0$ small enough so that $f = Q$ on M_ϵ.

(4) We claim that there is a fibered diffeomorphism $\varphi : \{f = -\epsilon\} \to \{Q = -\epsilon\}$ such that $\psi = \mathrm{id}$ on $N_\epsilon = \{f = -\epsilon\} \cap M_\epsilon = \{Q = -\epsilon\} \cap M_\epsilon$. This is the main step. In fact it may not hold directly, we further assume that the inequality $2i+1 \leq k$ holds. This can be achieved by stabilizing f (and therefore Q) by a positive definite quadratic form (i.e. set $f'(w, v, q) = |w|^2 + f(v, q)$ where $w \in \mathbf{R}^N$ for some sufficiently large N). Let us prove the claim. We first prove it for a fixed q and the show how to do it globally for $q \in V$.

By assumption f coincides with a quadratic form P outside of a compact set. We have $H^*(f_q, Y_q) = H^*_P(\mathbf{R}^k \times V)$ and hence P has the same index as Q. In particular for $c > 0$, the levet sets $\{P = -c\}$ and $\{Q = -c\}$ are diffeomorphic. Also, for large $c > 0$, we have $\{f = -c\} = \{P = -c\}$. Following the trajectories of Y, we find a diffeomorphism $\{f = -c\} \simeq \{f = -\epsilon\}$. In total we find that a diffeomorphism $\psi : \{f = -\epsilon\} \to \{Q = -\epsilon\}$. However ψ has no reason to be equal to the identity near N_ϵ. This N_ϵ is a tubular neighborhood $S^{i-1} \times D^{k-i}$ of a $(i-1)$-dimensional sphere in the manifold $\{Q = -\epsilon\}$ which is diffeomorphic to $S^{i-1} \times \mathbf{R}^{k-i}$. We claim that the inclusion $N_\epsilon \to \{Q = -\epsilon\}$ is of degree ± 1, i.e. is a homology isomorphism. Indeed the homology of $\mathbf{R}^k$ is the homology of a chain complex $0 \to \mathbf{Z} \to 0 \to \cdots \to 0 \to \mathbf{Z} \to \mathbf{Z} \to 0$ where the $\mathbf{Z}$ are in degree 0, $i-1$ and i, and the map $\mathbf{Z} \to \mathbf{Z}$ is the degree of the above map. Since $H_*(\mathbf{R}^k) = 0$ if $* \neq 0$, we obtain that the degree is ± 1. From Lemma 5.11, the inequality $2i+1 \leq k$ and Lemma 5.12, we deduce that the embedding $\varphi : S^{i-1} \to \{Q = -\epsilon\}$ is isotopic to the inclusion or a reflection σ about a hyperplane. From the isotopy uniqueness of tubular neighborhoods (Alexander's trick) we find that the embedding $\varphi : N_\epsilon = S^{i-1} \times D^{k-i} \to \{Q = -\epsilon\} = S^{i-1} \times \mathbf{R}^{k-i}$ is isotopic to the embedding $(x, y) \to (x, A_x(y))$ or $(\sigma(x), A_x(y))$ for some map $A : S^{i-1} \to O(k-i)$. This isotopy of embeddings extend to a compactly supported isotopy of $\{Q = -\epsilon\}$ and we obtain a diffeomorphism $\theta : \{Q = -\epsilon\} \to \{Q = -\epsilon\}$ such that $\theta \circ \psi(x, y)$ has one of the above form near $N_\epsilon = S^{i-1} \times D^{k-i}$. Since those formula define diffeomorphisms of $S^{i-1} \times \mathbf{R}^{k-i}$, postcomposing $\theta \circ \psi$ by their inverse gives the required diffeomorphism $\varphi : \{f = -\epsilon\} \to \{Q = -\epsilon\}$ which is the identity near N_ϵ.

