THE ISING MODEL
Exercises

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The Fortuin–Kasteleyn (also called Random Cluster) model

Definition of the Fortuin-Kasteleyn model

Let $\Lambda \Subset \mathbb{Z}^d$ be a finite connected subgraph of $\mathbb{Z}^d$. Consider the graph $G = (V_\Lambda, \mathcal{E}_\Lambda)$, where V_Λ and $\mathcal{E}_\Lambda$ are the vertices and edges of $\mathbb{Z}^d \cap \Lambda$. The configuration space is $\Omega_\Lambda := \{0,1\}^{\mathcal{E}_\Lambda}$. A configuration in Ω_Λ is denoted $\omega = \{\omega(e)\}_{e \in \mathcal{E}_\Lambda}$. It associates to each edge e a number in $\{0,1\}$. The edge e is called *open* in the configuration ω if $\omega(e) = 1$, it is called *closed* if $\omega(e) = 0$. A connected component (also called *cluster*) of ω is a set of vertices such that any two elements can be linked by a path of open edges.
Let $\overline{\omega} \in \Omega := \{0,1\}^{\mathcal{E}_{\mathbb{Z}^d}}$ be a boundary condition, we write $\Omega_\Lambda^{\overline{\omega}} := \left\{\omega \in \Omega \ tq \ \omega \equiv \overline{\omega} \text{ on } \mathbb{Z}^d \backslash \Lambda\right\}$.
Let $\kappa_\Lambda(\omega)$ be the number of connected components of the configuration $\omega \in \Omega_\Lambda^{\overline{\omega}}$ which intersect Λ, including isolated sites. We define a probability measure on $\Omega_\Lambda^{\overline{\omega}}$, which depends on two real parameters $p \in [0,1]$ and $q \in (0, \infty)$:

$$\mathbb{P}^{\overline{\omega}}_{\Lambda,p,q}(\omega) = \frac{1}{Z^{\overline{\omega}}_{\Lambda,p,q}} \prod_{e \in \mathcal{E}_\Lambda} \left[p^{\omega(e)}(1-p)^{1-\omega(e)}\right] q^{\kappa_\Lambda(\omega)}$$

Remarks :

- For $q = 1$, we recover the percolation model. It is the only parameter for which the variables $\omega(e)$ are independant. In the sequel $q \in \mathbb{N}^\star$ will play a special role.
- We define two "extremal" boundary conditions : $\overline{\omega} \equiv 0$ is called *free* boundary condition, we write $\Omega_\Lambda^{\mathbf{0}}$, and $\mathbb{P}^{\mathbf{0}}_{\Lambda,p,q}$, whereas $\overline{\omega} \equiv 1$ is called "wired" boundary condition, we write $\Omega_\Lambda^{\mathbf{1}}$ and $\mathbb{P}^{\mathbf{1}}_{\Lambda,p,q}$.

Definition of the Potts model

Let $\Lambda \Subset \mathbb{Z}^d$ and $G = (V_\Lambda, \mathcal{E}_\Lambda)$ as before. The configuration space is $\Sigma_\Lambda := \{1,...,q\}^{V_\Lambda}$. A configuration is written $\sigma = \{\sigma_i\}_{i \in V_\Lambda}$. It associates to each vertex $i \in \Lambda$ a number in $\{1,...,q\}$, usually called "color". Let $\overline{\sigma} \in \Sigma := \{1,...,q\}^{\mathbb{Z}^d}$ be a boundary condition, we write $\Sigma_\Lambda^{\overline{\sigma}} := \left\{\sigma \in \Sigma \ tq \ \sigma \equiv \overline{\sigma} \text{ on } \mathbb{Z}^d \backslash \Lambda\right\}$.

