Majoration et minoration du sup

Critère de CVU

$$\sup_{x\in I} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

$$\Leftrightarrow \quad \exists (a_n),\ a_n \xrightarrow[n\to\infty]{} 0,\ \forall n \quad \sup_{x\in I} |f_n(x) - f(x)| \leq a_n$$

$$\boxed{\begin{array}{l} f_n \xrightarrow[n\to\infty]{U} f \text{ sur } I \\ \Leftrightarrow \quad \exists (a_n),\ a_n \to 0,\ \forall n \ \boxed{\forall x\in I \ |f_n(x) - f(x)| \leq a_n} \end{array}}$$

majoration uniforme en x pour n donné.

Critère de non CVU

$$\exists c>0 \quad \forall n \quad \sup_{x\in I} |f_n(x) - f(x)| > c$$

$$\Rightarrow \quad \sup_{x\in I} |f_n(x) - f(x)| \underset{n\to\infty}{\not\longrightarrow} 0$$

$$\boxed{\begin{array}{l} \exists c>0 \quad \forall n \quad \exists x \quad |f_n(x) - f(x)| > c \\ \Rightarrow \quad f_n \overset{U}{\underset{n\to\infty}{\not\longrightarrow}} f \text{ sur } I \end{array}}$$

dépend en général de n
$x = x_n$

On cherche $c>0$ et (x_n) t.q. $\forall n \quad |f_n(x_n) - f(x_n)| > c$.

CVU

1) $$\left(f_n : \begin{array}{rcl} \mathbb{R} & \longrightarrow & \mathbb{R} \\ x & \longmapsto & \frac{1}{n}\cos\left(\frac{x}{n^2x^2+n+1}\right) \end{array} \right)_{n\geq 1}$$

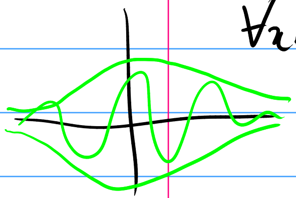
$$\forall n\geq 1 \quad \|f_n\|_\infty \leq \frac{1}{n}, \quad \frac{1}{n} \xrightarrow[n\to\infty]{} 0$$

Donc $f_n \xrightarrow[n\to\infty]{U} 0$.

2) $\left(f_n : \begin{array}{l} \mathbb{R} \longrightarrow \mathbb{R} \\ x \longmapsto \dfrac{1}{n} \cdot \dfrac{x\cos(nx)}{1+n^2x^2} \end{array} \right)_{n \geq 1}$.

$\|f_n\|_\infty$?

Soit $n \geq 1$.



$$\forall x \in \mathbb{R}, \quad \left| \frac{1}{n} \cdot \frac{x\cos(nx)}{1+n^2x^2} \right| \leq \frac{1}{n} \frac{|x|}{1+n^2x^2} \leftarrow \sup_{x\in\mathbb{R}} = \frac{1}{2\sqrt{n}}$$

$$\leq \frac{1}{n} \frac{|x|}{1+x^2}$$

$$\leq \begin{cases} \dfrac{1}{n} \cdot \dfrac{|x|}{x^2} = \dfrac{1}{n} \cdot \dfrac{1}{|x|} \leq \dfrac{1}{n} & \text{si } |x| \geq 1 \\ \dfrac{1}{n} \cdot |x| \leq \dfrac{1}{n} & \text{si } |x| \leq 1 \end{cases}$$

Donc $\sup_{x \in \mathbb{R}} |f_n(x)| \leq \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

Donc $f_n \xrightarrow[n\to\infty]{U} 0$.

3) $f_n : \begin{array}{l} \mathbb{R}_+ \longrightarrow \mathbb{R} \\ x \longmapsto \dfrac{n^2x + nx\cos x + 1}{1+n^2(1+x)} \end{array}$

CVS. Soit $x \in \mathbb{R}$.

si $x > 0$, $\dfrac{n^2x + nx\cos x + 1}{1+n^2(1+x)} \underset{n\to\infty}{\sim} \dfrac{n^2x}{n^2(1+x)} = \dfrac{x}{1+x}$

Donc $f_n(x) \xrightarrow[n\to\infty]{} \dfrac{x}{1+x}$.

si $x = 0$, $f_n(x) = \dfrac{1}{1+n^2} \xrightarrow[n\to\infty]{} 0$

Donc $f_n \xrightarrow[n\to\infty]{S} f : \left| \begin{array}{l} \mathbb{R}_+ \longrightarrow \mathbb{R} \\ x \longmapsto \dfrac{x}{1+x} \end{array} \right.$.