To find the diffeomorphism φ globally with $q \in V$. Consider the set of diffeomorphisms $\varphi : \{f = -\epsilon\} \to \{Q = -\epsilon\}$ equal to the identity near N_ϵ as a bundle over V. Our problem is then to find a global section of this bundle. We have just seen that the fibers of this bundle are not empty. We will prove now that the fibers are in fact contractible. The existence of a global section then follows from general theory of bundles. The fiber corresponds to the set of diffeomorphism $\varphi : S^{i-1} \times \mathbf{R}^{k-i} \to S^{i-1} \times \mathbf{R}^{k-i}$ with $\varphi(x, y) = (x, y)$ for y in a neighorhood of D^{k-i}. For $t \in (0, 1)$, set $\varphi_t(x, y) = \frac{1}{t}\varphi(x, ty)$ where the multiplication acts on the second coordinate. We have $\varphi_1 = \varphi$ and $\varphi_t = \mathrm{id}$ on $S^{i-1} \times \frac{1}{t}D^{k-i}$. Hence φ_t converges to id uniformly on compact sets as when t goes to 0. This defines a deformation retraction of this space to point id.

(5) The last step is to extend the diffeomorphism φ to the whole $\mathbf{R}^k$. We claim that there exists a unique fibered diffeomorphism $\varphi : \mathbf{R}^k \times V \to \mathbf{R}^k \times V$ such that $f = Q \circ \varphi$, $\varphi_* Y = X$, $\varphi = \mathrm{id}$ on N_ϵ and φ coincides with a diffeomorphism obtained in the previous step on $\{f = -\epsilon\}$. Indeed, φ is obtained concretely as follows: for $v \in \mathbf{R}^k$, there exists $T \in \mathbf{R}$ such that $\varphi_Y^T(v) \in N_\epsilon \cup \{f = -\epsilon\}$, then there exists a unique $T' \in \mathbf{R}$ such that $Q(\varphi_X^{T'}(\varphi(\varphi_Y^T(v)))) = f(v)$, we set $\varphi(v) = \varphi_X^{T'}(\varphi(\varphi_Y^T(v)))$.

This finishes the proof in the case of the zero-section.

Reduction to the case of the zero-section

Let f, g be two generating function quadratic at infinity for $\varphi_1(L)$. Theorem 5.9 provides generating functions f_t and g_t for $\varphi_t(L)$ such that f_1 is equivalent to f and g_1 is equivalent to g. By the uniqueness statement for the zero-section, f_0 and g_0 are equivalent. Hence we may assume $f_0 = g_0$. Consider now the path $(f_{1-t})_{t\in[0,1]}$ concatenated with $(g_t)_{t\in[0,1]}$, the Legendrian immersion associated to this path is the loop $\varphi_{1-t}(L)$ concatenated with its inverse $\varphi_t(L)$. Theorem 5.9 applied to a contraction of this loop provides a family $f_{t,s}$ of generating function for $\varphi_t(L)$ with $f_{t,0}$ equivalent to f_t and $f_{t,1}$ equivalent to g_t. Finally since $f_{t,s}$ generate L_{f_t} for all $s \in [0, 1]$, Lemma 5.13 implies that $f_{t,1}$ is equivalent to $f_{t,0}$. The result follows. □

5.4 Back to spectral invariants

We recall some results of the previous sections.

Let V be a closed manifold. A function $f : \mathbf{R}^k \times V \to \mathbf{R}$ is called quadratic at infinity if there exist a non-degenerate quadratic form $Q : \mathbf{R}^k \to \mathbf{R}$ and a function $g : V \to \mathbf{R}$ such that $f(v, x) = Q(v) + g(x)$ outside of some compact set.

There exists a notion of Morse pair (f, X) in this situation and we can define a Morse complex $C(f, X)$ and cohomology groups $H_Q^*(\mathbf{R}^k \times V)$. The basis of the spaces $C^*(f, X)$ only depend on f and the differential depends on X. The spaces $H_Q^*(-)$ are actually independent of X and f. Moreover if the index of Q is i, the Künneth morphism gives $H^k(V) = H^k(V) \otimes H_Q^i(\mathbf{R}^k) \xrightarrow{\sim} H_Q^{k+i}(\mathbf{R}^k \times V)$. We also have a filtered version $H_{Q,(a,b)}(f, X) = H_Q(\{f < b\}, \{f < a\})$ together with a restriction map $r_{c,f} \colon H_Q^*(\mathbf{R}^k \times V) \to H_{Q,(-\infty,c)}(f, X)$. For $\alpha \in H(V)$ we set

$$c(f, \alpha) = \sup\{c \in \mathbf{R} \setminus \mathrm{vcrit}(f);\ r_{c,f}(\alpha) = 0\},$$

Let 1 be the generator of $H^0(V)$ and μ the generator of $H^n(V)$. We define $c_-(f) = c(f, 1)$ and $c_+(f) = c(f, \mu)$. If V is reduced to a point, then $1 = \mu$ and we write $c(f) = c_+(f) = c_-(f)$. We remark that Lemma 4.38 holds in this situation (V replaced by $\mathbf{R}^k \times V$) with the same proof. We also recall Proposition 4.46.