We associate to each configuration $\sigma \in \Sigma_\Lambda^{\overline{\sigma}}$ its Hamiltonian in Λ, which depends on a real parameter: the inverse temperature $\beta \in [0, \infty]$:

$$H_{\Lambda,\beta}(\sigma) = -\beta \sum_{\{i,j\} \in \mathcal{E}_{\mathbb{Z}^d} : \{i,j\} \cap \Lambda \neq \varnothing} \delta_{\sigma_i, \sigma_j}$$

where $\delta_{x,y} = 1$ if $x = y$ and 0 otherwise.
We define a probability measure on $\Sigma_\Lambda^{\overline{\sigma}}$, depending on β and $q \in \mathbb{N}^*$:

$$\mathcal{P}^{\overline{\sigma}}_{\Lambda,\beta,q}(\sigma) = \frac{1}{\mathcal{Z}^{\overline{\sigma}}_{\Lambda,\beta,q}} \exp\left(-H_{\Lambda,\beta}(\sigma)\right) \quad \text{where} \quad \mathcal{Z}^{\overline{\sigma}}_{\Lambda,\beta,q} = \sum_{\sigma \in \Sigma_\Lambda^{\overline{\sigma}}} \exp\left(-H_{\Lambda,\beta}(\sigma)\right)$$

Remark : The boundary condition $\overline{\sigma} \equiv i \in \{1, \ldots, q\}$ is called boundary condition "i", we write Σ_Λ^i and $\mathcal{P}^i_{\Lambda,\beta,q}$. Note that all the colors play the same role.

QUESTION 1.
For $q = 2$, show that the Potts model at inverse temperature β coincides (if we rename the colors 1 and 2 in $+1$ and -1) with the Ising model at inverse temperature $\beta/2$. The Potts model can thus be viewed as a generalization of the Ising model.

The Edwards-Sokal coupling

We will study the link between the Potts model with boundary condition "1", and the FK model with wired boundary condition. Let $q \in \{2, 3, 4, ...\}$, $p \in [0, 1]$, and $G = (V_\Lambda, \mathcal{E}_\Lambda)$ as before.
Consider the product space

$$\Sigma_\Lambda^1 \times \Omega_\Lambda^{\mathbf{1}} = \{1, 2, 3, ...q\}^{V_\Lambda} \times \{0, 1\}^{\mathcal{E}_\Lambda}$$

We define a probability measure on this space (called Edwards-Sokal coupling) :

$$\mu_{\Lambda,p}(\sigma, \omega) = \frac{1}{Z} \prod_{e \in \mathcal{E}_\Lambda} \big((1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)\big) \tag{1}$$

where $\delta_e(\sigma) = \delta_{\sigma_i,\sigma_j}$ for $e = \{i, j\} \in \mathcal{E}_\Lambda$ and $Z = \sum_{(\sigma,\omega)\in\Sigma_\Lambda^1\times\Omega_\Lambda^{\mathbf{1}}} \prod_{e\in\mathcal{E}_\Lambda} (1-p)\delta_{\omega(e),0} + p\delta_{\omega(e),1}\delta_e(\sigma)$.

Note that $\mu_{\Lambda,p}$ can be seen as the product measure Uniform$(\Sigma_\Lambda) \times$ Bernoulli$(p)^{\mathcal{E}_\Lambda}$ conditionned on the event $C := \{\delta_e(\sigma) = 1 \text{ for all } e \text{ such that } \omega(e) = 1\}$.

QUESTION 2. *Marginals of the measure* $\mu_{\Lambda,p}$

Let $p = 1 - e^{-\beta} \in [0, 1)$, prove the following statements :

1. The marginal of $\mu_{\Lambda,p}$ on Σ_Λ^1 is the Potts measure, i.e.:

$$\mu_1(\sigma) := \sum_{\omega\in\Omega_\Lambda^{\mathbf{1}}} \mu_{\Lambda,p}(\sigma, \omega) = \frac{1}{\mathcal{Z}_{\Lambda,\beta,q}^1} \exp\left(\beta \sum_{e\in\mathcal{E}_{\mathbb{Z}^d}\,:\, e\cap\Lambda\neq\varnothing} \delta_e(\sigma)\right)$$

