Explicit Noether-Lefschetz for arbitrary threefolds

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Abstract

We study the Noether-Lefschetz locus of a very ample line bundle L on an arbitrary smooth threefold Y. Building on results of Green, Voisin and Otwinowska, we give explicit bounds, depending only on the Castelnuovo-Mumford regularity properties of L, on the codimension of the components of the Noether-Lefschetz locus of $|L|$.

1 Introduction.

It is well-known in algebraic geometry that the geometry of a given variety is influenced by the geometry of its subvarieties. It less common, but not unusual, that a given ambient variety forces to some extent the geometry of its subvarieties. A particularly nice case of the latter is given by line bundles, whose properties do very much influence the geometry.
If Y is a smooth variety and $i : X \hookrightarrow Y$ is a smooth divisor, there is then a natural restriction map

$$i^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

given by pull-back of line bundles.
Now suppose that X is very ample. By the Lefschetz theorem i^* is injective if $\mathrm{dim} Y \geq 3$. On the other hand, it was already known to the Italian school (Severi [18], Gherardelli [6]), that i^* is surjective when $\mathrm{dim} Y \geq 4$. Simple examples show that in the case where $\mathrm{dim} Y = 3$ we cannot hope for surjectivity unless a stronger restriction is considered.
For the case $Y = \mathbb{P}^3$, this is also a classical problem, first posed by Noether and solved in the case of generic X by Lefschetz who showed that

Theorem (Noether-Lefschetz) *For X a generic surface of degree $d \geq 4$ in $\mathbb{P}^3$ we have $\mathrm{Pic}(X) \cong \mathbb{Z}$.*

Here and below by generic we mean outside a countable union of proper subvarieties.

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Suppose now that a smooth threefold Y and a line bundle L on Y are given. We will say that a Noether-Lefschetz theorem holds for the pair (Y, L), if

$$i^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$$

is a surjection for a generic smooth surface $X \subset Y$ such that $\mathcal{O}_Y(X) = L$.

The following result of Moishezon ([14], see also the argument given in Voisin [21, Thm. 15.33]) establishes the exact conditions under which a Noether-Lefschetz theorem holds for (Y, L).

Theorem (Moishezon) *If (Y, L) are such that L is very ample and*

$$h^{0,2}_{ev}(X, \mathbb{C}) \neq 0$$

for a generic smooth X such that $\mathcal{O}_Y(X) = L$, then a Noether-Lefschetz theorem holds for the pair (Y, L).

Here, $h^{0,2}_{ev}$ denotes the evanescent $(2, 0)$-cohomology of X: see below for a precise definition.

More precisely, we denote by $U(L)$ the open subset of $\mathbb{P}H^0(L)$ parameterizing smooth surfaces in the same equivalence class as L. We further denote by $\mathrm{NL(L)}$ (*the Noether-Lefschetz locus of L*) the subspace parameterizing surfaces X equipped with line bundles which are not produced by pull-back from Y. The above theorem then admits the following alternative formulation.

Theorem (Moishezon) *If (Y, L) are such that L is very ample and*

$$h^{0,2}_{ev}(X, \mathbb{C}) \neq 0$$

for a generic smooth X such that $\mathcal{O}_Y(X) = L$, then the Noether-Lefschetz locus $\mathrm{NL(L)}$ is a countable union of proper algebraic subvarieties of $U(L)$.

These proper subvarieties will henceforth be referred to as *components of the Noether-Lefschetz locus*.

A Noether-Lefschetz theorem for a pair (Y, L) essentially says that for a generic surface X such that $\mathcal{O}_Y(X) = L$, the set of line bundles on X is well-understood and as simple as possible. A natural follow-up question is: how rare are surfaces with badly behaved Picard groups? Or alternatively: how large can the components of the Noether-Lefschetz locus be in comparison with $U(L)$? This leads us to attempt to prove what we will call *explicit Noether-Lefschetz theorems*. An explicit Noether-Lefschetz theorem (the terminology is due to Green) says that the codimension of $\mathrm{NL(L)} \subset U(L)$ is bounded below by some number n_L depending non-trivially on the positivity of L. The first known example of these was the following theorem, established independently by Voisin and Green, [8], [20], which gives an explicit Noether-Lefschetz theorem for $\mathbb{P}^3$.

Theorem (Green, Voisin) *Let $Y = \mathbb{P}^3$ and $L = \mathcal{O}_{\mathbb{P}^3}(d)$. Let $\Sigma_L \subset U(L)$ be any component of the Noether-Lefschetz locus. Then* $\operatorname{codim} \Sigma_L \geq d - 3$, *with equality being achieved only for the component of surfaces containing a line.*

In this theorem we see also another of the reigning principles of the study of components of the Noether-Lefschetz locus, namely that components of small codimension should parameterize surfaces containing low-degree curves.

Recently, the subject has been much advanced by the following result of Otwinowska, ([17], see also [15] and [16]) which implies an explicit Noether-Lefschetz theorem for analogues of Noether-Lefschetz loci for highly divisible line bundles on varieties of arbitrary odd dimension. (For ease of presentation, we give a weakened version of the result proved).

Theorem (Otwinowska) *Let Y be a projective variety of dimension $2n+1$, let $\mathcal{O}_Y(1)$ be a very ample line bundle on Y and let $\Sigma_L \subset U(\mathcal{O}_Y(d))$ be any component of the Noether-Lefschetz locus. Let X be a hypersurface contained in Σ_L. For d large enough, if*

$$\operatorname{codim} \Sigma_L \leq \frac{d^n}{n!}$$

then X contains a n-dimensional linear space.

In fact, Otwinowska also gives a numerical criterion on d and the codimension of Σ_L under which X necessarily contains a degree-b n-dimensional subvariety. We recall also the results of Joshi [13] and Ein-Lazarsfeld [5, Prop. 3.4].

The aim in this paper will be to shed light on the fact that it is the *Castelnuovo-Mumford regularity properties* of a line bundle that insure that an explicit Noether-Lefschetz theorem holds, independently on the divisibility properties.
To state our first result we suppose that Y is a smooth threefold and H is a very ample line bundle on Y. We define numbers α_Y and β_Y as follows.

Definition 1 *The integer α_Y is defined to be the minimal positive integer such that $K_Y + \alpha_Y H$ is very ample. The integer β_Y is defined to be the minimal integer such* *that $(\beta_Y - \alpha_Y)H - K_Y$ is nef.*

We recall that, by the results of adjunction theory [19], if $(Y, H) \neq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, we have that $\alpha_Y \leq 4$ with equality if and only if either Y is a $\mathbb{P}^2$-bundle over a smooth curve and the restriction of H to the fibers is $\mathcal{O}_{\mathbb{P}^2}(1)$ (*we will refer later to this case as a linear $\mathbb{P}^2$-bundle*) or $(Y, H) = (Q, \mathcal{O}_Q(1))$ where $Q \subset \mathbb{P}^4$ is a smooth quadric hypersurface. On the other hand $\beta_Y \geq 1$ with equality if Y is subcanonical and nonpositive (that is if $K_Y = eH$ for some integer $e \leq 0$).
We have

Theorem 1 *Let Y be a smooth threefold, $Y \neq \mathbb{P}^3$ and let H be a very ample divisor on Y. Let L be a (-d)-regular line bundle with respect to H. We suppose that either $H^1(\Omega^2_Y \otimes L) = 0$ or $d \geq 3\beta_Y - 3\alpha_Y + 13$. Let Σ_L be any component of the Noether-Lefschetz locus* $\mathrm{NL(L)}$. *The following bounds hold:*

(i) If (Y, H) is not a linear $\mathbb{P}^2$-bundle then

$$\operatorname{codim} \Sigma_L \geq \begin{cases} d - 5 + \alpha_Y - 2\beta_Y & \text{if } \beta_Y \geq 2 \text{ and } d \geq \frac{\beta_Y^2(\beta_Y+5)}{2} \\ d - 6 + \alpha_Y & \text{if } \beta_Y = 1 \end{cases}.$$

(ii) If (Y, H) is a linear $\mathbb{P}^2$-bundle then

$$\operatorname{codim} \Sigma_L \geq \begin{cases} d - 2 - 2\beta_Y & \text{if } \beta_Y \geq 2 \text{ and } d \geq \frac{\beta_Y^2(\beta_Y+5)}{2} \\ d - 3 & \text{if } \beta_Y = 1 \end{cases}.$$

We can do a little bit better in the case of the Noether-Lefschetz locus of adjoint line bundles.

