Commun. math. Phys. 35, 87—92 (1974)

GHS and other Inequalities*

Joel L. Lebowitz
Belfer Graduate School of Science, Yeshiva University, New York, N.Y., USA

Received September 16, 1973

Abstract. We use a transformation due to Percus to give a simple derivation of the Griffiths, Hurst, and Sherman, and some other new inequalities, for Ising ferromagnets with pair interactions. The proof makes use of the Griffiths, Kelly, and Sherman and the Fortuin, Kasteleyn, and Ginibre inequalities.

1. Introduction

We consider an Ising spin system with ferromagnetic pair interactions; $\sigma_i = \pm 1$, $i \in \Lambda$, $i = 1, \ldots, |\Lambda|$,

$$H(\boldsymbol{\sigma}) = -1/2 \sum_{i \neq j} J_{ij} \sigma_i \sigma_j - \sum_i h_i \sigma_i, \qquad J_{ij} \geqq 0, \tag{1.1}$$

where H is the energy of the system and h_i are external magnetic fields.

The Ursell, or cluster function, $u_l(i_1, \ldots, i_l)$ play a central role in statistical mechanics. They are given for spin systems, by the relations [1]

$$u_l(i_1, \ldots, i_l) = \frac{\partial^l}{\partial h_{i_1} \ldots \partial h_{i_l}} \ln Z(\Lambda; \boldsymbol{h}, \boldsymbol{J}), \qquad l = 1, \ldots, |\Lambda| \tag{1.2}$$

where we have written $\boldsymbol{h}$ and $\boldsymbol{J}$ for the collections $\{h_i\}$, $\{J_{ij}\}$ and

$$Z(\Lambda; \boldsymbol{h}, \boldsymbol{J}) = \mathrm{Tr}_{\{\sigma\}} \exp[-H(\boldsymbol{\sigma})]. \tag{1.3}$$

Thus

$$\begin{aligned} u_1(i) &= \langle \sigma_i \rangle, \qquad u_2(i,j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle, \\ u_3(i,j,k) &= \langle \sigma_i \sigma_j \sigma_k \rangle - \langle \sigma_i \rangle \langle \sigma_j \sigma_k \rangle - \langle \sigma_j \rangle \langle \sigma_i \sigma_k \rangle \\ &\quad - \langle \sigma_k \rangle \langle \sigma_i \sigma_j \rangle + 2 \langle \sigma_i \rangle \langle \sigma_j \rangle \langle \sigma_k \rangle, \ldots \end{aligned} \tag{1.4}$$

where $\langle\ \rangle$ denotes expectations with respect to the measure $\mu(\boldsymbol{\sigma}) = Z^{-1} \exp[-H]$; (we have set the temperature $\beta^{-1} = 1$).

The Griffiths, Kelly, and Sherman [2] (GKS) inequalities for this system apply when the h_i have the same sign for all i; say $h_i \geqq 0$, $\boldsymbol{h} \geqq 0$, (similar results hold by symmetry when $h_i < 0$, all i). They state

$$\langle \sigma_A \sigma_B \rangle \geqq \langle \sigma_A \rangle \langle \sigma_B \rangle, \qquad \boldsymbol{h} \geqq 0 \tag{1.5}$$

* Work supported in part by USAFOSR-73-2430.

where $A, B \subset \Lambda$ and $\sigma_A = \prod_{i \in A} \sigma_i$. The more recent Fortuin, Kasteleyn, and Ginibre [3] (FKG) inequalities apply whatever the signs of the h_i (assuming always of course that $J_{ij} \geqq 0$). FKG consider general increasing functions $f(\sigma_{i_1}, \ldots, \sigma_{i_n})$, $g(\sigma_{j_1}, \ldots, \sigma_{j_m})$, i.e. $f(\sigma_{i_1}, \ldots, \sigma_{i_l} = 1, \ldots, \sigma_{i_n}) \geqq f(\sigma_{i_1}, \ldots, \sigma_{i_l} = -1, \ldots, \sigma_{i_n})$. For such functions FKG prove that

$$\langle fg \rangle \geqq \langle f \rangle \langle g \rangle . \tag{1.6}$$

Note that if g is an increasing function $-g$ is a decreasing function and so the inequality in (1.6) is reversed if f is increasing and g is decreasing.

The more general form of the FKG inequalities, which we shall need later, state that if X is a "lattice", e.g. the subsets of Λ ordered by inclusion, and if $P(x)$, $x \in X$ is a normalized measure satisfying

$$P(x)\, P(y) \leqq P(x \vee y)\, P(x \wedge y), \quad \text{for all} \quad x, y \in X \tag{1.7}$$

then $\langle fg \rangle \geqq \langle f \rangle \langle g \rangle$ whenever $f(x)$ and $g(x)$ are increasing functions, $f(x) \geqq f(y)$ if $x \geqq y$. The expectations are now with respect to the measure P.

