Chow groups of surfaces with $h^{2,0} = 1$.

Abstract 1 *We will investigate the geometry of rational equivalence classes of points on a surface S. We will show that if S is a general projective K3 surface then these equivalence classes are dense in the complex topology. We will also show that if S has the property that these equivalence classes are Zariski dense, then $h^{2,0}(S) \leq 1$.*

1 Introduction and statement of results

The connection between the Chow group $CH_0(S)$ of 0-cycles on a surface S and $h^{2,0}(S)$ has been an object of interest since Mumford's 1968 paper[5], in which he proved the following result.

Theorem 1.1 (Mumford) *If $CH_0(S)$ is representable, then $h^{2,0}(S) = 0$.*

Bloch [1] conjectured that the converse is also true.

Conjecture 1 (Bloch) *If S is a smooth projective surface and $h^{2,0}(S) = 0$ then $CH_0(S)$ is representable.*

Bloch, Kas and Liebermann proved the Bloch conjecture for surfaces not of general type in [2]. This conjecture has also been shown to hold for various surfaces of general type such that $h^{2,0}(S) = 0$— see, for example, [7].

Our aim is to show there is also a close connection between the condition $h^{2,0}(S) = 1$ and the geometry of 0-cycles on S. In particular, we will show the following result.

Theorem 1.2 *Let S be a general smooth projective K3 surface. Then for general $x \in S$, the set*

$$\{y \in S | y \equiv x\}$$

is dense in S (for the complex topology).

Here $\equiv$ denotes rational equivalence between points. We will also prove a partial converse to this result.

Theorem 1.3 *Let S be a smooth complex surface, such that for a generic point x of S the set*

$$\{y \in S | y \equiv x\}$$

is Zariski dense in S. Then $h^{2,0}(S) \leq 1$.

2 Proof of Theorem 1.2

The proof of this theorem relies on three fundamental facts:

1. If E is an elliptic curve and $x \in E$, then the set $\{y \in E | ny \equiv nx$ for some integer $n\}$ is dense in E,
2. There are many families of elliptic curves on a K3 surface,

3. By a theorem of Roitman's, [6], the Chow group of a K3 surface is torsion-free.

What we actually need to prove the theorem is two one-dimensional families of elliptic curves which intersect transversally. It is well-known that such things exist, but finding a precise reference is harder. Chen proved the following theorem in [3].

Theorem 2.1 (Chen) *For any integers $n \geq 3$ and $d > 0$, the linear system $|\mathcal{O}_S(d)|$ on a general K3 surface S in $\mathbb{P}^n$ contains an irreducible nodal rational curve.*

From this we can deduce— as Griffiths and Green did in [4]— the following result.

Proposition 2.1 *The linear system $|\mathcal{O}_S(d)|$ contains a 1-dimensional family of curves of geometric genus ≤ 1 whose general element is irreducible and nodal.*

The space of nodal curves of geometric genus ≤ 1 is of codimension $g-1$ in the moduli space M_g of stable curves of genus g. There is a map from an open subset (corresponding to nodal curves) of $|\mathcal{O}_S(d)|$ towards M_g. The image of this space meets the subvariety corresponding to nodal curves of genus ≤ 1, since it contains a nodal irreducible rational curve.

The space of nodal curves of genus ≤ 1 in $|\mathcal{O}_S(d)|$ therefore has a component of dimension $\leq \dim |\mathcal{O}_S(d)| + 1 - g$, whose general element is irreducible. It is enough to show that if C is a generic (smooth, genus g) curve in $|\mathcal{O}_S(d)|$, then $h^0(\mathcal{O}_S(C)) = g+1$ and hence $\dim |\mathcal{O}_S(d)| = g$.
By the Kodaira vanishing theorem, we have

$$h^1(\mathcal{O}_S(C)) = h^2(\mathcal{O}_S(C)) = 0.$$

It will hence be enough to show that $\chi(\mathcal{O}_S(C)) = g+1$. By Riemann Roch and the adjunction formula,

$$\chi(\mathcal{O}_S(C)) = \chi(\mathcal{O}_S) + \frac{1}{2}C^2 = g+1.$$

The proposition follows. We now choose two distinct irreducible 1-dimensional families,

$$\pi_1 : F_1 \to B_1, \;\; \pi_2 : F_2 \to B_2$$

which are in the linear systems $|\mathcal{O}_S(1)|$ and $|\mathcal{O}_S(2)|$ respectively and whose general elements are integral nodal curves of geometric genus ≤ 1.There are surjective maps $\phi_i : F_i \to S$.

We consider those $x \in S$ such that x is not contained in the image under ϕ_2 of any non-integral fibre of π_2. This is the only condition needed to prove the theorem for x.

Choose $y \in F_1$ such that $\phi_1(y) = x$ and denote $\pi_1(y)$ by z. Denote the curve $\pi_1^{-1}(z)$ by D. There is a surjective map from a nodal curve of genus ≤ 1 to D, $r : \overline{D} \to D$.