Proposition 5.14 (Lem. 4.38 and Prop. 4.46). *(i) The spectral invariant $c(f, \alpha)$ is a critical value of f, for any $\alpha \in H(V)$.*

(ii) Let $x_0 \in V$. The function $f_{x_0}(v) = f(v, x_0)$ is quadratic at infinity on $\mathbf{R}^k$ and we have

$$c_-(f) \leq c(f_{x_0}) \leq c_+(f).$$

We could in fact define $H^*_Q(-)$ and the filtered version when Q is only fiberwise quadratic. We mean that $Q\colon \mathbf{R}^k \times V \to \mathbf{R}$ satisfies $Q_x = Q(-, x)$ is non-degenerate quadratic for each $x \in V$ and $f(v, x) = Q(v, x)$ outside of some compact set. (Remark that the index of Q_x is independent of x if V is connected.)

For any given function we can find an arbitrarily small modification which is Morse. By the Lipschitz property (see Lemma 4.41) we can extend the definition of $c(f, \alpha)$ to any function which is fiberwise quadratic at infinity. Proposition 5.14 still holds in this generality.

If $Q'\colon \mathbf{R}^l \times V \to \mathbf{R}$ is fiberwise non-degenerate quadratic and f is quadratic at infinity, then $f + Q'$ is fiberwise quadratic at infinity and, up to a shift by the index of Q', we have $C^*(f, X) \simeq C^*(f + Q', X + X')$, for a vector field X' adapted to Q'. We deduce $c(f, \mu) = c(f + Q, \mu)$. In the same way we have $c(f, \mu) = c(f \circ \varphi, \mu)$, if $\varphi\colon \mathbf{R} \times V \to \mathbf{R} \times V$ is a diffeomorphism preserving the projection to V. Hence, by the uniqueness theorem 5.10 for generating functions, it makes sense to give the following definition.

Definition 5.15. Let V be a closed connected manifold and let $L \subset J^1(V)$ be a Legendrian manifold which is contact isotopic to the zero section. Let $f\colon \mathbf{R}^l \times V \to \mathbf{R}$ be a generating function for L which is quadratic at infinity (which exists by Chekanov-Sikorav's theorem). For $\alpha \in H(V)$ we set

$$c(L, \alpha) = c(f, \alpha) = \sup\{c \in \mathbf{R} \setminus \mathrm{vcrit}(f);\ r_{c,f}(\alpha) = 0\}.$$

It only depends on L and not on f. We also set $c_\pm(L) = c_\pm(f)$.

Proposition 5.16. *Let V, V' be closed connected manifolds of dimension n, n' and μ, μ' the generators of $H^n(V)$, $H^{n'}(V')$. Let $f\colon \mathbf{R}^k \times V \to \mathbf{R}$, $f'\colon \mathbf{R}^{k'} \times V' \to \mathbf{R}$ be quadratic at infinity. By the Künneth isomorphism (μ, μ') is a generator of $H^{n+n'}(V \times V')$. Then $c(f + f', (\mu, \mu')) \leq c(f, \mu) + c(f', \mu')$ with equality when $f'(v', q') = Q'(v')$ for some quadratic function Q'.*