We now define numbers a_Y and b_Y as follows.

Definition 2 *The integer a_Y is defined to be the minimal integer such that $K_Y + a_Y H$ is very ample. The integer b_Y is defined to be the minimal integer such that $(b_Y - a_Y)H - K_Y$ is nef.*

As above, if $(Y, H) \neq (\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(1))$, we have that $a_Y \leq 4$ with equality if and only if either (Y, H) is a linear $\mathbb{P}^2$-bundle or $(Y, H) = (Q, \mathcal{O}_Q(1))$ and again $b_Y \geq 1$ with equality if Y is subcanonical.

Theorem 2 *Let Y be a smooth threefold, $Y \neq \mathbb{P}^3$ and let H be a very ample divisor on Y. Let*

$$L = K_Y + dH + A,$$

where A is numerically effective. We suppose that either $H^1(\Omega^2_Y \otimes L) = 0$ or $d \geq 2b_Y - 2a_Y + 13$. Let Σ_L be any component of the Noether-Lefschetz locus $\mathrm{NL(L)}$. *The following bounds hold:*

(i) If (Y, H) is not a linear $\mathbb{P}^2$-bundle then

$$\text{codim}\,\Sigma_L \geq \begin{cases} d-5-b_Y & \text{if } b_Y \geq 2 \text{ and } d \geq \frac{b_Y(b_Y^2+7b_Y-6)}{2} \\ d-5 & \text{if } b_Y = 1 \end{cases}.$$

(ii) If (Y, H) is a linear $\mathbb{P}^2$-bundle then

$$\text{codim}\,\Sigma_L \geq \begin{cases} d-6-b_Y & \text{if } b_Y \geq 2 \text{ and } d \geq \frac{b_Y(b_Y-1)(b_Y+8)}{2} \\ d-6 & \text{if } b_Y = 1 \end{cases}.$$

We also note the following application that generalises [2] (see also [3]).

Corollary 1 *Let Y be a smooth threefold such that $Y \neq \mathbb{P}^3$ and $\text{Pic}(Y) \cong \mathbb{Z}H$ where H is a very ample line bundle and let $K_Y = eH$. We suppose that either $H^1(\Omega_Y^2(d)) = 0$ or $d \geq 3e + 13$. Let $P_1, \ldots, P_k$ be k general points in Y and $\pi : \widetilde{Y} \to Y$ be the blow-up of Y at these points with exceptional divisors $E_1, \ldots, E_k$. If $d \geq 7 + e$ then*

$$d\pi^*(H) - E_1 - \ldots - E_k \quad \text{is ample on } \widetilde{Y} \;\Leftrightarrow\; d^3H^3 > k.$$

We outline our approach to the study of the Noether-Lefschetz locus.
In section 2, we will give the standard expression of this problem in terms of variation of Hodge structure of X. We will then recall the classical results of Griffiths, Carlson et. al. which allow us to express variation of Hodge structure of X in terms of multiplication of sections of line bundles on X.

We define σ to be the section of L defining X. The tangent space of a component of the Noether-Lefschetz locus is naturally a subspace of $H^0(L)/\langle\sigma\rangle$, and we will denote its preimage in $H^0(L)$ by T. If we suppose that $H^1(\Omega_Y^2 \otimes L) = 0$, then T has the following property: The natural multiplication map

$$T \otimes H^0(K_Y \otimes L) \to H^0(K_Y \otimes L^2) \tag{1}$$

is not surjective.
A full proof of this fact is given in section 3.
In section 3, we also explain Green's methods for proving the explicit Noether-Lefschetz theorem for $\mathbb{P}^3$ using Koszul cohomology to prove that equation (1) cannot be satisfied if T is too large. Green's method does not immediately apply to our case, since it requires T to be base-point free— which is only guaranteed if the tangent bundle of Y is globally generated, hence only for a few threefolds.

However, we show in section 4 that there exists $W \subset H^0(K_Y \otimes L(3))$ such that W is base-point free and

$$\{T \otimes H^0(K_Y \otimes L)\} \oplus \{W \otimes H^0(L(-3))\} \to H^0(K_Y \otimes L^2)$$

is not surjective. Results of Ein and Lazarsfeld [5] then imply a lower bound on the codimension of

$$\{T \otimes H^0(K_Y(3))\} \oplus W \subset H^0(K_Y \otimes L(3))$$

and more particularly on the codimension of

$$T \otimes H^0(K_Y(3)) \subset H^0(K_Y \otimes L(3)).$$

In introducing W, we get around the base-point free problems, but introduce others. In particular, we now need a method for extracting a lower bound on $\operatorname{codim} T$ from a lower bound for $\operatorname{codim} (T \otimes H^0(K_Y(3)))$. When $Y = \mathbb{P}^3$, this is a simple application of a classical inequality in commutative algebra due to Macaulay and Gotzmann. In section 5 we extend the Macaulay-Gotzmann inequality to sections of any Castelnuovo-Mumford regular sheaf. In section 6, we pull all of the above together to prove the theorem.

2 Preliminaries.

In this section we recall the classical results of Griffiths, Carlson et. al. on which our work will be based. We will show how a component Σ_L of the Noether-Lefschetz locus $\mathrm{NL(L)}$ can be locally expressed as the zeros of a certain section of a vector bundle over $U(L)$. We will then use this expression— together with the work of Griffiths from the 60s, relating variation of Hodge structure with deformations of X to multiplication of sections of line bundles on X— to relate the codimension of Σ_L to cohomological questions on X.

2.1 NL expressed as the zero locus of a vector bundle section.

We note first that by the Lefschetz theorem the map $\mathrm{Pic}_0(Y) \to \mathrm{Pic}_0(X)$ is necessarily an isomorphism. It follows that the map $i^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ fails to be surjective if and only if the $(1,1)$ integral evanescent cohomology is non-trivial: $H^{1,1}_{\text{ev}}(X, \mathbb{Z}) \neq 0$. (We recall that the subspace $H^{1,1}_{\text{ev}}(X, \mathbb{C}) \subset H^{1,1}(X, \mathbb{C})$ is defined by $\gamma \in H^{1,1}_{\text{ev}}(X, \mathbb{C}) \Leftrightarrow \langle i^*\beta, \gamma \rangle = 0$ for all $\beta \in H^2(Y, \mathbb{C})$.)

In particular, we can therefore define $\mathrm{NL(L)}$ as follows

$$X \in \mathrm{NL(L)} \Leftrightarrow H^{1,1}_{\text{ev}}(X, \mathbb{Z}) \neq 0.$$

This is the definition of $\mathrm{NL(L)}$ which we will use henceforth, since it is much more manageable. In particular, it is this description which will allow us to write any component of $\mathrm{NL(L)}$ as the zero locus of a special section of a vector bundle.

