THE ISING MODEL
Exercises

Loren Coquille (loren.coquille@univ-grenoble-alpes.fr)

THE ISING MODEL IN DIMENSION 1

Consider the graph $V_N := \{1, 2\ldots, N\} \subset \mathbb{Z}$, on which live the random variables $\sigma_i \in \{-1, +1\}$, $i \in V_N$ denoted "spins". We write $\boldsymbol{\sigma} := (\sigma_i)_{i\in V_N}$.

Let $\beta \in \mathbb{R}^+$ be the inverse temperature, and $h \in \mathbb{R}$ be the magnetic field. The energy (called "Hamiltonian") of the system is:

$$\mathcal{H}_{N,\beta,h}(\boldsymbol{\sigma}) := -\beta \sum_{\substack{i,j\in V_N \\ i\sim j}} \sigma_i\sigma_j - h\sum_{i=1}^{N}\sigma_i$$

We introduce the following probability measure on $\Omega_N := \{-1,+1\}^{V_N}$:

$$\mathbb{P}_{N,\beta,h}(\boldsymbol{\sigma}) := \frac{1}{Z_{N,\beta,h}} \exp\left(-\mathcal{H}_{N,\beta,h}(\boldsymbol{\sigma})\right)$$

where $Z_{N,\beta,h} = \sum_{\boldsymbol{\sigma}\in\Omega_N} \exp(-\mathcal{H}_{N,\beta,h}(\boldsymbol{\sigma}))$ is the normalizing constant called "partition function". The following limit, if it exists, is called the "free energy":

$$f_\beta(h) = \lim_{N\to\infty} f_{N,\beta}(h) := \lim_{N\to\infty} \frac{1}{N}\log Z_{N,\beta,h}$$

QUESTION 1. Show that for all $\beta > 0$ and all $h \in \mathbb{R}$,

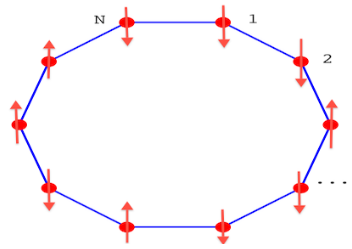
1. the free energy exists and equals:

$$f_\beta(h) = \log\left(e^\beta\cosh(h) + \sqrt{e^{2\beta}\cosh^2(h) - 2\sinh(2\beta)}\right).$$

Hint :

- *Calculate the partition function $\tilde{Z}_{N,\beta,h}$ and then the free energy $\tilde{f}_\beta(h)$ for the graph V_N where we add an edge between 1 and N (periodic boundary condition): show that $\tilde{Z}_{N,\beta,h}$ can be rewritten as*

$$\tilde{Z}_{N,\beta,h} = Tr(T^N) \text{ where } T = \begin{pmatrix} e^{\beta+h} & e^{-\beta+h} \\ e^{-\beta-h} & e^{\beta-h}\end{pmatrix} \text{ is called "transfert matrix".}$$



- *Show that* $\tilde{f}_\beta(h) = f_\beta(h)$ *by upper-bounding* $|\mathcal{H}_{N,\beta,h} - \tilde{\mathcal{H}}_{N,\beta,h}|$.

2. $f_\beta(h)$ is a convex function of h.

Hint : Show that $\log Z_{N,\beta,h}$ *is convex in* h *for all* N *and use the following theorem:*

Let $(g_n)_{n\geq 1}$ *be a sequence of convex functions from an open interval* $\mathcal{I} \subset \mathbb{R}$ *into* $\mathbb{R}$ *converging pointwise towards a function* g*, then* g *is also convex.*

QUESTION 2. Show that, for $h = 0$, the average magnetization of a system of size N concentrates exponentially (in N) on 0, i.e. show that for all $\beta \in (0, \infty)$ and for all $\varepsilon > 0$, there exists $c(\beta, \varepsilon) > 0$ such that for all N :

$$\mathbb{P}_{N,\beta,0}\left[\frac{1}{N}\sum_{i=1}^{N}\sigma_i \in (-\varepsilon, \varepsilon)\right] \geq 1 - 2e^{-c(\beta,\varepsilon)N}$$

in particular $\frac{1}{N}\sum_{i=1}^{N}\sigma_i \to 0$ in probability. The first bound implies also the almost sure convergence. This hence prove a strong law of large numbers for the (non independant) random variables σ_i.

Hint :

- *Use the Chebychev inequality to show that* $\mathbb{P}_{N,\beta,0}\left[\frac{1}{N}\sum_{i=1}^{N}\sigma_i \geq \varepsilon\right] \leq e^{-h\varepsilon N}\mathbb{E}_{N,\beta,0}\left[e^{h\sum_{i=1}^{N}\sigma_i}\right]$, $\forall h \geq 0$.
- *Show that* $\sup_{h\geq 0}\{h\varepsilon - \Lambda_\beta(h)\} > 0$ *by writing* $\Lambda_\beta(h)$ *in terms of* $f_\beta(h)$ *and using Question 1.*

QUESTION 3. On the graph V_N with periodic boundary condition, calculate the finite volume magnetization, i.e.

$$\tilde{m}(\beta, h) := \lim_{N\to\infty} \tilde{\mathbb{E}}_{N,\beta,h}[\sigma_1].$$

Draw the graph of $\tilde{m}(\beta, h)$ as a function of h for two values of β.

Hint : Show that $\tilde{\mathbb{E}}_{N,\beta,h}[\sigma_1] = \tilde{\mathbb{E}}_{N,\beta,h}[\frac{1}{N}\sum_{i=1}^{N}\sigma_i] = \frac{\partial}{\partial h}f_{N,\beta}(h)$*, and use the following theorem:*

Let $(g_n)_{n\geq 1}$ *be a sequence of convex functions from an open interval* $\mathcal{I} \subset \mathbb{R}$ *into* $\mathbb{R}$ *converging pointwise towards a function* g*. If* g *is differentiable at* x*, then*

$$\lim_{n\to\infty} \partial^+ g_n(x) = \lim_{n\to\infty} \partial^- g_n(x) = g'(x)$$

QUESTION 4. Show that for $k \leq N$,

$$\mathbb{C}\mathrm{ov}_{N,\beta,0}[\sigma_1\sigma_k] = \mathbb{E}_{N,\beta,0}[\sigma_1\sigma_k] = th(\beta)^{k-1}$$

i.e., when $h = 0$, the covariance of two spins decays exponentially in the distance between the spins.

Hint :

- *Rewrite the Hamiltonien* $\mathcal{H}_{N,\beta,0}(\boldsymbol{\sigma}) := -\beta\sum_{i=1}^{N-1} J_i\sigma_i\sigma_{i+1}$ *with* $J_i = 1$ *for all* i*, and show that*

$$\mathbb{E}_{N,\beta,0}[\sigma_1\sigma_k] = \frac{1}{\beta^{k-1}Z_{N,\beta,0}}\frac{\partial^{k-1}Z_{N,\beta,0}}{\partial J_1\ldots\partial J_{k-1}}\Big|_{J_i=1\ \forall i}.$$

- *Calculate* $Z_{N,\beta,0}$ *in terms of* $Z_{N-1,\beta,0}$ *to show that* $Z_{N,\beta,0} = \prod_{i=1}^{N} 2ch(\beta J_i)$ *and conclude.*