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Optimality conditions for a relaxed layout optimization problem

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Abstract. In this Note, we consider a layout optimization problem, where the dependence of the material coefficient with respect to the density is nonlinear. The particular case when the density only depends on one space variable is studied. We show that the original design problem can be relaxed into an optimization that only involves three shape variables. The structure of the optimal designs is characterized by expliciting optimality conditions of a relaxed functional. © Académie des Sciences/Elsevier, Paris

Conditions d'optimalité pour un problème d'optimisation de formes relaxé

Résumé. *On considère un problème d'optimisation de formes où la relation entre les valeurs des coefficients du matériau et la densité est non linéaire. Dans cette Note, on étudie le cas particulier où les formes ne dépendent que d'une seule variable d'espace. Nous caractérisons la structure des formes optimales de la fonctionnelle relaxée.* © Académie des Sciences/Elsevier, Paris

Version française abrégée

Dans cette Note, on considère des problèmes d'optimisation de la distribution d'un matériau de densité a pour minimiser, sous une contrainte de volume, une fonctionnelle de coût (3), qui dépend des solutions de l'edp (2). La conductivité du milieu est de la forme a^p et donc est une fonction non linéaire de la densité.

Ce type de problème est mal posé, car en général, il n'y a pas de minimum. On est amené à généraliser l'ensemble des distributions du matériau et à relaxer la fonctionnelle. Nous considérons le cas où a ne dépend que d'une variable d'espace, pour lequel on peut trouver une relaxation explicite de la fonctionnelle (théorème 1.1). Le théorème 1.5 montre que l'ensemble des distributions de matériau généralisées peut-être décrit par 3 variables de forme.

Note présentée par Jacques-Louis Lions.

En introduisant un multiplicateur de Lagrange associé à la contrainte de volume, nous écrivons des conditions d'optimalité sur ces variables de forme pour la fonctionnelle relaxée. Les conditions d'optimalité fournissent une caractérisation complète de la structure des formes généralisées optimales et permettent de construire un algorithme numérique.

Des résultats de relaxation similaires ont été obtenus auparavant par les auteurs dans le cadre de l'optimization de la rigidité d'une plaque d'épaisseur variable (*voir* [2], [3]).

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1. The optimization problem

We consider an open set Ω in $\mathbb{R}^2$, contained in the strip $[x_0, x_1] \times \mathbb{R}$, which is filled with a conducting material. The density of the material is represented by a L^∞ function $a(x, y)$, while its conductivity is measured by some power, i.e. it has the form a^p, for some positive constant p.

The conductivity is assumed to depend on one variable only and to be uniformely bounded and elliptic. In other words, we assume that for two given constants $a_{\min}$, $a_{\max}$,

$$a \in \mathcal{A} \equiv \{b(x, y) \equiv b(x) \in L^\infty([x_0, x_1]) \mid 0 < a_{\min} \leq b(x) \leq a_{\max} < \infty \text{ for a.e. } x\}. \tag{1}$$

Given a function $f \in H^{-1}(\Omega)$, let u denote the unique potential that satisfies

$$-\operatorname{div}(a^p \nabla u) = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{2}$$

Given a function $g \in L^2(\Omega)$, we consider the functional

$$L(a) = \int_\Omega g(x, y) u(x, y) \mathrm{d}x \mathrm{d}y, \tag{3}$$

and we minimize $L(a)$ among all the conductivities that satisfy (1) and a volume constraint:

$$\int_\Omega a(x) \mathrm{d}x \mathrm{d}y = V. \tag{4}$$

The seminal work of Murat and Tartar, [7], [9], has shown that, in this form, the minimization problem may not have a solution, since the functional L is not weakly lower-semi-continuous for the L^∞ weak * topology. The remedy is to perform a relaxation. This process consists in enlarging the space of admissible designs $a(x)$, in order to cope with the effects of rapidly oscillating minimizing sequences of original designs $a(x)$, and in extending the functional L to these generalized designs. The goal is to produce a functional which is weakly lower-semi-continuous on a compact set of generalized designs. For our problem at hand, the answer is completely explicit, as the admissible conductivities only depend on one variable.