Henceforth, we will assume that X is contained in $\mathrm{NL(L)}$ and γ will be a non-trivial element of $H^{1,1}_{\mathrm{ev}}(X,\mathbb{Z})$. The point in $U(L)$ corresponding to X will be denoted by 0. We will now define what we mean by the *Noether-Lefschetz locus associated to* γ, which we denote by $NL(\gamma)$. Since we will be interested in the local geometry of $\mathrm{NL(L)}$, we fix for simplicity a contractible neighbourhood of 0, O. Henceforth, all our calculations will be made over O. We form a vector bundle $\mathcal{H}^2_{\mathrm{ev}}$ over O, defined by

$$\mathcal{H}^2_{\mathrm{ev}}(u) = H^2_{\mathrm{ev}}(X_u,\mathbb{C}).$$

The vector bundle contains holomorphic sub-bundles $\mathcal{F}^i(\mathcal{H}^2_{\mathrm{ev}})$ given by

$$\mathcal{F}^i(\mathcal{H}^2_{\mathrm{ev}})(u) = F^i(H^2_{\mathrm{ev}}(X_u,\mathbb{C})).$$

We define bundles $\mathcal{H}^{i,2-i}_{\mathrm{ev}}$ by $\mathcal{H}^{i,2-i}_{\mathrm{ev}} = \mathcal{F}^i(\mathcal{H}^2_{\mathrm{ev}})/\mathcal{F}^{i+1}(\mathcal{H}^2_{\mathrm{ev}})$.

(The fibre of $\mathcal{H}^{i,2-i}_{\mathrm{ev}}$ at the point u is isomorphic to $H^{i,2-i}_{\mathrm{ev}}(X_u)$: however, $\mathcal{H}^{i,2-i}_{\mathrm{ev}}$ does not embed naturally into $\mathcal{H}^2_{\mathrm{ev}}$ as a holomorphic sub-bundle.) The bundle $\mathcal{H}^2_{\mathrm{ev}}$ is equipped with a natural flat connexion, the Gauss-Manin connexion, which we denote by ∇. We now define $\overline{\gamma}$ to be the section of $\mathcal{H}^2_{\mathrm{ev}}$ produced by flat transport of γ.

We define $\overline{\gamma}^{0,2}$, a section of $\mathcal{H}^{0,2}_{\mathrm{ev}}$, to be the image of $\overline{\gamma}$ under the projection

$$\pi : \mathcal{H}^2_{\mathrm{ev}} \to \mathcal{H}^{0,2}_{\mathrm{ev}}.$$

We are now in a position to define $\mathrm{NL}(\gamma)$.

Definition 3 *The Noether-Lefschetz locus associated to* γ, $\mathrm{NL}(\gamma)$*, is given by*

$$\mathrm{NL}(\gamma) = \mathrm{zero}(\overline{\gamma}^{0,2}).$$

Informally, $\mathrm{NL}(\gamma)$ parameterizes the small deformations of X on which γ remains of Hodge type $(1,1)$. Any component of $\mathrm{NL(L)}$ is locally equal to $\mathrm{NL}(\gamma)$ for some γ.

The tangent space $T\mathrm{NL}(\gamma)$ at X is a subspace of $H^0(L)/\langle\sigma\rangle$, where σ is the section of L defining X. We will denote its preimage in $H^0(L)$ by T.

2.2 IVHS and residue maps.

We will now explain the classical work of Griffiths which makes the section $\overline{\gamma}^{0,2}$ particularly manageable.

Let $\mathcal{H}^2_{\text{ev}}$ be as above. For the purposes of this section we will consider the holomorphic subvector bundle $\mathcal{F}^i_{\text{ev}}$ to be a holomorphic map $\mathcal{F}^i_{\text{ev}} : O \to \text{Grass}(f_i, \mathcal{H}^2_{\text{ev}})$ where f_i is the dimension of $F^i H^2_{\text{ev}}(X, \mathbb{C})$. The Gauss-Manin connexion gives us a canonical isomorphism $\mathcal{H}^2_{\text{ev}} \cong H^2_{\text{ev}}(X, \mathbb{C}) \times O$, from which we deduce a canonical isomorphism

$$\text{Grass}(f_i, \mathcal{H}^2_{\text{ev}}) \cong O \times \text{Grass}(f_i, H^2_{\text{ev}}(X, \mathbb{C})).$$

In particular, $\mathcal{F}^i_{\text{ev}}$ is now expressed as a map from O to the constant space $\text{Grass}(f_i, H^2_{\text{ev}}(X, \mathbb{C}))$, and as such can be derived. We obtain a derivation map, which we denote by IVHS (for Infinitesimal Variation of Hodge Structure)

$$\text{IVHS}^i : TO \to \text{Hom}(F^i(H^2_{\text{ev}}), H^2_{\text{ev}}/F^i(H^2_{\text{ev}})).$$

Griffiths proved the following result in [10].

Theorem (Griffiths' Transversality) *The image of* IVHS^i *is contained in* $\text{Hom}(H^{i,2-i}_{\text{ev}}, H^{i-1,3-i}_{\text{ev}})$.

The importance of this work for our purposes is the following lemma.

Lemma 1 *For any $v \in TO$, we have that $d_v(\overline{\gamma}^{0,2}) = -\text{IVHS}^1(v)(\gamma)$.*

Proof. The isomorphism $f : T_W\text{Grass}(\text{n}, \text{V}) \cong \text{Hom}(W, V/W)$ is given by

$$f(v) : w \to \frac{\partial}{\partial v}(\tilde{w})_{|V/W}$$

where $w \in W$ and $\tilde{w}$ is any local section of the tautological bundle over the Grassmannian such that $\tilde{w}_W = w$.
In the case in hand, we choose a lifting of $\overline{\gamma}^{0,2}$ to a section of $\mathcal{H}^2_{\text{ev}}$, which we denote by $\overline{\gamma}^{0,2}_{\text{lift}}$. By definition of $\overline{\gamma}^{0,2}$, we then have that $\overline{\gamma} - \overline{\gamma}^{0,2}_{\text{lift}} \in \mathcal{F}^1(\mathcal{H}^2_{\text{ev}})$ and it follows that $\text{IVHS}^1(v)(\gamma) = \frac{\partial}{\partial v}(\overline{\gamma} - \overline{\gamma}^{0,2}_{\text{lift}})_{|_{H^{0,2}_{\text{ev}}}}$ and now, since by definition $\overline{\gamma}$ is constant, $\text{IVHS}^1(v)(\gamma) = -d_v(\overline{\gamma}^{0,2}_{\text{lift}})_{|_{H^{0,2}_{\text{ev}}}} = -d_v(\overline{\gamma}^{0,2})$. □

We will also need the work of Carlson and Griffiths relating the residue maps to Hodge structure of varieties ([1]). Suppose given, for $i = 1, 2$, a section

$$s \in H^0(K_Y \otimes L^i).$$

This can be thought of as a holomorphic 3-form on Y with a pole of order i along X, and as such defines a cohomology class in $H^3(Y \setminus X, \mathbb{C})$. The group $H^3(Y \setminus X, \mathbb{C})$ maps to $H^2_{\text{ev}}(X, \mathbb{C})$ via residue, and hence there is an induced residue map

$$\mathrm{res_i} : H^0(K_Y \otimes L^i) \to H^2_{\text{ev}}(X, \mathbb{C}).$$

The relevance of this map to variation of Hodge structure comes from the following theorem, which is proved by Griffiths in [11].

Theorem *The image of* $\mathrm{res_i}$ *is contained in* $F^{3-i}(H^2_{\text{ev}})$.

Henceforth, we will denote by π_i the induced projection map

$$\pi_i : H^0(K_Y \otimes L^i) \to H^{3-i,i-1}_{\text{ev}}(X, \mathbb{C}).$$

In this representation, the map IVHS^{3-i} has a particularly nice form ([1], page 70).