Proof. We set $M = \mathbf{R}^k \times V$ and $M_t = \{f < t\}$. We define M', M'_t in the same way and $U_t = M_t \times M'$, $V_t = M \times M'_t$. We choose $t < c(f, \mu)$ and $u < c(f', \mu')$. Then $\{f + f' < t + u\} \subset W = U_t \cup V_u$. The Künneth isomorphism gives $H^{n+n'}(U_t) \simeq 0$ and $H^{n+n'}(V_u) \simeq 0$. Since $U_t \cap V_u = M_t \times M'_u$ we also have $H^{n+n'-1}(U_t \cap V_u) \simeq 0$. The Mayer-Vietoris sequence

$$\cdots \to H^{n+n'-1}(U_t \cap V_u) \to H^{n+n'}(W) \to H^{n+n'}(U_t) \oplus H^{n+n'}(V_u) \to 0$$

implies $H^{n+n'}(W) \simeq 0$ and we obtain the result. $\square$

5.5 Viterbo invariants for Hamiltonian isotopies

We will consider the spectral invariants $c_\pm(L)$ in the case where $L \subset J^1\mathbf{R}^{2n}$ is a Legendrian lift of the graph of a Hamiltonian isotopy of $\mathbf{R}^{2n}$. Since $\mathbf{R}^{2n}$ is non compact we will compactify it into S^{2n} but the compactification of L (which could be the diagonal $L = \Delta_{\mathbf{R}^{2n}}$ of $\mathbf{R}^{4n}$) should be isotopic to the zero section. We first make a change of variables which identifies $\mathbf{R}^{2n}$ with $T^*\mathbf{R}^n$ and turns the diagonal into the zero section.

On $\mathbf{R}^{2n} = \mathbf{R}^n \times \mathbf{R}^n$, we consider the Liouville form $\lambda = \frac{1}{2}(xdy - ydx)$. We will use the following contact diffeomorphism $\tau\colon \mathbf{R} \times \mathbf{R}^{2n} \times \mathbf{R}^{2n} \to J^1(\mathbf{R}^{2n})$ defined by

$$\tau(z,x,y,X,Y) = (u,p_1,p_2,q_1,q_2) = \Big(z + \frac{1}{2}(xY - yX),\, Y - y,\, x - X,\, \frac{x+X}{2},\, \frac{y+Y}{2}\Big).$$

It satisfies $\tau^*(du - p_1 \wedge dq_1 - p_2 \wedge dq_2) = dz - \lambda + \Lambda$ and maps the diagonal $\{z = 0, x = X, y = Y\}$ to the zero-section $\{u = 0, p_1 = 0, p_2 = 0\}$. More precisely we quote

$$\tau(c,x,y,x,y) = (c,0,0,x,y). \tag{5.1}$$

The graph of a symplectomorphism φ of $\mathbf{R}^{2n}$ is a Lagrangian submanifold of $\mathbf{R}^{4n}$. There exists a Legendrian lift of this graph (well-defined up to a translation in the variable z):

$$\Lambda_\varphi = \{(k(x,y), x, y, \varphi(x,y));\ (x,y) \in \mathbf{R}^{2n}\},$$

where the function $k\colon \mathbf{R}^{2n} \to \mathbf{R}$ is defined up to a constant by $\varphi^*\lambda - \lambda = -dk$. If φ has compact support, we can (and will) normalize k by setting $k = 0$ outside a compact set. We fix also a diffeomorphism $S^{2n} \setminus \{\infty\} = \mathbf{R}^{2n}$. Then $\tau(\Lambda_\varphi)$ is naturally extended by adding the point ∞ of the zero-section and we obtain a closed Legendrian submanifold of J^1S^{2n}:

$$\Gamma_\varphi = \overline{\tau(\Lambda_\varphi)} \subset J^1S^{2n}.$$

For $\varphi = \mathrm{id}$, Γ_φ is the zero-section.

Now we assume that φ is Hamiltonian isotopic to the identity. We can check that Γ_φ is Hamiltonian isotopic to the zero-section. Hence, according to Sikorav's theorem, Γ_φ admits a generating function quadratic at infinity and we may define its spectral invariants.

Definition 5.17. For $\varphi \in \mathrm{Ham}^c(\mathbf{R}^{2n})$, we define $c_\pm(\varphi) = c_\pm(\Gamma_\varphi)$ and $\gamma(\varphi) = c_+(\varphi) - c_-(\varphi)$.

Remark 5.18. By (5.1) the fixed points of φ are in bijection with $\Gamma_\varphi \cap (\mathbf{R} \times 0_{\mathbf{R}^{2n}})$. Moreover, if $(x,y) \in \mathbf{R}^{2n}$ is fixed by φ, the corresponding point in Γ_φ is $(k(x,y),0,0,x,y) \in J^1\mathbf{R}^{2n}$.

