An elliptic regularity result for a composite medium with "touching" fibers of circular cross-section.

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Abstract

In this paper we consider the elliptic equation $\nabla \cdot a \nabla u = 0$ in a two dimensional domain Ω, which contains a finite number of circular inhomogeneities (cross-sections of fibers). The coefficient, a, takes two constant values, one in all the inhomogeneities and one in the part of Ω which lies outside the inhomogeneities. A number of the inhomogeneities may possibly touch, and in spite of this we prove that any variational solution u (with sufficiently smooth boundary data) is in $W^{1,\infty}$. For this very interesting, particular type of coefficient, our result improves a classical regularity result due to DeGiorgi and Nash, which asserts that the solution is in the Hölder class C^γ for some positive exponent γ.

1 Introduction

Consider a domain $\Omega \subset \mathbb{R}^2$, representing the cross-section of a three dimensional body. We suppose the three dimensional body is occupied by a fiber-reinforced composite and we suppose the cross-section is taken perpendicular to the finitely many (identical) cylindrical fibers. Frequently in composites, the fibers are very closely spaced and may even touch. We suppose the strength of the fibers is different from that of the material between the fibers (the so-called matrix). When we talk about strength this could for instance refer to the shear modulus (for a problem of antiplane shear) or the conductivity (for a problem of heat or electric conduction). In all three cases the corresponding scalar variable, u, (the out of plane displacement, the temperature or the voltage potential) satisfies the elliptic equation

$$\nabla \cdot a \nabla u = 0 \quad \text{in } \Omega \quad , \tag{1}$$

with, for instance a given Dirichlet boundary condition

$$u = \phi \quad \text{on } \partial\Omega \quad . \tag{2}$$

The coefficient $0 < a < \infty$ takes two constant values, one in the fibers and one in the matrix. The aim of this paper is to study the behaviour of u and, in particular, its gradient near points where the fibers touch. Because the cross-section is perpendicular to the fibers, these appear as disks of identical radii. We may without loss of generality restrict attention to a situation

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with only two touching disks (fibers); for simplicity we take these to lie strictly inside Ω. We may also rescale the strength, a, so that

$$\begin{aligned} a(x) &= 1 \quad \text{for } x \text{ outside the two disks} \\ a(x) &= a_0 \quad \text{for } x \text{ inside the two disks} \quad . \end{aligned} \tag{3}$$

In the context of anti-plane shear it is probably physically most relevant to think of a_0 as being larger than 1 – after all, the fibers are there for reinforcement. However, in the context of heat- or electrical conduction there are no physical reasons why we might not have $a_0 < 1$ as well. We may, without loss of generality, suppose that the point where the two disks touch is located at the origin. We may also suppose that the domain Ω is of class C^∞ and symmetric in the x_1-axis, and that the boundary data ϕ is in $C^\infty(\partial\Omega)$. If not, we can simply take such a smooth domain inside Ω, containing the two fiber cross-sections, and rely on elliptic regularity to get that the (new) Dirichlet data is C^∞. The geometric situation is illustrated in Figure 1. For simplicity of illustration, we have drawn $\partial\Omega$ in the shape of a circle – a convention we shall follow throughout this paper. The solution u is defined variationally, by the requirements that u be in $H^1(\Omega)$ with $u|_{\partial\Omega} = \phi$ and

$$\int_\Omega a(x)\nabla u \nabla v \, dx = 0 \quad \forall v \in H_0^1(\Omega) \quad .$$

In two dimensions (as we are here) Sobolev's imbedding theorem states that elements of $H^s(\Omega)$, $1 < s$, are automatically continuous. But this is not true for all elements of $H^1(\Omega)$. However, a regularity result of DeGiorgi and Nash (cf. [3], [9] or [4]) asserts that any H^1 solution to a divergence form, scalar elliptic equation with bounded measurable coefficients such as (1), is indeed Hölder continuous inside Ω. Near $\partial\Omega$, the coefficent a is constant, and the boundary, as well as the boundary data ϕ are C^∞, so standard elliptic boundary regularity results immediately imply that u is C^∞ there. Indeed, away from the origin (where the two disks touch) standard elliptic regularity results (for operators with constant, or piecewise constant coefficients) very easily imply that the gradient of u is bounded. The origin, however, presents a serious problem. Neither standard elliptic regularity results, nor the DeGiorgi-Nash result assert anything about the boundedness of the gradient. Such boundedness is guaranteed by the main result of this paper:

Theorem The solution u is in $W^{1,\infty}(\Omega)$ for any fixed $0 < a_0 < \infty$.

Since the gradient of the solution u is generically discontinuous at the interfaces between the fibers and the matrix, this theorem is optimal in terms of global regularity.

We have assumed that the circular fiber cross-sections have same radius. It is worthwhile to point out that this assumption is for convenience only. A configuration with two touching disks of different radii (say, r_1 and r_2) may quite easily be mapped conformally to a configuration consisting of two identical touching disks: pick the x_1-axis to be the common tangent for the two disks (they touch at the origin) and let $z = \Phi(x)$ denote the conformal mapping $\Phi(x_1, x_2) = (x_1/(x_1^2+x_2^2), x_2/(x_1^2+x_2^2))$. Let T_c denote the vertical translation $T(z_1, z_2) = (z_1, z_2+c)$. With an appropriate choice of c $(= \pm(r_2 - r_1)/4r_1r_2)$ the conformal mapping $\Psi = \Phi^{-1} \circ T_c \circ \Phi$ maps the configuration with the two different disks to a configuration with two identical touching disks (of radius $2r_1r_2/(r_1+r_2)$). This mapping is furthermore smooth at the origin. The validity

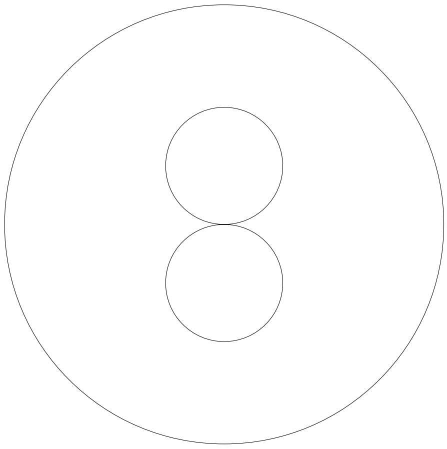

Figure 1: Two touching fibers.

of the above theorem for two identical disks now immediately implies its validity for different disks as well: the "push-forward" $w = u \circ \Psi^{-1}$ of the function u is in $W^{1,\infty}$ near the origin (since it pertains to a configuration of two identical disks) and due to the regularity of Ψ the same can therefore be said about u. This "push-forward" technique works for any configuration which is conformally equivalent to two touching disks. It should be extremely interesting to study the regularity issue for configurations that are not conformally equivalent to two touching disks.

In the context of antiplane shear, $a\nabla u$ represents the stresses (internal forces). Most linear fracture models suppose that fracturing will occur at points with extreme stress concentrations. The fact that the (shear) stresses remain bounded, even near points where the fibers touch, strongly indicates that separation between circular fibers and the matrix is not a likely mechanism for the onset of fracture. The result proven in this paper only applies to a scalar equation, it would be of utmost interest to extend this to the full system of linear elasticity.

To indicate that the behaviour of u near the origin is not entirely obvious (and not always the same) let us change to the (conformally) different situation shown in Figure 2 :two conical shapes symmetrically touching, with $a = a_0$ inside the two shapes, $a = 1$ outside. For a fixed a_0, the solution is in C^γ with γ uniformly bounded away from zero (independently of the size of the interior angle of the conical shapes). This is consistent with the result of DeGiorgi and Nash, which asserts that u is of a minimal Hölder class that only depends on the aspect ratio of the coefficient. The Hölder exponent γ is smallest when the conical shapes have interior angles $\alpha = \pi/2$. For $a_0 \neq 1$ Figure 3 shows the "generic γ" as a function of a_0 for the two possible symmetries of the solution u: one where u is odd with respect to the x_1 axis, and one where u is even with respect to the x_1 axis. Corresponding to this "generic γ", there is locally near the origin an H^1 solution to (1), which in polar coordinates has the form

$$r^\gamma(a\cos\gamma\theta + b\sin\gamma\theta) \quad ,$$

with a different pair of constants (a, b) in each of the four sectors. This γ is the smallest exponent for which a solution of this form exists in H^1. γ is very easily characterized as the smallest positive solution to a certain determinant identity (expressing that the system of linear equations for the coefficient pairs (a, b) has a nontrivial solution). An analysis very

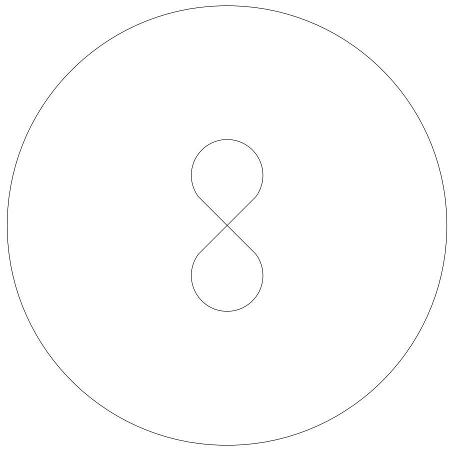

Figure 2: Two touching conical shapes.

similar to that found in [5] would show that near the origin any solution has, generically, the same behaviour as this special solution. It is clear from the graph, that a general solution u, which contains elements of both symmetries, is never in $W^{1,\infty}$ (except when $a_0 = 1$) indeed, depending on a_0, its regularity may not be better than C^γ for any arbitrarily small positive γ.

The fact that u is not in $W^{1,\infty}(\Omega)$ in this situation can be explained in terms of a "corner effect". Consider the geometric situation where the two convex shapes are $\epsilon > 0$ apart vertically. The solution will then have two singularities which arise due to the corners in the interfaces. Each corner has an angle of $\pi/2$. Figure 4 shows the generic Hölder coefficient for any solution corresponding to an interface with angle $\pi/2$ (and conductivity ratio $1 : a_0$). Figure 4 corresponds to the lower half of Figure 3; if we had considered solutions with a special symmetry, the curve on the right side (of $a_0 = 1$) would continue smoothly into the left quadrant, and vice versa. It is clear that, for $\epsilon > 0$, a general solution is never in $W^{1,\infty}(\Omega)$ (except when $a_0 = 1$), and it is quite natural to expect that this does not in any way improve in the limit $\epsilon \to 0$, when the two shapes touch.

But the "corner effect" is not the whole story in Figure 3: the regularity situation has clearly become significantly worse when compared to that in Figure 4. For fixed $\epsilon > 0$, Figure 4 clearly indicates that any solution will at least be in $C^{2/3}(\Omega)$, independently of a_0. However, when the shapes touch (for $\epsilon = 0$) Figure 3 clearly indicates that the regularity may not be better than C^γ for any arbitrarily small positive γ, depending on the size of a_0. The touching may thus induce a loss of almost a factor of 2/3 in terms of differentiability.

We now return to the case of (identical) circular fibers. The extreme situations that formally correspond to $a_0 = 0$ and $a_0 = \infty$ are somewhat particular. The corresponding solutions are now (essentially) only defined in $\Omega \setminus \{$ the fibers $\}$. If the boundary point, where the two fibers touch (the origin) is thought of as two different boundary points of $\Omega \setminus \{$ the fibers $\}$, then these solutions are always C^∞ in the interior and up to the boundary of $\Omega \setminus \{$ the fibers $\}$. With this convention we also have that all the derivatives of order ≥ 1 vanish at the origin. In the case $a_0 = 0$, and for a certain symmetry of the data, we generically have that the solution u^0 is multivalued at the origin, i.e., it has a different limit when approaching the origin from the cusp on the left than when approaching the origin from the cusp on the right. For $a_0 = \infty$, the

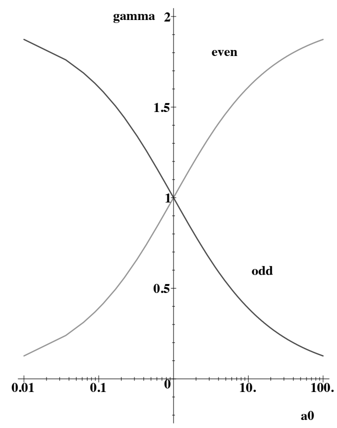

Figure 3: The generic γ as a function of a_0 for symmetrically touching conical shapes of interior angle $\pi/2$

solution u^∞ is always single valued at the origin. We refer the reader to the appendix, where these issues are discussed in more detail.

