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INFLUENCE OF DISTORTION IN THE HOMOGENIZATION OF FIBER-REINFORCED COMPOSITES

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We consider fiber-reinforced composites, where the fibers present some distortion in their longitudinal direction. The length scale of this distorsion is much larger than the spacing of the fibers. We derive asymptotic formulas for a conduction problem. Reiterated homogenization, i.e. homogenizing with respect to the smallest length scale, then with respect to the largest does not capture the effects of distortion. Instead, we use a representation formula to show how distortion influences the correction terms.

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1. Introduction

Fiber reinforced composites are most in use among composite materials. Automobiles, aircrafts, off-shore structures, skis, etc. have parts that are built with fiber composites, in order to combine lightweight and strength. The mechanics of such composites are very complex, involve several phenomena, at several length scales. Decohesion, debonding are the results of the local strength of stress fields and depend on the constituents of the fibers and the matrix, but may also depend on defects in the manufacturing such as the spacing between the fibers (some of them may touch), possible kinks or singularities in the fibers (for example at their extremities).

The theory of homogenization[6,8,15] idealizes a "real-life" fiber-reinforced composite as a periodic layered medium, where the material properties do not depend on the longitudinal direction of the fibers. Assuming the period to be small, the layered medium is replaced by an effective homogeneous material, that has the same response, in an average sense.

It is natural to question the validity of the homogenized model for prediction and design purposes. For one thing, the scale of variations in "real-life" is small, but not infinitesimal. Periodicity is not an assumption that the current manufacturing processes can guarantee. The relevance of the homogenized model also depends on the smoothness of the domain and of the loads. Homogenization may be regarded as a first step in the modeling of composites, and there is a great need for more understanding of the influence of "defects" in non-idealized situations.

In this paper, we attempt to take into account one category of defects, in the particular case of fiber-reinforced composites: the distortion of fibers. We study a model problem of a material where fibers are layered in the x_2 direction, but longitudinal distortion (in the x_2 variable) affects the geometry of the layers.

Distortion is present in all "real-life" fiber-composites and its effect is of high practical importance. In aeronautics, fibers are used that have a diameter of several micrometers, and that have longitudinal oscillations over distances of a few meters.[1] Several experimental and computational works show how theoretical bounds obtained for distortion-free models are affected (see e.g. Ref. 10 and the references therein).

We study a two-dimensional model problem where we make the following simplifying assumptions. Firstly, the medium is considered to be periodic, in both directions, but with different length scales. Distortion is modeled as smooth oscillations in the longitudinal direction of the fibers. Secondly, we deal with a conduction problem with a zeroth-order term (however, the results could be extended to elasticity). Thirdly, we consider the problem on the whole space $\mathbb{R}^2$.

This third assumption allows us to follow the approach developed by Babuška and Morgan in a series of papers.[2–4] These authors gave an integral representation of the solution u_ε to a conduction equation for a periodic medium with period $\varepsilon\,(0, 2\pi)^n$ filling the whole of $\mathbb{R}^n$, and studied the properties of the associated kernel. More precisely, they consider the following PDE in $H^1(\mathbb{R}^n)$

$$-\mathrm{div}(a_\varepsilon(x)\nabla u_\varepsilon) + a_{0,\varepsilon}(x)u_\varepsilon = f(x) \quad \text{in } \mathbb{R}^n, \tag{1.1}$$

where $(a_\varepsilon, a_{0,\varepsilon})(x) = (a, a_0)(x/\varepsilon)$ for some functions a, a_0, bounded above and below, and periodic on the cube $[0, 2\pi]^n$. Assuming $f \in L^2(\mathbb{R}^n)$, the solution to (1.1) has the representation

$$u_\varepsilon(x) = \int_{\mathbb{R}^n} \hat{f}(t)\,\phi(x/\varepsilon, \varepsilon, t)\,e^{it\cdot x}\,dt, \tag{1.2}$$

where $\hat{f}(t), t \in \mathbb{R}^2$, denotes the Fourier transform of f. The kernel $\phi(y, \varepsilon, t)$ solves a PDE in the reference cell of periodicity. The interest of this representation stems from the properties of analycity of the kernel, with respect to the period ε and with respect to the Fourier variable t. Approximations for u_ε can be obtained by expanding the kernel in powers of ε under the integral sign, if the load f is smooth. In this way, one recovers the results of classical homogenization.[2] In particular, corrector terms are easily found at any order of ε. One could also expand the kernel with respect to t (in particular for a nonsmooth load). Thus, different approximations of the kernel lead to different types of approximations of u_ε. Numerical comparisons of such different homogenizations are given in Ref. 4.

The representation (1.2) has the flavor of a Floquet–Bloch representation. However, determining the kernel ϕ, for a fixed Fourier frequency t, only requires the resolution of a PDE with a particular right-hand side, instead of the whole spectral information for that PDE. We refer to the work of Conca and Vanninathan[7] for the relationship between Bloch waves and homogenization (see also Ref. 17). In spite of being established only for unbounded domains, the representation (1.2) has been used also to construct special bases of finite elements, for elliptic problems with locally periodic microstructure. These bases guarantee that the computational cost is bounded independently of the scale of the periodic structure.[11,12]

In this work, we establish a similar integral representation for the potential. We are dealing however with two length scales: the period of the distortion (along the x_2 variable) is much larger than the spacing of the fibers (in the x_1 direction). More precisely, we assume that the coefficients have the form

$$(a_\nu, a_{0,\nu})(x_1, x_2) = (a, a_0)(x_1/\nu^2, x_2/\nu),$$

where a and a_0 are $[0, 2\pi]^2$-periodic scalar functions. The relationship between the x_1 and x_2 scales is consistent with the physical situation. Scales (ν^r, ν) with a different value for r could also be treated, but would lead to different asymptotic approximations. Let $f \in L^2(\mathbb{R}^2)$, we seek $u_\nu \in H^1(\mathbb{R}^2)$, solution to the elliptic differential equation

$$-\mathrm{div}(a_\nu(x)\nabla u_\nu) + a_{0,\nu}(x)u_\nu = f(x). \tag{1.3}$$

In this paper, we show that u_ν has a representation, similar to (1.2),

$$u_\nu(x) = \int_{\mathbb{R}^n} \hat{f}(t)\, \phi_\nu(x_1/\nu^2, x_2/\nu, t)\, e^{it\cdot x}\, dt. \tag{1.4}$$

In Sec. 3, we indicate how the results of Ref. 3 are easily generalized to show that ϕ is analytic on a complex neighborhood of any point (ν, t) with $\nu > 0, t \in \mathbb{R}^2$.

The situation concerning $\nu = 0$ is different. To explain why, let us examine the case of a single scale. Let $L^2_\#$ denote the space of square integrable functions which are periodic on $[0, 2\pi]^n$, and by $T(\varepsilon, t)$, the operator defined on $L^2_\#$ by

$$T(\varepsilon, t)u = -\mathrm{div}\big(a(y)\nabla(u(y)e^{i\varepsilon t\cdot y})\big)\, e^{-i\varepsilon t\cdot y} + \varepsilon^2 a_{0,\varepsilon}(x)u(y). \tag{1.5}$$

In Ref. 3, the kernel $\phi(\cdot, \varepsilon, t)$ is found as the periodic solution to $T_{\varepsilon,t}\phi(y, \varepsilon, t) = \varepsilon^2$. As ε tends to 0, the limiting PDE degenerates: 0 is an eigenvalue of $T_{0,t}$, with an eigenspace of dimension 1. A theorem of separation of spectrum[9] allows one to write the kernel as

$$\phi(\cdot, \varepsilon, t) = \frac{\varepsilon^2}{\lambda(\varepsilon, t)} P_{\varepsilon,t}(1) + \varepsilon^2 R_{\varepsilon,t}(1),$$

where $P_{\varepsilon,t}$ is a projection onto the eigenspace of dimension 1 associated with an eigenvalue $\lambda(\varepsilon, t)$, such that $\lambda(0, t) = 0$. Moreover, the operators P_ε, R_ε and the eigenvalue $\lambda(\varepsilon, t)$ depend analytically on (ε, t) in a complex neighborhood of any point $(0, t_0), t_0 \in \mathbb{R}^2$.

The analycity of $\phi(\cdot, \varepsilon, t)$ follows upon showing that $\lambda(\varepsilon, t) \sim \varepsilon^2$ as $\varepsilon \to 0$, so that $\varepsilon^2/\lambda(\varepsilon, t)$ is analytic.

In the case of two scales, the situation is more complicated when $\nu = 0$. The corresponding operator $T(0, t)$ has a kernel of infinite dimension and the theorem of separation of spectrum quoted above cannot be applied uniformly with respect to ν. Nevertheless, we show that ϕ_ν can be expanded in powers of ν in a sector of points (ν, t) that contains $[0, \infty) \times \mathbb{R}^2$.

The paper is organized as follows: Sec. 2 describes the setting and the notations of the problem and states the main results. Following Ref. 3, we briefly indicate in Sec. 3 how one can prove analytic of ϕ_ν in the neighborhood of a point (ν, τ) with $\nu > 0$. In Sec. 4, we proceed to a formal asymptotic expansion of ϕ_ν around $\nu = 0$. Section 5 is devoted to showing the convergence of the expansion in a neighborhood of points $(0, t)$. In view of these results, the representation formula (1.4) is justified in Sec. 6. Finally, we show in Appendix A why the strategy of Ref. 3 cannot be used to show analytic around $\nu = 0$ and in Appendix B we derive the form of the constant term in the first corrector.

2. Setting of the Problem and Statement of the Main Results

Throughout the paper, we use the following notations. The reference period $[0, 2\pi]^2$ is called Y. A subscript # refers to spaces of Y-periodic functions and $\|u\|_{p,\Omega}$ denotes the norm of the Sobolev space $H^p(\Omega)$. The norm of $H^1(Y)$ may simply be denoted by $\|u\|_p$, when the context is unambiguous. The symbols ∂_i and $\int_{y_j} f$ respectively stand for $\partial/\partial y_i$ and $\int_0^{2\pi} f(y_j)dy_j$.