Let $f\colon \mathbf{R}^k \times \mathbf{R}^{2n} \to \mathbf{R}$ be a generating function for Γ_φ. If (v,x,y) is a critical point of f, then $(f(v,x,y),0,0,x,y)$ is a point of Γ_φ and (x,y) is fixed point of φ. Moreover $k(x,y) = f(v,x,y)$. In particular the spectral invariants of Γ_φ are values of k at fixed points of φ. We have the following more precise result.

Lemma 5.19. *Let $f\colon \mathbf{R}^k \times \mathbf{R}^{2n} \to \mathbf{R}$ be a generating function for Γ_φ. The projection $\mathbf{R}^k \times \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ induces a bijection between the set,* $\mathrm{crit}(f)$*, of critical points of f and the fixed point set,* $\mathrm{Fix}(\varphi)$*, of φ. Moreover, for $(v,x,y) \in \mathrm{crit}(f)$, we have $k(x,y) = f(v,x,y)$ and $k(\mathrm{Fix}(\varphi)) = \mathrm{vcrit}(f)$ is a subset of $\mathbf{R}$ of measure 0.*

Proof. Recall that we defined $\Sigma \subset \mathbf{R}^k \times \mathbf{R}^{2n}$ by $\Sigma = \{(v,q);\ \frac{\partial f}{\partial v}(v,q) = 0\}$ and $i\colon \Sigma \to J^1(\mathbf{R}^{2n})$ by $i(v,q) = (f(v,q), \frac{\partial f}{\partial q}(v,q), q)$. We have seen that i is an embedding and gives a diffeomorphism from Σ to Γ_φ. It is clear that $\mathrm{crit}(f)$ is contained in Σ and, by the remark 5.18, i induces a bijection between $\mathrm{crit}(f)$ and $\mathrm{Fix}(\varphi)$. The equality $k(x,y) = f(v,x,y)$ for $(v,x,y) \in \mathrm{crit}(f)$ also follows from the same remark. The last assertion follows from Sard's theorem. $\square$

Lemma 5.20. *We assume that $\varphi = \varphi_h^1$ for some function $h\colon \mathbf{R}^{2n} \to \mathbf{R}$ with compact support and independent of time. We assume that 0 is a zero of dh (hence a fixed point of φ). Then $k(0) = h(0)$.*

Proof. Let X_h be the Hamiltonian vector field of h. We have $\iota_{X_h} d\lambda = -dh$. For $t \in [0,1]$ we define k_t by the condition $(\varphi_h^t)^*\lambda - \lambda = -dk_t$ and k_t is zero at infinity. We deduce

$$-d(\frac{\partial k_t}{\partial t}) = (\varphi_h^t)^*(\mathcal{L}_{X_h}\lambda) = (\varphi_h^t)^*(d(\iota_{X_h}\lambda) + \iota_{X_h} d\lambda) = -d((\varphi_h^t)^*(h - \iota_{X_h}\lambda)).$$

Hence $\frac{\partial k_t}{\partial t} = (\varphi_h^t)^*(h - \iota_{X_h}\lambda) + c_t$, for some constant c_t. Since h has compact support and k_t vanishes at infinity, we have $c_t = 0$. Since h is independent of time, it is preserved by φ_h^t, that is $(\varphi_h^t)^*(h) = h$. Our choice of λ implies $\lambda_0 = 0$. Finally we have $\frac{\partial k_t}{\partial t}(0) = h(0)$. Integrating between $t = 0$ and $t = 1$ gives $k_1(0) = h(0)$, as required. $\square$

Proposition 5.21. *For $\varphi \in Ham^c(\mathbf{R}^{2n})$, we have*

$$c_-(\varphi) \leq 0 \leq c_+(\varphi).$$

Moreover $c_+(\varphi) = c_-(\varphi)$ if and only if $\varphi = \mathrm{id}$.

Proof. Let $f\colon \mathbf{R}^k \times S^{2n} \to \mathbf{R}$ be a generating function for Γ_φ which is quadratic at infinity. The inequalities follow from Proposition 5.14-(ii) applied to the point ∞.