2. The marginal of $\mu_{\Lambda,p}$ on $\Omega_\Lambda^{\mathbf{1}}$ is the FK measure, i.e.:

$$\mu_2(\omega) := \sum_{\sigma\in\Sigma_\Lambda^1} \mu_{\Lambda,p}(\sigma, \omega) = \frac{1}{Z_{\Lambda,p,q}^{\mathbf{1}}} \left(\prod_{e\in\mathcal{E}_\Lambda} p^{\omega(e)}(1-p)^{1-\omega(e)}\right) q^{\kappa_\Lambda(\omega)}$$

3. Moreover, $Z_{\Lambda,p,q}^{\mathbf{1}} = e^{-\beta|\mathcal{E}_\Lambda|}\mathcal{Z}_{\Lambda,\beta,q}^1$.

QUESTION 3. *Reformulation of the previous results*
Using the previous question, show the following statements :

1. For $\omega \in \Omega_\Lambda^w$, the conditional distribution $\mu_{\Lambda,p}(\cdot|\omega)$ on Σ_Λ^1 is obtained by coloring the vertices as follows :

$$\begin{cases} \text{constant color} \in \{1, \ldots, q\} \text{ on each cluster} \\ \text{independant between clusters} \\ \text{uniformly distributed in } \{1, \ldots, q\} \end{cases}$$

2. For $\sigma \in \Sigma^1_\Lambda$, the conditional distribution $\mu_{\Lambda,p}(\cdot|\sigma)$ on $\Omega^{\mathbf{1}}_\Lambda$ is obtained as follows : Independently for each edge $e \in \mathcal{E}_\Lambda$,
if $e = [i,j] \in \mathcal{E}_\Lambda$ is such that $\sigma_i \neq \sigma_j$, then $\omega(e) = 0$,
and if $\sigma_i = \sigma_j$, then

$$\omega(e) = \begin{cases} 1 \text{ with probability } p \\ 0 \text{ with probability } 1-p \end{cases}$$

QUESTION 4. *Correlations in the Potts model are connexions in the FK model*
For $x, y \in \mathbb{Z}^d$, let $\{x \leftrightarrow y\} = \{$there exists some path of open edges connecting x to $y\}$.
Show that

$$\forall x, y \in V_\Lambda, \qquad \frac{\mathcal{P}^1_{\Lambda,\beta,q}(\sigma_x = \sigma_y) - 1/q}{1 - 1/q} = \mathbb{P}^{\mathbf{1}}_{\Lambda,p,q}(x \leftrightarrow y)$$

Note that the left-hand side above quantifies how $\mathcal{P}^1_{\Lambda,\beta,q}$ differs from $\otimes_{i\in\Lambda}\text{Uniform}_{\{1\ldots q\}}$ on the event $\{\sigma_x = \sigma_y\}$.

QUESTION 5. *FKG inequality and existence of the infinite volume measure* $\mathbb{P}^{\mathbf{1}}_{p,q}$.
There is a natural partial order on the set of configurations of the FK model:

$$\omega \leq \omega' \iff \omega(e) \leq \omega'(e) \quad \forall e \in \mathcal{E}_\Lambda.$$

Therefore, a function $f : \Omega_\Lambda \longrightarrow \mathbb{R}$ is called increasing if $\omega \leq \omega' \Longrightarrow f(\omega) \leq f(\omega')$.
The FKG inequality, seen in the course in terms of the spin variables for the Ising model, is valid in the framework of the FK model: Let $\Lambda \Subset \mathbb{Z}^d$, $p \in [0,1]$, $q \in [1,\infty)$ and $\overline{\omega} \in \Omega$, and let $f, g : \Omega^{\overline{\omega}}_\Lambda \longrightarrow \mathbb{R}$ be two increasing functions. Then

$$\langle fg \rangle^{\overline{\omega}}_{\Lambda,p,q} \geq \langle f \rangle^{\overline{\omega}}_{\Lambda,p,q} \langle g \rangle^{\overline{\omega}}_{\Lambda,p,q}$$

where we write $\langle . \rangle^{\overline{\omega}}_{\Lambda,p,q}$ the expectation under the measure $\mathbb{P}^{\overline{\omega}}_{\Lambda,p,q}$.