Theorem (multiplication) *Consider* $v \in TO$. *Let* $\tilde{v}$ *be a lifting of* v *to* $H^0(L)$. *Then for any* $P \in H^0(K_X \otimes L^i)$, *we have that*

$$\mathrm{IVHS}^{3-i}(v)(\pi_i(P)) = \pi_{i+1}(\tilde{v} \otimes P)$$

up to multiplication by some nonzero constant.

The only fly in the ointment is that in general we cannot be sure that the map π_i is surjective onto $H^{3-i,i-1}_{\text{ev}}(X, \mathbb{C})$. It is precisely for this reason that we will be obliged to suppose that $H^1(\Omega^2_Y \otimes L) = 0$.
The following lemma will be crucial.

Lemma 2 *Consider* $\gamma \in H^{1,1}_{\text{ev}}(X)$ *and* $\omega \in H^{2,0}_{\text{ev}}(X)$. *For any vector* $v \in TO$ *we have*

$$\langle \mathrm{IVHS}^1(v)(\gamma), \omega \rangle + \langle \gamma, \mathrm{IVHS}^2(v)(\omega) \rangle = 0.$$

Proof. We note that $d_v(\langle \overline{\gamma}, \overline{\omega} \rangle) = 0$. We note that we can write

$$\overline{\gamma} = \overline{\gamma}^1 + \overline{\gamma}^2$$

where $\overline{\gamma^1} \in \mathcal{F}^1_{\text{ev}}$ and $\overline{\gamma}^2(0) = 0$. Similarly, we can write $\overline{\omega} = \overline{\omega}^1 + \overline{\omega}^2$ where $\overline{\omega^1} \in \mathcal{F}^2_{\text{ev}}$ and $\overline{\omega}^2(0) = 0$. We note that for Hodge theoretic reasons $< \overline{\omega}^1, \overline{\gamma}^1 >= 0$ and hence

$$d_v(\langle \overline{\gamma}, \overline{\omega} \rangle) = \langle d_v(\overline{\gamma}^2), \omega \rangle + \langle \gamma, d_v(\overline{\omega}^2) \rangle.$$

Here, of course, it makes sense to talk about $d_v(\overline{\omega}^2)$ and $d_v(\overline{\gamma}^2)$ only because $\overline{\omega}^2(0) = 0$ and $\overline{\gamma}^2(0) = 0$. Since $\langle \mathcal{F}^1, \mathcal{F}^2 \rangle = 0$, we have that

$$\langle d_v(\overline{\gamma}^2), \omega \rangle = \langle d_v(\overline{\gamma}^2)^{0,2}, \omega \rangle = \langle -\mathrm{IVHS}^1(v)(\gamma), \omega \rangle$$

and similarly

$$\langle \gamma, d_v(\overline{\omega}^2)\rangle = \langle \gamma, (d_v\overline{\omega}^2)^{1,1}\rangle = \langle \gamma, -\mathrm{IVHS}^2(v)(\omega)\rangle.$$

So it follows immediately from $d_v(\langle \overline{\gamma}, \overline{\omega}\rangle) = 0$ that

$$\langle \mathrm{IVHS}^1(v)(\gamma), \omega\rangle + \langle \gamma, \mathrm{IVHS}^2(v)(\omega)\rangle = 0. \qquad \square$$

3 Strategy and overview.

The basic idea of this proof is that used by Green in [8]. We summarise his proof and explain why it cannot be immediately applied to the situation in hand.

First some notation. Given any pair of coherent sheaves on X, L and M we denote by $\mu_{L,M}$ the multiplication map

$$\mu_{L,M} : H^0(L) \otimes H^0(M) \to H^0(L \otimes M).$$

Where there is no risk of confusion, we will write μ for $\mu_{L,M}$.

The starting point of Green's work is the following lemma.

Lemma 3 *Suppose that $T \subset H^0(\mathcal{O}_{\mathbb{P}^3}(d))$ is the preimage of $TNL(\gamma)$. Then the inclusion*

$$\mu(T \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4))) \subset H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2d-4))$$

is a strict inclusion.

Proof. In the case of $Y = \mathbb{P}^3$, we have that $\pi_i : H^0(K_Y \otimes L^i) \to H^{3-i,i-1}_{\mathrm{ev}}(X)$ is a surjection. (See, for example, [21, proof of Thm. 18.5, page 420]). By Lemma 2, if $v \in TNL(\gamma)$ and $P \in H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4))$ then

$$\langle \gamma, \mathrm{IVHS}^2(v)(\pi_1(P))\rangle = -\langle \mathrm{IVHS}^1(v)(\gamma), \pi_1(P)\rangle = 0$$

from which we conclude that $\mathrm{IVHS}^2(v)(\pi_1(P)) \in \gamma^\perp$, where $\gamma^\perp$ is the orthogonal to γ, and in particular is a proper subspace. By the multiplication theorem it follows that $\pi_2(\mu(\tilde{v} \otimes P)) \in \gamma^\perp$ or alternatively

$$\mu(\tilde{v} \otimes P) \in \pi_2^{-1}(\gamma^\perp).$$

Since π_2 is surjective, $\pi_2^{-1}(\gamma^\perp)$ is a proper subspace. $\square$

Green then proves the following theorem via the vanishing of certain Koszul cohomology groups.

Theorem (Green) *Let $T \subset H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ be a base-point free linear system of codimension c. Then the Koszul complex*

$$\bigwedge^{p+1} T \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(k-d)) \to \bigwedge^{p} T \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(k)) \to \bigwedge^{p-1} T \otimes H^0(\mathcal{O}_{\mathbb{P}^r}(k+d))$$

is exact in the middle provided that $k \geq p + d + c$.

In the case in hand, on setting $r = 3, p = 0$ and $k = 2d - 4$ we see that the multiplication map

$$T \otimes H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(d-4)) \to H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2d-4))$$

is surjective if $2d - 4 \geq d + c$. But we have already observed that this multiplication map is necessarily non-surjective, from which it follows that $c \geq d - 3$.

In Lemma 4 below we will see that, provided $H^1(\Omega^2_Y \otimes L) = 0$, it is still true that the multiplication map $T \otimes H^0(K_Y \otimes L) \to H^0(K_Y \otimes L^2)$ is non-surjective. One might therefore reasonably entertain the hope of adapting Green's methods to arbitrary varieties. The difficulty is that in order to apply Green's result, T must be base-point free. This was immediate when $Y = \mathbb{P}^3$, since, if X was given by $F \in H^0(\mathcal{O}_{\mathbb{P}^3}(d))$, T then automatically contained $H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \times \langle \frac{\partial F}{\partial X_i} \rangle$. However if T_Y is not globally generated, there is no reason why this should hold in general. The rest of this paper will be concerned with finding ways around this difficulty.

Lemma 4 *Let L be very ample and such that $H^1(\Omega^2_Y \otimes L) = 0$.*
Let $T \subset H^0(L)$ be the preimage in $H^0(L)$ of the tangent space to $\mathrm{NL}(\gamma)$. Then

$$\mu(T \otimes H^0(K_Y \otimes L)) \subset H^0(K_Y \otimes L^2)$$

is a strict inclusion.

Proof. We note that by the argument given in the proof of Lemma 3,

$$\pi_2(\mu(T \otimes H^0(K_Y \otimes L))) \neq H^{1,1}_{\text{ev}}(X, \mathbb{C}).$$

Now it just remains to observe that, by [21, proof of Thm. 18.5, page 420],

$$\pi_2 : H^0(K_Y \otimes L^2) \to H^{1,1}_{\text{ev}}(X, \mathbb{C})$$

is a surjection, since $H^1(\Omega^2_Y(X)) = 0$. □

So, we would now like to apply Green's argument; unfortunately, T may have base points. Our strategy for getting around this problem will be as follows.

