A finite-volume version of the AIZENMAN-HIGUCHI THEOREM for the 2d Ising model

FNSNF Swiss National Science Foundation

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Abstract

In the late 1970s, in two celebrated papers, Aizenman and Higuchi independantly established that all infinite-volume Gibbs measures of the two-dimensional ferromagnetic nearest-neighbor Ising model at inverse temperature $\beta \geq 0$ are of the form $\alpha\mu^+_\beta + (1-\alpha)\mu^-_\beta$, where μ^+_β and μ^-_β are the two pure phases and $0 \leq \alpha \leq 1$.

We present here a new approach to this result, with a number of advantages:

1. *We obtain a finite-volume, quantitative analogue (implying the classical claim);*
2. *the scheme of our proof seems more natural and provides a better picture of the underlying phenomenon;*
3. *this new approach seems substantially more robust.*

Known results for Gibbs measures and the 2d Ising model

- The set $\mathcal{G}_\beta$ of infinite volume Gibbs measures at inverse temperature β is a Choquet simplex. In particular, any $\mu \in \mathcal{G}_\beta$ has a unique decomposition onto a set of extremal measures $\mathcal{G}^{ex}_\beta$.
- $\mu^\pm_\beta$ (infinite volume limits with + or - boundary conditions) are extremal measures.
- 1975 [Miracle-Sole, Messager] :
  If $\mu \in \mathcal{G}_\beta$ is translation invariant, then $\mu = \alpha\mu^+_\beta + (1-\alpha)\mu^-_\beta$.
- 1980 [Aizenman, Higuchi (independantly)]
  Any $\mu \in \mathcal{G}_\beta$ is translation invariant.

The theorem

Let $\beta > \beta_c$.
For every $\xi < 1/2$ and $0 < \delta < 1/2 - \xi$, as n tends to infinity, there exists a constant $\alpha^{n,\omega}(\beta) \in [0,1]$ such that,

$$\mu^\omega_{\Lambda_n,\beta}(f) = \alpha^{n,\omega}\mu^+_\beta(f) + (1-\alpha^{n,\omega})\mu^-_\beta(f) + O(\|f\|_\infty n^{-\delta})$$

where the O notation is uniform in the boundary condition ω and in function f having support in Λ_{n^ξ}

Key tools

- **FKG inequality :** Increasing functions are positively correlated.
- **Kramers-Wannier duality :**
  Strict positivity of the surface tension at inverse temperature $\beta \Leftrightarrow$ Strict positivity of the inverse correlation length at dual inverse temperature $\beta^\star$. $(\tanh \beta^\star = e^{-2\beta})$

$$\frac{Z^{\pm,(x,y)}_{\Lambda,\beta}}{Z^+_{\Lambda,\beta}} = \langle \sigma_x \sigma_y \rangle^{free}_{\Lambda^\star,\beta^\star}$$

- **BK inequality :** Writing $\frac{Z^\omega_{\Lambda,\beta}}{Z^+_{\Lambda,\beta}} = \sum_{\gamma\sim\omega} q_{\Lambda,\beta}(\gamma)$ We have

$$\sum_{\substack{\gamma_1:x_1\to y_1\\ \gamma_2:x_2\to y_2}} q(\gamma_1,\gamma_2) \leq \sum_{\gamma_1:x_1\to y_1} q(\gamma_1) \sum_{\gamma_2:x_2\to y_2} q(\gamma_2)$$

- **Finite volume estimates for the 2-points function (Ornstein-Zernike prefactor):**
  For all $x, y \in \partial\Lambda_n$ such that $\overline{xy} \cap \Lambda_{n/2} \neq \emptyset$, we have

$$\frac{C_1}{n^{1/2}} e^{-\tau_\beta(x-y)} \leq \frac{Z^{\pm(x,y)}_{\Lambda_n,\beta}}{Z^+_{\Lambda_n,\beta}} \leq \frac{C_2}{|x-y|^{1/2}} e^{-\tau_\beta(x-y)}$$

- **Exponential relaxation in pure phases :**
  For every $\beta > \beta_c$, uniformly in the local function f,