We will use the following definition of analytic functions of several complex variables, with value in a Banach space $\mathcal{B}$ (see Ref. 9).

Definition 2.1. A function $f \in \mathcal{G} \subset \mathbb{C}^3 \to \mathcal{B}$ is analytic if for each $z_0 \in \mathcal{G}$, there exists an open ball $B(z_0, \rho), \rho > 0$, and coefficients $(f_\alpha) \subset \mathcal{B}$, such that $\sum_{0 \leq |\alpha|} f_\alpha (z - z_0)^\alpha$ converges in $\mathcal{B}$ to $f(z)$, for each $z \in B(z_0, \rho)$.

We assume that the coefficients a_ν and $a_{0,\nu}$ describe a medium layered in the x_1-direction, with layers that are oscillating in the x_2-direction, on a scale much larger than the spacing of the fibers. More precisely, the material properties are defined by

$$(a_\nu, a_{0,\nu})(x_1, x_2) = (a, a_0)(x_1/\nu^2, x_2/\nu) \\ = (\tilde{a}, \tilde{a_0})(x_1/\nu^2 + \gamma(x_2/\nu)), \tag{2.1}$$

where $\tilde{a}$ and $\tilde{a}_0$ are bounded periodic functions and γ is a smooth 2π-periodic function such that $\int_{y_2} \gamma = 0$. We further assume uniform ellipticity, i.e. that there exist constants $0 < \alpha < M$, such that $\alpha \leq \tilde{a}(y), \tilde{a}_0(y) \leq M$, for a.e. y in $[0, 2\pi]$.

The problem of finding approximations for u_ν could be treated as a problem of reiterated homogenization: one would homogenize first with respect to the fastest scale, then homogenize the resulting problem with respect to the slower one. However, for this particular medium, homogenization with respect to the fastest scale yields a homogeneous operator with respect to both variables (x_1, x_2) where distortion has disappeared. Although one could obtain results about correctors in this manner, we think that their convergence is easier to establish using a representation formula like (1.2).

We seek u_ν in the form

$$u_\nu(x_1, x_2) = \int_{\mathbb{R}^2} \hat{f}(t)\, \phi_\nu(x_1/\nu^2, x_2/\nu, t)\, e^{it\cdot x}\, dt, \tag{2.2}$$

where $\phi_\nu(y_1, y_2, t)$ is Y-periodic with respect to (y_1, y_2). Introduction of this expression into (2.1) and performing the change of variables $(x_1/\nu^2, x_2/\nu) = (y_1, y_2)$ shows that ϕ formally solves

$$T(\nu, t)\phi_\nu = -\partial_1 \left[a(y_1, y_2)\partial_1 \left(\phi_\nu(y_1, y_2, t)e^{i\xi\cdot y}\right)\right] e^{-i\xi\cdot y} \\ - \nu^2\partial_2 \left[a(y_1, y_2)\partial_2 \left(\phi_\nu(y_1, y_2, t)e^{i\xi\cdot y}\right)\right] e^{-i\xi\cdot y} \\ + \nu^4 a_0\phi_\nu(y_1, y_2, t) = \nu^4, \tag{2.3}$$

where $\xi = \xi(\nu, t) = (\nu^2 t_1, \nu t_2)$.

We consider $T(\nu, t)$, defined by (2.3), as an unbounded operator on $L^2_\#(Y)$ with domain $H^2_\#(Y)$. In addition, we consider the following sesquilinear form on $H^1_\#(Y) \times H^1_\#(Y)$, with values in $\mathbb{C}$.

$$\mathbf{A}(\nu, t)[v, w] = \int_Y a(y)\partial_1 \left(ve^{i\xi\cdot y}\right) \partial_1 \left(\overline{w}e^{-i\xi\cdot y}\right) dy \\ + \nu^2 \int_Y a(y)\partial_2 \left(ve^{i\xi\cdot y}\right) \partial_2 \left(\overline{w}e^{-i\xi\cdot y}\right) dy \\ + \int_Y a_0(y)v\overline{w}\, dy. \tag{2.4}$$

As usual in homogenization, we introduce auxiliary functions: Let $\tilde{\chi}$ denote the solution in $H^1_\#(0,2\pi)$ to the 1-D variational problem

$$\forall\, v \in H^1_\#(0,2\pi), \qquad \int_0^{2\pi} \tilde{a}\left(\frac{d\chi}{dz}+1\right)\frac{dv}{dz} = 0,$$

normalized to have 0 mean value on $[0,2\pi]$, and let

$$\begin{cases} \chi_1(y_1,y_2,t) = it_1\,\tilde{\chi}(y_1+\gamma(y_2)), \\ \chi_2(y_1,y_2,t) = it_2\,\gamma'(y_2)\,\tilde{\chi}(y_1+\gamma(y_2)). \end{cases} \tag{2.5}$$

It is easy to check that $\chi_j \in H^1_\#(Y)$ solves

$$-\partial_1(a(y_1,y_2)\partial_1\chi_j(y_1,y_2,t)) = it_j\,\partial_j a(y_1,y_2), \tag{2.6}$$

and that its Y-mean value is equal to 0.

This paper is devoted to proving the following results:

Theorem 2.1. *There exists a sector $\mathcal{G}$ in $\mathbb{C}^3$, that contains $(0,\infty)\times\mathbb{R}^2$, such that, for $(\nu,t)\in\mathcal{G}$, there exists a unique solution $\phi_\nu(\cdot,t)\in H^1_\#(Y)$ to* (2.3). *Moreover, the mapping $(\nu,t)\to\phi_\nu(\cdot,t)$ is analytic in $\mathcal{G}$.*

Theorem 2.2. *For any $t\in\mathbb{R}^2$, the function $\phi_\nu(\cdot,t)$ can be expanded as*

$$\phi_\nu(y_1,y_2,t) = \phi_0(y_1,y_2,t) + \sum_{j\geq 2} \nu^j\,\phi_j(y_1,y_2,t), \tag{2.7}$$

where each function $\phi_j(y_1,y_2,t)\in H^1_\#(Y)$ is analytic in t. The expansion is convergent in a neighborhood of $(0,t)\cap\mathcal{G}$ and the following estimate holds:

$$\left\|\phi_\nu(y_1,y_2,t) - \phi_0(y_1,y_2,t) - \sum_{2\leq j\leq k} \nu^j\,\phi_j(y_1,y_2,t)\right\|_{1,Y} \leq C_k(t)\nu^{k-2}$$

where $C_k(t)$ is a constant independent of ν. The function ϕ_0 does not depend on distortion

$$\phi_0(t) = \frac{4\pi^2}{-it_1\int_Y a\partial_1\chi^1 + \int_Y |t|^2 a + a_0}. \tag{2.8}$$

Influence of distortion is only felt at the order ν^2, and ϕ_2 has the expression $\phi_2(y_1,y_2,t) = \phi_0(t)\,\chi^1(y_1,y_2,t) + \rho_2(y_2,t) + \sigma_2(t)$, where ρ_2 and σ_2, defined by (4.17), (4.22) *depend on γ.*

Remark. The form of the zeroth-order term was expected as the zeroth-order term of the homogenization of a stratified medium. With a different relative scaling between the spacing of the fibers and distortion, influence of the latter will affect terms of different orders.

With these two theorems, the proof of the following representation theorem can be easily adapted from the proof of theorem 11 of Ref. 2. Let $H^1_\eta(\mathbb{R}^2)$ denote the weighted Sobolev space equal to the completion of $\mathcal{C}^\infty$ functions with compact support in $\mathbb{R}^2$ for the norm

$$\|u\|_{1,\eta}^2 = \int_{\mathbb{R}^2} \left(|u|^2 + |\nabla u|^2\right) e^{2\eta|x|} dx.$$

.

Theorem 2.3. *Let $f \in L^2(\mathbb{R}^2)$. There exists $\eta_0 > 0$ such that for $\eta < \eta_0$, the solution u_ν of* (1.3) *has the following representation*:

$$u_\nu(x) = \lim_{N\to\infty} \frac{1}{2\pi} \int_{|t|\le N} \hat{f}(t)\phi_\nu(x_1/\nu^2, x_2/\nu, t)e^{it\cdot x} dt,$$

where the integral is defined as a Bochner integral of $H^1_{-\eta}$-valued functions.

Theorem 2.4. *Let $f \in L^2(\mathbb{R}^2)$ and u_ν be the solution to* (1.3)*. Let $\eta < \eta_0$ given by Theorem* 2.3*. Then*

$$\left\| u_\nu(x) - \frac{1}{2\pi} \int_{\mathbb{R}^2} \hat{f}(t)\left(\phi_0(t) + \nu^2\phi_2(x_1/\nu^2, x_2/\nu, t)\right) e^{it\cdot x} dt \right\|_{0,-\eta} \to 0,$$

as $\nu \to 0, \nu \in \mathcal{G}$.