These inequalities which apply to more general (than pair) ferromagnetic type interactions have found many uses in statistical mechanics [1] (and more recently also in field theory [4]). They can be applied directly to the first two Ursell functions, $u_1(i) \geqq 0$, if $\boldsymbol{h} \geqq 0$, by GKS $u_2(i,j) \geqq 0$, for arbitrary fields by FKG. For simple proofs of GKS and FKG see [5] and [6] respectively.

There exists also another extremely useful inequality due to Griffiths, Hurst, and Sherman [7] (GHS) which *is* restricted to ferromagnetic pair interactions,

$$u_3(i,j,k) \leqq 0, \quad \text{for} \quad \boldsymbol{h} \geqq 0 . \tag{1.8}$$

(By symmetry $u_3 \geqq 0$ for $\boldsymbol{h} \leqq 0$.) The derivation of this inequality which is much more specialized than either the GKS or the FKG inequalities, is in many ways also more complicated involving combinatorial analysis, etc. It is the purpose of this note to derive the GHS inequality as part of a whole set of inequalities which are in turn derived, in a fairly simple way from the GKS and FKG inequalities.

2. Proof of GHS and other Inequalities

Let $\{s_i\}$ be a "duplicate" set of variables to the $\{\sigma_i\}$ with the same energy as in (1) and let $\langle\ \rangle'$ denote expectations with respect to the measure

$$\mu(\boldsymbol{\sigma}, s) = Z^{-2}(\Lambda, \boldsymbol{h}, \boldsymbol{J}) \exp[H(\boldsymbol{\sigma}) + H(s)] . \tag{2.1}$$

Define $q_i=\frac{1}{2}(\sigma_i-s_i)$, $t_i=\frac{1}{2}(\sigma_i+s_i)$, $q_A=\prod_{i\in A} q_i$, $t_A=\prod_{i\in A} t_i$. q_i and t_i can take on the values, $-1, 0, 1$ with the constraint that $q_i=0\Rightarrow t_i=\pm 1$, $t_i=0\Rightarrow q_i=\pm 1$. We then readily find that (dropping the primes on $\langle\ \rangle'$)

$$u_1(i)=\langle t_i\rangle, \quad u_2(i,j)=2\langle q_i q_j\rangle$$
$$u_3(i,j,k)=4[\langle q_i q_j t_k\rangle-\langle q_i q_j\rangle\langle t_k\rangle], \tag{2.2}$$

$$u_4(i,j,k,l)=8[\langle q_i q_j t_k t_l\rangle-\langle q_i q_j\rangle\langle t_k t_l\rangle]$$
$$-2[u_3(i,j,k)\langle t_l\rangle+u_3(i,j,l)\langle t_k\rangle]. \tag{2.3}$$

Theorem. *For Ising system with ferromagnetic pair interactions*

$$\langle q_C\rangle\geqq 0, \quad C\subset\Lambda, \tag{2.4}$$

$$\langle q_C t_D\rangle\leqq\langle q_C\rangle\langle t_D\rangle, \tag{2.5a}$$
$$\text{if}\quad \boldsymbol{h}\geqq 0,\ C,D\subset\Lambda,$$
$$\langle q_C q_D\rangle\geqq\langle q_C\rangle\langle q_D\rangle. \tag{2.5b}$$

Remarks. (i) $\langle t_A\rangle\geqq 0$, and $\langle t_A t_B\rangle\geqq\langle t_A\rangle\langle t_B\rangle$, if $\boldsymbol{h}\geqq 0$, is a direct consequence of the GKS inequalities.

(ii) The proof of (2.4) is based on the GKS inequalities alone and is due to Percus [8] who introduced the "duplicate" set of $\boldsymbol{s}$ variables. (2.4) and (2.2) imply $u_2(i,j)\geqq 0$ for arbitrary h_i without the use of the FKG inequalities [8]. The proof of (2.5) is new and makes use of the GKS and the general form of the FKG inequalities discussed in Section 1. (2.5) together with (2.3) imply the GHS inequality.