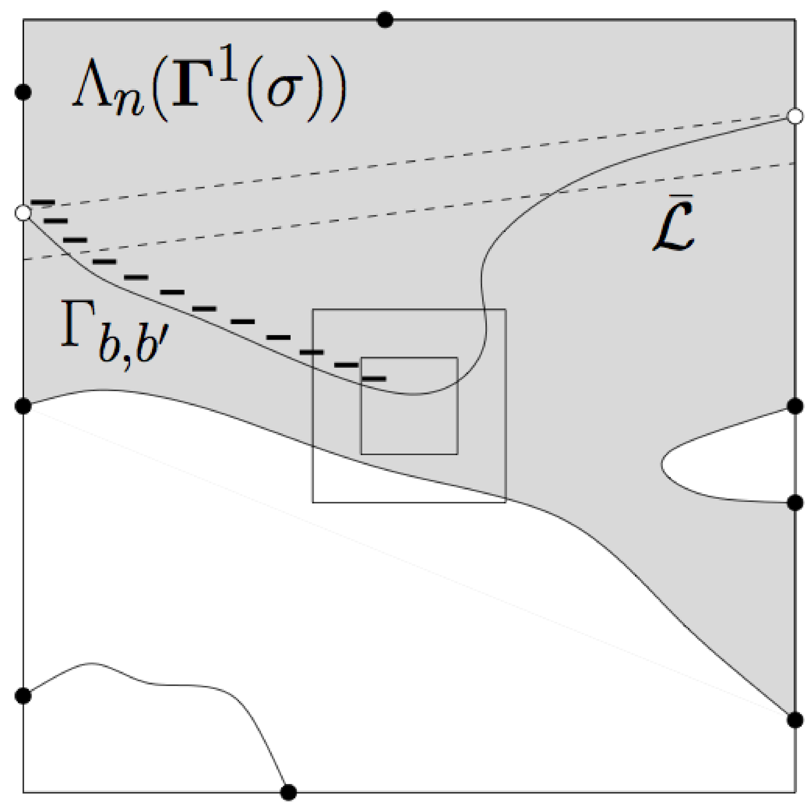
$$|\mu^+_{\Lambda,\beta}(f) - \mu^+_\beta(f)| \leq \|f\|_\infty |Supp(f)| e^{-C\cdot d(Supp(f),\Lambda^c)}$$

Sketch of the proof

Lemma 1

Take $a < 1$. If the interface $\Gamma_{bb'}$ reaches the box Λ_{n^a}, then $\overline{bb'}$ must intersect Λ_{2n^a} with probability $1 - o_n(1)$.
More precisely,

$$\mu^\omega_{\Lambda_n,\beta}\left(\begin{matrix}\exists\, \Gamma_{bb'} \text{ such that } \Gamma_{bb'}\cap\Lambda_{n^a}\neq\emptyset \\ \text{and } \overline{bb'}\cap\Lambda_{2n^a}=\emptyset\end{matrix}\right) \leq e^{-C\, n^{2a-1}}$$



Idea of the proof : Condition on interfaces above $\overline{bb'}$. Notice that the event "$\Gamma_{bb'}$ reaches the box Λ_{n^a}" is decreasing (wlog there exists a path of - spins reaching the little box) and use FKG inequality and Markov property to work in the box Λ_n with $\pm(b,b')$ boundary condition. Work with surface tension :