3. Proof of Theorem 2.1

In this section, we follow closely Ref. 3 and only sketch the argument. Let

$$\mathcal{G} = \{(\nu, t) \in \mathbb{C}^3, \text{ such that } 0 < |\mathrm{Im}(\nu)| < (\sqrt{2}-1)\mathrm{Re}(\nu)\}.$$

Lemma 3.1. *For $(\nu,t) \in \mathcal{G}$, there exists a pair of real-valued functions $\delta(\nu,t) > 0$, $M(\nu,t) > 0$, and a continuous real-valued function $\mu(\nu,t)$ such that, for all $(u,v) \in (H^1_\#(Y))^2)$,*

$$\begin{aligned} |\mathbf{A}(\nu,t)[u,v]| &\le M(\nu,t)\|u\|_1\|v\|_1 \\ \delta(\nu,t)\|v\|_1^2 &\le \mathrm{Re}(\mathbf{A}(\nu,t)[v,v]) + \mu(\nu,t)\|v\|_0^2. \end{aligned} \tag{3.1}$$

Proof. Using the hypothesis on the coefficients a, a_0 and the Cauchy–Schwartz inequality, we can easily check that, for $(u,v) \in (H^1_\#(Y))^2$,

$$\begin{aligned} |\mathbf{A}(\nu,t)[u,v]| = \bigg| &\int_Y a\partial_1 u\, \partial_1\overline{v} + it_1\nu^2 \int_Y a(u\partial_1\overline{v} - \overline{v}\partial_1 u) \\ &+ \nu^2 \int_Y a\partial_2 u\, \partial_2\overline{v} + it_2\nu^3 \int_Y a(u\partial_2\overline{v} - \overline{v}\partial_2 u) \\ &+ \nu^4 \int_Y (a|t|^2 + a_0)u\overline{v}\bigg| \\ \le\ & M[1 + (1+|t_1|)|\nu|^2 + |t_2||\nu^3| + (1+|t|^2)|\nu|^4]\|u\|_1\|v\|_1. \end{aligned}$$

We also have for all $\delta_1 > 0, \delta_2 > 0$

$$\mathrm{Re}(\mathbf{A}(\nu,t)[v,v]) \geq (1-\delta_1)\int_Y a|\partial_1 v|^2 + (\mathrm{Re}(\nu^2)-\delta_2)\int_Y a|\partial_2 v|^2$$
$$-\left(\frac{|\nu^2 t_1|^2}{\delta_1} + \frac{|\nu^3 t_2|^2}{\delta_2}\right)\int_Y a|v|^2 + \mathrm{Re}(\nu^4)\int_Y (a|t|^2 + a_0)|v|^2,$$

and choosing $\delta_1 = 1/2, \delta_2 = |\nu|^2/2$, we get

$$\mathrm{Re}(\mathbf{A}(\nu,t)[v,v]) + 2M\left(\frac{|\nu^2 t_1|^2}{2} + \frac{|\nu|^4|t_2|^2}{2}\right)\int_Y |v|^2$$
$$\geq \alpha/2\int_Y a|\partial_1 v|^2 + \alpha(Re(\nu^2) - |\nu|^2/2)\int_Y a|\partial_2 v|^2$$
$$+\alpha\,\mathrm{Re}(\nu^4)(1+|t|^2)\int_Y |v|^2.$$

If $(\nu,t)\in\mathcal{G}$, $\delta = \min\left(\alpha/2, \alpha(\mathrm{Re}(\nu^2) - |\nu|^2/2), \mathrm{Re}(\nu^4)(1+|t|^2)\right) > 0$ verifies the claim. □

Lemma 3.2. *When $(\nu,t)\in\mathcal{G}\cap\mathbb{R}^3$ with $\nu > 0$, the sesquilinear form $\mathbf{A}(\nu,t)$ satisfies the hypothesis of the Lax–Milgram lemma.*

Proof. Using the notation $\xi = (\nu t_1, \nu^2 t_2)\in\mathbb{R}^2$, we notice first that for $w\in H^1(Y)$,

$$\int_Y |\nabla(we^{-i\xi\cdot y})|^2 + |we^{-i\xi\cdot y}|^2 \leq \int_Y |\nabla w|^2 - 2\xi\cdot\mathrm{Re}(w\nabla\overline{w}) + (1+|\xi|^2)|w|^2$$
$$\leq (1+2|\xi|^2)\|w\|_{1,Y}^2 \leq 2(1+|\xi|)^2\|w\|_{1,Y}^2.$$

Applying this inequality to $w = ve^{i\xi\cdot y}$ shows that

$$\forall\, v\in H^1_{\#}(Y),\quad \frac{1}{\sqrt{2}(1+|\xi|)}\|v\|_1 \leq \|ve^{i\xi\cdot y}\|_1. \tag{3.2}$$

Next, for $v\in H^1_{\#}(Y)$, the above inequality yields

$$\mathbf{A}(\nu,t)[v,v] \geq \alpha\nu^2\int_Y |\nabla(ve^{i\xi\cdot y})|^2 + \nu^4\int_Y a_0|v|^2$$
$$\geq \min\left(\frac{\alpha\nu^2}{2(1+|\xi|)^2}, \alpha\nu^4\right)\|v\|_1,$$

which shows the coercivity of $\mathbf{A}(\nu,t)$. □

According to Theorem 15 in Ref. 3, for each $(\nu,t)\in\mathcal{G}$, the above lemma allows us to define a closed operator $T(\nu,t)\in\mathcal{C}(H^1_{\#}(Y), L^2_{\#}(Y))$ with domain $D(T(\nu,t))$ dense in $H^1_{\#}(Y)$, such that

$$\begin{cases}\forall\, u\in D(T(\nu,t))\\ \forall\, v\in H^1_{\#}(Y),\end{cases}\qquad \mathbf{A}(\nu,t)[u,v] = \int_Y T(\nu,t)u\,\overline{v}.$$

Equation (2.3) for the kernel ϕ_ν can be rewritten in the form

$$T(\nu,t)\phi_\nu(\cdot,\cdot,t)=\nu^4.$$

When $(\nu_0,t_0)\in\mathcal{G}\cap\mathbb{R}^3$ with $\nu>0$, Lemma 3.2 shows that $T(\nu_0,t_0)$ is invertible, i.e. 0 is in the resolvent set of $T(\nu_0,t_0)$. Results of analytic perturbation theory[3,9,16] imply that $T(\nu,t)$ is invertible in a complex neighborhood of (ν_0,t_0), on which $\phi_\nu=\nu^4T(\nu,t)^{-1}(1)$ is analytic with respect to (ν,t).

4. Formal Expansion

4.1. *Preliminaries*

In this section, we formally derive an expansion of $\phi_\nu(\cdot,\cdot,t)$ with respect to $\nu>0$, as $\nu\to 0$. Inserting

$$\phi_\nu(y_1,y_2,\nu,t)=\sum_j \nu^j\phi_j(y_1,y_2,\nu,t), \tag{4.1}$$

in Eq. (2.3), and regrouping the terms with the same powers of ν we obtain

$$\forall\, v\in H^1_\#(Y),\quad \mathbf{A}_0[\phi_j,v]+\mathbf{A}_2[\phi_{j-2},v]+\mathbf{A}_3[\phi_{j-3},v]+\mathbf{A}_4[\phi_{j-4},v]=\delta_{j,4}\int_Y\overline{v}, \tag{4.2}$$

where we have used the following notations

$$\begin{cases}\mathbf{A}_0[\phi,v]=\displaystyle\int_Y a\partial_1\phi\,\partial_1\overline{v}\\ \mathbf{A}_2[\phi,v]=\displaystyle\int_Y a\partial_2\phi\,\partial_2\overline{v}+it_1\int_Y a\phi\partial_1\,\overline{v}-a\partial_1\phi\,\overline{v}\\ \mathbf{A}_3[\phi,v]=it_2\displaystyle\int_Y a\phi\partial_2\,\overline{v}-a\partial_2\phi\,\overline{v}\\ \mathbf{A}_4[\phi,v]=\displaystyle\int_Y (a|t|^2+a_0)\phi\,\overline{v},\end{cases} \tag{4.3}$$

and where $\phi_j\equiv 0$ for $j<0$.

We notice that if $c\equiv c(t)$ does not depend on y_1 and y_2, we have

$$\forall\, v\in H^1_\#(Y),\quad \begin{cases}\mathbf{A}_2[c,v]=it_1\displaystyle\int_Y a\partial_1\overline{v}=-c\mathbf{A}_0[\chi_1,v]\\ \mathbf{A}_3[c,v]=it_1\displaystyle\int_Y a\partial_2\overline{v}=-c\mathbf{A}_0[\chi_2,v].\end{cases} \tag{4.4}$$

Furthermore, since $\int_{y_1} a$ is independent of y_2, these relations simplify if $w\in H^1_\#(Y)$ only depends on y_2

$$\begin{cases}A_2[c,w]=0,\\ A_3[c,w]=0.\end{cases} \tag{4.5}$$

Each of the Eqs. (4.2) can be viewed as a system of equations of the form

$$\mathbf{A}_0[\phi_j, v] = R_j(v), \tag{4.6}$$

in the variable y_1, parametrized by y_2. A necessary condition for the existence of a y_1-periodic solution $\phi_j(\cdot, y_2)$ is

$$\forall\, w = w(y_2) \in H^1_\#(0, 2\pi), \quad R_j(w) = 0. \tag{4.7}$$

This condition imposes constraints on the form of the expansion, but is not sufficient for the solvability of (4.6). We will nevertheless be able to construct the correctors, due to the special form of the coefficient a. To this end, we use the following lemmas.

Lemma 4.1. *Let $\phi = \phi(y_1) \in H^1_\#(0, 2\pi)$. Then the function $\Phi(y_1, y_2) - \phi(y_1 + \gamma(y_2))$ is in $H^1_\#(Y)$ and*

$$\nabla\Phi = \phi'(y_1 + \gamma(y_2)) \begin{pmatrix} 1 \\ \gamma'(y_2) \end{pmatrix}.$$

Proof. (a) The conclusion is obvious when $\phi \in \mathcal{C}^\infty_\#(0, 2\pi)$.

