

COMPUTING THE EQUILIBRIUM CONFIGURATION OF EPITAXIALLY STRAINED CRYSTALLINE FILMS*

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Abstract. We study a model for shape instabilities of heteroepitaxial crystalline films. Lattice misfits between the substrate and the film induce elastic stresses in the film, which adjusts the shape of its free surface to reduce its total energy, sum of an elastic and a surface energy. We give a precise framework that guarantees the existence of solutions to this variational problem. We show that equilibrium states can be approximated using a two-phase model for representing the surface energy. Numerical results, obtained via this approximation, are presented.

Key words. Γ -convergence, epitaxially stressed films, shape instabilities

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1. Introduction. This paper is a contribution to the mathematical formulation of morphological instabilities of interfaces induced by stress rearrangement. Stress driven rearrangement instabilities (SDRI) are observed in many branches of material sciences, such as fracture, crystal growth, or corrosion. They occur, for instance, in the epitaxial growth of thin layers of highly strained hetero systems such as In-GaAs/GaAs or SiGe/Si for applications to electronic device structures. Because of these instabilities, controlling the growth of such systems is a significant challenge for the applied physics community. (See, for instance, the special issue of the Materials Research Society Bulletin on this topic [25].) There is also a big economic incentive, as SiGe systems, for instance, would provide low-cost high-performance technology.

When an epitaxial film is grown on a (flat) substrate, if kinetic effects are neglected, the free surface of the film is flat until a critical value of the thickness is reached, after which the free surface becomes corrugated. Atomic-force microscopy images show ripples, or pits and islands of pyramidal shapes, depending on the type of alloy.

The basic mechanism that explains this behavior is the following. The lattice misfits between the substrate and the film induce strains in the film. To release some of the elastic energy due to these strains, the atoms on the free surface of the film have the ability to move, and the resulting morphology is energetically more economical.

The explanation put forward is that competition takes place between two forms of energy, the surface energy and the bulk elastic energy. The former is roughly proportional to the area of the free surface, thus favoring flat configurations. A simple asymptotic computation [18] shows that a flat free surface is unstable with respect to minimizing the bulk elastic energy of a linear elastic solid.

The literature on the shape of epitaxial islands due to misfit strains is considerable. (See the pioneering works [4, 17] and [14] for a review.) In most of this work, the surface flux of atoms is proportional to the gradient of a diffusion potential [27],

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the form of which is derived from the thermodynamics of stressed solids [22]. This potential is a sum of a (local) surface strain energy density and a term of surface energy, which is equal to a constant times the curvature of the free surface in the reference state. The film/substrate interaction is modeled by an induced misfit strain in the film. The free surface is described by a function $h(x, y, t)$, and one ends up with a nonlinear evolution equation which involves 4th order derivatives of h . In the steady-state case, integrating this equation shows that the potential should be constant along the interface.

Linear stability around a uniform profile has been addressed for the steady-state and time-dependent problems in various configurations: when the film is modeled as a semi-infinite region [38, 32, 7], and when the interaction between film and substrate (rigid or nonrigid) is taken into account [34, 35]. The stabilizing influence of kinetic (deposition) effects has also been studied with this model [34, 35]. A similar model has been investigated in the context of alloy thin films [19, 37, 36].

We are interested in a variational formulation of the problem of finding the equilibrium film shapes. A model problem has been studied in [5] in order to understand how the stability of the uniform (flat) free surface depends on the mean thickness. In this one-dimensional (1-d) formulation, the film occupies a strip $\Omega = \{0 < x < 1, 0 < y < h(x)\}$, and stable equilibria are defined as global minimizers of an energy functional that depends on the thickness h and on the displacement $u(x)$ (in the x direction) in the strip. The problem takes the form

$$(1.1) \quad \inf_{u \in V, h \in H} \mathcal{E}(u, h) = \inf_{u \in V, h \in H} K \int_0^1 h(x) [u'(x)]^2 dx + \int_0^1 \sqrt{1 + [h'(x)]^2} dx.$$

The first term in the energy expression models an elastic bulk energy induced by $u(x)$, while the second represents the length of the curve h . The parameter K is related to the scalings of the physical constants (length of the specimen, mean thickness, elasticity constants). The larger K becomes, the larger is the mean thickness. (All inertial effects have been neglected and the model is quasi-static.)