Let us assume $c_+(\varphi) = c_-(\varphi) = 0$. For any $q \in S^{2n}$, $c(f_q)$ is a critical value of f_q by Proposition 5.14-(i) and we have $c(f_q) = 0$ by Proposition 5.14-(ii) again. Hence the projection of Γ_φ to J^0S^{2n} contains the graph of the zero function. Therefore Γ_φ contains the 1-graph of the zero function which is $\{0\} \times 0_{S^{2n}}$. Since Γ_φ is a connected submanifold, it is equal to $\{0\} \times 0_{S^{2n}}$ and we obtain $\varphi = \mathrm{id}$. $\square$

5.6 Capacities

We say that a Hamiltonian isotopy $\varphi = \varphi^1$ displaces a set K if $\varphi(K) \cap K = \emptyset$. We say that it is supported in K if the support of φ^t is contained in K for all $t \in [0,1]$. We recall that $\gamma(\varphi) = c_+(\varphi) - c_-(\varphi) \geq c_+(\varphi)$.

Definition 5.22. For an open subset U and a compact subset K of $\mathbf{R}^{2n}$, we define

- $c(U)$ to be the supremum of $c_+(\varphi)$ where $\varphi \in \mathrm{Ham}^c(\mathbf{R}^{2n})$ is supported in U,
- $\gamma(K)$ to be the infimum of $\gamma(\varphi)$ where $\varphi \in \mathrm{Ham}^c(\mathbf{R}^{2n})$ displaces K,
- $\gamma(U)$ to be the supremum of $\gamma(K)$ where K runs over the compact subsets of U.

If $U \subset V$ we have $c(U) \leq c(V)$ and $\gamma(U) \leq \gamma(V)$.

Proposition 5.23. *Let φ_t, $t \in [0,1]$, be a Hamiltonian isotopy supported in a compact set K such that $\mathbf{R}^{2n} \setminus K$ is connected, and let $\psi \in \mathrm{Ham}^c(\mathbf{R}^{2n})$ be an isotopy which displaces K. Then $c_+(\psi \circ \varphi_t) = c_+(\psi)$.*

Proof. (i) We have $\psi^*\lambda = \lambda + dk$ with $k = 0$ at infinity. On $\mathbf{R}^{2n} \setminus K$, we have $\psi \circ \varphi_t = \psi$ and thus, $(\psi \circ \varphi_t)^*\lambda = \lambda + dk_t$ with $dk_t = dk$ on $\mathbf{R}^{2n} \setminus K$, and $k_t = 0$ at infinity. By connectedness, we obtain $k_t = k$ on $\mathbf{R}^{2n} \setminus K$. The fixed points of $\psi \circ \varphi_t$ are outside of K and are precisely the fixed points of ψ.

(ii) Let f_t be a family of gfqi for $\Gamma_{\psi\circ\varphi_t}$. By the remark 5.18 the critical values of f_t are values of k_t at fixed points of $\psi \circ \varphi_t$. These are the same as the fixed points of ψ, they are outside of K and $k_t = k$ outside of K. Hence $k_t(\mathrm{Fix}(\psi \circ \varphi_t)) = k(\mathrm{Fix}(\psi))$ and, by Lemma 5.19, this is a set of measure 0. Since $t \mapsto c_+(\psi \circ \varphi_t)$ is continuous and takes values in $k(\mathrm{Fix}(\psi))$, it is constant. □

Proposition 5.24. *Let $\varphi, \psi \in Ham^c(\mathbf{R}^{2n})$. Then $c_\pm(\psi \circ \varphi \circ \psi^{-1}) = c_\pm(\varphi)$.*

Proof. Let k_t be the function associated with $\psi_t \circ \varphi \circ \psi_t^{-1}$. It is defined by $dk_t = \lambda - (\psi_t \circ \varphi \circ \psi_t^{-1})^*\lambda$ and $k_t = 0$ at infinity. Let l_t, l'_t be associated in the same way with ψ_t and ψ_t^{-1}. We have

$$d(\varphi \circ \psi_t^{-1})^* l_t = (\varphi \circ \psi_t^{-1})^*\lambda - (\psi_t \circ \varphi \circ \psi_t^{-1})^*\lambda, \qquad d(\psi_t^{-1})^* k_0 = (\psi_t^{-1})^*\lambda - (\varphi \circ \psi_t^{-1})^*\lambda$$

and $d(\psi_t^{-1})^* l_t = (\psi_t^{-1})^*\lambda - \lambda$. Summing up we have $k_t = k_0 \circ \psi_t^{-1} + l_t \circ \varphi \circ \psi_t^{-1} - l_t \circ \psi_t^{-1}$.