(b) Let $\phi \in H^1_\#(0, 2\pi)$. Since γ is smooth, the functions $\phi(y_1+\gamma(y_2)), \phi'(y_1+\gamma(y_2))$ and $\gamma'(y_2)\phi'(y_1+\gamma(y_2))$ are in $L^2_\#(Y)$. Moreover, if $(\phi_n) \subset \mathcal{C}^\infty_\#(0, 2\pi)$ approximates ϕ in $H^1_\#(0, 2\pi)$, it follows from the Fubini and the Lebesgue theorem that for all $w \in H^1_\#(Y)$,

$$\begin{aligned} \int_Y \phi(y_1 + \gamma(y_2))\, \partial_i \overline{w} &= \lim_n \int_{y_2} \int_{y_1} \phi_n(y_1 + \gamma(y_2))\, \partial_i \overline{w} \\ &= -\lim_n \int_{y_2} \int_{y_1} \partial_i \gamma(y_2) \phi_n'(y_1 + \gamma(y_2))\, \overline{w} \\ &= -\int_Y \partial_i \gamma(y_2) \phi'(y_1 + \gamma(y_2))\, \overline{w}. \end{aligned}$$

□

Lemma 4.2. *Let $\kappa \in L^2_\#(0, 2\pi), \lambda \in H^1_\#(0, 2\pi)$ and $w \in H^1_\#(Y)$. Then*

$$\begin{aligned} &\int_Y a(y_1 + \gamma(y_2))\kappa(y_1 + \gamma(y_2))\lambda(y_2)\, \partial_2 \overline{w} \\ &\quad = \int_Y a(y_1 + \gamma(y_2))\kappa(y_1 + \gamma(y_2))\gamma'(y_2)\lambda(y_2)\, \partial_1 \overline{w} \\ &\qquad - \int_Y a(y_1 + \gamma(y_2))\kappa(y_1 + \gamma(y_2))\lambda'(y_2)\, \overline{w}. \end{aligned}$$

Proof. If κ is smooth, the relations (2.6) give for any $w \in H^1_\#(Y)$

$$\begin{aligned} \int_Y a(y_1 + \gamma(y_2))\, \partial_2 \overline{w} &= \int_Y a(y_1 + \gamma(y_2)) \partial_1 \left[\gamma'(y_2) \tilde{\chi}(y_1 + \gamma(y_2))\right] \partial_1 \overline{w} \\ &= \int_Y a(y_1 + \gamma(y_2))\, \partial_1 \left[\gamma'(y_2) \overline{w}\right]. \end{aligned}$$

Thus, we have

$$\begin{aligned}\int_Y & a(y_1+\gamma(y_2))\kappa(y_1+\gamma(y_2))\lambda(y_2)\,\partial_2\overline{w} \\ &= \int_Y a\partial_2(\kappa\lambda\overline{w}) - \int_Y a(\kappa'\gamma'\lambda+\kappa\lambda')\,\overline{w} \\ &= \int_Y a\partial_1(\kappa\gamma'\lambda\overline{w}) - \int_Y a(\kappa'\gamma'\lambda+\kappa\lambda')\,\overline{w} \\ &= \int_Y a\kappa\gamma'\lambda\,\partial_1\overline{w} - \int_Y a\kappa\lambda'\,\overline{w}.\end{aligned}$$

The general case follows by approximating κ by smooth functions. $\square$

We now consider the equation for $\psi \in H^1_\#(Y)$

$$\begin{cases}\forall\, v \in H^1_\#(Y), \\ \mathbf{A}_0[\psi, v] + \mathbf{A}_2[\psi_2, v] + \mathbf{A}_3[\psi_3, v] + \mathbf{A}_4[\psi_4, v] = \displaystyle\int_Y g\overline{v},\end{cases} \tag{4.8}$$

where the ψ_i's $1 \le i \le 3$, and g are given functions which satisfy the following assumptions:

(i) Each function ψ_i has the form

$$\psi_i(y_1, y_2) = \sum_{j=1}^{N(i)} \kappa_i^j(y_1+\gamma(y_2))\lambda_i^j(y_2) + \rho_i(y_2) + \sigma_i$$

where $\kappa_i^j \in H^1_\#(0, 2\pi)$, where λ_i^j, ρ_i are smooth and 2π-periodic with $\int_0^{2\pi} \rho_i(y) = 0$, and where σ_i does not depend on y_1 and y_2.

(ii) The function $g = g(y_1+\gamma(y_2), y_2)$ is smooth with respect to its second variable, and for a.e. y_2 it defines a $L^2_\#$-periodic function of y_1.

(iii) For all $w = w(y_2) \in H^1_\#(0, 2\pi)$, we have

$$\mathbf{A}_2[\psi_2, w] + \mathbf{A}_3[\psi_3, w] + \mathbf{A}_4[\psi_4, w] = \int_Y g\overline{w}. \tag{4.9}$$

Lemma 4.3. *There exists a unique solution $\psi \in H^1_\#(Y)$ to* (4.8) *determined up to a function of y_2 only. It has the form*

$$\psi(y_1, y_2) = \tau(y_1+\gamma(y_2), y_2) + \rho(y_2) + \sigma, \tag{4.10}$$

where τ is a finite sum of products of a $H^1_\#(0, 2\pi)$-function of $y_1+\gamma(y_2)$ times a smooth function of y_2, ρ has average 0, *and σ does not depend on y_1, y_2.*

Proof. We show that (4.8) can be interpreted as an equation for functions of y_1, parametrized by y_2. To this end, we transform all the terms that contain $\partial_2\overline{v}$ in terms that only involve $\overline{v}$ or $\partial_1\overline{v}$. Using Lemma 4.2, we compute for $v \in H^1_\#(Y)$ and for κ, λ functions of one variable only

$$\begin{aligned}
&\mathbf{A}_2[\kappa(y_1+\gamma(y_2))\lambda(y_2), v] \\
&\quad = \int_Y a\partial_2(\kappa\lambda)\,\partial_2\overline{v} + it_1\int_Y a\left[\kappa\lambda\,\partial_1\overline{v} - \overline{v}\,(\partial_1\kappa)\lambda\right] \\
&\quad = \int_Y a\partial_1\left[\kappa\lambda(\gamma')^2\right]\,\partial_1\overline{v} + \int_Y a\left[\kappa\lambda'\gamma'\,\partial_1\overline{v} - \overline{v}\,\partial_1(\kappa\lambda'\gamma')\right] \\
&\qquad - \int_Y a\left[\partial_1\kappa(\lambda\gamma'') + \kappa\lambda''\right]\overline{v} + it_1\int_Y a\left[\kappa\lambda\,\partial_1\overline{v} - \overline{v}\,(\partial_1\kappa)\lambda\right].
\end{aligned}$$

In a similar fashion,

$$\begin{aligned}
&\mathbf{A}_3[\kappa(y_1+\gamma(y_2))\lambda(y_2), v] \\
&\quad = it_2\int_Y a\left[\kappa\lambda\,\partial_2\overline{v} - \overline{v}\,\partial_2(\kappa\lambda)\right] \\
&\quad = it_2\int_Y a\left[\kappa\lambda\gamma'\,\partial_1\overline{v} - \overline{v}\,\partial_1\kappa\lambda\gamma'\right] - it_2\int_Y a\kappa\lambda'\overline{v}.
\end{aligned}$$

Consequently, (4.8) can be rewritten in the form

$$\int_Y a\partial_1\psi\,\partial_1\overline{v} = \int_Y R_1\,\partial_1\overline{v} + R_2\,\overline{v},$$

where R_1 and R_2 are sums of products of a periodic function of $y_1+\gamma(y_2)$ times a smooth periodic function of y_2. Choosing $v = \varphi(y_1)w(y_2)$, we see that we must have for a.e. y_2 and for all $\varphi \in H^1_\#(0, 2\pi)$

$$\int_{y_1} a\partial_1\psi\,\partial_1\overline{\varphi} = \int_{y_1} R_1\,\partial_1\overline{\varphi} + R_2\,\overline{\varphi}. \tag{4.11}$$

Moreover, the condition (4.9) shows that $\int_{y_1} R_2 = 0$, so that (4.11) can be solved uniquely in $H^1_\#(0, 2\pi)$ for a.e. y_2. Since the right-hand side depends smoothly on y_2 and is y_2-periodic, we can construct $\psi \in H^1_\#(Y)$ with the form (4.10), solution to (4.8). □

4.2. *Construction of the correctors*

We now construct the first terms in the asymptotic expansion (4.1), i.e. solutions to (4.2) that satisfy the constraint (4.7). We easily check that we can choose ϕ_0, ϕ_1 independent of y_1. Since the average of $a(y_1, y_2)$ with respect to y_1 does not depend on y_2, the condition (4.7) for $j = 2$ reduces to

$$\begin{aligned}
\forall\, w = w(y_2) \in H^1_\#(0, 2\pi), \quad A_2[\phi_0, w] &= \int_Y a\partial_2\phi_0\,\partial_2\overline{w} \\
&= \left(\int_{y_1} a\right)\int_{y_2} \partial_2\phi_0\,\partial_2\overline{w},
\end{aligned} \tag{4.12}$$

from which we deduce that $\phi_0 \equiv \phi_0(t)$ does not depend on y_2 either. In view of (4.5), when $j = 3$ the condition (4.7) takes the form

$$\forall\, w = w(y_2) \in H^1_\#(0, 2\pi), \quad \mathbf{A}_2[\phi_1, w] = 0,$$

and thus $\phi_1 \equiv \phi_1(t)$ is also independent of y_1, y_2.

With this form of ϕ_0, ϕ_1, Eq. (4.2) for $j = 2$ and $j = 3$ can be solved: using (4.5), they reduce to

$$\forall\, v \in H^1_\#(0, 2\pi), \quad \begin{cases} \mathbf{A}_0[\phi_2 - \phi_0\chi_1, v] = 0, \\ \mathbf{A}_0[\phi_3 - \phi_1\chi_1 - \phi_0\chi_2, v] = 0, \end{cases}$$

and the general form of solutions is

$$\begin{cases} \phi_2 = \phi_0\chi_1 + \rho_2(y_2, t) + \sigma_2(t), \\ \phi_3 = \phi_1\chi_1 - \phi_0\chi_2 + \rho_3(y_2, t) + \sigma_3(t), \end{cases}$$

where each ρ_j is a y_2-periodic function with average 0. Since we are seeking ϕ_j in $H^1_\#(Y)$, we may assume $\rho_j \in H^1_\#(0, 2\pi)$.