Let $\widetilde{\mathcal{A}}$ denote the set of functions $(a, m, c) \in (L^\infty([x_0, x_1]))^3$ such that

$$(a, m, c^{-1}) = w^* - \lim(a_n, a_n^p, a_n^{-p}), \tag{5}$$

for some sequence of designs $a_n \in \mathcal{A}$. For a triple $(a, m, c^{-1}) \in \widetilde{\mathcal{A}}$, we consider the anisotropic conductivity $A(x) = \begin{pmatrix} c & 0 \\ 0 & m \end{pmatrix}$ and let $\widetilde{u}$ denote the solution to

$$-\operatorname{div}(A \nabla \widetilde{u}) = f \quad \text{in } \Omega, \qquad \widetilde{u} = 0 \quad \text{on } \partial\Omega. \tag{6}$$

Finally, let $\widetilde{L}$ denote the functional $\widetilde{L}(a, m, c) = \int_\Omega g \widetilde{u} \mathrm{d}x \mathrm{d}y$.

THEOREM 1.1. – *The pair* $(\widetilde{\mathcal{A}}, \widetilde{L})$ *is a relaxation of* $(\mathcal{A}, L)$.

The proof is based on explicit formulae for the limit, in the sense of H-convergence, of sequences of conductivities tensors that only depend on one variable (*see* [6], [1], [2], [3]).

Equivalently, the set of generalized designs $\widetilde{\mathcal{A}}$ is the set of moments $(\langle\nu_x,\lambda\rangle, \langle\nu_x,\lambda^p\rangle, \langle\nu_x,\lambda^{-p}\rangle)$, of all Young measures with support in $[a_{\min}, a_{\max}]$ (*see* [2]). For fixed $x \in [x_0, x_1]$, the set of moments $M(\langle\nu_x,\lambda\rangle, \langle\nu_x,\lambda^p\rangle, \langle\nu_x,\lambda^{-p}\rangle)$ is convex and every point inside M can be represented by a convex combination of images of Dirac masses, which are its extreme points. In other words, given a positive Radon measure ν_x with support in $[a_{\min}, a_{\max}]$, there exists N points $a_i \in [a_{\min}, a_{\max}]$ and weights $\theta_i \in [0,1]$ such that

$$\langle\nu_x,(1,\lambda,\lambda^p,\lambda^{-p})\rangle = \sum_{i=1}^{N}\theta_i(1,a_i,a_i^p,a_i^{-p}).$$

Such a representation furnishes a parametrization of M, and we seek the simplest one. In particular, we seek the minimal number of points necessary to span the entire set M. A classical quadrature result [8] answers this question for the moments of the polynomials λ, λ^2, $\lambda^3, \ldots, \lambda^{2n-1}$. Only n points are required in this case. theorem 1.3 [4] below, shows that it remains true for more general moments:

DEFINITION 1.2. – *A* set of $n+1$ continuous functions $\phi_0, \ldots, \phi_n$ is said to form a *Tchebychev system* if and only if

$$\det\begin{vmatrix}\phi_0(a_0) & \cdots & \phi_0(a_n)\\ \vdots & & \vdots\\ \phi_n(a_0) & \cdots & \phi_n(a_n)\end{vmatrix} \neq 0,$$

whenever $a_{\min} \leq a_0 < a_1 < \cdots < a_n \leq a_{\max}$.

THEOREM 1.3. – *If* $\phi_0 = 1, \ldots, \phi_{2n-1}$ *forms a Tchebychev system over an interval* $[a_{\min}, a_{\max}]$, *then the set of moments* $(1, \langle\nu,\phi_1\rangle, \ldots, \langle\nu,\phi_{2n-1}\rangle)$ *is spanned by convex combinations of at most* n *points.*

It turns out that the functions $(1, a, a^p, a^{-p})$, that are relevant in the optimization problem, form a Tchebychev system.