An interesting project would be to consider the case where the fibers are very close, but not touching. Say, the circular cross-sections are ϵ apart vertically. For the case of $a_0 = 0$ (as well as $a_0 = \infty$) the discontinuity mentioned above gives the existence of solutions, the gradients of which become unbounded at the origin as the distance ϵ approaches zero. This phenomenon has been noted and studied in detail by several authors (cf. [2], [8] and [6]). We again refer to the appendix, where some of this work is discussed in a little more detail. For fixed $0 < a_0 < \infty$, we conjecture that the gradient near the origin stays bounded independently of ϵ. We base this conjecture, among other things, on some very accurate calculations communicated to us by Börje Andersson of the Aeronautical Research Institute of Sweden [1]. For special boundary conditions, such as those corresponding to the solutions considered in [2] and [8], it is not unlikely that the mapping technique used there would make it possible to establish this uniform boundedness. However, for general boundary conditions we have at the moment no proof of this conjecture.

2 A reduced problem

Since Ω is symmetric in the x_1-axis, the boundary condition $u|_{\partial\Omega} = \phi(x)$ may also be written: $u|_{\partial\Omega} = \frac{1}{2}(\phi(x) - \phi(\overline{x})) + \frac{1}{2}(\phi(x) + \phi(\overline{x}))$, where $\overline{x} = (x_1, -x_2)$ denotes the reflection of the point $x = (x_1, x_2)$. As a consequence, we may separate our boundary value problem into one of two cases: 1) the solution u is odd in the x_1-axis, 2) the solution u is even in the x_1-axis.

In case the boundary data (and thus the solution, u) is even in the x_1-axis, consider the

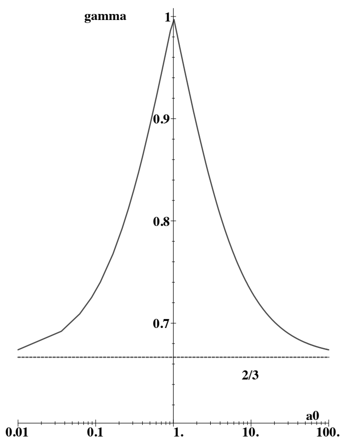

Figure 4: The generic γ as a function of a_0 for a single conical shape of interior angle $\pi/2$

a-harmonic conjugate to u. This function, v, is related to u by

$$a\nabla u = \nabla v^\perp = \begin{pmatrix} -\frac{\partial v}{\partial x_2} \\ \frac{\partial v}{\partial x_1} \end{pmatrix} \ , \tag{4}$$

and it solves

$$\nabla \cdot a^{-1}(x)\nabla v = 0 \quad \text{in } \Omega \ .$$

From (4) it follows immediately that on $\partial\Omega$

$$\begin{aligned} a^{-1}\nabla v \cdot n &= -a^{-1}\nabla v \cdot \tau^\perp \\ &= a^{-1}\nabla v^\perp \cdot \tau \\ &= \frac{\partial u}{\partial \tau} \ , \end{aligned}$$

where τ denotes the counter-clockwise tangent. Since u is even it now follows that v (normalized by $\int_\Omega v \ dx = 0$) is odd in the x_1-axis.

In the rest of this paper we shall concentrate on the case where u is odd in the x_1-axis, and prove that the gradient of u is bounded. The exact same argument that we present could of course be applied to the (odd) function v (the only difference being that a_0, the conductivity inside the two fibers, gets replaced by a_0^{-1}) thus proving that the gradient of v stays bounded in Ω. Because of the relationship (4) this immediately implies that ∇u is also bounded in Ω in the case where u is even in the x_1-axis. By means of the splitting introduced at the beginning of this section, this now verifies the boundedness of ∇u in the general case (without any symmetry assumptions).

Under the assumption that u is odd in the x_1-axis it indeed suffices to consider the boundary value problem in the half-domain, $\Omega_+ = \{ (x_1, x_2) \in \Omega \ : \ 0 < x_2 \}$, with the additional boundary condition $u = 0$ at $x_2 = 0$ (cf. Figure 5). For simplicity we shall from now on assume that the fiber has radius 1, so that its boundary is given by the equation $x_1^2 + (x_2^2 - 1)^2 = 1$.

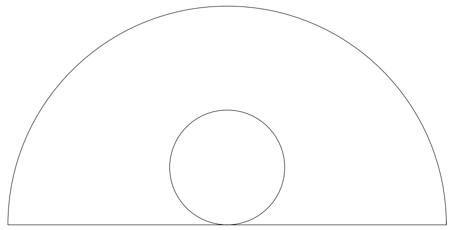

Figure 5: The reduced geometric setup.

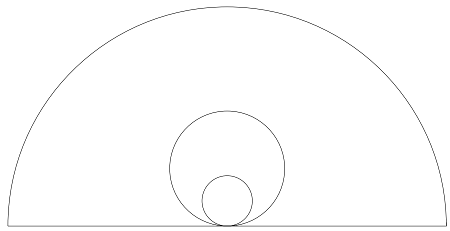

Figure 6: The geometric situation with the "auxiliary" boundary.

3 An auxiliary boundary value problem

Let D_ϵ denote the disk $D_\epsilon = \{(x_1, x_2) \ : \ x_1^2 + (x_2 - \epsilon)^2 = \epsilon^2\}$, centered at $(0, \epsilon)$, with radius ϵ. For small, positive ϵ we introduce an auxiliary function u_ϵ as the solution to the boundary value problem

$$\begin{aligned} \nabla \cdot a(x) \nabla u_\epsilon &= 0 \quad \text{in } \Omega_+ \ , \\ u_\epsilon &= \phi \quad \text{on } \partial\Omega_+ \setminus \{x_2 = 0\} \ , \\ u_\epsilon &= 0 \text{ on } \{x_2 = 0\} \text{ and on } \partial D_\epsilon \ . \end{aligned}$$

For a geometric illustration, see Figure 6.

The conductivity, a, is as given before. The solution, u, whose gradient we are trying to bound, solves the corresponding "limiting" boundary value problem

$$\begin{aligned} \nabla \cdot a(x) \nabla u &= 0 \quad \text{in } \Omega_+ \ , \\ u &= \phi \quad \text{on } \partial\Omega_+ \setminus \{x_2 = 0\} \ , \\ u &= 0 \text{ on } \{x_2 = 0\} \ . \end{aligned}$$

A fairly direct argument shows that

Proposition 3.1 *Let u_ϵ and u be as defined above, with u_ϵ extended to all of Ω_+ by setting it to zero on D_ϵ. Then $u_\epsilon \to u$ in $H^1(\Omega_+)$ as $\epsilon \to 0$. Let K denote a compact subset of $\overline{\Omega_+} \setminus \{ \ x_1^2 + (x_2 - 1)^2 = 1 \ \}$. Then $u_\epsilon \to u$ in $C^\infty(K)$ as $\epsilon \to 0$.*

Proof Let $0 \leq \psi \leq 1$ be in $C^\infty(\mathbb{R}^2)$ with

$$\psi(y) = 0 \quad \text{for } |y| \leq 1 \ , \quad \text{and } \psi(y) = 1 \quad \text{for } |y| \geq 2 \ ,$$

and define the function v_ϵ by

$$v_\epsilon(x) = u(x)\psi(\frac{x_1}{\epsilon}, \frac{x_2-\epsilon}{\epsilon}) \quad .$$

The function $v_\epsilon \in H^1(\Omega_+)$ satisfies

$$v_\epsilon = \phi \quad \text{on } \partial\Omega_+ \setminus \{ \ x_2 = 0 \ \} \quad ,$$
$$v_\epsilon = 0 \quad \text{on } \{ \ x_2 = 0 \ \} \text{ and on } \partial D_\epsilon \quad .$$

Thus, due to Dirichlet's principle

$$\begin{aligned}
\left[\int_{\Omega_+\setminus D_\epsilon} a|\nabla u_\epsilon|^2 \, dx\right]^{1/2} &\leq \left[\int_{\Omega_+\setminus D_\epsilon} a|\nabla v_\epsilon|^2 \, dx\right]^{1/2} \\
&\leq \left[\int_{\Omega_+\setminus D_\epsilon} a|\nabla u|^2 \, dx\right]^{1/2} + \frac{1}{\epsilon}\left[\int_{\Omega_+\setminus D_\epsilon} a|u|^2|\nabla\psi(\frac{x_1}{\epsilon}, \frac{x_2-\epsilon}{\epsilon})|^2 \, dx\right]^{1/2} \\
&= \left[\int_{\Omega_+\setminus D_\epsilon} a|\nabla u|^2 \, dx\right]^{1/2} + o(1) \quad .
\end{aligned}$$

For the last estimate we have used the fact that $\nabla\psi(y)$ vanishes for $|y| \geq 2$, and the fact that u is continuous at 0 with $u(0) = 0$ (due to the result of DeGiorgi and Nash) to conclude that

$$\left[\int_{\Omega_+\setminus D_\epsilon} a|u|^2|\nabla\psi(\frac{x_1}{\epsilon}, \frac{x_2-\epsilon}{\epsilon})|^2 \, dx\right]^{1/2} \leq C\epsilon \max_{\Omega_+\cap\{x_1^2+(x_2-\epsilon)^2\leq 4\epsilon^2\}} |u| = o(\epsilon) \quad .$$

Since u_ϵ has been extended to be zero on D_ϵ, and since u is in $H^1(\Omega_+)$, it follows that

$$\int_{\Omega_+} a|\nabla u_\epsilon|^2 \, dx \leq \int_{\Omega_+} a|\nabla u|^2 \, dx + o(1) \quad . \tag{5}$$

On the other hand, Dirichlet's principle also gives

$$\int_{\Omega_+} a|\nabla u|^2 \, dx \leq \int_{\Omega_+} a|\nabla u_\epsilon|^2 \, dx \quad . \tag{6}$$

A combination of (5) and (6) yields

$$\int_{\Omega_+} a|\nabla u_\epsilon|^2 \, dx = \int_{\Omega_+} a|\nabla u|^2 \, dx + o(1) \quad . \tag{7}$$

It is easy to see that

$$\int_{\Omega_+} a|\nabla(u_\epsilon - u)|^2 \, dx = \int_{\Omega_+} a|\nabla u_\epsilon|^2 \, dx - \int_{\Omega_+} a|\nabla u|^2 \, dx \quad ,$$

and therefore by insertion of (7)

$$\int_{\Omega_+} a|\nabla(u_\epsilon - u)|^2 \, dx \to 0 \quad \text{as } \epsilon \to 0 \quad .$$

Since $u_\epsilon = u = 0$ on $\{ \ x_2 = 0 \ \}$, it follows immediately that $u_\epsilon \to u$ in $H^1(\Omega_+)$.

The statement about C^∞ convergence follows from elliptic regularity theory and from the fact that ϕ comes from an odd C^∞ function on all of $\partial\Omega$. We have to exclude the curve $\{ \ x_1^2 + (x_2-1)^2 = 1 \ \}$, since the coefficient a is discontinuous across it.

□

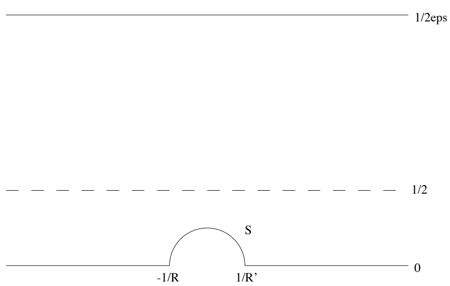

Figure 7: The geometric situation in the z-coordinates.