One of the issues that stems from [5] is the choice of spaces of admissible displacements and thicknesses. In that work, which was carried out in order to understand computational results obtained for the above energy [12], H was chosen to be the space of piecewise C^∞ positive functions, which satisfied the volume constraint

$$(1.2) \quad \int_0^1 h(x) dx = 1,$$

reflecting conservation of mass during the rearrangement process. The space V was simply chosen to be $x + H^1(\mathbf{S}^1)$. (\mathbf{S}^1 denotes the torus $\mathbf{R}/(0, 1)$.) It turns out that these spaces are adequate neither for the analysis of the problem nor for computing purposes. In particular, if K is large, the equilibrium configuration has a vertical crack that runs from the free surface to the bottom of the film. The computations did not show anything close to such a configuration and proved (ironically) highly unstable. A sound mathematical formulation of the minimization of the energy functional should involve spaces of functions which can hence be somewhat rough; the difficulty lies in then defining the corresponding energy.

These questions are at the heart of the present paper, where we study a physically more meaningful formulation: We consider the film as a full two-dimensional (2-d) elastic solid. Its displacement is thus a vector-valued function $u(x, y)$. The main change concerns the modeling of the contact between the substrate and the film.

In the 1-d model, strain was created by imposing boundary conditions on u at the endpoints. Here, the contact affects the film in a more realistic way: A Dirichlet boundary condition is imposed at the interface between film and substrate, which models the case of a film growing on an infinitely rigid substrate. The enforcement of this boundary condition is what causes the film to be strained, i.e., what generates elastic energy.

Additionally, we make the following assumptions. First, we assume that the admissible free surfaces are graphs of lower semicontinuous functions. This space is endowed with a natural topology, for which sequences of free surfaces with uniformly bounded length are compact. Second, the configuration is supposed to be 1-periodic in the x -direction, and the displacements are taken to be periodic up to a linear displacement (the displacement in the substrate). Third, we assume as in [5] that the film is made of a linear elastic material with homogeneous Hooke's law \mathbf{A} . Finally, we assume that the substrate is infinitely large with respect to the film and occupies the region $\mathbf{S}^1 \times (-\infty, 0]$. The contact between the substrate and the film costs surface energy, and the corresponding surface tension is denoted by σ_s .

This work is devoted to giving a “sound mathematical formulation” for finding the equilibrium configuration. We define an energy $E(\Omega, u)$ for graphs Ω of l.s.c. functions and for displacements u . We show that a minimizer exists and that it can be approximated by “smooth” thicknesses, namely, by thicknesses which are Lipschitz. It was observed in [5] that when piecewise smooth thicknesses converge to a configuration with a vertical crack, the length of the crack has to be counted twice in the limiting energy. Our present formulation conveys the same feature. Furthermore, if the variational problem (2.4) has a smooth solution (u, Ω) , a simple argument of variation of domain yields that

$$\gamma\kappa + KAe(u) : e(u) = \text{constant}$$

on the interface, where κ is the curvature and γ depends on the surface tension. In this sense, our model contains that described by [32, 35, 33] (at equilibrium).