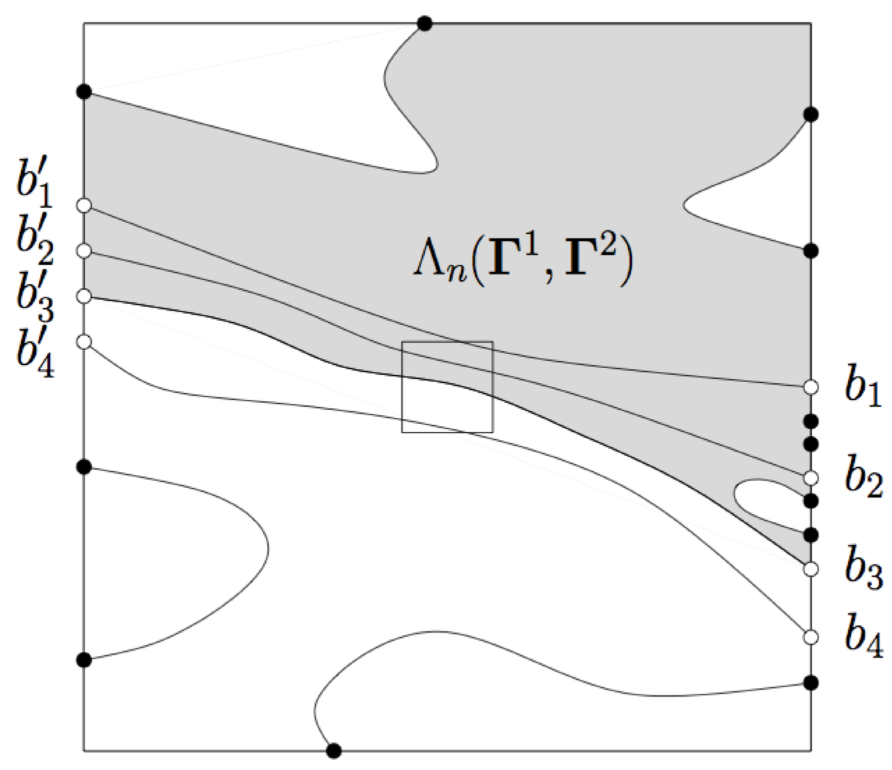
$$\mu^{\pm(b,b')}_{\Lambda_n,\beta}(\Gamma_{bb'} \cap \mathcal{L} \neq \emptyset) = \underbrace{\frac{Z^{\pm,(b,b')}_{\Lambda_n,\beta}(\Gamma_{bb'} \cap \mathcal{L} \neq \emptyset)}{Z^+_{\Lambda_n,\beta}}}_{(1)} \underbrace{\frac{Z^+_{\Lambda_n,\beta}}{Z^{\pm(b,b')}_{\Lambda_n,\beta}}}_{(2)}$$

Use BK inequality to bound (1) from above, and a soft finite volume estimate to bound $(2)^{-1}$ from below.

Lemma 2

The number N_{cross} of interfaces intersecting Λ_{n^a} is either 0 or 1 with probability $1 - o_n(1)$.
More precisely,