4 Preliminary estimates

It turns out to be somewhat simpler to work with the variables $(z_1, z_2) = \Phi(x_1, x_2)$, given by the conformal transformation

$$\Phi(x_1, x_2) = \left(\frac{x_1}{x_1^2 + x_2^2}, \frac{x_2}{x_1^2 + x_2^2} \right) \quad .$$

The geometric situation is now as illustrated in Figure 7. The "outer" boundary $\partial\Omega \cap \{x_2 > 0\}$ maps to the "inner" boundary S. The circle $\{x_1^2 + (x_2 - 1)^2 = 1\}$ (the fiber) and the circle $\{x_1^2 + (x_2 - \epsilon)^2 = \epsilon^2\}$ (the additional boundary for u_ϵ) map to the horizontal straight lines $z_2 = 1/2$ and $z_2 = 1/2\epsilon$ respectively. The inside of these circles map to the halfplanes $z_2 > 1/2$ and $z_2 > 1/2\epsilon$ respectively. The lower boundary $\{x_2 = 0 \ , \ -R < x_1 < R'\}$ maps to the straight part of the lower boundary, $\{z_2 = 0 \ , \ z_1 < -1/R \text{ or } 1/R' < z_1\}$. Since Φ is a conformal mapping, it follows immediately that

$$\int_{\Omega_+} a(x) |\nabla_x v|^2 \, dx = \int_{\Phi(\Omega_+)} A(z) |\nabla_z V|^2 \, dz \quad , \forall v \in H^1(\Omega_+) \quad . \tag{8}$$

Here A and V are related to a and v by

$$V(z) = v \circ \Phi^{-1}(z) \ , \quad A(z) = a \circ \Phi^{-1}(z) \ .$$

The transformed solutions $U(z) = u \circ \Phi^{-1}(z)$ and $U_\epsilon(z) = u_\epsilon \circ \Phi^{-1}(z)$ satisfy the differential equations

$$\nabla \cdot A(z) \nabla U = 0 \ \text{ for } z \in \Phi(\Omega_+)$$

and

$$\nabla \cdot A(z) \nabla U_\epsilon = 0 \ \text{ for } z \in \Phi(\Omega_+) \ , \quad z_2 < 1/2\epsilon \quad .$$

The transformed conductivity $A(z) = A(z_2)$ has the form

$$A(z) = 1 \text{ for } z_2 < 1/2, \quad A(z) = a_0 \text{ for } z_2 > 1/2 \quad .$$

On the "inner" boundary S the functions U and U_ϵ satisfy the boundary conditions

$$U(z) = U_\epsilon(z) = \phi \circ \Phi^{-1}(z) \quad .$$

Furthermore, U satisfies

$$U = 0 \text{ on the lower straight boundary } \{z_2 = 0, \quad z_1 < -1/R \text{ or } 1/R' < z_1\} \ ,$$

and U_ϵ satisfies

$$U_\epsilon = 0 \text{ on the lower straight boundary } \{z_2 = 0, \ z_1 < -1/R \text{ or } 1/R' < z_1\} \ , \text{ and on } \{z_2 = 1/2\epsilon\} \, .$$

From the energy identity (8) and the fact that $u, u_\epsilon \in H^1(\Omega_+)$, with $u_\epsilon \to u$, it follows immediately that

$$\int_{\Phi(\Omega_+)} |\nabla U|^2 \, dz < \infty \quad \text{and} \quad \int_{\Phi(\Omega_+)\cap\{z_2<1/2\epsilon\}} |\nabla U_\epsilon|^2 \, dz < C \ , \text{ independently of } \epsilon \ . \tag{9}$$

The function $U(z)$ tends to zero as $|z| \to \infty$ (this follows from the fact that u is continuous and has the value 0 at the origin). A simple calculation, using separation of variables, shows that the auxiliary function, U_ϵ, has the expansion

$$U_\epsilon(z) = \sum_{n=1}^{\infty} \beta_{n,\epsilon}\phi_{n,\epsilon}(z_2) e^{-\sqrt{\lambda_{n,\epsilon}} \ z_1} \quad \text{for } z_1 \text{ sufficiently positive} \ , \tag{10}$$

$$U_\epsilon(z) = \sum_{n=1}^{\infty} \beta'_{n,\epsilon}\phi_{n,\epsilon}(z_2) e^{\sqrt{\lambda_{n,\epsilon}} \ z_1} \quad \text{for } z_1 \text{ sufficiently negative} \ , \tag{11}$$

where $\lambda_{n,\epsilon} > 0$ and $\phi_{n,\epsilon}(\cdot)$ are the eigenvalues and the eigenvectors of the two point boundary value problem

$$\begin{aligned} -(A(\cdot)\phi')' = \lambda A(\cdot)\phi \ , \quad & \text{in } (0, 1/2\epsilon) \\ \phi(0) = \phi(1/2\epsilon) = 0 \ . & \end{aligned} \tag{12}$$

Since the coefficient A is bounded from above and is bounded away from zero, it is not difficult to see that

$$d(n\epsilon)^2 \le \lambda_{n,\epsilon} \le D(n\epsilon)^2 \ ,$$

for some constants $0 < d < D$, independent of n and ϵ. We assume the $\phi_{n,\epsilon}$ are normalized by $||\phi_{n,\epsilon}||_{L^2(0,1/2\epsilon)} = 1$. Note that there are no exponentially increasing terms in either of the representations (10) and (11) due to the second inequality from (9). Using the same inequality from (9) (and a standard trace theorem) it follows that there exists $z_1^* > 0$, such that

$$\begin{aligned} \sum_{n=1}^{\infty} (\beta_{n,\epsilon})^2 e^{-2\sqrt{\lambda_{n,\epsilon}} \ z_1^*} &\le C ||U_\epsilon|_{z_1=z_1^*}||^2_{L^2(0,1/2\epsilon)} \\ &\le C_\epsilon \int_{\Phi(\Omega_+)\cap\{0\le z_1\le z_1^* \ , \ 0\le z_2\le 1/2\epsilon \ \}} |\nabla U_\epsilon|^2 \, dz \\ &< \infty \ , \end{aligned}$$

for any fixed $\epsilon > 0$. Similarly we get that

$$\sum_{n=1}^{\infty} (\beta'_{n,\epsilon})^2 e^{-2\sqrt{\lambda_{n,\epsilon}} \ z_1^*} < \infty$$

for any fixed $\epsilon > 0$. As a consequence of these two bounds and the fact that $d(n\epsilon)^2 \leq \lambda_{n,\epsilon} \leq D(n\epsilon)^2$, it follows immediately that

$$|U_\epsilon(z)| \leq C_\epsilon e^{-c_\epsilon |z_1|}$$

for $|z_1|$ sufficiently large, uniformly in $0 < z_2 < 1/2\epsilon$. That is, for a fixed ϵ, the function U_ϵ converges exponentially to zero as $|z_1| \to \infty$.

Consider the restriction of U_ϵ to the horizontal line $z_2 = 1$ (supposing $\epsilon < 1/2$). By simple integration by parts and Hölder's inequality

$$\int_{z_2=1} U_\epsilon^2 \; dz_1 \leq C \left(\int_{\Phi(\Omega_+)\cap\{z_2<1\}} \left(\frac{\partial}{\partial z_2} U_\epsilon \right)^2 \; dz \; + \int_S U_\epsilon^2 \; ds_z \right) \quad .$$

Using the energy bound (9) for U_ϵ and the boundedness of the boundary values of U_ϵ on S, we now obtain

$$\int_{z_2=1} U_\epsilon^2 \; dz_1 \leq C \left(\int_{\Phi(\Omega_+)\cap\{z_2<1\}} |\nabla U_\epsilon|^2 \; dz + \int_S \left(\phi \circ \Phi^{-1}\right)^2 \; ds_z \right) \;\leq\; C \quad . \tag{13}$$

The fact that $U_\epsilon(z)$ approaches 0 exponentially fast as $z_1 \to \pm\infty$ along the line $z_2 = 1$, translates into the fact that the function $u_\epsilon(x)$, restricted to the corresponding circle $\{ \; x_1^2 + (x_2 - 1/2)^2 = 1/4 \; \}$, is C^∞, with its value and all the values of its tangential derivatives vanishing at zero. Let v_ϵ denote the solution to

$$\begin{aligned} \triangle v_\epsilon &= 0 \quad \text{in } \{ \; x_1^2 + (x_2 - 1/2)^2 < 1/4 \; \} \quad , \\ v_\epsilon &= u_\epsilon \quad \text{on } \{ \; x_1^2 + (x_2 - 1/2)^2 = 1/4 \; \} \quad . \end{aligned}$$

We introduce one more auxiliary function $V_\epsilon(z)$, the "push-forward" of the function v_ϵ

$$V_\epsilon(z) = v_\epsilon \circ \Phi^{-1}(z) \quad .$$

V_ϵ is defined on the halfplane $\{z_2 > 1\}$. V_ϵ, together with all its derivatives with respect to z_1, converges to zero as $|z| \to \infty$, and it satisfies

$$\triangle V_\epsilon = 0 \quad \text{in } \{ \; z_2 > 1 \; \} \quad , \; V_\epsilon = U_\epsilon \quad \text{on } \{ \; z_2 = 1 \; \} \quad .$$

We have the following representation for V_ϵ

$$V_\epsilon(z_1, z_2) = \frac{(z_2 - 1)}{\pi} \int_{-\infty}^{\infty} \frac{U_\epsilon(s, 1)}{(z_1 - s)^2 + (z_2 - 1)^2} \; ds \quad ,$$

from which it immediately follows that

$$\begin{aligned} |V_\epsilon(z_1, z_2)| &\leq C(z_2 - 1) ||U_\epsilon(\cdot, 1)||_{L^p(-\infty,\infty)} || \frac{1}{(z_1 - \cdot)^2 + (z_2 - 1)^2} ||_{L^q(-\infty,\infty)} \\ &\leq C(z_2 - 1)^{-\frac{1}{p}} ||U_\epsilon(\cdot, 1)||_{L^p(-\infty,\infty)} \quad . \end{aligned} \tag{14}$$

Here $1 \leq p \leq \infty$, and $\frac{1}{p} + \frac{1}{q} = 1$. Note that we are not asserting that $||U_\epsilon(\cdot, 1)||_{L^p(-\infty,\infty)}$ is finite for all $1 \leq p \leq \infty$. For any integer $k \geq 1$, we similarly get

$$\left| \left(\frac{\partial}{\partial z_1} \right)^k V_\epsilon(z_1, z_2) \right| \leq C(z_2 - 1)^{-k-\frac{1}{p}} ||U_\epsilon(\cdot, 1)||_{L^p(-\infty,\infty)} \quad . \tag{15}$$

The estimates (13), (14) and (15) (the latter two for $p = 2$) lead to

Lemma 4.1 *Let $0 < \alpha$. Then there exists a constant C, independent of ϵ, such that*

$$|U_\epsilon(z)| \le C|z|^{-1/2} \ , \quad \left|\frac{\partial}{\partial z_1}U_\epsilon(z)\right| \le C|z|^{-3/2} \ , \quad \text{and} \ \left|\left(\frac{\partial}{\partial z_1}\right)^2 U_\epsilon(z)\right| \le C|z|^{-5/2} \ ,$$

for $z \in \{ \ \max(2, \alpha|z_1|) \le z_2 \le 1/2\epsilon \ \}$.