We remark that $\mathrm{Fix}(\psi_t \circ \varphi \circ \psi_t^{-1}) = \psi_t(\mathrm{Fix}(\varphi))$ and, for $z \in \psi_t(\mathrm{Fix}(\varphi))$, we have $\varphi \circ \psi_t^{-1}(z) = \psi_t^{-1}(z)$ and thus $k_t(z) = k_0(\psi_t^{-1}(z))$. Hence $k_t(\mathrm{Fix}(\psi_t \circ \varphi \circ \psi_t^{-1})) = k_0(\mathrm{Fix}(\varphi))$. We conclude with the same continuity argument as in the second part of the proof of Proposition 5.23. □

Proposition 5.25. *Let $\varphi, \psi \in Ham^c(\mathbf{R}^{2n})$. Then $c_+(\psi \circ \varphi) \leq c_+(\psi) + c_+(\varphi)$.*

Sketch of proof. We define Hamiltonian isotopies a, b on $\mathbf{R}^{2n} \times \mathbf{R}^{2n}$ by $a(z, z') = (\varphi(z'), \psi(z))$ and $b(z, z') = (\varphi(z), z')$. We set $a' = b^{-1} \circ a \circ b$. Then we have $a'(z, z') = b^{-1}(\varphi(z'), \psi(\varphi(z))) = (z', \psi(\varphi(z)))$. We let $s\colon \mathbf{R}^{2n} \times \mathbf{R}^{2n} \to \mathbf{R}^{2n} \times \mathbf{R}^{2n}$ be the switch $s(z, z') = (z', z)$.

We can apply $\tau \times \tau$ to the graphs of $s \circ a$, $s \circ a'$ and compactify $\mathbf{R}^{2n} \times \mathbf{R}^{2n}$ into $S^{2n} \times S^{2n}$ instead of S^{4n}. We obtain Legendrian submanifolds Γ_a, $\Gamma_{a'}$ of $J^1(S^{2n} \times S^{2n})$. Then $\Gamma_a = \Gamma_\psi \times \Gamma_\varphi$ and $\Gamma_{a'} = \Gamma_{\psi\circ\varphi} \times 0_{S^{2n}}$.

Let (μ, μ') be the generator of $H^{4n}(S^{2n} \times S^{2n})$. Since a' is obtained from a by conjugating with the isotopy $b_t = s \circ (\mathrm{id} \times \varphi_t)$, the same proof as in Proposition 5.24 gives $c(\Gamma_{a'}, (\mu, \mu')) = c(\Gamma_a, (\mu, \mu'))$. Now we have $c(\Gamma_{a'}, (\mu, \mu')) = c_+(\psi \circ \varphi)$ and $c(\Gamma_a, (\mu, \mu')) \leq c_+(\psi) + c_+(\varphi)$, by Proposition 5.16 and this gives the result. □

Proposition 5.26. *With the same hypothesis as in Proposition 5.23 we have $c_+(\varphi_t) \leq \gamma(\psi)$.*

Proof. First observe that $c_+(\psi^{-1}) = -c_-(\psi)$. Indeed, if f is a generating function for Γ_ψ, then $-f$ is a generating function for $\Gamma_{\psi^{-1}}$. Using Poincaré duality we can prove that $c_+(-f) = -c_-(f)$. By Propositions 5.25 and 5.23 we have

$$c_+(\varphi_t) = c_+(\psi^{-1} \circ \psi \circ \varphi_t) \leq c_+(\psi^{-1}) + c_+(\psi \circ \varphi_t) = -c_-(\psi) + c_+(\psi) = \gamma(\psi).$$

□

Corollary 5.27. *For any U and $\varphi \in Ham^c(\mathbf{R}^{2n})$ we have $c(\varphi(U)) = c(U)$, $\gamma(\varphi(U)) = \gamma(U)$ and $c(U) \leq \gamma(U)$.*

We set $B = \{(x,y) \in \mathbf{R}^{2n};\ ||(x,y)|| < 1\}$ and $Z = \{(x,y) \in \mathbf{R}^{2n};\ ||(x_1,y_1)|| < 1\}$.