Proof. The function $H(\boldsymbol{\sigma})+H(\boldsymbol{s})$ appearing in the measure $\mu(\boldsymbol{\sigma},\boldsymbol{s})$ in (2.1) becomes in terms of the variables $\boldsymbol{q}$ and $\boldsymbol{t}$,

$$H_\Lambda(\boldsymbol{\sigma})+H_\Lambda(\boldsymbol{s})=-\left[1/2\sum_{i\neq j}2J_{ij}q_iq_j\right]-\left[1/2\sum_{i\neq j}2J_{ij}t_it_j+\Sigma\, 2h_it_i\right]$$
$$=\hat{H}_\Lambda(\boldsymbol{q};0,2\boldsymbol{J})+\hat{H}_\Lambda(\boldsymbol{t};2\boldsymbol{h},2\boldsymbol{J}) \tag{2.6}$$

where we have indicated explicitly the dependences of the energy on $\boldsymbol{h}$ and $\boldsymbol{J}$ and use the cap to remind us that q_i and t_i in (2.6) are *not* Ising variables since they take the values $-1, 0, 1$ with the restrictions mentioned earlier. Using (2.6) it is easy to see that for any functions $\phi(\boldsymbol{q})$ and $\Psi(\boldsymbol{t})$,

$$\langle\phi(\boldsymbol{q})\,\Psi(\boldsymbol{t})\rangle=\sum_{A\subset\Lambda}P(A)\,f(A)\,q(A), \tag{2.7}$$

where

$$P(A)=\{\mathrm{Tr}_{q_I}\exp[-H_A(\boldsymbol{q};0,2\boldsymbol{J})]\}$$
$$\cdot\{\mathrm{Tr}_{t_I}\exp[-H_{\tilde{A}}(\boldsymbol{t};2\boldsymbol{h},2\boldsymbol{J})]\}/Z^2(\Lambda;\boldsymbol{h},\boldsymbol{J}), \tag{2.8}$$

$$f(A)=\{\mathrm{Tr}_{q_I}\phi(\boldsymbol{q})\exp[-H_A(q;0,2\boldsymbol{J})]\}/Z(A;0,2\boldsymbol{J}), \tag{2.9}$$

$$g(A)=\{\mathrm{Tr}_{t_I}\Psi(\boldsymbol{t})\exp[-H_{\tilde{A}}(\boldsymbol{t};2\boldsymbol{h},2\boldsymbol{J})]\}/Z(\tilde{A};0,2\boldsymbol{J}). \tag{2.10}$$

Here $\tilde{A}$ denotes the complement of A in Λ and the subscripts I as well as the absence of a cap on H_A and $H_{\tilde{A}}$ indicate that in taking traces in (2.8)–(2.10) over the q_i, $i \in A$, and t_i, $i \in \tilde{A}$, the q_i and t_i are to be treated as Ising variables taking on the values ± 1 only, while $q_i = 0$ for $i \in \tilde{A}$, $t_i = 0$ for $i \in A$. Clearly $P(A)$ is the probability with respect to the measure $\mu(\boldsymbol{\sigma}, \boldsymbol{s})$ that $q_i = \pm 1$, $t_i = 0$ for $i \in A$, while $f(A)$ is the expectations value of $\phi(q)$ in an Ising system in A with energy $H_A(q; 0, 2\boldsymbol{J})$. Similarly $g(A)$ is the expectation value of $\Psi(\boldsymbol{t})$ in an Ising system in $\tilde{A}$ with energy $H_{\tilde{A}}(\boldsymbol{t}; 2\boldsymbol{h}, 2\boldsymbol{J})$.

It is now readily seen that (2.4) holds since in this case $f(A) \geqq 0$ by GKS and $g(A) = 1$. To prove (2.5) we note first that it follows from the GKS inequalities that $\langle \sigma_B \rangle$, $B \subset \Lambda$ is non decreasing as Λ is increased, i.e. $\langle \sigma_B \rangle_{\Lambda'} \geqq \langle \sigma_B \rangle_{\Lambda}$ for $\Lambda' \supset \Lambda$ (and $\boldsymbol{h} \geqq 0$). Hence for $\phi(\boldsymbol{q}) = q_C$, $\Psi(\boldsymbol{t}) = t_D$, $f(A)$ is an increasing and $g(A)$ a decreasing function of the "size" of A, i.e. $f(A') \geqq f(A)$, $g(A') \leqq g(A)$ if $A' \supset A$. Similarly

$$\langle q_C q_B \rangle = \sum_{A \subset \Lambda} P(A)\, \chi(A) \geqq \sum_{A \subset \Lambda} P(A)\, \chi_1(A)\, \chi_2(A) \tag{2.11}$$

where

$$\chi(A) = \mathrm{Tr}_{q_I} \{q_C q_D \exp[-H_A(\boldsymbol{q}; 0, 2\boldsymbol{J})]\} / Z(A; 0, 2\boldsymbol{J}), \tag{2.12}$$

and $\chi_1(A)$, $\chi_2(A)$ are the corresponding Ising expectations of q_C and q_B respectively. The inequality in (2.11) follows from GKS which also shows that both $\chi_1(A)$ and $\chi_2(A)$ are increasing functions on the sets A.