Proof Let V_ϵ be as above. The function $U_\epsilon - V_\epsilon$ satisfies

$$\begin{aligned} \triangle(U_\epsilon - V_\epsilon) &= 0 \quad \text{in } \{ \ 1 < z_2 < 1/2\epsilon \ \} \ , \\ U_\epsilon - V_\epsilon &= 0 \quad \text{on } \{ \ z_2 = 1 \ \} \ , \\ U_\epsilon - V_\epsilon &= O(\sqrt{\epsilon}) \quad \text{on } \{ \ z_2 = 1/2\epsilon \ \} \ . \end{aligned} \tag{16}$$

The estimate of $U_\epsilon - V_\epsilon$ on $\{ \ z_2 = 1/2\epsilon \ \}$ follows from (13) and (14) with $p = 2$. For any fixed ϵ we also have that $(U_\epsilon - V_\epsilon)(z) \to 0$ as $|z_1| \to \infty$ (uniformly on $1 \le z_2 \le 1/2\epsilon$). An application of the maximum principle to (16) now yields

$$|(U_\epsilon - V_\epsilon)(z)| \le C\sqrt{\epsilon} \quad \text{on } \{ \ 1 \le z_2 \le 1/2\epsilon \ \} \ ,$$

and therefore, based on (13) and (14) with $p = 2$

$$|U_\epsilon(z)| \le C(\sqrt{\epsilon} + z_2^{-1/2}) \quad \text{on } \{ \ 2 \le z_2 \le 1/2\epsilon \ \} \ .$$

It follows from this that

$$|U_\epsilon(z)| \le C|z|^{-1/2} \quad \text{on } \{ \ \max(2, \alpha|z_1|) \le z_2 \le 1/2\epsilon \ \} \ ,$$

as desired. A similar argument may be used to derive the desired estimates for $\frac{\partial}{\partial z_1}U_\epsilon$ and $\left(\frac{\partial}{\partial z_1}\right)^2 U_\epsilon$, the only difference being that one applies (15) with $k = 1$ and 2 in place of (14).

□

The following result will prove convenient both here and later.

Lemma 4.2 *Let $0 < M_0$, $0 < \gamma$ and $0 < \alpha$, with $\arctan\alpha < \frac{\pi}{2(\gamma+1)}$. Suppose W_ϵ is continuous on $\{ \ M_0 \le z_1 \ , \ 0 \le z_2 \le \frac{1}{2\epsilon} \ \}$, continuously differentiable on each of the sets $\{ \ M_0 \le z_1 \ , \ \frac{1}{2} \le z_2 \le \frac{1}{2\epsilon} \ \}$ and $\{ \ M_0 \le z_1 \ , \ 0 \le z_2 \le \frac{1}{2} \ \}$, and satisfies, in a weak sense*

$$\begin{aligned} \nabla \cdot A(z)\nabla W_\epsilon &= 0 \quad \text{in } \{ \ M_0 < z_1 \ , \ 0 < z_2 < 1/2\epsilon \ \} \\ W_\epsilon &= 0 \ \text{on } \{ \ M_0 \le z_1 \ , \ z_2 = 1/2\epsilon \ \} \ \text{and} \ \{ \ M_0 \le z_1 \ , \ z_2 = 0 \ \} \ . \end{aligned}$$

Suppose furthermore that $|W_\epsilon(M_0, z_2)| \le K_0$ for $0 \le z_2 \le \alpha M_0$, and that, for fixed ϵ, $W_\epsilon(z)z_1^\gamma \to 0$ as $z_1 \to \infty$, uniformly on $0 \le z_2 \le 1/2\epsilon$. Then

$$|W_\epsilon(z)| \le C_0|z|^{-\gamma} \ \text{on } \{ \ M_0 \le z_1 \ , \ \alpha M_0 \le z_2 \le 1/2\epsilon \ \} \cap \{ \ z_2 = \alpha z_1 \ \}$$

implies that

$$|W_\epsilon(z)| \le C_0'|z|^{-\gamma} \ \text{on } \{ \ M_0 \le z_1 \ , \ 0 \le z_2 \le \min(\alpha z_1, \ 1/2\epsilon) \ \} \ ,$$

with C_0' depending on C_0, M_0, K_0, α and γ, but otherwise independent of W_ϵ (and ϵ).

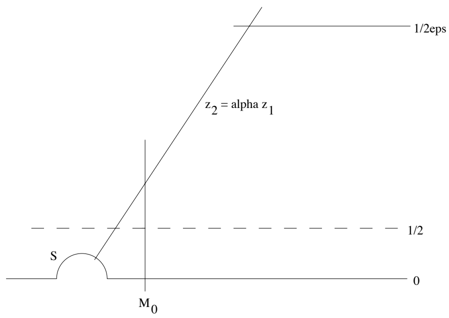

Figure 8: The semi-infinite strip. The dashed line represents the conductivity discontinuity at $z_2 = 1/2$. The "inner" boundary S is illustrated as a semi-circle.

Proof The proof differs slightly depending on whether $a_0 < 1$ or $a_0 > 1$. We start by considering the case $a_0 < 1$. Let $\mu(z)$ denote the function

$$\mu(z) = r^{-\gamma} \sin \gamma(\theta + \delta) \ ,$$

where (r, θ) denotes polar coordinates centered at the origin. Let P_ϵ denote the semi-infinite strip bounded by the four lines $\{ \ z_2 = 0 \ \}$, $\{ \ z_1 = M_0 \ \}$, $\{ \ z_2 = \alpha z_1 \ \}$, and $\{ \ z_2 = 1/2\epsilon \ \}$ (see Figure 8).

The function μ is clearly harmonic in P_ϵ. We also calculate

$$\begin{aligned} \frac{\partial \mu}{\partial z_2} = \sin\theta \frac{\partial \mu}{\partial r} + \frac{\cos\theta}{r}\frac{\partial \mu}{\partial \theta} &= -\gamma r^{-\gamma-1}(\sin\theta \sin\gamma(\theta+\delta) - \cos\theta\cos\gamma(\theta+\delta) \ ) \\ &= \gamma r^{-\gamma-1}\cos(\ (\gamma+1)\theta + \gamma\delta) \ . \end{aligned}$$

Due to the condition $\arctan\alpha < \frac{\pi}{2(\gamma+1)}$, it follows that $0 \le \theta \le \arctan\alpha < \frac{\pi}{2(\gamma+1)}$ on $\overline{P_\epsilon}$. By selecting $0 < \delta$ sufficiently small we may thus obtain

$$0 < (\gamma+1)\theta + \gamma\delta < \pi/2 \quad \text{on } \overline{P_\epsilon} \ .$$

It follows immediately that

$$\mu(z) > 0 \quad \text{for } z \in \overline{P_\epsilon} \ , \ \text{and} \ \ \frac{\partial \mu}{\partial z_2}(z) > 0 \quad \text{for } z \in P_\epsilon \ . \tag{17}$$

Now consider the function

$$w_\epsilon(z) = \frac{W_\epsilon(z)}{\mu(z)} \ .$$

A simple calculation gives that

$$\triangle w_\epsilon + 2\frac{1}{\mu}\nabla\mu \cdot \nabla w_\epsilon = 0 \quad \text{in } P_\epsilon \setminus \{ \ z_2 = 1/2 \ \} \ , \tag{18}$$

and

$$a_0 \frac{\partial w_\epsilon}{\partial z_2}^+ - \frac{\partial w_\epsilon}{\partial z_2}^- = (1-a_0)\frac{1}{\mu}\frac{\partial \mu}{\partial z_2} w_\epsilon \quad \text{on the half-line } \{ \ z_2 = 1/2 \ \} \cap P_\epsilon \ . \tag{19}$$

The equation (18) , the fact that w_ϵ attains the value 0 on ∂P_ϵ, and the fact that $w_\epsilon(z) \to 0$ as $z_1 \to \infty$, imply that w_ϵ attains it extremal values (min and max) on ∂P_ϵ or on the half-line $\{ z_2 = 1/2 \} \cap P_\epsilon$. The condition (19) rules out the posibility that an extremal value can be attained on $\{ z_2 = 1/2 \} \cap P_\epsilon$ unless w_ϵ is constant $(= 0)$: if a maximum was attained at $z_0 \in \{ z_2 = 1/2 \} \cap P_\epsilon$ then $w_\epsilon(z_0) \geq 0$ and thus, according to (17) and (19),

$$a_0 \frac{\partial w_\epsilon}{\partial z_2}^+ (z_0) - \frac{\partial w_\epsilon}{\partial z_2}^- (z_0) \geq 0 \quad .$$

(Remember we are in the case $a_0 < 1$). However, Hopf's version of the maximum principle asserts that if w_ϵ is not constant $(= 0)$ then $\frac{\partial w_\epsilon}{\partial z_2}^+(z_0) \leq 0$ and $\frac{\partial w_\epsilon}{\partial z_2}^-(z_0) \geq 0$, and at least one of these values is nonzero. Consequently

$$a_0 \frac{\partial w_\epsilon}{\partial z_2}^+ (z_0) - \frac{\partial w_\epsilon}{\partial z_2}^- (z_0) < 0 \quad ,$$

and this represents a contradiction. Corresponding to a minimum we would have $w_\epsilon(z_0) \leq 0$, and the same argument as above would lead to a contradiction (unless $w_\epsilon = 0$).

We may therefore conclude that the extremal values of w_ϵ are always attained on ∂P_ϵ. Let d_0 denote the constant

$$d_0 = \sin \gamma\delta \quad .$$

It follows that $0 < d_0$, and that d_0 only depends on γ and δ. It is quite easy to see that

$$|w_\epsilon(z)| \leq K_0 d_0^{-1} M_0^\gamma (1 + \alpha^2)^{\gamma/2}$$

on $\{z_1 = M_0 \ , \quad 0 \leq z_2 \leq \alpha M_0\}$, and that

$$|w_\epsilon(z)| \leq C_0 d_0^{-1}$$

on $\{M_0 \leq z_1 \ , \quad \alpha M_0 \leq z_2 \leq 1/2\epsilon\} \cap \{z_2 = \alpha z_1\}$. Together these two estimates show that

$$|w_\epsilon(z)| \leq C_0' \quad \text{on } \partial P_\epsilon \quad ,$$

with $C_0' = d_0^{-1} \max(C_0, K_0 M_0^\gamma (1 + \alpha^2)^{\gamma/2})$. Since the extremal values of w_ϵ are attained on the boundary of P_ϵ, we immediately conclude that $|w_\epsilon(z)| \leq C_0'$ in P_ϵ, and thus

$$|W_\epsilon(z)| \leq C_0' |z|^{-\gamma} \quad \text{in } P_\epsilon \quad ,$$

exactly as desired (for $a_0 < 1$).

In the case $a_0 > 1$ we introduce the funtion $\mu = r^{-\gamma} \cos \gamma\theta$. We now calculate

$$\begin{aligned} \frac{\partial \mu}{\partial z_2} = \sin\theta \frac{\partial \mu}{\partial r} + \frac{\cos\theta}{r} \frac{\partial \mu}{\partial \theta} &= -\gamma r^{-\gamma-1} (\sin\theta \cos\gamma\theta + \cos\theta \sin\gamma\theta \ ) \\ &= -\gamma r^{-\gamma-1} \sin(\gamma + 1)\theta \quad . \end{aligned}$$

Therefore

$$\mu(z) > 0 \quad \text{for } z \in \overline{P_\epsilon} \ , \quad \text{and} \quad \frac{\partial \mu}{\partial z_2} < 0 \quad \text{for } z \in P_\epsilon \quad .$$

The argument from before works in an identical fashion with this function, since the signs of $1-a_0$ and $\frac{\partial}{\partial z_2}\mu$ have both changed, so that $\frac{1}{\mu}(1-a_0)\frac{\partial}{\partial z_2}\mu$ stays positive. The constant d_0 gets replaced by

$$d_0' = \cos\gamma\theta_\alpha > 0 \quad ,$$

with $\theta_\alpha = \arctan\alpha$. This shows that

$$|W_\epsilon(z)| \leq C_0'|z|^{-\gamma} \quad \text{in } P_\epsilon \quad ,$$

exactly as desired (for $a_0 > 1$ as well).