1. First of all, we will construct $W \subset H^0(K_Y \otimes L(3))$ with the following good properties.
   (a) W is base-point free,
   (b) $\pi_2(\mu(W \otimes H^0(L(-3)))) = 0$.
2. The result proved by Ein and Lazarsfeld in [5] then gives us a lower bound on the codimension of $\mu(T \otimes H^0(K_Y(3)))$ in $H^0(K_Y \otimes L(3))$.
3. We will then extract from the lower bound on $\operatorname{codim} \mu(T \otimes H^0(K_Y(3)))$ a lower bound on the codimension of T in $H^0(L)$.

4 Constructing W.

We henceforth let Y be a smooth threefold, $Y \neq \mathbb{P}^3$ and H be a very ample divisor on Y.

Proposition 1 *There is a subspace $W \subset H^0(K_Y \otimes L(3))$ such that*

1. *The map $\pi_2 \circ \mu : W \otimes H^0(L(-3)) \to H^{1,1}_{\mathrm{ev}}(X, \mathbb{C})$ is identically zero.*
2. *W is base-point free.*

Proof. We denote the image of $\mu : W \otimes H^0(L(-3)) \to H^0(K_Y \otimes L^2)$ by $\langle W \rangle$. Consider the map

$$d : H^0(\Omega^2_Y \otimes L) \to H^0(K_Y \otimes L^2)$$

which sends a two-form on Y with a simple pole along X to its derivation. We note that for any $\omega \in H^0(\Omega^2_Y \otimes L)$ we have that $d\omega \in \mathrm{Ker}(\mathrm{res}_2)$, because $d\omega$, being exact, defines a null cohomology class on $Y \setminus X$.
The space W will be chosen in such a way that

$$\langle W \rangle_{|X} \subset \mathrm{Im}(d)_{|X}.$$

The map d is difficult to deal with because it is not a map of $\mathcal{O}_Y$-modules: the value of $d\omega$ at a point x is not determined by the value of ω at x. In particular, it is not possible to form a tensor product map

$$d \otimes (L^{-1}(3)) : H^0(\Omega^2_Y(3)) \to H^0(K_Y \otimes L(3)).$$

Our first step will be to show that, even if d does not come from an underlying map of $\mathcal{O}_Y$-modules, the restriction

$$d_X : H^0(\Omega^2_Y \otimes L) \to H^0(K_X \otimes L_{|X}).$$

does.

Lemma 5 *Let the map $r : \Omega^2_Y \otimes L \to K_X \otimes L$ be given by tensoring with L the pull-back $i^* : \Omega^2_Y \to \Omega^2_X (\cong K_X)$. Then we have that $d_X = -H^0(r)$.*

Proof. We calculate in analytic complex coordinates near a point $p \in X$. Let f be a function defining X in a neighbourhood of p and let x, y be coordinates chosen in such a way that (f, x, y) form a system of coordinates for Y close to p. If $\nu \in H^0(\Omega^2_Y \otimes L)$, then in a neighbourhood of p we can write

$$\nu = \frac{f_1 dx \wedge dy + f_2 dx \wedge df + f_3 dy \wedge df}{f}$$

where f_1, f_2, f_3 are holomorphic functions on a neighbourhood of p.
Differentiating and restricting to X, we get that

$$d\nu_{|X} = \frac{-f_1 dx \wedge dy \wedge df}{f^2}.$$

As an element of $H^0((K_Y \otimes L) \otimes L)$, this is represented by

$$\frac{-f_1 dx \wedge dy \wedge df}{f} \otimes 1/f.$$

Under the canonical isomorphism $(K_Y \otimes L)_{|X} \to K_X$, we have that

$$\frac{-f_1 dx \wedge dy \wedge df}{f} \to -f_1 dx \wedge dy.$$

Hence, under the canonical isomorphism $(K_Y \otimes L^2)_{|X} \to K_X \otimes L_{|X}$, we have that

$$(d\nu)_{|X} \to \frac{-f_1 dx \wedge dy}{f} = -r(\nu).$$

This concludes the proof of Lemma 5. $\square$

We now proceed with the proof of Proposition 1.
The map d_X, which is a map of $\mathcal{O}_Y$-modules, has the advantage that we can form tensor products. We consider the map induced by tensor product with $L^{-1}(3)$

$$d_X^{L^{-1}(3)} : H^0(\Omega^2_Y(3)) \to H^0(K_X(3)).$$

We define W by

$$W = \{w \in H^0(K_Y \otimes L(3)) : w_{|X} \in \mathrm{Im}(d_X^{L^{-1}(3)})\}.$$

We will prove first that

Lemma 6 *For any $w \in W$ and $P \in H^0(L(-3))$, we have that*

$$\pi_2(\mu(P \otimes w)) = 0.$$

Proof. Since $w \in W$ there exists $s \in H^0(\Omega^2_Y(3))$ such that $w_{|X} = d_X^{L^{-1}(3)} s$ and hence

$$(Pw)_{|X} = d_X(Ps) = d(Ps)_{|X}.$$

From this it follows that there exists $s' \in H^0(K_Y \otimes L)$ such that

$$Pw = d(Ps) + \sigma s'.$$

We observed above that $\pi_2(d(Ps)) = 0$. We note that $\operatorname{res}_2(\sigma s') = \operatorname{res}_1(s')$ and hence $\operatorname{res}_2(\sigma s') \in F^2 H^2_{\mathrm{ev}}(X, \mathbb{C})$, from which it follows that $\pi_2(\sigma s') = 0$. Whence $\pi_2(Pw) = 0$. This concludes the proof of Lemma 6. □

To conclude the proof of Proposition 1 it remains only to show that W is base-point free. Since $Y \neq \mathbb{P}^3$ we have ([4]) that $K_Y(3)$ is globally generated. Also

$$\mu(\mathbb{C}\sigma \otimes H^0(K_Y(3))) \subset W$$

therefore the only possible base points of W are the points of X. Consider an arbitrary point $p \in X$. Now if $\mathbb{P}^N = \mathbb{P}H^0(Y, H)$ we have that $\Omega^2_Y(3)$ is globally generated since $\Omega^2_{\mathbb{P}^N}(3)$ is such and there is a surjection $\Omega^2_{\mathbb{P}^N}(3) \twoheadrightarrow \Omega^2_Y(3)$. Whence there exists a section $s \in H^0(\Omega^2_Y(3))$ such that $d_X^{L^{-1}(3)}(s)(p) \neq 0$. From the short exact sequence

$$0 \to K_Y(3) \to K_Y \otimes L(3) \to K_X(3) \to 0$$

and Kodaira vanishing we see that there exists $w \in H^0(K_Y \otimes L(3))$ such that $w_{|X} = d_X^{L^{-1}(3)}(s)$. It follows that $w \in W$, and

$$w(p) = d_X^{L^{-1}(3)}(s)(p) \neq 0.$$

Hence p is not a base-point of W. This completes the proof of Proposition 1. □

To get lower bounds on the codimension we will apply the following result of Ein and Lazarsfeld, [5, Prop. 3.1].

Theorem (Ein, Lazarsfeld) *Let H be a very ample line bundle and B, C be nef line bundles on a smooth complex projective n-fold Z. We set*

$$F_f = K_Z + fH + B \text{ and } G_e = K_Z + eH + C.$$

Let $V \subset H^0(Z, F_f)$ be a base-point free subspace of codimension c and consider the Koszul-type complex

$$\bigwedge^{p+1} V \otimes H^0(G_e) \to \bigwedge^{p} V \otimes H^0(F_f + G_e) \to \bigwedge^{p-1} V \otimes H^0(2F_f + G_e).$$

If $(Z, H, B) \neq (\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n})$, $f \geq n+1$ and $e \geq n+p+c$, then this complex is exact in the middle.