PROPOSITION 1.4. – *If* a, b, c, d *are distinct positive reals, then*

$$\det\begin{vmatrix}1 & 1 & 1 & 1\\ a & b & c & d\\ a^p & b^p & c^p & d^p\\ a^{-p} & b^{-p} & c^{-p} & d^{-p}\end{vmatrix} \neq 0 \quad \forall p \neq 1. \tag{7}$$

Proof. – We assume that (a,b,c,d) are distinct and that the determinant vanishes. Subtracting the first column to the remaining three and developing the resulting determinant gives

$$\begin{vmatrix}1 & 1 & 1\\ \dfrac{b^p-a^p}{b-a} & \dfrac{c^p-a^p}{c-a} & \dfrac{d^p-a^p}{d-a}\\ \dfrac{b^{-p}-a^{-p}}{b-a} & \dfrac{c^{-p}-a^{-p}}{c-a} & \dfrac{d^{-p}-a^{-p}}{d-a}\end{vmatrix} = 0.$$

In order words, the points

$$B = \begin{pmatrix}\dfrac{b^p-a^p}{b-a}\\ \dfrac{b^{-p}-a^{-p}}{b-a}\end{pmatrix}, \qquad C = \begin{pmatrix}\dfrac{c^p-a^p}{c-a}\\ \dfrac{c^{-p}-a^{-p}}{c-a}\end{pmatrix}, \qquad D = \begin{pmatrix}\dfrac{d^p-a^p}{d-a}\\ \dfrac{d^{-p}-a^{-p}}{d-a}\end{pmatrix}$$

are on the same line: for instance, C can be assumed to lie on the segment $[B, D]$, and we can infer that, for some $0 \leq \theta \leq 1$,

$$\theta B + (1 - \theta) D = C. \tag{8}$$

It is convenient to define, for all $x > 0$, the following functions:

$$\begin{cases} k(x) = -(p+1)x^{-p} + apx^{-p-1} + a^{-p}, \\ h(x) = (p-1)x^p - apx^{p-1} + a^p. \end{cases} \tag{9}$$

We remark that $h'(x) = p(p-1)x^{p-2}(x-a)$ has the sign of $(p-1)(x-a)$. Also, h only vanishes at $x = a$ and is positive (resp. negative) if $p > 1$ (resp. if $p < 1$).

Let us further assume that $p > 1$ (the case $p < 1$ can be proved in the same way). The functions $\phi(x) = \dfrac{x^p - a^p}{x - a}$ and $\psi(x) = \dfrac{x^{-p} - a^{-p}}{x - a}$ are increasing and define bijections from $]0, \infty[$ onto $]0, \infty[$ and $]-\infty, 0[$ respectively. Thus, the function $F(y) = \psi \circ \phi^{-1}(y)$ maps $]a^{p-1}, \infty[$ onto $]-\infty, 0[$, and condition (8) is equivalent to

$$\theta F(\phi(b)) + (1-\theta) F(\phi(d)) = F(\theta\phi(b) + (1-\theta)\phi(d)).$$

We now show that F is strictly concave except at one point, which contradicts this last condition. Indeed,

$$F''(y) = \frac{\mathrm{d}}{\mathrm{d}y}\left[\frac{\psi'(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}\right] = \frac{1}{\phi'(x)}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\psi'(x)}{\phi'(x)}\right] = \frac{1}{\phi'(x)}\frac{k'h - h'k}{h^2}.$$

Since h only vanishes at $x = a$ and since $\phi' > 0$, the sign of F'' is the sign of

$$k'h - h'k = p(x-a)x^{-2}H(a/x).$$

where $H(X) = 2(p+1)(p-1) - 2p^2X + (p+1)X^p - (p-1)X^{-p}$. One easily checks that $H(1) = 0$ and that $H'(X) = G(X^{p-1})$, with $G(Y) = -2p^2 + p(p+1)Y + p(p-1)Y^{-\frac{p+1}{p-1}}$. Since $G'(Y) = p(p+1)\left(1 - Y^{-\frac{p+1}{p-1}-1}\right)$ has the same sign as $Y - 1$ and since $G(1) = 0$, the function H has the same sign as $X - 1$. We conclude that $k'h - h'k$ is negative and only vanishes at $x = a$, which proves the claim about the convexity of F. □

To wrap up this section, we apply Theorem 1.3 and Proposition 1.4, to get a simple parametrization of the set of generalized designs.