□

Based on the two previous lemmas it is now fairly simple to prove

Proposition 4.3 *There exists a constant C, independent of ϵ, such that the functions U_ϵ satisfy*

$$|U_\epsilon(z)| \leq C|z|^{-1/2} \ , \quad \left|\frac{\partial}{\partial z_1}U_\epsilon(z)\right| \leq C|z|^{-3/2} \ , \ \textit{and} \ \left|\left(\frac{\partial}{\partial z_1}\right)^2 U_\epsilon(z)\right| \leq C|z|^{-5/2} \quad ,$$

for $z \in \Phi(\Omega_+) \cap \{ \ z_2 < 1/2\epsilon \ \}$.

Proof Select $0 < \alpha$ so that $\arctan\alpha < \pi/3$, and select $0 < M$ so that $2 < \alpha M$, and so that the line $\{ \ z_1 = M \ \}$ does not intersect the "inner" boundary S. Elliptic regularity results (in combination with the uniform energy bound for U_ϵ) easily give

$$|U_\epsilon(M, z_2)| \leq K \quad \text{for } 0 \leq z_2 \leq \alpha M \quad ,$$

with K independent of ϵ. We also have that $U_\epsilon(z)z_1^{1/2} \to 0$ as $z_1 \to \infty$ (uniformly on $0 \leq z_2 \leq 1/2\epsilon$). Indeed, U_ϵ decreases exponentially as $z_1 \to \infty$. From Lemma 4.1 we know that

$$|U_\epsilon(z)| \leq C|z|^{-1/2} \quad \text{on } \{ \ M \leq z_1, \ \alpha M \leq z_2 \leq 1/2\epsilon \ \} \cap \{ \ z_2 = \alpha z_1 \ \} \quad .$$

Application of Lemma 4.2 with $\gamma = 1/2$ now gives

$$|U_\epsilon(z)| \leq C'|z|^{-1/2} \quad \text{on } \{M \leq z_1, \ 0 \leq z_2 \leq \min(\alpha z_1, \ 1/2\epsilon) \ \} \quad ,$$

with C' independent of ϵ. For $0 \leq z_1$, $\max(\alpha M, \alpha z_1) \leq z_2 \leq 1/2\epsilon$ it follows immediately from Lemma 4.1 that $|U_\epsilon(z)| \leq C|z|^{-1/2}$. For z outside S, $0 \leq z_1 \leq M$ and $0 \leq z_2 \leq \alpha M$ (the remainder of $\Phi(\Omega_+) \cap \{ \ 0 \leq z_1, \ z_2 \leq 1/2\epsilon \ \}$) elliptic regularity results yield that $|U_\epsilon(z)| \leq C \leq C|z|^{-1/2}$. In summary, we have thus verified that

$$|U_\epsilon(z)| \leq C|z|^{-1/2} \quad \text{in } \Phi(\Omega_+) \cap \{ \ 0 \leq z_1, \ z_2 < 1/2\epsilon \ \} \quad .$$

A similar argument (e.g. using $U_\epsilon(-z_1, z_2)$ in place of $U(z_1, z_2)$) proves the same estimate in $\Phi(\Omega_+) \cap \{ \ z_1 \leq 0, \ z_2 < 1/2\epsilon \ \}$, thus completing the proof of the first assertion of this proposition. Almost identical arguments with $\gamma = 3/2$ and $\gamma = 5/2$ lead to the desired estimates for $\frac{\partial}{\partial z_1}U_\epsilon$

and $\left(\frac{\partial}{\partial z_1}\right)^2 U_\epsilon$ respectively. Note that these functions also solve the type of boundary value problem required in Lemma 4.2.

$\square$

Since $U_\epsilon \to U$, $\frac{\partial}{\partial z_1} U_\epsilon \to \frac{\partial}{\partial z_1} U$ and $\left(\frac{\partial}{\partial z_1}\right)^2 U_\epsilon \to \left(\frac{\partial}{\partial z_1}\right)^2 U$ pointwise inside $\Phi(\Omega_+) \setminus \{z_2 = 1/2\}$ (Proposition 3.1) and since U, $\frac{\partial}{\partial z_1} U$, and $\left(\frac{\partial}{\partial z_1}\right)^2 U$ are all continuous in $\Phi(\Omega_+)$, we derive from the above proposition

Corollary 4.4 *There exists a constant C such that*

$$|U(z)| \le C|z|^{-1/2} \ , \quad \left|\frac{\partial}{\partial z_1} U(z)\right| \le C|z|^{-3/2} \ , \ \textit{and} \ \left|\left(\frac{\partial}{\partial z_1}\right)^2 U(z)\right| \le C|z|^{-5/2} \ ,$$

for $z \in \Phi(\Omega_+)$.

Based on the use of Proposition 4.3 we are able to establish improved estimates for U_ϵ and U that immediately lead to a proof of our main theorem.

5 The improved estimates

We extend the function A to all of the halfplane $0 < z_2$ by setting it to 1 in the domain bounded by $S = \Phi(\partial\Omega \cap \{x_2 > 0\})$ and $\{z_2 = 0\}$, and we introduce the new variables $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$ as follows

$$\tilde{z}_1 = z_1 \quad , \quad \tilde{z}_2 = a_0 \int_0^{z_2} \frac{1}{A(s)} \, ds \ = \begin{cases} z_2 + \frac{a_0 - 1}{2} \ , & z_2 > 1/2 \\ a_0 z_2 \ , & z_2 < 1/2 \ . \end{cases} \tag{20}$$

The transformed function $\tilde{U}(\tilde{z}) = U(z)$ solves

$$\left[\left(\frac{\tilde{A}(\tilde{z}_2)}{a_0}\right)^2 \left(\frac{\partial}{\partial \tilde{z}_1}\right)^2 + \left(\frac{\partial}{\partial \tilde{z}_2}\right)^2\right] \tilde{U} = 0 \quad \text{for } \tilde{z} \in \tilde{\Phi}(\Omega_+) \ ,$$

with the domain $\tilde{\Phi}(\Omega_+)$ defined by

$$\tilde{z} \in \tilde{\Phi}(\Omega_+) \quad \text{if and only if } z \in \Phi(\Omega_+) \ .$$

See Figure 9. We have illustrated the stretched "inner" boundary $\tilde{S}$ as the upper half of an ellipse.

The function $\tilde{A}$ is defined by $\tilde{A}(\tilde{z}_2) = A(z_2)$, where $\tilde{z}_2$ and z_2 are related by the second formula in (20). From the above definitions it follows immediately that

$$\triangle_{\tilde{z}} \tilde{U} = 0 \quad \text{for } a_0/2 < \tilde{z}_2 \ ,$$

and

$$\triangle_{\tilde{z}} \tilde{U} = (1 - \frac{1}{a_0^2}) \left(\frac{\partial}{\partial \tilde{z}_1}\right)^2 \tilde{U} \quad \text{for } \tilde{z} \in \tilde{\Phi}(\Omega_+) \cap \{\tilde{z}_2 < a_0/2\} \ .$$

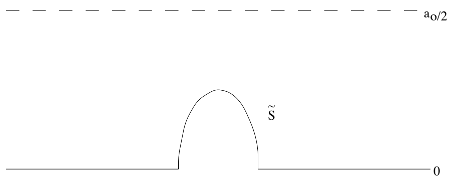

Figure 9: The streched geometry. The dashed line represents the location of the discontinuity of the coefficient $\tilde{A}$.

Similarly,

$$\triangle_{\tilde z}\tilde U_\epsilon = 0 \quad \text{for } a_0/2 < \tilde z_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon} \quad ,$$

and

$$\triangle_{\tilde z}\tilde U_\epsilon = (1-\frac{1}{a_0^2})\left(\frac{\partial}{\partial \tilde z_1}\right)^2 \tilde U_\epsilon \quad \text{for } \tilde z \in \tilde\Phi(\Omega_+)\cap\{\tilde z_2 < a_0/2\} \quad .$$

Let $\tilde E$ denote the domain bounded by the inner boundary $\tilde S$ and $\{\tilde z_2 = 0\}$. In Figure 9 it is represented by the inside of the half-ellipse. We now extend both $\tilde U$ (and $\tilde U_\epsilon$) to all of the halfplane $0 < \tilde z_2$ (the strip $0 < \tilde z_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon}$) in such a way that the extensions are zero on $\tilde z_2 = 0$ and are $C^{2,\beta}$ bounded in a neighborhood of $\tilde E$ (independently of ϵ). This may be done since, near $\tilde S$, $\tilde U$ and $\tilde U_\epsilon$ are C^∞, and uniformly bounded in all C^k norms (due to elliptic regularity results and the uniform energy estimate). As a consequence we get that

$$\begin{aligned}\triangle_{\tilde z}\tilde U &= 0 \quad \text{for } a_0/2 < \tilde z_2 \quad , \\ \triangle_{\tilde z}\tilde U &= (1-\frac{1}{a_0^2})\left(\frac{\partial}{\partial \tilde z_1}\right)^2 \tilde U + f(\tilde z) \quad \text{for } 0 < \tilde z_2 < a_0/2 \quad ,\end{aligned}$$

where f is uniformly bounded and supported in the closure of $\tilde E$. We also get that

$$\begin{aligned}\triangle_{\tilde z}\tilde U_\epsilon &= 0 \quad \text{for } a_0/2 < \tilde z_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon} \quad , \\ \triangle_{\tilde z}\tilde U_\epsilon &= (1-\frac{1}{a_0^2})\left(\frac{\partial}{\partial \tilde z_1}\right)^2 \tilde U_\epsilon + f_\epsilon(\tilde z) \quad \text{for } 0 < \tilde z_2 < a_0/2 \quad ,\end{aligned}$$

where the f_ϵ are uniformly bounded (independently of ϵ) and supported in the closure of $\tilde E$. The functions $\tilde U$, $\tilde U_\epsilon$ and the gradients $\nabla_{\tilde z}\tilde U$, $\nabla_{\tilde z}\tilde U_\epsilon$ are continuous across the line $\tilde z_2 = a_0/2$, so the above piecewise formulas entirely describe (the distributions) $\triangle_{\tilde z}\tilde U$ and $\triangle_{\tilde z}\tilde U_\epsilon$. Let g_ϵ denote the function $g_\epsilon = (1-\frac{1}{a_0^2})\left(\frac{\partial}{\partial \tilde z_1}\right)^2 \tilde U_\epsilon + f_\epsilon$ (in $\tilde E$) and define

$$\begin{aligned}\tilde V_\epsilon^*(\tilde z) &= \frac{a_0^2-1}{4\pi a_0^2}\int_{\{(s,t)\,:\,0<t<a_0/2\}\setminus\tilde E}\Big[\log(|\tilde z_1 - s|^2 + |\tilde z_2 - t|^2) \\ &\qquad -\log(|\tilde z_1 - s|^2 + |\tilde z_2 + t|^2)\Big]\left(\frac{\partial}{\partial s}\right)^2 \tilde U_\epsilon(s,t)\,dsdt \\ &\quad +\frac{1}{4\pi}\int_{\tilde E}\Big[\log(|\tilde z_1 - s|^2 + |\tilde z_2 - t|^2)\end{aligned}$$

$$- \log(|\tilde z_1 - s|^2 + |\tilde z_2 + t|^2)\Big] g_\epsilon(s,t)\; dsdt \quad .$$

It is not difficult to see that $\tilde V_\epsilon^*$ satisfies

$$\begin{aligned}
\triangle_{\tilde z} \tilde V_\epsilon^* &= 0 \quad \text{for } a_0/2 < \tilde z_2 \;, \\
\triangle_{\tilde z} \tilde V_\epsilon^* &= (1 - \frac{1}{a_0^2}) \left(\frac{\partial}{\partial \tilde z_1}\right)^2 \tilde U_\epsilon + f_\epsilon(\tilde z) \quad \text{for } 0 < \tilde z_2 < a_0/2 \;, \quad \text{and} \\
\tilde V_\epsilon^* &= 0 \quad \text{at } \tilde z_2 = 0 \;.
\end{aligned}$$

$\tilde V_\epsilon^*$ and $\nabla_{\tilde z} \tilde V_\epsilon^*$ are continuous across the line $\tilde z_2 = a_0/2$, so the above piecewise formula entirely describe (the distribution) $\triangle_{\tilde z} V_\epsilon^*$. The second order derivative $\left(\frac{\partial}{\partial \tilde z_1}\right)^2 \tilde V_\epsilon^*$ is also continuous across the line $\tilde z_2 = a_0/2$, but the second order derivative $\left(\frac{\partial}{\partial \tilde z_2}\right)^2 \tilde V_\epsilon^*$ is in general discontinuous.