In order to apply this to our situation, we set $p = 0$, and, in case $L = K_Y + dH + A$ we choose $f = d, e = d-3$, $B = A + K_Y + 3H$ (note that B is nef since $K_Y + 3H$ is globally generated) and $C = A$. In the case L (-d)-regular we have $L = M(d)$ for a Castelnuovo-Mumford regular line bundle M and we choose $f = d+3, e = d-3+\alpha_Y - \beta_Y$, $B = M$ and $C = M + (\beta_Y - \alpha_Y)H - K_Y$, so that B is nef since M is globally generated and also C is nef by definition of α_Y and β_Y (see Definition 1). We then have that

$$F_f = K_Y \otimes L(3) \text{ and } G_e = L(-3)$$

and the theorem in this particular case says that:

Proposition 2 *Suppose that $d \geq 4$ and $Y \neq \mathbb{P}^3$. Let V be a base-point free linear system in $H^0(K_Y \otimes L(3))$ with the property that*

$$\mu(V \otimes H^0(L(-3))) \subset H^0(K_Y \otimes L^2)$$

is a strict inclusion. Then the codimension c of V satisfies the inequality

$$c \geq \begin{cases} d-5+\alpha_Y-\beta_Y & \text{if L is } (-\text{d})-\text{regular} \\ d-5 & \text{if } L = K_Y + dH + A \end{cases}.$$

In general, pulling together the results of sections 3 and 4, we have the following bound.

Proposition 3 *Suppose that $Y \neq \mathbb{P}^3$ and $H^1(\Omega^2_Y \otimes L) = 0$. Then the codimension of the image of*

$$\mu : T \otimes H^0(K_Y(3)) \to H^0(K_Y \otimes L(3))$$

is at least $d-5+\alpha_Y-\beta_Y$ if L is (-d)-regular or at least $d-5$ if $L = K_Y + dH + A$.

Proof. For simplicity, we set

$$\tilde{T} := W + \mu(T \otimes H^0(K_Y(3))) \subset H^0(K_Y \otimes L(3)).$$

Notice that the multiplication map

$$\tilde{\mu} : \tilde{T} \otimes H^0(L(-3)) \to H^0(K_Y \otimes L^2)$$

cannot be surjective, otherwise, as in the proof of Lemma 4, we get that

$$\pi_2 \circ \tilde{\mu}(\tilde{T} \otimes H^0(L(-3))) = H^{1,1}_{\mathrm{ev}}(X, \mathbb{C})$$

and, given the first property of W, the latter equality implies the contradiction

$$\pi_2 \circ \mu(T \otimes H^0(K_Y \otimes L))) = H^{1,1}_{\mathrm{ev}}(X, \mathbb{C}).$$

Now, by Proposition 2, we get that

$$\operatorname{codim} \mu(T \otimes H^0(K_Y(3))) \geq \begin{cases} d - 5 + \alpha_Y - \beta_Y & \text{if L is } (-\mathrm{d}) - \text{regular} \\ d - 5 & \text{if } L = K_Y + dH + A \end{cases}.$$

□

Therefore it will be enough to devise a mechanism for extracting codimension bounds for T from codimension bounds for $\mu(T \otimes H^0(K_Y(3)))$. This is the subject of the next section.
We end the section by studying the vanishing of $H^1(\Omega^2_Y \otimes L)$.

Remark 1 *If* $d \geq 3\beta_Y - 3\alpha_Y + 13$ *and* L *is (-d)-regular or if* $d \geq 2b_Y - 2a_Y + 13$ *and* $L = K_Y + dH + A$*, then* $H^1(\Omega^2_Y \otimes L) = 0$.
Proof. We just apply Griffiths' vanishing theorem [12] to the globally generated vector bundle $E = \Omega^2_Y(3)$. We write

$$\Omega^2_Y \otimes L = E(\det E + K_Y + B)$$

whence we just need to prove that $B = L - 12H - 3K_Y$ is ample. By definition of a_Y, b_Y, α_Y and β_Y we can write $-K_Y = (a - b)H + A'$, where A' is nef and $a = \alpha_Y, b = \beta_Y$ if L is (-d)-regular, while $a = a_Y, b = b_Y$ if $L = K_Y + dH + A$. Hence $B = (d - 12 - ub + ua)H + A''$, where A'' is nef and $u = 2$ if $L = K_Y + dH + A$, $u = 3$ if L is (-d)-regular. Therefore B is ample. □

Remark 2 Notice that if Y is a quadric hypersurface in $\mathbb{P}^4$, since $K_Y = -3H$, if $L = (d - 3)H$, we have that $H^1(\Omega^2_Y \otimes L) = 0$ for $d \geq 7$, whence

$$\operatorname{codim} T \geq d - 5.$$

5 Macaulay-Gotzmann for CM regular sheaves.

We start by reviewing the situation for $\mathbb{P}^n$, which we will then generalise to arbitrary varieties.

Definition of $c^{<d>}$ and $c_{<d>}$. Given integers $c \geq 1, d \geq 1$, there exists a unique sequence of integers $k_d, k_{d-1}, \ldots, k_f$ with $d \geq f \geq 1$ (f is uniquely determined by c and d) such that

1. $k_d > k_{d-1} > \ldots > k_f \geq f$,
2. $c = \sum_{i=d}^{f} \binom{k_i}{i}$.

Here and below we use the convention $\binom{m}{p} = 0$ if $m < p$. We define

$$c^{<d>} := \sum_{i=d}^{f} \binom{k_i + 1}{i + 1}, \; c_{<d>} := \sum_{i=d}^{f} \binom{k_i - 1}{i}.$$

When $c = 0$ we set $c^{<d>} = c_{<d>} = 0$.
We have the following result of Macaulay and Gotzmann, which can be found in [7], pages 64-65.

Theorem (Macaulay, Gotzmann) *Let $V \subset H^0(\mathcal{O}_{\mathbb{P}^n}(d))$ be a subspace of codimension c. Then the subspace*

$$\mu(V \otimes H^0(\mathcal{O}_{\mathbb{P}^n}(1))) \subset H^0(\mathcal{O}_{\mathbb{P}^n}(d+1))$$

is of codimension at most $c^{<d>}$.

Gotzmann proved the Macaulay-Gotzmann inequality using combinatorial algebraic techniques. Green gave a geometric proof in [9]. We will now generalise the argument given by Green in order to prove that the Macaulay-Gotzmann inequality is valid for arbitrary Castelnuovo-Mumford regular sheaves.

Theorem 3 *Let M be a Castelnuovo-Mumford regular coherent sheaf on a projective space $\mathbb{P}^N$. For $d \geq 1$ let $V \subset H^0(M(d))$ be a subspace of codimension c, and define $V^{d+1} \subset H^0(M(d+1))$ by $V^{d+1} = \mu(V \otimes H^0(\mathcal{O}_{\mathbb{P}^N}(1)))$. Then*

$$\operatorname{codim} V^{d+1} \leq c^{<d>}.$$

The Theorem will follow from the following proposition.

Proposition 4 *Suppose that V, M and d are as above. Let H be a generic hyperplane of $\mathbb{P}^N$ and denote by M_H the restriction of M to H. We further denote the restriction of V to $H^0(M_H(d))$ by V_H. Then*

$$\text{codim } V_H \leq c_{<d>}.$$

Proof. We shall proceed by a double induction on the dimension of the support of M and the number d. We assume now that $d \geq 2, \dim\text{Supp}(M) \geq 1$. The proof of the Proposition for $d = 1$ or for sheaves with zero-dimensional supports is to be found in subsections 5.0.1 and 5.0.2.