The value of ϕ_0 is obtained from (4.7) when $j = 4$

$$\forall\, w \in H^1_\#(0, 2\pi), \quad \mathbf{A}_2[\phi_2, w] + \mathbf{A}_3[\phi_1, w] + \mathbf{A}_4[\phi_0, w] = \int_Y \overline{w}. \tag{4.13}$$

Noticing that $\mathbf{A}_3[\phi_1, w(y_2)] = 0$, we choose $w \equiv 1$ and substitute the expression of ϕ_2 to get

$$\mathbf{A}_2[\phi_2, 1] + \mathbf{A}_4[\phi_0, 1] = \mathbf{A}_2[\phi_0\chi_1, 1] + \mathbf{A}_4[\phi_0, 1] = |Y|. \tag{4.14}$$

We conclude that

$$\phi_0 = \frac{|Y|}{\mathbf{A}_4[1,1] + \mathbf{A}_2[\chi_1, 1]} = \frac{4\pi^2}{\int_Y (a|t|^2 + a_0) - it_1 \int_Y a\partial_1\chi_1}. \tag{4.15}$$

The above expression is well-defined, as its denominator is positive: indeed,

$$\int_Y (a|t|^2 + a_0) - it_1 \int_Y a\partial_1\chi_1 = \int_Y (at_2^2 + a_0) + \mathbf{A}_0[\chi_1 + it_1y_1, \chi_1 + it_1y_1] > 0.$$

With this choice of ϕ_0, Eq. (4.13), viewed as an equation for $\rho_2(y_2, t)$ in the variable y_2, becomes

$$\forall\, w = w(y_2) \in H^1_\#(0, 2\pi),$$
$$\mathbf{A}_2[\rho_2, w] = \int_Y \overline{w} - \mathbf{A}_2[\phi_0\chi_1 + \sigma_2, w] - \mathbf{A}_4[\phi_0, w].$$

Using (4.4), we obtain

$$\left(\int_{y_1} a\right) \int_{y_2} \partial_2\rho_2\, \partial_2\overline{w} = \int_Y \overline{w} - \mathbf{A}_2[\phi_0\chi_1, w] - \mathbf{A}_4[\phi_0, w], \tag{4.16}$$

where the right-hand side vanishes when $w \equiv 1$. It follows that ρ_2 is uniquely determined in $H^1_\#(0, 2\pi)$, with the normalization $\int_Y \rho_2 = 0$. Setting W equal to w

minus its average, the above equation reduces to

$$\left(\int_{y_1} a\right)\int_{y_2} \partial_2\rho_2\,\partial_2\overline{W} = -\int_{y_2}\left(\int_{y_1} a\phi_0\partial_1\chi_1\right)\gamma'(y_2)\,\partial_2\overline{W},$$

and since we assumed that $\int_{y_2}\gamma = 0$, we conclude that

$$\rho_2(y_2) = -\phi_0\frac{\int_{y_1} a\partial_1\chi_1}{\int_{y_1} a}\,\gamma(y_2). \tag{4.17}$$

Finally, we consider Eq. (4.2) with $j = 4$:

$$\begin{aligned}&\forall\, v \in H^1_\#(Y),\\ &\mathbf{A}_0[\phi_4, v] + \mathbf{A}_2[\phi_0\chi_1 + \rho_2 + \sigma_2, v] + \mathbf{A}_3[\phi_1, v] + \mathbf{A}_4[\phi_0, v] = \int_Y \overline{v}.\end{aligned} \tag{4.18}$$

The relations (4.4) allow us to group the unknown quantities to obtain

$$\begin{aligned}&\forall\, v \in H^1_\#(Y),\\ &\mathbf{A}_0[\phi_4 - \sigma_2\chi_1 - \phi_1\chi_2, v] = -\mathbf{A}_2[\phi_0\chi_1 + \rho_2, v] - \mathbf{A}_4[\phi_0, v] + \int_Y \overline{v},\end{aligned} \tag{4.19}$$

where all the terms on the right-hand side have been determined. Since condition (4.9) reduces to (4.13), it follows from Lemma 4.3 that $\phi_4 - \sigma_2\chi_1 - \phi_1\chi_2$ can be determined from this equation in $H^1_\#(Y)$, up to a function $\rho_4(y_2, t) + \sigma_4(t)$ (where ρ_4 has average 0).

An induction argument gives the general form of the ϕ_j's and the following:

Theorem 4.1. *The jth term in the expansion of ϕ can be constructed with the form*

$$\phi_j(y_1, y_2, t) = \tau_j(y_1 + \gamma(y_2), y_2, t) + \rho_j(y_2, t) + \sigma_j(t), \tag{4.20}$$

where τ_j is a sum of products of smooth functions of y_2 times H^1-periodic functions of $(y_1 + \gamma(y_2))$ with y_1-average 0, and where $\rho_j \in H^1_\#(0, 2\pi)$ has y_2-average 0.

Proof. Let $j \geq 4$ and assume that we have determined

$$\begin{cases}\tau_k & \text{for } 0 \leq k \leq j,\\ \rho_k & \text{for } 0 \leq k \leq j-2,\\ \sigma_k & \text{for } 0 \leq k \leq j-4,\end{cases}$$

where the functions τ_k, ρ_k and the constants σ_k are as in Lemma 4.3 and where $\phi_k(y_1, y_2) = \tau_k(y_1 + \gamma(y_2)) + \rho_k(y_2) + \sigma_k$ satisfies (4.2) and (4.7) for $0 \leq k \leq j$ and for $0 \leq k \leq j-2$ respectively.

We seek ρ_{k-1} by imposing (4.7) for $k = j-1$, i.e. for any $w \in H^1_{\#}(0, 2\pi)$

$$\mathbf{A}_2[\rho_{j-1}, w] = -\mathbf{A}_2[\tau_{j-1}, w] - \mathbf{A}_3[\tau_{j-2} + \rho_{j-2}, w] - \mathbf{A}_4[\tau_{j-2} + \rho_{j-2} + \sigma_{j-3}, w],$$

which can be rewritten

$$\begin{aligned}\left(\int_{y_1} a\right) \int_{y_2} \partial_2 \rho_{j-1} \partial_2 \overline{w} &= \int_{y_2} \left(\int_{y_1} R_1(y_1 + \gamma(y_2), y_2)\right) \partial_2 \overline{w} \\ &\quad + \int_{y_2} \left(\int_{y_1} R_2(y_1 + \gamma(y_2), y_2)\right) \overline{w},\end{aligned} \tag{4.21}$$

where R_1, R_2 are known functions. We can thus determine $\rho_{j-1} \in H^1_{\#}(0, 2\pi)/\mathbb{R}$ uniquely provided that

$$\mathbf{A}_2[\tau_{j-1}, 1] - \mathbf{A}_3[\tau_{j-2} + \rho_{j-2}, 1] - \mathbf{A}_4[\tau_{j-2} + \rho_{j-2} + \sigma_{j-3}, 1] = 0,$$

which determines σ_{j-3}. We then seek ϕ_{j+1} from (4.2): for any $v \in H^1_{\#}(Y)$ we must have

$$\mathbf{A}_0[\phi_{j+1}, v] + \mathbf{A}_2[\phi_{j-1}, v] + \mathbf{A}_3[\phi_{j-2}, v] + \mathbf{A}_4[\phi_{j-3}, v] = 0,$$

which, using (4.4) rewrites

$$\begin{aligned}\mathbf{A}_0[\phi_{j+1} - \sigma_{j-1}\chi_1 - \sigma_{j-2}\chi_2, v] &= \mathbf{A}_2[\tau_{j-1} + \rho_{j-1}, v] + \mathbf{A}_3[\tau_{j-2} + \rho_{j-2}, v] \\ &\quad + \mathbf{A}_4[\phi_{j-3}, v],\end{aligned}$$

where the right-hand side only contains known quantities. Applying Lemma 4.3 shows that ϕ_{j+1} is uniquely determined up to $\rho_{j+1}(y_2) + \sigma_{j+1}$. Thus, the ϕ_j's can be determined inductively. □

Remark. Carrying on the computations yields the following form for the first correctors in the expansion:

$$\begin{aligned}\phi_1 &= 0, \\ \phi_2 &= \phi_0(t)\chi_1(y_1 + \gamma(y_2)) + \rho_2(y_2, t) + \sigma_2(t),\end{aligned}$$

with

$$\begin{aligned}\rho_2(y_2, t) &= -\phi_0(t) \frac{\int_{y_1} a \partial_1 \chi_1}{\int_{y_1} a} \gamma(y_2), \\ \sigma_2(t) &= -\frac{\mathbf{A}_2[\phi_0\chi_1 + \rho_2, \chi_1] + \mathbf{A}_3[\phi_0\chi_2, 1]}{\mathbf{A}_4[1, 1] - \mathbf{A}_0[\chi_1, \chi_1]}.\end{aligned} \tag{4.22}$$

The derivation of σ_2 is detailed in Appendix B. These expressions and (4.15) show that

$$|\phi_0(t)| \leq \frac{C}{1+|t|^2}, \tag{4.23}$$

$$|\phi_2(t)| \leq \frac{C|t|}{1+|t|^2}, \tag{4.24}$$

uniformly in t, for some constant C.

5. Convergence of the Expansion as $\nu \to 0$

This section is devoted to proving convergence results for the expansion (4.1) as $\nu \to 0$. We first estimate the difference of ϕ_ν with the first term:

Theorem 5.1. *There exists a function* $C_0(t) > 0$, *continuous in* t, *such that*

$$\|\phi_\nu - \phi_0\|_1 \leq C_0(t)\nu. \tag{5.1}$$

Proof. (a) Multiplying (2.3) by ϕ_ν, integrating and using the ellipticity of a, we easily derive the following *a priori* estimates: there exists a constant C, independent of ν and t, such that

$$\begin{cases} \|\phi_\nu\|_0 \leq C, \\ \|\partial_1(\phi_\nu e^{i\xi\cdot y})\|_0 \leq C\alpha^{-1/2}\nu^2, \\ \|\partial_2(\phi_\nu e^{i\xi\cdot y})\|_0 \leq C\alpha^{-1/2}\nu, \end{cases} \tag{5.2}$$

where again $\xi = (\nu^2 t_1, \nu t_2)$. Expanding $\partial_1(\phi_\nu e^{i\xi\cdot y})$, we get

$$\begin{aligned} \frac{1}{\nu^2}\|\partial_1\phi_\nu\|_0 &\leq \frac{1}{\nu^2}\left(\|\partial_1(\phi_\nu e^{i\xi\cdot y})\|_0 + \nu^2|t_1|\,\|\phi_\nu\|_0\right) \\ &\leq C(\alpha^{-1/2} + |t_1|). \end{aligned} \tag{5.3}$$