Lemma 5.1 *For any fixed ϵ we have that*

$$|\tilde V_\epsilon^*(\tilde z)| \to 0 \;, \qquad \left|\frac{\partial}{\partial \tilde z_1} \tilde V_\epsilon^*(\tilde z)\right| \to 0 \;, \qquad \textit{and} \; \left|\left(\frac{\partial}{\partial \tilde z_1}\right)^2 \tilde V_\epsilon^*(\tilde z)\right| \to 0 \;,$$

as $|\tilde z_1| \to \infty$, uniformly for $\tilde z \in \{\; 0 < \tilde z_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon} \;\}$.

Proof Since the function g_ϵ is uniformly bounded independently of ϵ, and since

$$\log(|\tilde z_1 - s|^2 + |\tilde z_2 - t|^2) - \log(|\tilde z_1 - s|^2 + |\tilde z_2 + t|^2) = -2t \frac{2(\tilde z_2 + \theta)}{|\tilde z_1 - s|^2 + |\tilde z_2 + \theta|^2} \quad ,$$

where $-a_0/2 < \theta < a_0/2$ (when $0 < t < a_0/2$) it follows immediately that the last integral in the definition of $\tilde V_\epsilon^*$,

$$\frac{1}{4\pi} \int_{\tilde E} \Big[\log(|\tilde z_1 - s|^2 + |\tilde z_2 - t|^2) - \log(|\tilde z_1 - s|^2 + |\tilde z_2 + t|^2) \Big] g_\epsilon(s,t)\; dsdt \quad ,$$

converges to 0 as $|z_1| \to \infty$ (uniformly for $0 < z_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon}$) . Let $\tilde E^c$ denote the set

$$\tilde E^c = \{\; (s,t) \;:\; 0 < t < a_0/2 \;\} \setminus \tilde E \quad .$$

The first integral in the definition of $\tilde V_\epsilon^*$ may be bounded by

$$\begin{aligned}
C \int_{\tilde E^c \cap \{\; |\tilde z_1 - s| < |\tilde z_1|/2 \;\}} &\Big[\left|\log(|\tilde z_1 - s|^2 + |\tilde z_2 - t|^2)\right| \\
&\qquad + \left|\log(|\tilde z_1 - s|^2 + |\tilde z_2 + t|^2)\right| \Big] \left| \left(\frac{\partial}{\partial s}\right)^2 \tilde U_\epsilon(s,t) \right| \; dsdt \\
+ C \int_{\tilde E^c \cap \{\; |\tilde z_1 - s| \geq |\tilde z_1|/2 \;\}} & t \frac{|\tilde z_2 + \theta|}{|\tilde z_1 - s|^2 + |\tilde z_2 + \theta|^2} \left| \left(\frac{\partial}{\partial s}\right)^2 \tilde U_\epsilon(s,t) \right| \; dsdt \quad . \qquad (21)
\end{aligned}$$

As $|\tilde{z}_1| \to \infty$ (for $0 < \tilde{z}_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon}$) the second integral in (21) is bounded by

$$C_\epsilon \frac{1}{|\tilde{z}_1|^2} \int_{\tilde{E}^c} |\left(\frac{\partial}{\partial s}\right)^2 \tilde{U}_\epsilon(s,t)|\; dsdt \quad ,$$

which clearly approaches 0 (the integral in this estimate is uniformly bounded in ϵ according to Proposition 4.3). For $0 < \tilde{z}_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon}$, $0 < t < a_0/2$ we have

$$|\log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 - t|^2)| \leq |\log(|\tilde{z}_1 - s|^2)| + K_\epsilon \quad ,$$

and

$$|\log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 + t|^2)| \leq |\log(|\tilde{z}_1 - s|^2)| + K_\epsilon \quad ,$$

so that the first integral in (21) becomes bounded by

$$\begin{aligned} C_\epsilon \int_{\tilde{E}^c \cap \{\ |\tilde{z}_1 - s| < |\tilde{z}_1|/2\ \}} & (|\log(|\tilde{z}_1 - s|^2)| + 1) | \left(\frac{\partial}{\partial s}\right)^2 \tilde{U}_\epsilon(s,t)|\; dsdt \\ & \leq C_\epsilon e^{-c_\epsilon |\tilde{z}_1|} \int_{\tilde{E}^c \cap \{\ |\tilde{z}_1 - s| < |\tilde{z}_1|/2\ \}} (|\log(|\tilde{z}_1 - s|^2)| + 1)\; dsdt \\ & \leq C_\epsilon e^{-c_\epsilon |\tilde{z}_1|} |\tilde{z}_1|\, |\log(|\tilde{z}_1|^2)| \quad \text{as } |\tilde{z}_1| \to \infty \quad . \end{aligned}$$

Here we used that $|\tilde{z}_1 - s| < |\tilde{z}_1|/2 \Rightarrow |s| > |\tilde{z}_1|/2$, and that $\left|\left(\frac{\partial}{\partial s}\right)^2 \tilde{U}_\epsilon(s,t)\right|$ decreases exponentially in $|s|$ (uniformly in t, for fixed ϵ). This proves that the first integral in the definition of $\tilde{V}_\epsilon^*$ converges to 0 as $|\tilde{z}_1| \to \infty$ (uniformly for $0 < z_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon}$) and it thus verifies the asymptotic statement concerning $\tilde{V}_\epsilon^*$. For the first and second order derivatives of $\tilde{V}_\epsilon^*$ with respect to $\tilde{z}_1$ we write (for $|\tilde{z}_1|$ large)

$$\begin{aligned} \left(\frac{\partial}{\partial \tilde{z}_1}\right)^k \tilde{V}_\epsilon^*(\tilde{z}) \;=\; & \frac{a_0^2 - 1}{4\pi a_0^2} \int_{\{(s,t)\ :\ 0<t<a_0/2\} \setminus \tilde{E}} \Big[\log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 - t|^2) \\ & \qquad - \log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 + t|^2) \Big] \left(\frac{\partial}{\partial s}\right)^{2+k} \tilde{U}_\epsilon(s,t)\; dsdt \\ & + \frac{1}{4\pi} \int_{\tilde{E}} \Big[\left(\frac{\partial}{\partial \tilde{z}_1}\right)^k \log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 - t|^2) \\ & \qquad - \left(\frac{\partial}{\partial \tilde{z}_1}\right)^k \log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 + t|^2) \Big] g_\epsilon(s,t)\; dsdt \quad , \end{aligned}$$

and apply an argument very similar to that above.

□

Let $H(x,y)$ denote the function

$$H(x,y) = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} \quad .$$

For $a_0/2 < \tilde{z}_2$ a fairly straightforward computation gives

$$\left(\frac{\partial}{\partial \tilde{z}_1}\right)^2 \tilde{V}_\epsilon^*(\tilde{z}_1,\tilde{z}_2) = \frac{a_0^2-1}{4\pi a_0^2}\int_{\{(s,t)\ :\ 0<t<a_0/2\}\setminus\tilde{E}} \Big[H(\tilde{z}_1-s,\tilde{z}_2-t) - H(\tilde{z}_1-s,\tilde{z}_2+t)\Big]\left(\frac{\partial}{\partial s}\right)^2 \tilde{U}_\epsilon(s,t)\ dsdt + \frac{1}{4\pi}\int_{\tilde{E}}\left[H(\tilde{z}_1-s,\tilde{z}_2-t)-H(\tilde{z}_1-s,\tilde{z}_2+t)\right]g_\epsilon(s,t)\ dsdt\ .$$

By use of Taylor's formula we now get

$$\left(\frac{\partial}{\partial \tilde{z}_1}\right)^2 \tilde{V}_\epsilon^*(\tilde{z}_1,\tilde{z}_2) = -\frac{a_0^2-1}{4\pi a_0^2}\int_{\{(s,t)\ :\ 0<t<a_0/2\}\setminus\tilde{E}} 2t\ H_y(\tilde{z}_1-s,\tilde{z}_2+\theta)\left(\frac{\partial}{\partial s}\right)^2 \tilde{U}_\epsilon(s,t)\ dsdt - \frac{1}{4\pi}\int_{\tilde{E}} 2t\ H_y(\tilde{z}_1-s,\tilde{z}_2+\theta)g_\epsilon(s,t)\ dsdt\ , \tag{22}$$

where θ lies between $-a_0/2$ and $a_0/2$ (and depends on $\tilde{z}_1-s$, $\tilde{z}_2$ and t). The derivative H_y is given by

$$|H_y(x,y)| = |\frac{y(12x^2-4y^2)}{(x^2+y^2)^3}| \le Cy^{-3} \quad 1<y\ . \tag{23}$$

From Proposition 4.3 we know that

$$|\left(\frac{\partial}{\partial s}\right)^2 \tilde{U}_\epsilon(s,t)| \le C(s^2+t^2)^{-5/4}$$

for $(s,t)\in\tilde{E}^c$. We also know that g_ϵ is uniformly bounded on $\tilde{E}$, independently of ϵ. Combining these two facts with (22) and (23) we now conclude that

$$\left|\left(\frac{\partial}{\partial \tilde{z}_1}\right)^2 \tilde{V}_\epsilon^*(\tilde{z}_1,\tilde{z}_2)\right| \le C\tilde{z}_2^{-3} \quad 1+a_0/2<\tilde{z}_2\ , \tag{24}$$

with C independent of ϵ. By an entirely similar argument (taking just one derivative, or no derivative at all) we obtain the estimates

$$\left|\frac{\partial}{\partial \tilde{z}_1}\tilde{V}_\epsilon^*(\tilde{z}_1,\tilde{z}_2)\right| \le C\tilde{z}_2^{-2}\ , \quad \text{and} \quad \left|\tilde{V}_\epsilon^*(\tilde{z}_1,\tilde{z}_2)\right| \le C\tilde{z}_2^{-1} \quad \text{for } 1+a_0/2<\tilde{z}_2\ . \tag{25}$$

The estimates (24) and (25) are stronger than the corresponding estimates (14) and (15) (with $p=2$) by a factor of $\tilde{z}_2^{-1/2}$. Not surprisingly, these estimates for $\tilde{V}_\epsilon^*$ lead to improvements of the results of Lemma 4.1 by a factor of $|z|^{-1/2}$.

Lemma 5.2 *Let* $0<\alpha$. *Then there exists* C, *independent of* ϵ, *such that*

$$|U_\epsilon(z)| \le C|z|^{-1}\ , \quad \left|\frac{\partial}{\partial z_1}U_\epsilon(z)\right| \le C|z|^{-2}\ , \quad \textit{and} \quad \left|\left(\frac{\partial}{\partial z_1}\right)^2 U_\epsilon(z)\right| \le C|z|^{-3}\ ,$$

for $z\in\{\ \max(2,\alpha|z_1|)\le z_2\le 1/2\epsilon\ \}$.

Proof The function $\tilde{U}_\epsilon - \tilde{V}_\epsilon^*$ satisfies

$$\begin{aligned} \triangle(\tilde{U}_\epsilon - \tilde{V}_\epsilon^*) &= 0 \quad \text{for } 0 < \tilde{z}_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon} \quad , \\ \tilde{U}_\epsilon - \tilde{V}_\epsilon^* &= 0 \quad \text{at } \tilde{z}_2 = 0 \quad , \\ \tilde{U}_\epsilon - \tilde{V}_\epsilon^* &= O(\epsilon) \quad \text{at } \tilde{z}_2 = \frac{1+\epsilon(a_0-1)}{2\epsilon} \quad , \end{aligned}$$

and $\tilde{U}_\epsilon - \tilde{V}_\epsilon^* \to 0$ as $|z_1| \to \infty$ (uniformly for $0 < \tilde{z}_2 < \frac{1+\epsilon(a_0-1)}{2\epsilon}$). The desired estimate for U_ϵ follows by an application of the maximum principle just as in the proof of Lemma 4.1 (possibly with a smaller coefficient α') and then a return to the z coordinates. The estimates for $\frac{\partial}{\partial z_1} U_\epsilon$ and $\left(\frac{\partial}{\partial z_1}\right)^2 U_\epsilon$ follow in a completely similar manner.