Using the FKG inequality, prove the existence of the infinite volume measure $\mathbb{P}^{\mathbf{1}}_{p,q}$.
(*Hint : Proceed the same way as in the course for the Ising model and* μ^+_β*!*)

QUESTION 6. *Phase transition in the Potts model ⇔ Percolation transition in the FK model*
Let us define the magnetization of the Potts model with boundary condition 1 :

$$m_\Lambda(\beta) = \frac{\langle \frac{1}{|\Lambda|} \sum_{i\in\Lambda} \delta_{\sigma_i,1} \rangle^1_{\Lambda,\beta,q} - 1/q}{1 - 1/q}$$

Note that $m_\Lambda = 1$ if all the vertices of Λ have the color 1, and $m_\Lambda = 0$ if each vertex has a uniform color in $\{1, \ldots q\}$ (it is the case for $\beta = \infty$). We say that the system is in an ordered phase if

$$m(\beta) := \frac{\langle \delta_{\sigma_0,1} \rangle^1_{\beta,q} - 1/q}{1 - 1/q} = \lim_{\Lambda \uparrow \mathbb{Z}^d} m_\Lambda(\beta) > 0.$$

On the other hand, we say that percolation occurs in the (infinite volume) FK model if

$$\mathbb{P}^w_{p,q}(0 \leftrightarrow \infty) = \lim_{\Lambda \uparrow \mathbb{Z}^d} \mathbb{P}^w_{\Lambda,p,q}(0 \leftrightarrow \Lambda^c) > 0,$$

where $\{0 \leftrightarrow \infty\} = \{0$ belongs to an infinite cluster$\}$.

In this exercise you will prove that these two limits are well defined and coincide, which shows that the Potts model is ordered if and only if the FK model percolates.

1. Let $p = 1 - e^{-\beta}$, show that

$$\mathbb{P}^{\mathbf{1}}_{\Lambda,p,q}(i \leftrightarrow \Lambda^c) = \frac{\langle \delta_{\sigma_i,1}\rangle^1_{\Lambda,\beta,q} - 1/q}{1 - 1/q}$$

2. Show that

$$\mathbb{P}^{\mathbf{1}}_{p,q}(0 \leftrightarrow \infty) = \lim_{\Lambda \nearrow \mathbb{Z}^d} \mathbb{P}^{\mathbf{1}}_{\Lambda,p,q}(0 \leftrightarrow \Lambda^c)$$

3. Show that

$$\langle \delta_{\sigma_0,1}\rangle^1_{\beta,q} = \lim_{\Lambda \uparrow \mathbb{Z}^d} \langle \frac{1}{|\Lambda|} \sum_{i \in \Lambda} \delta_{\sigma_i,1}\rangle^1_{\Lambda,\beta,q}$$

(*Hint : Proceed the same way as in the course for the Ising model and* $\mu^+_\beta(\sigma_0)$.)

QUESTION 7. *Comparison inequalities*
Let $p_1, p_2 \in [0,1]$ and $q_1, q_2 \in [1,\infty)$ such that either

- $q_1 \geq q_2$ and $p_1 \leq p_2$, or
- $q_2 \geq q_1$ and $\frac{p_1}{q_1(1-p_1)} \leq \frac{p_2}{q_2(1-p_2)}$

1. Show that for any increasing function f,

$$\langle f\rangle^{\mathbf{1}}_{\Lambda,p_1,q_1} \leq \langle f\rangle^{\mathbf{1}}_{\Lambda,p_2,q_2}.$$

(*Hint : Use the FKG inequality.*)

2. Deduce that the function $p \mapsto \mathbb{P}^{\mathbf{1}}_{p,q}(0 \leftrightarrow \infty)$ is non decreasing. We can thus unambiguously introduce the following definition:

$$p_c(q) := \sup\{p \in [0,1] : \mathbb{P}^{\mathbf{1}}_{p,q}(0,\infty) = 0\}.$$

3. Use the previous question and the results proved in the course for the Ising model to show that $p_c(2) \in (0,1)$.

4. Deduce that $p_c(q) \in (0,1)$ for all $q > 2$.