In a similar fashion, we can estimate $\partial_2\phi_\nu$:

$$\frac{1}{\nu}\|\partial_2\phi_\nu\|_0 = C(\alpha^{-1/2} + |t_2|). \tag{5.4}$$

(b) Let l denote the following linear form on $H^1_\#(Y)$:

$$l(u) = \frac{\phi_0}{|Y|}\left(\mathbf{A}_2[u,\chi_1] + \mathbf{A}_4[u,1]\right).$$

The definition (2.8) of ϕ_0 shows that $\forall\, \lambda \in \mathbb{C}, l(\lambda) = \lambda$. We next compute

$$\begin{aligned} \nu^4\left(|Y| + \nu^2\int_Y \overline{\chi_1}\right) &= \mathbf{A}_\nu[\phi_\nu, 1+\nu^2\chi_1] \\ &= (\mathbf{A}_0 + \mathbf{A}_2 + \mathbf{A}_3 + \mathbf{A}_4)\,[\phi_\nu, 1+\nu^2\chi_1] \\ &= \nu^2\left(\mathbf{A}_2[\phi_\nu,1] + \mathbf{A}_0[\phi_\nu,\chi_1]\right) + \nu^3\mathbf{A}_3[\phi_\nu,1] \\ &\quad + \nu^4\left(\mathbf{A}_4[\phi_\nu,1] + \mathbf{A}_2[\phi_\nu,1]\right) + \nu^5\mathbf{A}_3[\phi_\nu,\chi^1] + \nu^6\mathbf{A}_4[\phi_\nu,\chi_1] \\ &= -\nu^3\mathbf{A}_0[\phi_\nu,\chi_2] + \frac{|Y|}{\phi_0}l(\phi_\nu) + \nu^5\mathbf{A}_3[\phi_\nu,\chi^1] + \nu^6\mathbf{A}_4[\phi_\nu,\chi_1], \end{aligned}$$

where we have used (4.4). It follows that

$$l(\phi_\nu) = \frac{\phi_0}{|Y|}\left\{|Y| + \nu^2 \int_Y \overline{\chi_1} + \frac{1}{\nu}\mathbf{A}_0[\phi_\nu, \chi_2] - \nu \mathbf{A}_3[\phi_\nu, \chi^1] - \nu^2 \mathbf{A}_4[\phi_\nu, \chi_1]\right\}.$$

In other words, we get

$$l(\phi_\nu) - l(\phi_0) = \frac{\phi_0}{|Y|}\left\{\nu^2 \int_Y \overline{\chi_1} + \frac{1}{\nu}\mathbf{A}_0[\phi_\nu, \chi_2] - \nu \mathbf{A}_3[\phi_\nu, \chi^1] - \nu^2 \mathbf{A}_4[\phi_\nu, \chi_1]\right\}.$$

Further, we note that the H^1 norms of χ_1 and χ_2 are bounded by an independent constant, times $|t_1|$ and $|t_2|$ respectively. Thus, the *a priori* estimates (5.3), (5.4) show that

$$\begin{aligned}
|\nu^{-1}\mathbf{A}_0[\phi_\nu, \chi_2]| &\le \frac{1}{\nu} M \|\partial_1 \phi_\nu\|_0 \, \|\partial_1 \chi_2\|_0 \\
&\le \nu C(\alpha^{-1/2} + |t_1|)\,|t_1| \\
|\nu \mathbf{A}_3[\phi_\nu, \chi_1]| &= \left|\nu i t_2 \int_Y a\phi_\nu \partial_2 \overline{\chi_1} - \nu i t_2 \int_y a \partial_2 \phi_\nu \overline{\chi_1}\right| \\
&\le \nu c\, C(1 + \alpha^{-1/2} + |t_2|)\,|t_1 t_2| \\
|\nu^2 \mathbf{A}_4[\phi_\nu, \chi_1]| &\le \nu^2(1 + |t|^2) C |t_1|,
\end{aligned}$$

for some constant C independent of t and ν. Next, the generalized Poincaré inequality ensures the existence of a constant $C_Y(t) > 0$, such that for any $v \in H^1_\#(Y)$,

$$\|v - l(v)\|_1 \le C_Y(t) \|\nabla v\|_0. \tag{5.5}$$

Using these estimates, we conclude that there exists $C_0(t) > 0$, that depends continuously on t, such that

$$\begin{aligned}
\|\phi_\nu - \phi_0\|_1 &\le \|\phi_\nu - l(\phi_\nu)\|_1 + \|l(\phi_\nu) - l(\phi_\nu)\|_0 \\
&\le C_Y(t)\|\nabla \phi_\nu\|_0 + |Y|\,|l(\phi_\nu) - l(\phi_0)| \\
&\le \nu C_0(t).
\end{aligned}$$

□

To prove finer estimates, we use Tartar's method as in standard homogenization.[6] To this end, let

$$\psi_\nu^{(k)} = \phi_0 + \nu^2 \phi_2 + \cdots + \nu^k \phi_k.$$

Theorem 5.2. *There exists a constant $C_k(t) > 0$, continuous in t such that for $k \ge 2$*

$$\|\phi_\nu - \psi_\nu^{(k)}\|_1^2 \le C_k(t) \nu^k. \tag{5.6}$$

Proof. (a) Let $k > 2$ and let $C(k)$ denote a generic constant that only depends on k. Using (4.4) and the recurrence relations that determine the ϕ_k's, we compute, for $v \in H^1_\#(Y)$,

$$
\begin{aligned}
\mathbf{A}_\nu[\psi_\nu^{(k)}, v] &= \sum_{j=0}^{k} \nu^j \left\{ \mathbf{A}_0[\phi_j, v] + \nu^2 \mathbf{A}_2[\phi_j, v] + \nu^3 \mathbf{A}_3[\phi_j, v] + \nu^4 \mathbf{A}_4[\phi_j, v] \right\} \\
&= \mathbf{A}_0[\phi_0, v] + \nu^2 \left(\mathbf{A}_0[\phi_2, v] + \mathbf{A}_2[\phi_0, v] \right) \\
&\quad + \nu^3 \left(\mathbf{A}_0[\phi_3, v] + \mathbf{A}_3[\phi_0, v] \right) \\
&\quad + \sum_{j=4}^{k} \nu^j \left\{ \mathbf{A}_0[\phi_j, v] + \mathbf{A}_2[\phi_{j-2}, v] + \mathbf{A}_3[\phi_{j-3}, v] + \mathbf{A}_4[\phi_{j-4}, v] \right\} \\
&\quad + \nu^{k+1} \left(\mathbf{A}_2[\phi_{k-1}, v] + \mathbf{A}_3[\phi_{k-2}, v] + \mathbf{A}_4[\phi_{k-3}, v] \right) \\
&\quad + \nu^{k+2} \left(\mathbf{A}_2[\phi_k, v] + \mathbf{A}_3[\phi_{k-1}, v] + \mathbf{A}_4[\phi_{k-2}, v] \right) \\
&\quad + \nu^{k+3} \left(\mathbf{A}_3[\phi_k, v] + \mathbf{A}_4[\phi_{k-1}, v] \right) + \nu^{k+4} \mathbf{A}_4[\phi_k, v] \\
&= \nu^4 \int_Y \overline{v} - \nu^{k+1} \mathbf{A}_0[\phi_{k+1}, v] - \nu^{k+2} \mathbf{A}_0[\phi_{k+2}, v] \\
&\quad + \nu^{k+3} \left(\mathbf{A}_3[\phi_k, v] + \mathbf{A}_4[\phi_{k-1}, v] \right) + \nu^{k+4} \mathbf{A}_4[\phi_k, v]. \qquad (5.7)
\end{aligned}
$$

(b) We then form

$$
\begin{aligned}
\mathbf{A}_\nu[\phi_\nu - \psi_\nu^{(k)}, \phi_\nu - \psi_\nu^{(k)}] &= \mathbf{A}_\nu[\phi_\nu, \phi_\nu] - \mathbf{A}_\nu[\phi_\nu, \psi_\nu^{(k)}] - \mathbf{A}_\nu[\psi_\nu^{(k)}, \phi_\nu] \\
&\quad - \mathbf{A}_\nu[\psi_\nu^{(k)}, \psi_\nu^{(k)}] \\
&= \nu^4 \int_Y \overline{\phi_\nu} - \nu^4 \int_Y \overline{\psi_\nu^{(k)}} - \mathbf{A}_\nu[\psi_\nu^{(k)}, \phi_\nu] - \mathbf{A}_\nu[\psi_\nu^{(k)}, \psi_\nu^{(k)}].
\end{aligned}
$$

In (5.7), we substitute ϕ_ν and $\psi_\nu^{(k)}$ for v to treat the last two terms in the above expression

$$
\mathbf{A}_\nu[\phi_\nu - \psi_\nu^{(k)}, \phi_\nu - \psi_\nu^{(k)}] = \nu^{k+1} \mathbf{A}_0[\phi_{k+1}, \phi_\nu] - \nu^{k+1} \mathbf{A}_0[\phi_{k+1}, \psi_\nu^{(k)}] + \nu^{k+2} R_k,
$$

where the remaining terms R_k are bounded independently of ν (the bound however depends on k). Theorem 5.1 implies that

$$
\begin{aligned}
\mathbf{A}_0[\phi_{k+1}, \phi_\nu] &= \mathbf{A}_0[\phi_{k+1}, \phi_\nu - \phi_0] \\
&\leq M \|\phi_{k+1}\|_1 \|\phi_\nu - \phi_0\|_1 \leq C(k)\nu \\
\mathbf{A}_0[\phi_{k+1}, \psi_\nu^{(k)}] &= \mathbf{A}_0[\phi_{k+1}, \psi_\nu^{(k)} - \phi_0] \\
&\leq \mathbf{A}_0[\phi_{k+1}, \nu^2 \phi_2 + \cdots + \nu^k \phi_k]_1 \leq C(k)\nu^2.
\end{aligned}
$$

It follows that $\mathbf{A}_\nu[\phi_\nu - \psi_\nu^{(k)}, \phi_\nu - \psi_\nu^{(k)}] \leq C(k)\nu^{k+2}$, from which we deduce that

$$
\begin{cases}
\|\partial_1(\phi_\nu - \psi_\nu^{(k)})\|_0^2 \leq C(k)\nu^{k+2}, \\
\|\partial_2(\phi_\nu - \psi_\nu^{(k)})\|_0^2 \leq C(k)\nu^{k}, \\
\|\phi_\nu - \psi_\nu^{(k)}\|_0^2 \qquad \leq C(k)\nu^{k-2},
\end{cases}
$$

and thus $\|\phi_\nu - \psi_\nu^{(k)}\|_1^2 \leq C(k)\nu^{k-2}$. Since ϕ_{k+1}, ϕ_{k+2} are independent of ν, we finally get

$$
\|\phi_\nu - \psi_\nu^{(k-2)}\|_1^2 \leq C(k)\nu^{k-2}. \qquad \square
$$

6. Proof of the Representation Formula and Approximation of u_ν

Again, we dwell on the work of Babuška and Morgan[2] and will only give details about the parts that are different from their proof. In this section, we assume that $0 < \nu < 1$.