THEOREM 1.5. – *Let $\mathcal{A}^*$ denote the set of functions (θ, a_1, a_2) such that $\theta \in \mathrm{L}^\infty([x_0, x_1]; [0,1])$ and $(a_1, a_2) \in \mathrm{L}^\infty([x_0, x_1]; [a_{\min}, a_{\max}])^2$. Let*

$$(a(x), m(x), c^{-1}(x)) = \theta(x)(a_1(x), a_1^p(x), a_1^{-p}(x)) + (1 - \theta(x))(a_2(x), a_2^p(x), a_2^{-p}(x)). \tag{10}$$

Let u_* denote the solution to (6) with the coefficients m and c of the conductivity tensor defined by (10) and let $L^*(\theta, a_1, a_2) = \displaystyle\int_\Omega g u_* \mathrm{d}x\mathrm{d}y$. Then, $(\mathcal{A}^*, L^*)$ is a relaxation of $(\mathcal{A}, L)$.

2. Optimality conditions for the energy

In this section, the function g, that appears in the cost functional, is chosen to be equal to the applied load f. We derive optimality conditions, that further reduce the form of the optimal generalized design.

We reformulate the problem, introducing a Lagrange multiplier l to take the volume constraint into account. In other words, instead of (3), we seek the infimum of

$$J(a) = \int_\Omega fu - la(x),$$

on the set of functions satisfying (1). With the notations of Theorem 1.5, this infimum is equal to the minimum of $J^*(\theta, a_1, a_2)$ over the set $\mathcal{A}^*$, where

$$\begin{aligned} J^*(\theta, a_1, a_2) &= \int_\Omega fu_* - l(\theta a_1 + (1-\theta)a_2) \\ &= \int_\Omega \begin{pmatrix} c & 0 \\ 0 & m \end{pmatrix} \nabla u_* \cdot \nabla u_* - l(\theta a_1 + (1-\theta)a_2). \end{aligned}$$

Using a standard duality argument, we set $\Sigma = \{\sigma \in \mathrm{L}^2(\Omega)^2 | \mathrm{div}(\sigma) = f\}$, and transform the minimization into

$$\min_{\mathcal{A}^*} J^*(\theta, a_1, a_2) = \min_{\mathcal{A}^*} \min_{\sigma \in \Sigma} \int_\Omega \begin{pmatrix} c^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix} \sigma \cdot \sigma - l(\theta a_1 + (1-\theta)a_2).$$

In this last expression, the minimization with respect to the shape parameters can be pulled inside the integral sign, as the functions (θ, a_1, a_2) are not tied by any integral constraint. Denoting $e(\theta, a_1, a_2, \sigma) = c^{-1}\sigma_1^2 + m^{-1}\sigma_2^2 - l(\theta a_1 + (1-\theta)a_2)$, we obtain that

$$\min_{\mathcal{A}^*} J^*(\theta, a_1, a_2) = \min_{\sigma \in \Sigma} \int_\Omega \min_{\theta, a_1, a_2} e(\theta, a_1, a_2, \sigma). \tag{11}$$

We now focus on the pointwise minimization of the energy density e with respect to the shape parameters. If (θ, a_1, a_2) is a minimum for a given σ, admissible perturbations $\delta\theta$, δa_1, δa_2 must satisfy

$$\begin{aligned} &-\theta\delta a_1\left[p\left(a_1^{-p-1}\sigma_1^2 + a_1^{p-1}\frac{\sigma_2^2}{m^2}\right) + l\right] - (1-\theta)\delta a_2\left[p\left(a_2^{-p-1}\sigma_1^2 + a_2^{p-1}\frac{\sigma_2^2}{m^2}\right) + l\right] \\ &-\delta\theta\left[-(a_1^{-p} - a_2^{-p})\sigma_1^2 + (a_1^p - a_2^p)\frac{\sigma_2^2}{m^2} + (a_1 - a_2)l\right] \geq 0. \end{aligned} \tag{12}$$

If $\theta = 0$, $\theta = 1$ or $a_1 = a_2$, the design at that point cprresponds to an original design with pure material. The interesting case is whether a composite can occur and how its structure is restricted by the optimality conditions.