Let H and H' be two generic hyperplanes. We define the spaces V^H (respectively $V^{H'}$) in the following way. Let L_H (resp. $L_{H'}$) be a linear polynomial defining H (resp. H'). We define $V^H \subset H^0(M(d-1))$ by

$$v \in V^H \Leftrightarrow L_H \times v \in V.$$

(Similarly, $V^{H'}$ is defined by $v \in V^{H'} \Leftrightarrow L_{H'} \times v \in V$.) We now consider the following exact sequence

$$0 \to H^0(M(d-1)) \stackrel{\times L_H}{\to} H^0(M(d)) \stackrel{\text{res}}{\to} H^0(M_H(d)) \to 0.$$

Here, of course, we have right exactness of the sequence only because M is a Castelnuovo-Mumford regular sheaf. There is an induced exact sequence

$$0 \to V^H \to V \to V_H \to 0$$

whence we see that

$$\text{codim } V = \text{codim } V^H + \text{codim } V_H.$$

We now consider the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (V^{H'})^H & \longrightarrow & V^{H'} & \longrightarrow & (V^{H'})_H & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \times L_{H'}} & & \downarrow{\scriptstyle \times L_{H\cap H'}} & & \\
0 & \longrightarrow & V^H & \longrightarrow & V & \longrightarrow & V_H & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \text{res}} & & \downarrow{\scriptstyle \text{res}} & & \\
0 & \longrightarrow & (V_{H'})^{H\cap H'} & \longrightarrow & V_{H'} & \longrightarrow & (V_{H'})_{H\cap H'} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

In the above diagram, all the rows are exact (since M_H is Castelnuovo-Mumford regular on H), as is the middle column. It is not immediate that the right-hand column is exact, but we will be able to show that it is close enough to exact for our purposes.

More precisely,

$$(V_{H'})_{H\cap H'} = V_{|_{H\cap H'}} = (V_H)_{H\cap H'}$$

and hence the restriction map $V_H \to (V_{H'})_{H\cap H'}$ is a surjection. We have automatically that $(V^{H'})_H \subset (V_H)^{H\cap H'}$ and hence the composition of the maps $\times L_{H\cap H'}$ and res is zero. It follows that

$$\mathrm{codim}\ V_H \leq \mathrm{codim}\ (V_{H'})_{H\cap H'} + \mathrm{codim}\ (V^{H'})_H.$$

We denote by c' the codimension of V_H for generic H. Hence, since H' has been chosen generic, $\mathrm{codim}\ V_{H'} = c'$. We have that $\mathrm{codim}\ V^{H'} = c - c'$. We note that

1. $V^{H'} \subset H^0(M(d-1))$ and hence by the induction hypothesis

$$\mathrm{codim}\ (V^{H'})_H \leq (c - c')_{<d-1>}.$$

2. The dimension of the support of $M_{H'}$ is strictly less than the dimension of the support of M and hence by the induction hypothesis

$$\mathrm{codim}\ (V_{H'})_{H\cap H'} \leq c'_{<d>}.$$

It follows that

$$c' \leq c'_{<d>} + (c - c')_{<d-1>}.$$

Green shows in [9], pages 77-78, that this inequality implies that $c' \leq c_{<d>}$.

It remains only to prove the Proposition for zero-dimensional sheaves or for $d = 1$.

5.0.1 The case d=1.

For any $c \neq 0$ we have that $c_{<1>} = c - 1$. We suppose first that $V \neq H^0(M(1))$. If for generic H we have $\mathrm{codim}\ V_H > c_{<1>}$, then, for generic H, $V^H = H^0(M)$. In other words, for generic H

$$L_H \times H^0(M) \subset V.$$

It follows that

$$\mu(H^0(M), H^0(\mathcal{O}_{\mathbb{P}^N}(1))) \subset V.$$

Since M is Castelnuovo-Mumford regular, it follows that $V = H^0(M(1))$ which contradicts our supposition that $V \neq H^0(M(1))$.

But if $c = 0$ then $c_{<1>} = 0$ and Proposition 4 is immediate. This completes the proof of the Proposition in the case where $d = 1$.

5.0.2 The case where the dimension of the support of M is zero.

In this case, for generic H, $H^0(M_H(d)) = 0$, and hence $\mathrm{codim}\ V_H = 0$. This completes the proof of the Proposition in the case where the dimension of the support of M is zero.

This completes the proof of Proposition 4. □

We now show how Proposition 4 implies Theorem 3. We proceed by induction on the dimension of the support of M. We consider the following exact sequence, where H is once again a generic hyperplane in $\mathbb{P}^N$,

$$0 \to (V^{d+1})^H \to V^{d+1} \to (V^{d+1})_H \to 0$$

from which it follows that

$$\mathrm{codim}\ V^{d+1} = \mathrm{codim}\ (V^{d+1})^H + \mathrm{codim}\ (V^{d+1})_H.$$

We note that $V \subset (V^{d+1})^H$ and $(V_H)^{d+1} \subset (V^{d+1})_H$ from which it follows that

$$\mathrm{codim}\ V^{d+1} \leq c + (c_{<d>})^{<d>} \leq c^{<d>}.$$

This completes the proof of Theorem 3. □

6 Proof of the main theorems.

We will now show how all this ties together to give a proof of the main theorems. We henceforth set

$$a = \begin{cases} \alpha_Y & \text{if L is } (-\mathrm{d}) - \text{regular} \\ a_Y & \text{if } L = K_Y + dH + A \end{cases}, \quad b = \begin{cases} \beta_Y & \text{if L is } (-\mathrm{d}) - \text{regular} \\ b_Y & \text{if } L = K_Y + dH + A \end{cases}$$

where α_Y, β_Y, a_Y and b_Y are as in Definitions 1 and 2.
It is now that we will use the supposition that (Y, H) is not a linear $\mathbb{P}^2$-bundle, hence $K_Y(3)$ is very ample, or, alternatively, that $a \leq 3$ (the case of the quadric is done by Remark 2). The case $a = 4$ will be dealt with at the end of the article.
We start with the following lemma.

Lemma 7 *Suppose $d \geq 5$ and let $T \subset H^0(L)$ be of codimension $c \leq d-4$. Define*

$$T' := \mu(T \otimes H^0(\mathcal{O}_Y(3-a))) \subset H^0(L(3-a)).$$

Then

$$\operatorname{codim} T' \leq c$$

Proof. When L is (-d)-regular we can write $L = M(d)$, where M is a Castelnuovo-Mumford regular sheaf. Also when $L = K_Y + dH + A$, since $M := K_Y + 4H + A$ is Castelnuovo-Mumford regular, we can write $L = M(d-4)$, where M is a Castelnuovo-Mumford regular sheaf. Applying Theorem 3, $(3-a)$-times, we obtain the result. □

We denote now by n the integer $\lfloor \frac{d+3-a}{b} \rfloor - 4$. We will also denote the very ample line bundle $K_Y(a)$ by P, and the bundle $L(3-a)$ by L'. We have the following lemma.

Lemma 8 *The line bundle L' can be written in the form*

$$L' = M_P + nP$$

where M_P is a sheaf which is Castelnuovo-Mumford regular with respect to the projective embedding defined by P.

Proof. We know by definition of a and b that there is a nef line bundle N such that $bH = K_Y + aH + N$, from which it follows that

$$(d+3-a)H = (n+4)P + (n+4)N + rH$$

for some $r \geq 0$, hence

$$(d+3-a)H = (n+4)P + A'$$

where A' is a nef line bundle. Now

$$M_P := L' - nP = \begin{cases} 4P + A_1 & \text{if L is } (-\text{d}) - \text{regular} \\ K_Y + 4P + A_2 & \text{if } L = K_Y + dH + A \end{cases}$$

for some nef line bundles A_1, A_2. This clearly implies, by Kodaira vanishing, that M_P is Castelnuovo-Mumford regular with respect to P in the case $L = K_Y + dH + A$. But also in the other case, for each $1 \leq i \leq 3$, we can write

$$M_P - iP = K_Y + aH + (3-i)P + A_1$$

whence again we have Castelnuovo-Mumford regularity by Kodaira vanishing since now $a = \alpha_Y > 0$ by definition. □

We are now in a position to prove the following proposition.