$$\mu^\omega_{\Lambda_n,\beta}(N_{cross} \geq 2) \leq e^{-C\, n^{2a-1}}$$

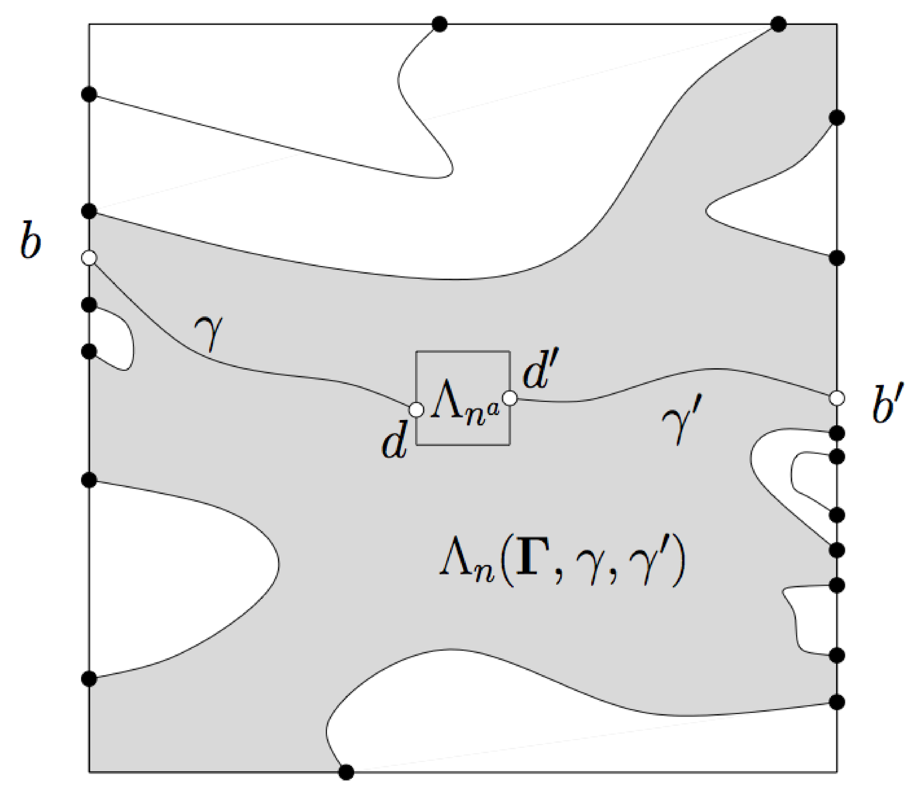


Idea of the proof : Use Lemma 1 to notice that endpoints of crossing interfaces are "quasi" diametraly opposed. Condition on interfaces above the first crossing. In the random box (colored in grey on the picture), compare, for the two remaining interfaces, the cost of crossing (soft infinite volume upper bound) and of following the boundary (soft estimate, equivalent to a partial change of boundary condition outside the box Λ_n).

Lemma 3

Take $b < a/2$. If there is exactly one crossing interface Γ, then it will miss a box Λ_{n^b} with probability $1 - o_n(1)$.
More precisely,

$$\mu^\omega_{\Lambda_n,\beta}(N_{cross} = 1 \text{ and } \Gamma \cap \Lambda_{2n^b} \neq \emptyset) \leq C\, n^{b-a/2}$$



Idea of the proof : Essentially the same as for Lemma 1, but applied to the box Λ_{n^a} : d and d' are "quasi" diametraly opposed with high probability. Use the refined finite volume estimate (Ornstein-Zernike prefactor) to get the right lower bound. Key point : $\Gamma_{dd'}$ has a Brownian bridge behavior in the scaling limit, so we have a good lower bound on the fluctuations of this interface.

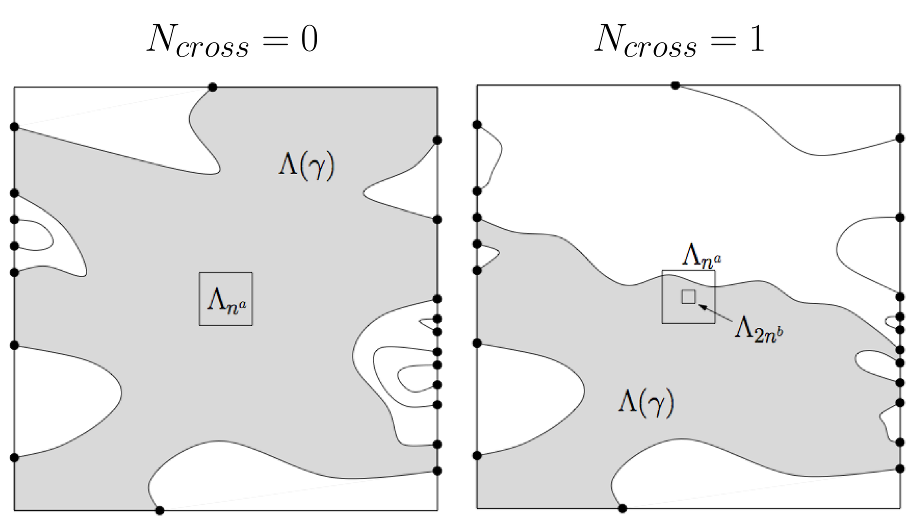
Conclusion

For any local function f having support in Λ_{n^b},

$$\mu^\omega_{\Lambda_n,\beta}(f) = \mu^\omega_{\Lambda_n,\beta}(\mathcal{I}^+)\mu^+_\beta(f) + \mu^\omega_{\Lambda_n,\beta}(\mathcal{I}^-)\mu^-_\beta(f) + O_\beta(\|f\|_\infty\, n^{b-a/2})$$

where $\mathcal{I}^\pm = \mathcal{I}^\pm_0 \cup \mathcal{I}^\pm_1$, with

$$\mathcal{I}^\pm_0 = \{0 \text{ crossing interface}, \Lambda_{n^b} \subset \pm \text{ phase }\}$$
$$\mathcal{I}^\pm_1 = \{1 \text{ crossing interface}, \Lambda_{n^b} \subset \pm \text{ phase }\}$$



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Combinatorics and Analysis in Spatial Probability, Eurandom, NL, December 12-17, 2010