Lemma 5.28. *We have $c(B) \geq \pi$.*

Proof. We define $h\colon \mathbf{R}^{2n} \to \mathbf{R}$ by $h(x,y) = f(||(x,y)||^2)$, where $f(r)$ is a function on $[0,\infty[$ which is 0 for $r \geq 1$ and is a smoothing of $r \mapsto \pi(1-r)$ on $[0,1]$ with $-\pi < f'(r) < 0$ on $[0,1[$. Then $f(0) = \pi - \varepsilon$ for some $\varepsilon > 0$ which can be made as small as desired.

Let $\varphi = \varphi_h^1$ be the Hamiltonian flow of h. Then φ has a support contained in B. Since $-\pi < f'(r) \leq 0$, the fixed point set of φ is $\{0\} \cup \overline{(\mathbf{R}^{2n} \setminus \mathrm{supp}(h))}$.

Defining k by $dk = \lambda - \varphi^*\lambda$, the value of k on $\overline{(\mathbf{R}^{2n} \setminus \mathrm{supp}(h))}$ is zero. By Lemma 5.20 its value at 0 is $k(0) = h(0) = \pi - \varepsilon$. Hence we only have two possible values for the spectral invariants of φ. Since $\varphi \neq \mathrm{id}$, Proposition 5.21 implies that $c_+(\varphi) = \pi - \varepsilon$. It follows that $c(B) \geq \pi - \varepsilon$. □

Lemma 5.29. *We have $\gamma(Z) \leq \pi$.*

Proof. We choose $r < 1$ and $R \gg 0$ and we set $K = B_r \times B'_R$ where B_r is the closed ball of radius r in $\mathbf{R}^2$ and B'_R the closed ball of radius R in $\mathbf{R}^{2n-2}$. We can find a Hamiltonian isotopy of $\mathbf{R}^2$ which sends B_r into $B_+ = \{||(x_1,y_1)|| \leq \sqrt{2};\ y_1 > 0\}$. Its product with $\mathrm{id}_{\mathbf{R}^{2n-2}}$ sends K into $K_+ = B_+ \times B_r$. It is thus enough to find a Hamiltonian isotopy φ which displaces K_+ and satisfies $\gamma(\varphi) \leq \pi + \varepsilon$ for $\varepsilon > 0$ arbitrarily small.

We define $h_1\colon \mathbf{R}^2 \to \mathbf{R}$ by $h_1(x,y) = f(||(x,y)||^2)$, where $f(r)$ is a function on $[0,\infty[$ which is 0 for $r \geq 3$, equals $\frac{\pi}{2}(2-r)+\epsilon$ on $[0,2]$ and $-\pi < f'(r) < 0$ on $]2,3[$. Then $f(0) = \pi + \varepsilon$ for some $\varepsilon > 0$ which can be made as small as desired.

We then define $h\colon \mathbf{R}^{2n} \to \mathbf{R}$ by $h(x,y) = h_1(x_1,y_1)\rho(x,y)$ for some bump function ρ with compact support which is 1 on $B_{\sqrt{2}} \times B'_R$. Then φ_h^1 displaces K_+. We can check that $c_+(\varphi_h^1) = h(0) = \pi + \varepsilon$, as in the proof of Lemma 5.28, and $c_-(\varphi_h^1) = 0$. Hence $\gamma(\varphi_h^1) = \pi + \varepsilon$ and we deduce the lemma. □

Proposition 5.30. *We have $c(B) = \gamma(B) = c(Z) = \gamma(Z) = \pi$.*

Proof. This follows from Lemmas 5.28, 5.29 and the inequalities $c(B) \leq \gamma(B) \leq \gamma(Z)$ and $c(B) \leq c(Z) \leq \gamma(Z)$. □

Theorem 5.31 (Gromov non-squeezing theorem)**.** *Let B_r be the ball of radius r in $\mathbf{R}^{2n}$. If there exists $\varphi \in \mathrm{Ham}^c(\mathbf{R}^{2n})$ such that $\varphi(B_r) \subset Z$, then $r \leq 1$.*

Proof. We can deduce from the previous proposition that $c(B_r) = \pi r^2$. We deduce $\pi r^2 = c(B_r) = c(\varphi(B_r)) \leq c(Z) = \pi$. □