□

Based on Lemma 4.2 and Lemma 5.2 it is now possible to establish

Proposition 5.3 *There exists a constant C, independent of ϵ, such that the functions U_ϵ satisfy*

$$|U_\epsilon(z)| \leq C|z|^{-1} \; , \quad \left|\frac{\partial}{\partial z_1} U_\epsilon(z)\right| \leq C|z|^{-2} \; , \; and \; \left|\left(\frac{\partial}{\partial z_1}\right)^2 U_\epsilon(z)\right| \leq C|z|^{-3} \quad ,$$

for $z \in \Phi(\Omega_+) \cap \{ \, z_2 < 1/2\epsilon \, \}$.

Proof The proof is exactly the same as that of Proposition 4.3.

□

Taking the limit $\epsilon \to 0$, we conclude just as before

Corollary 5.4 *There exists a constant C, such that*

$$|U(z)| \leq C|z|^{-1} \; , \quad \left|\frac{\partial}{\partial z_1} U(z)\right| \leq C|z|^{-2} \; , \; and \; \left|\left(\frac{\partial}{\partial z_1}\right)^2 U(z)\right| \leq C|z|^{-3} \quad ,$$

for $z \in \Phi(\Omega_+)$.

6 The proof of the main theorem

Going back to the function $\tilde{U}$, one can check that it has the representation formula

$$\begin{aligned} \tilde{U}(\tilde{z}) = \frac{a_0^2 - 1}{4\pi a_0^2} \int_{\{(s,t) \,:\, 0<t<a_0/2\}\setminus\tilde{E}} &\Big[\log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 - t|^2) \\ &- \log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 + t|^2) \Big] \left(\frac{\partial}{\partial s}\right)^2 \tilde{U}(s,t) \, ds dt \\ + \frac{1}{4\pi} \int_{\tilde{E}} &\Big[\log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 - t|^2) \\ &- \log(|\tilde{z}_1 - s|^2 + |\tilde{z}_2 + t|^2) \Big] g(s,t) \, ds dt \quad , \end{aligned} \tag{26}$$

where g denotes the function

$$g = (1 - \frac{1}{a_0^2}) \left(\frac{\partial}{\partial \tilde{z}_1}\right)^2 \tilde{U} + f \quad ,$$

in $\tilde{E}$. To see this, it suffices to notice that the right hand side of (26) satisfies the same equation and the same boundary condition as $\tilde{U}$, and to verify that it also converges to zero as $|\tilde{z}|$ tends to infinity. The fact that the two integrals in (26) converge to zero for $0 < \tilde{z}_2 < K$ as $|\tilde{z}_1| \to \infty$, follows from an argument very similar to that used in the proof of Lemma 5.1 (one compensates for the fact that $\tilde{U}$ does not necessarily decrease exponentially in z_1 by using the decay estimate of Corollary 4.4). As we shall observe later, it is not difficult to prove that these two integrals are also bounded by $C\tilde{z}_2^{-1}$ (uniformly in $\tilde{z}_1$) as $\tilde{z}_2 \to \infty$. A combination of these two facts yields that the right hand side in the formula (26) converges to zero as $|\tilde{z}| \to \infty$, which now implies that it is indeed a representation of $\tilde{U}$. For $a_0/2 < \tilde{z}$ we calculate

$$\begin{aligned}\frac{\partial}{\partial z_2}\tilde{U}(\tilde{z}) = \frac{a_0^2 - 1}{4\pi a_0^2} \int_{\{(s,t)\ :\ 0<t<a_0/2\}\setminus \tilde{E}} & \left[\frac{2(\tilde{z}_2 - t)}{|\tilde{z}_1 - s|^2 + |\tilde{z}_2 - t|^2}\right. \\ & \left. - \frac{2(\tilde{z}_2 + t)}{|\tilde{z}_1 - s|^2 + |\tilde{z}_2 + t|^2}\right] \left(\frac{\partial}{\partial s}\right)^2 \tilde{U}(s,t)\ dsdt \\ + \frac{1}{4\pi} \int_{\tilde{E}} & \left[\frac{2(\tilde{z}_2 - t)}{|\tilde{z}_1 - s|^2 + |\tilde{z}_2 - t|^2}\right. \\ & \left. - \frac{2(\tilde{z}_2 + t)}{|\tilde{z}_1 - s|^2 + |\tilde{z}_2 + t|^2}\right] g(s,t)\ dsdt \quad .\end{aligned}$$

Introducing $K(x,y) = \frac{2y}{x^2+y^2}$, we obtain

$$\begin{aligned}\frac{\partial}{\partial z_2}\tilde{U}(\tilde{z}) &= -\frac{a_0^2 - 1}{4\pi a_0^2} \int_{\{(s,t)\ :\ 0<t<a_0/2\}\setminus \tilde{E}} 2tK_y(\tilde{z}_1 - s, \tilde{z}_2 + \theta) \left(\frac{\partial}{\partial s}\right)^2 \tilde{U}(s,t)\ dsdt \\ & \quad - \frac{1}{4\pi} \int_{\tilde{E}} 2tK_y(\tilde{z}_1 - s, \tilde{z}_2 + \theta) g(s,t)\ dsdt \quad ,\end{aligned}$$

with θ lying between $-a_0/2$ and $a_0/2$ (and depending on $\tilde{z}_1 - s$, $\tilde{z}_2$ and t). Since $|K_y(x,y)| = |\frac{2(x^2-y^2)}{(x^2+y^2)^2}| \leq Cy^{-2}$, $1 < y$, the known decay of $\left(\frac{\partial}{\partial s}\right)^2 \tilde{U}$ (Corollary 4.4) and the boundedness of g imply

$$\left|\frac{\partial}{\partial \tilde{z}_2}\tilde{U}(\tilde{z})\right| \leq C\tilde{z}_2^{-2} \ , \quad 1 + a_0/2 < \tilde{z}_2 \quad . \tag{27}$$

An argument identical to that given just above (taking no derivative) would immediately yield that the two integrals in the right hand of the formula (26) are bounded by $C\tilde{z}_2^{-1}$ (we needed this fact earlier when we showed that (26) is indeed a representation of $\tilde{U}$). Rewritten in terms of the z coordinates, (27) gives

$$\left|\frac{\partial}{\partial z_2}U(z)\right| \leq Cz_2^{-2} \ , \quad 1/2 + c < z_2 \quad , \tag{28}$$

for some $0 < c$. Using the fact that

$$\left(\frac{\partial}{\partial z_2}A(z_2)\frac{\partial}{\partial z_2}U(z)\right) = -A(z_2)\left(\frac{\partial}{\partial z_1}\right)^2 U(z) = O(|z|^{-3}) \quad , \quad z \in \Phi(\Omega_+) \ , \tag{29}$$

as asserted by Corollary 5.4, we are now able to prove that

Lemma 6.1 *There exists a constant C, such that*

$$\left|\frac{\partial}{\partial z_2}U(z)\right| \leq C|z|^{-2} \quad ,$$

for $z \in \Phi(\Omega_+) \setminus \{z_2 = 1/2\}$.

Proof From (28) it follows immediately that

$$\left|\frac{\partial}{\partial z_2}U(z)\right| \leq C z_2^{-2} \leq C|z|^{-2} \quad \text{for } \max(1/2 + c, |z_1|) \leq z_2 \quad . \tag{30}$$

For $0 < z_2$ and $\max(K, z_2) \leq |z_1|$ (with K sufficiently large) we have that

$$\left(A\frac{\partial}{\partial z_2}U\right)(z_1, z_1) - \left(A\frac{\partial}{\partial z_2}U\right)(z_1, z_2) = \int_{z_2}^{z_1}\left(\frac{\partial}{\partial z_2}A\frac{\partial}{\partial z_2}U\right)(z_1, t)\, dt \quad ,$$

and therefore

$$\begin{aligned}\left|\left(A\frac{\partial}{\partial z_2}\right)U(z_1, z_2)\right| &\leq C|z|^{-2} + C\int_{z_2}^{z_1}(|z_1|^2 + |t|^2)^{-3/2}\, dt \\ &\leq C|z|^{-2} \quad \text{for } \max(K, z_2) \leq |z_1| \quad .\end{aligned} \tag{31}$$

Here we have used the estimates (29) and (30) (and the fact that we may select $K > 1/2 + c$) to derive the first inequality. Based on a combination of (30) and (31) we conclude that

$$\left|\frac{\partial}{\partial z_2}U(z)\right| \leq C|z|^{-2} \quad \text{for } |z| > \sqrt{2}K, \ z \notin \{z_2 = 1/2\} \quad .$$

The function A is discontinuous across $\{z_2 = 1/2\}$ and the derivative $\frac{\partial}{\partial z_2}U$ is not properly defined there; this is why we subtract the set $\{z_2 = 1/2\}$. The above estimate in combination with an elliptic regularity estimate (for $|z|$ small) immediately leads to the desired result.

□

Combining Corollary 5.4 and Lemma 6.1 we finally arrive at

Theorem 6.2 *The solution, $u \in H^1(\Omega)$, to the boundary value problem (1), with conductivity a given by (3), is in $W^{1,\infty}(\Omega)$ for any fixed $0 < a_0 < \infty$.*

Proof We already know (cf. [4]) that $u \in C^\beta(\Omega)$ for some $\beta > 0$. From standard elliptic regularity results we also know that u is smooth, and therefore bounded, near $\partial\Omega$. It thus suffices to prove that $\nabla u \in L^\infty(\Omega)$. As already explained earlier (in section two) we may restrict attention to u that are odd in the x_1-axis. For such u, it suffices to show that $|\nabla u| \le C$ in $\Omega_+ \setminus \{ x_1^2 + (x_2-1)^2 = 1 \} = (\Omega \cap \{ 0 < x_2 \}) \setminus \{ x_1^2 + (x_2-1)^2 = 1 \}$. The solution u has the form

$$u(x) = U \circ \Phi(x) \quad ,$$

where U has been studied in the preceeding three sections. We calculate

$$\nabla u(x) = D\Phi^t(x)(\nabla_z U)(\Phi(x)) \quad . \tag{32}$$

The matrix $D\Phi$ is given by

$$D\Phi = \begin{pmatrix} \frac{\partial z_1}{\partial x_1} & \frac{\partial z_1}{\partial x_2} \\ \frac{\partial z_2}{\partial x_1} & \frac{\partial z_2}{\partial x_2} \end{pmatrix} \quad ,$$

and a simple computation yields

$$\left| \frac{\partial z_i}{\partial x_j} \right| = \left| \frac{\delta_{ij}(x_1^2 + x_2^2) - 2x_i x_j}{(x_1^2 + x_2^2)^2} \right| \le C \frac{1}{x_1^2 + x_2^2} \qquad x \in \Omega_+ \quad . \tag{33}$$

At the same time, Corollary 5.4 and Lemma 6.1 give that

$$|\nabla_z U(\Phi(x))| \le C|\Phi(x)|^{-2} = C(x_1^2 + x_2^2) \quad \text{for } x \in \Omega_+ \setminus \{ x_1^2 + (x_2-1)^2 = 1 \} \quad . \tag{34}$$

Combining (32), (33), and (34) we finally obtain

$$|\nabla u(x)| \le C \qquad x \in \Omega_+ \setminus \{ x_1^2 + (x_2-1)^2 = 1 \} \quad ,$$

as desired.