If (θ, a_1, a_2) is an interior point, i.e., if $0 < \theta < 1$ and if $a_{\min} < a \neq a_2 < a_{\max}$, the factors of $\delta\theta$, δa_1, δa_2 in (12) must vanish. Thus, $\left(\sigma_1^2, \dfrac{\sigma_2^2}{m^2}, l\right)$ must be solution to a homogeneous 3×3 system, the determinant of which is

$$\begin{vmatrix} pa_1^{-p-1} & pa_1^{p-1} & 1 \\ pa_2^{-p-1} & pa_2^{p-1} & 1 \\ (a_2^{-p} - a_1^{-p}) & (a_1^p - a_2^p) & (a_1 - a_2) \end{vmatrix} = \frac{p(a_1^p - a_2^p)}{a_1^{p+1}a_2^{p+1}}[(a_1^p - a_2^p)(a_1 + a_2) - p(a_1^p + a_2^p)(a_1 - a_2)].$$

To study its zeroes, let $\xi(x, a) = (x^p - a^p)(x + a) - p(x^p + a^p)(x - a)$. Its x-derivative is equal to $-(p+1)$ times the functions h of (9), and so ξ only vanishes when $x = a$. Consequently, if $a_1 \neq a_2$, the determinant is different from zero, which forces $\sigma_1 = \sigma_2 = l = 0$. It follows, that the minimum cannot be attained at an interior point, if for instance $|\sigma| \neq 0$.

Assume now that $0 < \theta < 1$ and that $a_{\min} = a_1 < a_2 < a_{\max}$. It follows from (12), that the factors of δa_2 and $\delta\theta$ must vanish. Solving these equations for σ_1^2 and $\frac{\sigma_2^2}{m^2}$, then injecting the result in the factor of δa_1, which must be positive, we arrive at the condition

$$(p+1)(a_1 a_2^{2p} + a_2 a_1^{2p}) - (p-1)(a_1^{2p+1} + a_2^{2p+1}) - 2(a_1^{p+1} a_2^p + a_1^p a_2^{p+1}) \geq 0. \tag{13}$$

Here, we have used the fact that l is negative (since the factor of δa_2 vanishes). The left hand side can be rewritten as $a_1^{2p+1} g(a_2/a_1)$ with

$$\begin{aligned} g(X) &= (p+1)(X^{2p} + X) - (p-1)(X^{2p+1} + 1) - 2(X^{p+1} + X^p) \\ &= (X^p - 1)[(X^p - 1)(X+1) - p(X^p + 1)(X-1)] \\ &= (X^p - 1)\xi(X-1). \end{aligned}$$

Since $\frac{\partial \xi}{\partial x} = -(p+1)h(X)$, and recalling the properties of h, the left hand side of (13) turns out to be negative (resp. positive) if $p > 1$ (resp. $p < 1$). We conclude that a configuration such that $a_1 = a_{\min} < a_2 < a_{\max}$ is only possible at the minimum if $p < 1$.

A symmetric situation holds if for instance $a_{\min} < a_1 < a_2 = a_{\max}$. We summarize the results in the following

THEOREM 2.1. – *At the optimal configuration, the design is:*
- *either made of pure material, i.e.,* $(a, m, c) = (a_1, a_1^p, a_1^p)$ *for some* $a_{\min} \leq a_1 \leq a_{\max}$*;*
- *or made of a mixture of* $a_{\max}$ *and another conductivity* $a_{\min} \leq a_1 < a_{\max}$ *if* $p > 1$ *(if* $p < 1$*, a mixture of* $a_{\min}$ *and some* $a_{\min} < a_1 \leq a_{\max}$*).*

Remark 2.2. – An iterative numerical algorithm can be designed very easily using the formulation (11) and the optimality conditions. A related relaxation result has been obtained in the case of the optmization of the compliance for a plate with variable thickness [2], [3]. The underlying operator is then of fourth order. Optimality can be worked out in that case in the similar manner as here, leading to an analogous result about the structure of the optimizes. For the plate problem, another form of relaxation has been proposed by J. Muñoz and P. Pedregal [5].

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