Every component of $\overline{D}$ is of geometric genus ≤ 1. There is a component of $\overline{D}$ which intersects $(\phi_1 \circ r)^{-1}(x)$ and whose image under $\phi_1 \circ r$ is not a single point. Denote the image of this component by E. Since E has a normalisation of genus ≤ 1, the set

$$\{z \in E \text{ such that } z - x \text{ is torsion in } CH^0(E)\}$$

is dense in E. By a result of Roitman's, [6], the torsion part of $CH^0(S)$ is 0. Hence, the set

$$\{z \in E \text{ such that } z \equiv x \text{ in } CH^0(S)\}$$

is dense in E.

Our strategy is as follows. The curve E is transverse to general elements of the family F_2. Consider the curves in the family F_2 which are elliptic or rational and meet E in a point rationally equivalent to x. The set of such curves is dense in F_2. If E_2 is such a curve then the set {points of E_2 rationally equivalent to x} is dense in E_2.

More precisely, consider the variety $V = \phi_2^{-1}(E)$ which parameterises points of intersection of E with a curve in the family F_2. The projection of V onto B_2 is surjective. Let S_E be the set

$$\{y \in E \text{ such that } y \equiv x \text{ in } CH^0(S)\}.$$

The set S_E is dense in E for the complex topology. We denote by T the closure of

$$\{y \in S \text{ such that } y \equiv x \text{ in } CH^0(S)\}.$$

We define $\tilde{B_2}$ to be the open set in B_2 parameterising irreducible members of the family F_2. Consider

$$Z = \pi_2 \circ \phi_2^{-1}(S_E),$$

the set parameterising curves in the family F_2 which meet E in at least one point of $S(E)$. We denote by $\tilde{Z}$ the set $Z \cap \tilde{B_2}$. Once again, if $z \in \tilde{Z}$, then the set

$$\{y \in F_{2,z} \text{ such that } y \equiv x \text{ in } CH^0(S)\}$$

is dense in $F_{2,z}$, the fibre over z in F_2. Hence, T contains $\pi_2^{-1}(\tilde{Z})$. We now need the following lemma.

Lemma 1 *The set Z is dense in B_2.*

There is a component C of V mapping surjectively to E. Since x is not contained in any non-integral fibre of π_2, and E is not an element of $|\mathcal{O}_S(2)|$ for degree reasons, $\pi_2 : C \to B_2$ is surjective. Since C is irreducible and $\phi_2|_C$ is surjective onto E $\phi_2|_C^{-1}(S(E))$ is dense in C. It follows that, since $\pi_2|_C$ is surjective and continuous, Z is dense in B_2.

It immediately follows that T is dense in S. This completes the proof of Theorem 30. □

3 Proof of Theorem 1.3

Now suppose that S satisfies the hypothesis that for general $x \in S$ the set

$$\{y \in S | x \equiv y \in CH^0(S)\}$$

is Zariski dense in S. We want to show that $h^{2,0}(S) \leq 1$. Mumford proved the following result in [5].

Theorem 3.1 (Mumford) *There exists a countable union of maps of reduced algebraic schemes*

$$\phi_i : W_i \to S \times S$$

such that the following hold.

1. *$x \equiv y$ if and only if there exists i such that $(x, y) \in \phi_i(W_i)$,*
2. *Let pr^1 and pr^2 be the two projections from $S \times S$ onto S. Consider the maps*

   $$\pi_i^1 \text{ and } \pi_i^2 : W_i \to S$$

   given by $\pi_i^j = pr^j \circ \phi_i$. We then have for any 2 form on S, ω,

   $$\pi_i^{1*}(\omega) = \pi_i^{2*}(\omega).$$

We may restrict ourselves to the case where the images of all the maps ϕ_i are of dimension ≤ 2, since Mumford proved in [5] that

Proposition 3.1 (Mumford) *If there is an i such that the image of ϕ_i is of dimension ≥ 3 then $h^{2,0}(S) = 0$.*

We now choose y such that

1. $y \notin \pi_i^j(W_i)$ for any i such that $\dim(\text{Im } \phi_i) \leq 1$,
2. There do not exist x, i, j such that $(x, y) \in \text{Im}(\phi_{\text{i}})$ and π_i^j is not submersive at any point of $\phi_i^{-1}(x, y)$,
3. The set $\{x \in S | y \equiv x\}$ is Zariski dense in S.

Since the varieties described in 1 and 2 are of dimension ≤ 1 and, by assumption, 3 holds for general y, there exists such a y. The theorem follows from the following proposition.

Proposition 3.2 *There is no non-zero 2-form ω on S vanishing at y.*

Let ω be such a 2-form, and consider $x \in S$ such that $y \equiv x$. By the assumptions on y it follows that ω vanishes at x. Indeed, there is some W_i such that $(x, y) \in \phi_i(W_i)$. By assumption 2, there exists $p \in W_i$ such that $\phi_i(p) = (x, y)$ and π_i^1, π_i^2 are both submersive at p. We know that

$$\pi_i^{2*}(\omega)(p) = 0$$

since $\omega(y) = 0$. It follows that $\pi_i^{1*}(\omega)(p) = 0$. But by assumptions 1 and 2, this implies that $\omega(x) = 0$. Therefore, since the set of such points is Zariski dense, ω is identically 0. It follows immediately that

$$h^{2,0}(S) \leq 1.$$

This completes the proof of the theorem. I would like to express my gratitude to my thesis supervisor, Claire Voisin, for proposing this problem and for all her help.

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