Proposition 5 *Suppose $d \geq 5$ and let $T \subset H^0(L)$ be of codimension $c \leq d-4$. Define*

$$\overline{T} := \mu(T \otimes H^0(K_Y(3)) \subset H^0(K_Y \otimes L(3)).$$

Then

$$\operatorname{codim} \overline{T} \leq c^{<n>}.$$

Proof. With T' as in Lemma 7, we note that

$$\mu(T' \otimes H^0(K_Y(a)) \subset \overline{T}.$$

We know by Lemma 7 that $\operatorname{codim} T' \leq c$. We know further by Lemma 8 that $L' = M_P + nP$ and hence Theorem 3 applied to the map

$$\mu : T' \otimes H^0(P) \to H^0(K_Y \otimes L(3))$$

gives us that $\operatorname{codim} \mu(T' \otimes H^0(K_Y(a)) \leq c^{<n>}$. From this it follows that

$$\operatorname{codim} \overline{T} \leq c^{<n>}.$$

□

By Proposition 3 we know that

$$\operatorname{codim} \overline{T} \geq \begin{cases} d-5+\alpha_Y-\beta_Y & \text{if L is } (-\mathrm{d})-\text{regular} \\ d-5 & \text{if } L = K_Y + dH + A \end{cases}$$

and hence either $c \geq d-3$ or

$$c^{<n>} > \begin{cases} d-6+\alpha_Y-\beta_Y & \text{if L is } (-\mathrm{d})-\text{regular} \\ d-6 & \text{if } L = K_Y + dH + A \end{cases}.$$

The following elementary lemma will allow us to control the growth of $c^{<n>}$.

Lemma 9 *If there exists an integer $e \geq 0$ such that*

$$c < \sum_{i=0}^{e} (n+1-i)$$

then $c^{<n>} \leq c+e$.

Proof. The Lemma being obvious for $c=0$ we suppose $c \geq 1$ and $c = \sum_{i=n}^{f} \binom{k_i}{i}$. Observe that

$$\sum_{i=0}^{e} (n+1-i) \leq \frac{(n+1)(n+2)}{2}.$$

Now suppose $k_i = i$ for $f \leq i \leq f_1$ for some $f - 1 \leq f_1 \leq n$, $k_i = i + 1$ for $f_1 + 1 \leq i \leq f_2$ for some f_2 such that $f_1 \leq f_2 \leq n$ and $k_i \geq i + 2$ for $f_2 + 1 \leq i \leq n$ (the case $f - 1 = f_1$ simply means that no k_i is equal to i, and similarly for f_2). Then, if $f_2 < n$, we have

$$c \geq \binom{k_n}{n} \geq \binom{n+2}{2} = \frac{(n+1)(n+2)}{2}$$

contradicting the hypothesis. Therefore $f_2 = n$ and $c^{<n>} = c + n - f_1$ and it remains to show that $n - f_1 \leq e$. Since we can write $c = \sum_{i=0}^{n-f_1}(n+1-i) - f$ if $n - f_1 \geq e + 1$ we deduce the contradiction $c \geq \sum_{i=0}^{e}(n+1-i)$. □

In particular, it follows that

Lemma 10 *Suppose* $L = K_Y + dH + A$, $b_Y \geq 2$ *and* $d - 6 - b_Y < \sum_{i=0}^{b_Y}(n+1-i)$. *Then*

$$\operatorname{codim} T > d - 6 - b_Y.$$

If $b_Y = 1$, *then*

$$\operatorname{codim} T > d - 6.$$

Proof. By Lemma 9, if $b_Y \geq 2$, we have $d - 6 - b_Y < \sum_{i=0}^{b_Y}(n+1-i)$ and $c = \operatorname{codim} T \leq d - 6 - b_Y$ whence, by Proposition 5,

$$\operatorname{codim} \overline{T} \leq c^{<n>} \leq d - 6.$$

But this is impossible by Proposition 3. If $b_Y = 1$ and $c \leq d - 6$ we have $c \leq n$ hence

$$\operatorname{codim} \overline{T} \leq c^{<n>} = c \leq d - 6,$$

again impossible by Proposition 3. □

Similarly we have

Lemma 11 *Suppose* L *is (-d)-regular,* $\beta_Y \geq 2$ *and*

$$d - 6 + \alpha_Y - 2\beta_Y < \sum_{i=0}^{\beta_Y}(n+1-i).$$

Then

$$\operatorname{codim} T > d - 6 + \alpha_Y - 2\beta_Y.$$

If $\beta_Y = 1$, *then*

$$\operatorname{codim} T > d - 7 + \alpha_Y.$$

We now require only the following lemma.

Lemma 12 *If* $b_Y \geq 2$ *and* $d \geq \frac{b_Y(b_Y^2+7b_Y-6)}{2}$ *then*

$$d - 6 - b_Y < \sum_{i=0}^{b_Y}(n+1-i). \tag{2}$$

If $\beta_Y \geq 2$ *and* $d \geq \frac{\beta_Y^2(\beta_Y+5)}{2}$ *then*

$$d - 6 + \alpha_Y - 2\beta_Y < \sum_{i=0}^{\beta_Y}(n+1-i).$$

Proof. We note first that $n \geq \lfloor \frac{d}{b_Y} \rfloor - 4$ and it follows that $b_Y(n+1) > d - 4b_Y$. Hence we have that

$$\sum_{i=0}^{b_Y}(n+1-i) > d - 4b_Y + (n+1) - \frac{b_Y(b_Y+1)}{2}.$$

In particular, if

$$d - 6 - b_Y \leq d - 4b_Y + (n+1) - \frac{b_Y(b_Y+1)}{2}$$

then (2) is immediately satisfied. This inequality is equivalent to

$$-7 + 3b_Y \leq n - \frac{b_Y(b_Y+1)}{2}$$

and since $n \geq \lfloor \frac{d}{b_Y} \rfloor - 4$, (2) will be satisfied provided that

$$-7 + 3b_Y \leq \lfloor \frac{d}{b_Y} \rfloor - 4 - \frac{b_Y(b_Y+1)}{2}$$

which is equivalent to $-3 + 3b_Y + \frac{b_Y(b_Y+1)}{2} \leq \lfloor \frac{d}{b_Y} \rfloor$, which is equivalent to

$$\frac{b_Y(b_Y^2 + 7b_Y - 6)}{2} \leq d.$$

The second assertion of the Lemma is proved similarly. □

Completion of the proof of Theorems 1 and 2.
The results proved so far (together with Remark 2) give a proof of the Theorems under the hypothesis that (Y, H) is not a linear $\mathbb{P}^2$-bundle. In the latter case since

$K_Y(4)$ is very ample, repeating verbatim the whole proof replacing everywhere $K_Y(3)$ with $K_Y(4)$ and using $a_Y = \alpha_Y = 4$ we get the desired bound. $\square$

Proof of Corollary 1.
This is a straightforward generalisation of [2] given the following two facts :
1. a lower bound on the codimension on the components of the Noether-Lefschetz locus $\mathrm{NL}(\mathcal{O}_Y(d))$ that insures that they have codimension at least two (our hypothesis $d \geq 7 + e$);
2. the fact that, on a general surface X not in $\mathrm{NL}(\mathcal{O}_Y(d))$ we have that if a complete intersection of X with another surface in $|\mathcal{O}_Y(d)|$ is reducible then its irreducible components are also complete intersection of X with another surface in $|\mathcal{O}_Y(s)|$ for some s (this is needed in the proof of [2, Prop. 2.1] and is insured, in our case, by the hypothesis $\mathrm{Pic}(Y) \cong \mathbb{Z}$). $\square$

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