The solution u_ν to (1.3) has the variational characterization

$$\forall\, v \in H^1(\mathbb{R}^2), \quad \Psi(\nu)[u_\nu, v] := \int_{\mathbb{R}^2} a\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) \nabla u \cdot \nabla \overline{v} + a_0\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) u\, \overline{v}$$
$$= \int_{\mathbb{R}^2} f\overline{v}.$$

The existence of u_ν is easily established, as the bilinear $\Psi(\nu)$ satisfies the hypothesis of the Lax–Milgram lemma.

Any function $v(y_1, y_2) \in L^2_\#(Y)$, can be rescaled and extended by periodicity to a function $v(x_1/\nu^2, x_2/\nu)$ defined on $\mathbb{R}^2$. Formally the function $K(x,t) = \phi_\nu(x_1/\nu^2, x_2/\nu, t)e^{it\cdot x}$ solves the above equation when $f \equiv e^{it\cdot x}$. However, this function is not in $L^2(\mathbb{R}^2)$, so to make sense of this claim, we consider Ψ as a bilinear form defined on the weighted Sobolev spaces introduced in Sec. 2

$$\Psi(\nu) \colon H^1_{-\eta}(\mathbb{R}^2) \times H^1_{\eta}(\mathbb{R}^2) \longrightarrow \mathbb{C}.$$

The arguments of Lemma 3 in Ref. 2 show that $\Psi(\nu)$ satisfies the inf–sup condition in $H^1_{-\eta}(\mathbb{R}^2) \times H^1_{\eta}(\mathbb{R}^2)$, provided η is small enough. It is easily checked that the corresponding coercivity constant depends on η, but is bounded above 0 uniformly for $0 < \nu < 1$: there exists $\alpha(\eta) > 0$ such that

$$\inf_{\|u\|_{1,-\eta}=1} \sup_{\|v\|_{1,\eta}=1} |\Psi(\nu)[u,v]| \geq \alpha(\eta) > 0. \tag{6.1}$$

We proceed with some estimates on the kernel $K(x,t)$.

Lemma 6.1. *There exists a constant $C > 0$, such that for all $0 < \nu < 1$, $\eta > 0$ and $t \in \mathbb{R}^2$,*

$$\begin{aligned} &\forall\, v \in L^2_\#(Y), \quad \left\| v\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) e^{it\cdot x} \right\|_{0,-\eta} \leq C\|v\|_{0,Y}, \\ &\forall\, v \in H^1_\#(Y), \quad \left\| v\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) e^{it\cdot x} \right\|_{1,-\eta} \leq (1+\nu^{-2})C\|v\|_{1,Y} \end{aligned} \tag{6.2}$$

Proof. To compute the $H^1_{-\eta}$ norm on the left-hand side of the second inequality, let $Y(p)$ denote the rectangle $2\pi\nu^2(p_1 + (0,1) \times 2\pi\nu(p_2 + (0,1))$, $p \in \mathbb{Z}^2$. We have

$$\begin{aligned} &\left\| v\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) e^{it\cdot x} \right\|^2_{1,-\eta} \\ &\quad\leq \sum_{p\in\mathbb{Z}^2} \int_{Y(p)} \left\{ \nu^{-4} \left| \frac{\partial v}{\partial y_1}\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) + i\nu^2 t_1 v\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) \right|^2 \right. \\ &\qquad \left. + \nu^{-2} \left| \frac{\partial v}{\partial y_2}\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) + i\nu t_2 v\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) \right|^2 + \left| v\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}\right) \right|^2 \right\} e^{-2\eta|x|} dx \end{aligned}$$

$$\leq (1+\nu^{-4}) \sum_{p\in\mathbb{Z}^2} \int_Y \left\{ \left|\nabla_y (v(y_1,y_2)e^{i\xi\cdot y})\right|^2 \right.$$
$$\left. + \left|v(y_1,y_2)\right|^2 \right\} e^{-2\eta|(\nu^2(y_1+2\pi p_2),\nu(y_2+2\pi p_2))|}\ \nu^3 dy$$
$$\leq (1+\nu^{-4}) \left\{ \sum_p \nu^3 e^{-4\pi\eta(\nu^2|p_1|+\nu|p_2|)} \right\} \|ve^{i\xi\cdot y}\|_{1,Y}^2,$$

where $\xi = (\nu^2 t_1, \nu t_2)$. Since $\sqrt{1+\nu^{-4}} \leq (1+\nu^{-2})$ and since

$$\sum_p \nu^3 e^{-4\pi\eta(\nu^2|p_1|+\nu|p_2|)} \leq \nu^3 \left(2\sum_{p_1\geq 0} e^{-4\pi\eta\nu^2 p_1} \right) \left(2\sum_{p_2\geq 0} e^{-4\pi\eta\nu p_2} \right)$$
$$\leq \frac{4\nu^3}{(1-e^{-4\pi\eta\nu^2})(1-e^{-4\pi\eta\nu})},$$

is bounded uniformly for $0 \leq \nu \leq 1, 0 \leq \eta \leq \eta_0$, the proof of (6.2) follows from Lemma 1. The first inequality is proved similarly. $\square$

Lemma 6.2. *Let* $0 < \nu \leq 1, t \in \mathbb{R}^2$. *The solution* $\phi_\nu(\cdot,\cdot,t)$ *to* (2.3) *satisfies*

$$\begin{aligned} &\|\phi_\nu(y_1,y_2,t)\|_{0,Y} \leq 2\pi/\alpha, \\ &\|\phi_\nu(y_1,y_2,t)e^{i\xi\cdot y}\|_{1,Y} \leq 2\pi/\alpha, \end{aligned} \tag{6.3}$$

with $\xi = (\nu^2 t_1, \nu t_2)$.

Proof. Multiplying (2.3) by ϕ_ν and integrating yields

$$\int_Y a(y_1,y_2)\left(|\partial_1(\phi_\nu e^{i\xi\cdot t})|^2 + \nu^2|\partial_2(\phi_\nu e^{i\xi\cdot t})|^2\right) + \nu^4 a_0|\phi_\nu|^2 \leq \nu^4 \int_Y \overline{\phi_\nu},$$

and the estimates follow from the Cauchy–Schwartz inequality and the assumptions on the coefficients. $\square$

Lemma 6.3. *For* $0 < \nu \leq 1$ *and* $0 < \eta < \eta_0$, *the kernel* $\phi_\nu(y_1,y_2,t)$ *has the following properties.*

(i) $x \to \phi_\nu\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right) e^{it\cdot x} \in H^1_{-\eta}(\mathbb{R}^2)$,

(ii) $\Psi(\nu)\left[\phi_\nu\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right) e^{it\cdot x}, v\right] = \int_{\mathbb{R}^2} e^{it\cdot x}\,\overline{v(x)} \quad \forall\, v \in H^1_\eta(\mathbb{R}^2)$,

(iii) $\left\|\phi_\nu\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right) e^{it\cdot x}\right\|_{1,-\eta} \leq \frac{C}{\eta\alpha(\eta)}$.

The point (i) easily follows from Lemma 6.2. The proof of (ii) is the same as that of Theorem 8 in Ref. 2. Finally, (iii) follows from (ii), the inf–sup condition (6.1) and the estimate

$$\left|\int_{\mathbb{R}^2} e^{it\cdot x}\,\overline{v(x)}\right| \leq \frac{1}{\eta}\|v\|_{1,\eta}.$$

Lemma 6.4. *For all $\nu > 0$, the function*

$$t \in \mathbb{R}^2 \to \phi_\nu \left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu^2}, t\right) e^{it\cdot x} \in H^1_{-\eta}(\mathbb{R}^2)$$

is continuous.