CVU Soit $n \in \mathbb{N}$.

$$\forall x \in \mathbb{R}_+, \quad f_n(x) - f(x) = \frac{n^2x + nx\cos x + 1}{1+n^2(1+x)} - \frac{x}{1+x}$$

$$= \frac{nx(1+x)\cos x + 1 + x - x}{(1+x)(1+n^2(1+x))}$$

$$= \frac{nx(1+x)\cos x + 1}{(1+x)(1+n^2(1+x))}$$

$\forall x \in \mathbb{R}_+,$

$$|f_n(x) - f(x)| = \frac{|nx(1+x)\cos x + 1|}{(1+x)(1+n^2(1+x))}$$

$$\leq \frac{|nx(1+x)\cos x| + 1}{(1+x)(1+n^2(1+x))}$$

$$\leq \frac{nx(1+x) + 1}{(1+x)(1+n^2(1+x))}$$

$$\leq \frac{nx(1+x)}{(1+x)(1+n^2(1+x))} + \frac{1}{(1+x)(1+n^2(1+x))}$$

$$\leq \frac{nx}{n^2(1+x)} + \frac{1}{1+n^2}$$

$$\leq \frac{1}{n} + \frac{1}{1+n^2} \xrightarrow[n\to\infty]{} 0$$

Donc $f_n \xrightarrow[n\to\infty]{U} f$ sur $\mathbb{R}_+$.

<u>non CVU</u>

1) $f_n : \mathbb{R}_+ \longrightarrow \mathbb{R}$
$x \longmapsto nx e^{-nx}$

$$f_n \xrightarrow[n\to\infty]{S} 0 \quad \text{sur } \mathbb{R}_+$$

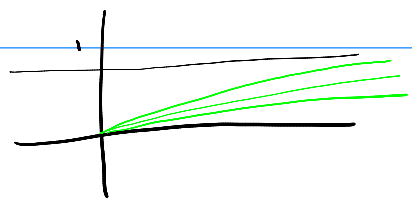
Soit $n \in \mathbb{N}$. $f_n\left(\frac{1}{n}\right) = e^{-1}$

Donc $\sup_{x\in\mathbb{R}_+} |f_n(x)| \geq e^{-1} > 0$

$$\|f_n\|_\infty \underset{n\to\infty}{\not\longrightarrow} 0, \quad f_n \overset{U}{\underset{n\to\infty}{\not\longrightarrow}} 0.$$

2) $\left(f_n : \mathbb{R}_+ \longrightarrow \mathbb{R},\ x \longmapsto \frac{x}{x+n} \right)_{n\geq 1}$

$$f_n \xrightarrow[n\to\infty]{S} 0$$



Soit $n \in \mathbb{N}^*$, $\quad f_n(n) = \dfrac{n}{n+n} = \dfrac{1}{2}$

Donc $\|f_n\|_\infty \geq \dfrac{1}{2}$,

$$f_n \underset{n\to\infty}{\overset{U}{\not\longrightarrow}} 0$$

3) $$f_n : \mathbb{R}_+ \longrightarrow \mathbb{R}$$
$$x \longmapsto \frac{2^n x}{1+n2^n x^2}$$

$$f_n \xrightarrow[n\to\infty]{S} 0.$$

Soit $n \in \mathbb{N}$ $\quad \left| \dfrac{2^n \frac{1}{n}}{1+2^n n \frac{1}{n^2}} \right| = \dfrac{\frac{2^n}{n}}{1+\frac{2^n}{n}} = \dfrac{2^n}{n+2^n} \xrightarrow[n\to\infty]{} 1$

donc $\quad \sup\limits_{x\in\mathbb{R}_+} |f_n(x)| \geq \dfrac{2^n}{n+2^n} \quad \left| \begin{array}{l} \geq \dfrac{2^n}{2\cdot 2^n} = \dfrac{1}{2} \\ \xrightarrow[n\to\infty]{} 1 \end{array} \right.$

Donc $\sup\limits_{x\in\mathbb{R}_+} |f_n(x)| \underset{n\to\infty}{\not\longrightarrow} 0$.

Donc $$f_n \underset{n\to\infty}{\overset{U}{\not\longrightarrow}} 0$$