Thus (2.5) will be a consequence of the FKG inequalities if we can show that the measure P satisfies (1.7), i.e.

$$P(A)\, P(B) \leqq P(A \cup B)\, P(A \cap B). \tag{2.13}$$

Using (2.8) we see that (2.13) is a consequence of the following lemma.

Lemma. *For an Ising spin system with general ferromagnetic interactions,*

$$H_\Lambda = -\sum_{Q \subset \Lambda} J_Q \sigma_Q, \quad J_Q \geqq 0 \tag{2.14}$$

$$Z(\Lambda_1)\, Z(\Lambda_2) \leqq Z(\Lambda_1 \cup \Lambda_2)\, Z(\Lambda_1 \cap \Lambda_2) \tag{2.15}$$

where $Z(\Lambda) = \mathrm{Tr}_{\sigma_\cdot} \exp[H_\Lambda(\boldsymbol{\sigma})]$.

Remark. In applying (2.15) to prove (2.13), $\Lambda_1 = A$, $\Lambda_2 = B$ or $\Lambda_1 = \tilde{A}$, $\Lambda_2 = \tilde{B}$.

Proof. Setting $\Lambda_1 \cap \Lambda_2 = K_1$, $\Lambda_1 \backslash K_1 = K_2$, $\Lambda_2 \backslash K_1 = K_3$ we want to show that

$$F \equiv \ln Z(K_1 \cup K_2 \cup K_3) - \ln[Z(K_1 \cup K_2)\, Z(K_1 \cup K_3)/Z(K_1)] \geqq 0. \tag{2.16}$$

Now it follows readily from the GKS inequalities that (2.16) holds when there is *no* interaction between spins on sites in K_1 and spins on sites in K_3. Consider now the change in F when an interaction term of the form $J\sigma_{Q_1}\sigma_{Q_3}$, $J>0$, $Q_1\subset K_1$, $Q_3\subset K_3$ is added to the energy of the system. Taking the derivative of F with respect of J we obtain

$$\frac{\partial F}{\partial J}=\langle\sigma_{Q_1}\sigma_{Q_3}\rangle_{K_{123}}-\langle\sigma_{Q_1}\sigma_{Q_3}\rangle_{K_{13}} \tag{2.17}$$

where we have used the abbreviations $K_{12}=K_1\cup K_2$, etc. The right side of (2.17) is $\geqq 0$ by GKS and hence $F\geqq 0$. The lemma and the theorem are thus proven.

Remarks. (i) It follows from (2.3) that $u_4(i,j,k,l)\leqq 0$ whenever $\langle\sigma_k\rangle$ and $\langle\sigma_l\rangle$ (or, by symmetry the expectation values of any pair of spins in the set $\{i,j,k,l\}$) vanish, e.g. $\boldsymbol{h}=0$. Furthermore using the symmetry of u_4 and the fact that $\langle t_k t_l\rangle=\langle q_k q_l\rangle$ when $\boldsymbol{h}=0$, we have in this case

$$|u_4(i,j,k,l)|\leqq 2u_2(i,j)\,u_2(k,l)\,,$$
$$|u_4(i,j,k,l)|\leqq 2[u_2(i,j)\,u_2(k,l)\,u_2(i,k)\,u_2(j,l)]^{1/2}\,,$$
$$|u_4(i,j,k,l)|\leqq 2[u_2(i,j)\,u_2(k,l)\,u_2(i,k)\,u_2(j,l)\,u_2(i,l)\,u_2(j,k)]^{1/3}\,.$$

This may be of some relevance to the question of the decay rates of the different Ursell functions [9].

(ii) For $\boldsymbol{h}\geqq 0$, $\langle q_A\rangle$ is a decreasing and $\langle t_A\rangle$ is an increasing function of the external fields $\{h_i\}$.

(iii) By combining (2.4), (2.5) and the subsequence remark we find that for $\boldsymbol{h}\geqq 0$,

$$\langle q_A t_B\rangle\,\langle q_C\rangle\leqq\langle q_A\rangle\,\langle t_B\rangle\,\langle q_C\rangle\leqq\langle q_A q_C\rangle\,\langle t_B\rangle\,,$$
$$\langle q_A t_B\rangle\,\langle t_C\rangle\leqq\langle q_A\rangle\,\langle t_B\rangle\,\langle t_C\rangle\leqq\langle q_A\rangle\,\langle t_B t_C\rangle\,.$$

Acknowledgement. I am greatly indebted to J. Percus for communicating to me his method of using the "duplicate" set of variables s prior to publication. I also want to thank D. Ruelle for some useful comments.

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Communicated by D. Ruelle

J. L. Lebowitz
Belfer Graduate School of Science
Yeshiva University
500 West 185th Street
New York, N.Y. 10033, USA