Proof. The choice $v = \phi_\nu \left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu^2}, \sigma\right) - \phi_\nu \left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu^2}, \tau\right)$ and $t = 0$ in the second statement of Lemma 6.1 yields

$$\begin{aligned}&\left\|\phi_\nu \left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, \sigma\right) - \phi_\nu \left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, \tau\right)\right\|_{1,-\eta} \\ &\quad \le (1 + \nu^{-2})C\|\phi_\nu(y_1, y_2, \sigma) - \phi_\nu(y_1, y_2, \tau)\|_{1,Y}.\end{aligned}$$

The analycity of $(\nu, t) \to \phi_\nu(\cdot, \cdot, t)$ in a neighborhood of any point (ν, t), $\nu > 0, t \in \mathbb{R}^2$, then shows that the above left-hand side tends to 0 as $\sigma \to \tau$, which concludes the proof since $t \to e^{it\cdot x}$ is continuous. □

For $f \in L^2(\mathbb{R}^2)$ let f_N denote the inverse Fourier transform of the function $\hat{f}(t)\mathbf{1}_{\{|t|\le N\}}$:

$$f_N(x) = \int_{|t\le N} \hat{f}(t)e^{it\cdot x}dt.$$

We also let

$$u_{\nu,N}(x) = \frac{1}{2\pi}\int_{|t|\le N} \hat{f}(t)\phi_\nu \left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right) e^{it\cdot x}dt,$$

which is easily seen to be in $H^1_{-\eta}(\mathbb{R}^2)$. Lemmas 3.2–6.3 provide all the ingredients in order to reproduce the arguments of Ref. 2 in our situation. In particular, $u_{\nu,N}$ can be interpreted as a Bochner integral of $H^1_{-\eta}(\mathbb{R}^2)$-valued functions,[14] which satisfies

$$\Psi(\nu)[u_{\nu,N}, v] = \int_{\mathbb{R}^2} f_N(x)\overline{v}(x)dx \quad \forall\ v \in H^1_\eta(\mathbb{R}^2)$$

and which converges to u_ν, solution to (1.3) as $\nu \to 0$ (in $H^1_{-\eta}(\mathbb{R}^2)$ and in $H^1(\mathbb{R}^2)$). This shows that

$$u_\nu(x) = \lim_{N\to\infty} \frac{1}{2\pi}\int_{|t|\le N} \hat{f}(t)\phi_\nu \left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right) e^{it\cdot x}dt, \tag{6.4}$$

where the right-hand side is a Bochner integral of a $H^1_{-\eta}(\mathbb{R}^2)$-valued function. Moreover, since the coercivity constant $\alpha(\eta)$ of $\Psi(\nu)$ is uniformly positive for $0 < \nu < 1$, we have

$$\|u_\nu - u_{\nu,N}\|_{1,-\eta} \le \frac{1}{\alpha(\eta)}\|f - f_N\|_0, \tag{6.5}$$

for $0 < \nu < 1$.

We conclude this section with an approximation result for u_ν, based on using aproximate kernels in the representation formula. To this end, let

$$u_0(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{f}(t)\phi_0(t)e^{it\cdot x}dt \tag{6.6}$$

$$u_2(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{f}(t)\phi_2\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right)e^{it\cdot x}dt. \tag{6.7}$$

Lemma 6.5. *The kernels $\phi_0(t)$ and $\phi_2(y_1, y_2, t)$ are continuous and uniformly bounded with respect to t. Consequently, the integrals* (6.6), (6.7) *are well-defined as Bochner integrals of $H^1_{-\eta}$-valued functions.*

The proof of the lemma follows from the continuity and boundedness properties (with respect to t) of $\phi_0(t)$ and $\phi_2(y_1, y_2, t)$, with which one can easily adapt the arguments of Ref. 2.

We are now ready to prove Theorem 2.4. Let $\varepsilon > 0$, and for $N > 0$ let

$$g_{\nu,N}(x,t) = \frac{1}{2\pi}\int_{|t|\leq N} \hat{f}(t)\left(\phi_0(t) + \nu^2\phi_2\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right)\right)e^{it\cdot x}dt.$$

From the estimates (4.23) and Lemma 6.1, we have

$$\begin{aligned}
&\|u_0 - \nu^2 u_2 - g_{\nu,N}\|_{0,-\eta} \\
&\quad \leq \lim_{M\to\infty}\int_{N<t<M} |\hat{f}(t)| \left\|\phi_0(t) + \nu^2\phi_2\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right)^{it\cdot x}\right\|_{0,-\eta} dt \\
&\quad \leq \int_{N<t} |\hat{f}(t)|\, C\|\phi_0(t) + \nu^2\phi_2(y_1, y_2, t)\|_{0,Y}\, dt \\
&\quad \leq \int_{N<t} |\hat{f}(t)|\, C\frac{1+\nu^2|t|}{1+|t|^2}dt \\
&\quad \leq C\left(\int_{N<t} |\hat{f}(t)|^2 dt\right)^{1/2}.
\end{aligned}$$

Recalling (6.5), we can choose N large enough, so that for all $0 < \nu < 1$, both $\|u_\nu - u_{\nu,N}\|_{0,-\eta}$ and $\|u_0 - \nu^2 u_2 - g_{\nu,N}\|_{0,-\eta}$ are smaller than ε. We then have

$$\begin{aligned}
&\|u_\nu - (u_0 + \nu^2 u_2)\|_{0,-\eta} \\
&\quad \leq 2\|u_\nu - u_{\nu,N}\|_{0,-\eta} + \|g_{\nu,N} - (u_0 + \nu^2 u_2)\|_{0,-\eta} + \|u_{\nu,N} - g_{\nu,N}\|_{0,-\eta} \\
&\quad \leq 2\varepsilon + \frac{1}{2\pi}\int_{|t|\leq N} \left\|\left(\phi_\nu\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right) - \left[\phi_0(t) + \nu^2\phi_2\left(\frac{x_1}{\nu^2}, \frac{x_2}{\nu}, t\right)\right]\right)\right\|_{0,-\eta} |\hat{f}(t)|dt \\
&\quad \leq 2\varepsilon + \frac{1}{2\pi}C\int_{|t|\leq N} \|\left(\phi_\nu(y_1, y_2, t) - [\phi_0(t) + \nu^2\phi_2(y_1, y_2, t)]\right)\|_{0,Y} |\hat{f}(t)|dt \\
&\quad \leq 2\varepsilon + \frac{1}{2\pi}C\nu^2\int_{|t|\leq N} |\hat{f}(t)|C_2(\nu, t)dt.
\end{aligned}$$

The last inequality follows from Theorem 5.2 and it shows that the norm inside the integrand can be made smaller than ε as $\nu \to 0$. We conclude that $\|u_\nu - (u_0 + \nu^2 u_2)\|_{0,-\eta} \to 0$.

Appendix A. Non-Separation of 0 from the Spectrum of $T(\nu, t)$

Here, we justify the fact that 0 cannot be separated from the spectrum of $T(\nu, t)$, uniformly with respect to ν in a sector around 0.

Recall that for $u \in H^1_\#(Y)$, we have

$$(T(\nu,t)u,u) = \int_Y a|\partial_1(ue^{i\xi\cdot t})|^2 + \nu^2 \int_Y a|\partial_2(ue^{i\xi\cdot t})|^2 + \nu^4 \int_Y a_0|u|^2.$$

The uniform boundedness and the ellipticity of the coefficients show that, for all $u \in H^1_\#(Y)$,

$$(S^\alpha(\nu)u,u) \le (T(\nu,t)u,u) \le (S^M(\nu)u,u), \tag{A.1}$$

where $S^\delta(\nu)$ denotes the operator with constant coefficients defined by

$$\begin{aligned}(S^\delta(\nu)u,v) = \delta \int_Y a\partial_1(ue^{i\xi\cdot t})\partial_1(\overline{ve^{i\xi\cdot t}}) \\ + \nu^2\delta \int_Y a\partial_2(ue^{i\xi\cdot t})\partial_2(\overline{ve^{i\xi\cdot t}}) + \nu^4 \int_Y a_0 u\overline{v}.\end{aligned}$$

The eigenvalues of $S^\delta(\nu)$, with periodic boundary conditions, are easily computed

$$\lambda^\delta_m = \delta[(m_1 + \nu^2 t_1)^2 + \nu^2(m_2 + \nu t_2)^2] + \nu^4 a_0 > 0$$

where $m \in \mathbb{Z}^2$. The estimates (A.1) and the min–max principle[13] show that the eigenvalues $\lambda_m(\nu)$ of $T(\nu,t)$ can be compared to those of $S^\alpha(\nu)$ and $S^M(\nu)$.

In particular, any open interval around 0 contains countably many eigenvalues of $T(\nu,t)$, when ν is small enough.

It would be very interesting to obtain more information about the spectrum of $T(\nu,t)$. Formally, setting $\nu = 0$ in the definition of $T(\nu,t)$, we obtain the degenerate operator

$$T_0 u = -\partial_1\left(a(y_1,y_2)\partial_1 u\right).$$

This operator, viewed as an operator defined on the space of $L^2_\#(Y)$-functions with a y_1-derivative in $L^2_\#(Y)$ has a spectrum which is easy to characterize. It consists of isolated points λ_m (each with infinite multiplicity) which are the eigenvalues of the one-dimensionnal operator $-\frac{d}{dx}\left(\tilde{a}(x)\frac{du}{dx}\right)$ defined on $L^2_\#(0,2\pi)$. It would be particularly interesting to show that, like the operator $S^\delta(\nu)$ defined above, one could define bundles of regular curves $\nu \to \lambda_m(\nu)$ stemming from each value λ_m.

Appendix B. Derivation of σ_2

The value of σ_2 is obtained from Eq. (4.7) when $j = 6$. Choosing $w \equiv 1$ in this equation yields

$$\mathbf{A}_2[\phi_4, 1] + \mathbf{A}_3[\phi_3, 1] + \mathbf{A}_4[\phi_2, 1] = 0. \tag{B.1}$$

Since $\mathbf{A}_2[\phi_4, 1] = \overline{\mathbf{A}_2[1, \phi_4]} = \overline{-A_0[\chi_1, \phi_4]} = -\mathbf{A}_0[\phi_4, \chi_1]$, (4.2) when $j = 4$, and the fact that χ_1 has 0-average, yield

$$\mathbf{A}_2[\phi_4, 1] = \mathbf{A}_2[\phi_2, \chi_1] + \mathbf{A}_4[\phi_0, \chi_1].$$

Furthermore, it is easy to check that from (4.2), $j = 3$ that $\phi_3 = \phi_0\chi_2 + \rho_3(y_2, t) + \sigma_3(t)$. The left-hand side of (B.1) rearranges thus as

$$\mathbf{A}_2[\phi_0\chi_1 + \rho_2 + \sigma_2, \chi_1] + \mathbf{A}_4[\phi_0, \chi_1] + \mathbf{A}_3[\phi_0\chi_2, 1] + \mathbf{A}_4[\phi_0\chi_1 + \rho_2 + \sigma_2, 1],$$

which implies that

$$\sigma_2 = -\frac{\mathbf{A}_2[\phi_0\chi_1 + \rho_2, \chi_1] + \mathbf{A}_3[\phi_0\chi_2, 1]}{\mathbf{A}_4[1, 1] - \mathbf{A}_0[\chi_1, \chi_1]}.$$

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