□

Remark In the appendix we shall see that the case which formally corresponds to $a_0 = 0$ admits solutions that are discontinuous at the origin. Thus, it would not be reasonable to expect the solution u (given fixed boundary data) to have a gradient that is uniformly bounded, independently of a_0. The L^∞ norm of $|\nabla u|$ may well become unbounded as a_0 approaches 0. By duality the same phenomenon may also occur as a_0 approaches ∞.

7 Appendix

In this appendix we give a short review of what happens in the two cases that at least formally correspond to $a_0 = 0$ and $a_0 = \infty$. In both cases the relevant boundary value problems live in $\Omega \setminus \{$ the fibers $\}$. They require that

$$\triangle u^0 = \triangle u^\infty = 0 \quad \text{in } \Omega \setminus \{ \text{ the fibers } \} \quad , \tag{35}$$

with

$$\frac{\partial}{\partial n} u^0 = 0 \quad \text{on the boundaries of the fibers} \quad , \tag{36}$$

and

$$u^\infty = \text{ constant on the boundary of each fiber } , \tag{37}$$

respectively. The constants in the boundary condition (37) are not arbitrary, they are those for which the energy expression attains its smallest value. On the boundary $\partial\Omega$, the two solutions satisfy the common boundary condition

$$u^0 = u^\infty = \phi \ .$$

Given smooth boundaries, the solutions u^0 and u^∞ would be obtained as limits of the solution to (1), as a_0 tends to 0 and as a_0 tends to ∞ respectively. We suspect that the same holds true for boundaries with cusps as here, but we have not carried out the analysis. This is why we use the terminology "formally corresponding to $a_0 = 0$ and $a_0 = \infty$".

In the transformed variables $z = \Phi(x)$, with Φ as before, the equations (35)–(37) become

$$\triangle U^0 = \triangle U^\infty = 0 \quad \text{in } \{ \ z \in \Phi(\Omega) \ , \ \ -1/2 < z_2 < 1/2 \ \} \ , \tag{38}$$

with

$$\frac{\partial}{\partial z_2} U^0 = 0 \quad \text{on } z_2 = \pm 1/2 \ , \tag{39}$$

and

$$U^\infty = c_\pm \quad \text{on } z_2 = \pm 1/2 \ , \tag{40}$$

respectively. The common boundary condition on $\partial\Omega$ transforms into

$$U^0 = U^\infty = \phi \circ \Phi^{-1} \quad \text{on } \Phi(\partial\Omega) \ .$$

For the moment we restrict attention to the boundary value problem for U^0. At the very end of this section, we return to make some remarks about the boundary value problem for U^∞. As mentioned previously, any solution to this boundary value problem may be written as a sum of two harmonic functions in $\{ \ z \in \Phi(\Omega) \ , \ -1/2 < z_2 < 1/2 \ \}$, one which is even in the z_1-axis and one which is odd. These two functions have somewhat different behaviour. We first consider the even function, which, when restricted to the interval $0 < z_2 < 1/2$, is a solution to

$$\begin{aligned} \triangle U^0 &= \quad \text{in } \{ \ z \in \Phi(\Omega) \ , \ \ 0 < z_2 < 1/2 \ \} \ , \\ \frac{\partial}{\partial z_2} U^0 &= 0 \quad \text{on } \{ \ z_2 = 1/2 \ \} \ \text{ and on } \{ \ z \in \Phi(\Omega) \ , \ \ z_2 = 0 \ \} \ , \\ U^0 &= \phi \circ \Phi^{-1} \text{ on } \{ z \in \Phi(\partial\Omega) \ , \ \ 0 < z_2 \ \} \ . \end{aligned} \tag{41}$$

Separation of variables now immediately gives that U^0 must have the form

$$U^0(z_1, z_2) = \beta_0 + \sum_{n=1}^{\infty} \beta_n \cos(2n\pi z_2) e^{-2n\pi z_1} \quad \text{for } z_1 \text{ sufficiently positive } , \tag{42}$$

$$U^0(z_1, z_2) = \beta_0' + \sum_{n=1}^{\infty} \beta_n' \cos(2n\pi z_2) e^{2n\pi z_1} \quad \text{for } z_1 \text{ sufficiently negative } . \tag{43}$$

Conversely any function, U^0, that is defined by (42) for $z_1 > 0$ and by (43) for $z_1 < 0$, is a solution to

$$\triangle U^0 = 0 \quad \text{in } \{ z_1 \neq 0 \}$$

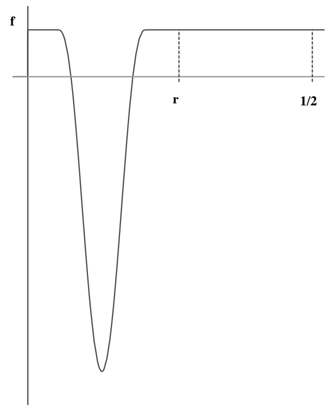

Figure 10: The function f.

with boundary conditions

$$\frac{\partial}{\partial z_2}U^0 = 0 \quad \text{on } \{z_2 = 0\} \text{ and on } \{z_2 = 1/2\} \ .$$

We shall now use this fact to construct a rather large class of solutions.

Select β_0 and β_0' arbitrarily, and let $f(z_2)$ denote any smooth, even and periodic function with period 1, such that $f(z_2) = \beta_0' - \beta_0$ for $r < z_2 < 1/2$ $(0 < r)$ and such that $\int_0^{1/2} f(s)\, ds = 0$. A graph of such a function on the interval $(0, 1/2)$ is illustrated in Figure 10. The value of r is selected small enough, so that the line segment $\{z_1 = 0 \ , \ \ 0 < z_2 < r\ \}$ lies inside $\Phi(\mathbb{R}^2 \setminus \Omega)$.

Let β_n, $n \geq 1$, be the cosine Fourier coefficients of the function $f/2$, i.e.

$$2\sum_{n=1}^{\infty} \beta_n \cos(2n\pi z_2) = f(z_2) \ . \tag{44}$$

Since the integral of f is zero, the expansion does not contain any 0'th order term . Since f is smooth, the β_n converge very fast to zero.

Let β_n', $n \geq 1$, be given by

$$\beta_n' = -\beta_n \ , \tag{45}$$

and consider U^0 defined by (42) for $z_1 > 0$, respectively by (43) for $z_1 < 0$. Due to (44), (45), and the fact that $f(z_2) = \beta_0' - \beta_0$ for $r < z_2 < 1/2$, we observe that U^0 is continuous across the line segment $\{\ z_1 = 0 \ , \ \ r < z_2 < 1/2\ \}$. The fact that $\beta_n' = -\beta_n$ insures that $\frac{\partial}{\partial z_2}U^0$ is even in z_1, and thus automatically continuous across the line segment $\{\ z_1 = 0 \ , \ \ r < z_2 < 1/2\ \}$. We conclude that this U^0 is indeed harmonic in all of $\Phi(\Omega) \cap \{\ 0 < z_2 < 1/2\ \}$, and satisfies the boundary conditions

$$\frac{\partial}{\partial z_2}U^0 = 0 \quad \text{on } \{\ z_2 = 1/2\ \} \ \text{ and on } \{\ z \in \Phi(\Omega) \ , \ \ z_2 = 0\ \} \ .$$

The values of U^0 on $\Phi(\partial\Omega) \cap \{ 0 < z_2 \}$ (the remainder of the boundary) naturally depend on f, and correspond to a particular choice of ϕ.

Since β_0 and β_0' were chosen arbitrarily we have $\beta_0' \neq \beta_0$ in general. In the z coordinates β_0 and β_0' are the limits of U^0 at $z_1 = +\infty$ and $z_1 = -\infty$ respectively. In the x-coordinates these are the limits of u^0 as we approach the origin through the cusp on the right and through the cusp on the left respectively.

We have thus constructed a family of solutions which are discontinuous at the origin (and even in the x_1-axis). This should be the typical behaviour of solutions, when the data ϕ is odd in the x_2-axis (and even in the x_1-axis).

If the data ϕ is even in both the x_2- and the x_1-axis, then we must necessarily have $\beta_n' = \beta_n$ for all $0 \leq n$ (in the expansion (42) and (43)) and so u^0 has to be continuous at the origin. If the data ϕ is odd in the x_1-axis then it is very easy, again by separation of variables, to see that u^0 is continuous at the origin (its value is zero).

In all the cases considered above the function u^0 is C^∞ inside each of the cusps and all its derivatives (of order ≥ 1) vanish at the origin.

Let us now briefly return to the case $a_0 = \infty$. When ϕ is even in the x_1-axis, separation of variables readily gives that $U^\infty(z)$ must have the form

$$\begin{aligned} U^\infty(z_1, z_2) - c_0 &= \sum_{n=1}^{\infty} \beta_n \cos((2n+1)\pi z_2) e^{-(2n+1)\pi z_1} \quad \text{for } z_1 \text{ sufficiently positive} \ , \\ U^\infty(z_1, z_2) - c_0 &= \sum_{n=1}^{\infty} \beta_n' \cos((2n+1)\pi z_2) e^{(2n+1)\pi z_1} \quad \text{for } z_1 \text{ sufficiently negative} \ , \end{aligned}$$

where c_0 is the common value attained on the fibers (there is just one constant value, due to the evenness of the solution). It follows immediately that u^∞ is continuous at 0 (in the x-coordinates), and that all its derivatives vanish at 0. When ϕ is odd in the x_1-axis, so are u^∞ and U^∞. Separation of variables thus yields

$$U^\infty(z_1, z_2) - 2c_0 z_2 = \sum_{n=1}^{\infty} \beta_n \sin(2n\pi z_2) e^{-2n\pi z_1} \quad \text{for } z_1 \text{ sufficiently positive} \ , \tag{46}$$

$$U^\infty(z_1, z_2) - 2c_0 z_2 = \sum_{n=1}^{\infty} \beta_n' \sin(2n\pi z_2) e^{2n\pi z_1} \quad \text{for } z_1 \text{ sufficiently negative} \ , \tag{47}$$

where c_0 is the value attained on the upper fiber. However, in this case the requirement that u^∞ be H^1 in the x-coordinates, implies that the gradients ∇u^∞ and ∇U^∞ must be L^2 in the x- and in the z-coordinates respectively. Thus, c_0 must be equal to 0. It follows, using the representation (46) and (47), that u^∞ is continuous at $x = 0$ (it has value 0), and that, similarly, all its derivatives vanish at $x = 0$.

The fact that all the solutions are C^∞, when regarded as functions in just each individual cusp, would also follow from the analysis in [7].

As mentioned earlier, it would be very interesting to analyse the geometric setting, when the fibers are close but not quite touching, say, the cross-sections are ϵ apart vertically. A few things can be said related to the calculations carried out above, as the distance ϵ tends to 0. When the boundary value ϕ is odd in the x_2-axis but even in the x_1-axis, then the singularity mentioned above for u^0 gives rise to a gradient (an x_1-derivative $\frac{\partial}{\partial x_1} u_\epsilon^0(0)$) which becomes unbounded as ϵ tends to zero. The solution u^∞ for the case of a ϕ, which is even in the x_2-axis but odd in the x_1-axis, is related to the previous solution by harmonic conjugation (rotation of the gradient by 90 degrees). We thus in general, in this case, also obtain a gradient (an x_2-derivative $\frac{\partial}{\partial x_2}u_\epsilon^\infty(0)$) which becomes unbounded as ϵ tends to zero. This in spite of the fact that there is no irregularity in the "limiting" solution when the fibers touch. The rate at which this gradient becomes unbounded has actually been calculated in [2], for a special solution corresponding to uniform antiplane shear (see also [8]). For this special solution, the rate turns out to be $\epsilon^{-1/2}$; we believe this is the generic rate for the above mentioned symmetries in the boundary data. It should also be mentioned that for two touching fibers and $0 < a_0 < \infty$, Budiansky and Carrier ([2]) calculate a finite value for the stress $\left(a\frac{\partial}{\partial x_2}u\right)(0)$ of the same special (antiplane shear) solution. This calculation relates $\left(a\frac{\partial}{\partial x_2}u\right)(0)$ to the "shear at infinity".

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