We also propose an approximation scheme for computing minimizers. It is based on a diffusive two-phase model, one phase representing the film while the other represents the void above the free surface. A Cahn–Hilliard energy [1, 6, 23] approximates the length of the free surface. The total energy E_ε depends on the displacement u and on a marker function v that takes the values $v \simeq 1$ in the film and $v \simeq 0$ in the void. The scale of the Cahn–Hilliard approximation ε controls the width of the transition zone between the two phases. We show that, as $\varepsilon \rightarrow 0$, a minimizing graph for E can be recovered as the set of points where the sequence of approximate markers $v_\varepsilon \rightarrow 1$. We give some numerical examples, based on the minimization of E_ε . This method is inspired by the approximation techniques introduced by [3] for a free discontinuity problem in the context of image processing, using Γ -convergence [1, 9]. The computations of elastically stressed binary alloys of Leo, Lowengrub, and Jou [21] use a related approach. (See also [20] for a solid-liquid interface.) In these works, both phases are nondegenerate, whereas here, one of the phases is void. Muller and Grant [26] introduce a similar Ginzburg–Landau approach to study numerically, in two and three dimensions, the Grinfeld instability of the free interface of a nonhydrostatically stressed solid. For a concise and comprehensive review about the Γ -convergence and the Γ -limit of the Cahn–Hilliard free energy, we refer to [1].

Several papers have also been devoted to the case of thin films, when strain relaxation induces the film to form isolated islands separated by a thin “wetting” layer.

In [30, 33], the wetting layer is assumed to be of negligible thickness and is modeled as effective boundary conditions for the solutions to the free surface equation. Equilibrium island shapes are determined by asymptotic expansion about a flat configuration. In another approach [8], the wetting layer is modeled by a boundary layer in which the surface energy depends on the film thickness.

In the model studied in this paper, the appearance of wetting layers is conditioned by the relative strength of the film and substrate surface tensions (see Figure 4). In particular, the form of our approximating energies is consistent with the boundary layer transitions studied by Spencer [31]: The smearing parameter used in this work plays the same role as our approximating parameter ε . (See, for instance, (2.8) and (2.12) in [31].) It would be interesting to investigate whether our approach gives the same qualitative results as those of [31], regarding the contact angle between wetting layer and island.

We do not consider in this paper the anisotropic or so-called crystalline case, in which the surface energy also depends on the orientation of the surface of the crystal. However, this can easily be done by introducing a convex 1-homogeneous function $\varphi(\nu)$ of the normal vector to the interface, as a weight in the lengths in section 2.1 (e.g., replacing $\mathcal{H}^1(\partial\Omega)$ with $\int_{\partial\Omega} \varphi(\nu(x)) d\mathcal{H}^1(x)$, where $\nu(x)$ is the normal vector to $\partial\Omega$ at x). In the same way, the approximation result of section 3 holds if we replace $|\nabla v(x, y)|^2$ with $\varphi(\nabla v(x, y))^2$ in (3.2). However, performing numerical computations is a much harder task in the crystalline case, since the physics require φ to be singular (only Lipschitz-continuous).

The paper is organized as follows. In section 2, the energy functional E is defined in detail and the lower semicontinuity of the surface energy is stated (Lemma 2.1). Existence of a minimizer is then proved (Theorem 2.2). In section 3, we introduce the approximating energies E_ε and give the main theorem of this paper, Theorem 3.1, which states Γ -convergence of the energies E_ε towards E . Numerical examples using the approximating energies E_ε are given in section 4. The proofs of Lemma 2.1 and Theorem 3.1 comprise sections 5 and 6, respectively. Finally, the appendix groups a few results about functions with bounded variation that are used in the text.

We do not address the issue of stability with respect to the mean thickness. These aspects will be treated subsequently.

2. Statement of the problem and existence of a minimizer. In the whole paper, Q denotes the 2-d space $\mathbf{S}^1 \times \mathbf{R}$ ($\mathbf{S}^1 = \mathbf{R}/\mathbf{Z}$), $Q^+ = \mathbf{S}^1 \times (0, +\infty)$, and for any $a > 0$, $Q^a = \mathbf{S}^1 \times (0, a)$. The canonical projection from \mathbf{R}^2 onto Q will be denoted by π ; however, in some nonambiguous situations it will not be explicitly mentioned.

We will denote by \mathcal{G} the set of all open subsets Ω of Q that are the subgraphs of a nonnegative l.s.c. function $h : \mathbf{S}^1 \rightarrow [0, +\infty)$:

$$\Omega \in \mathcal{G} \iff \exists h : \mathbf{S}^1 \rightarrow [0, +\infty) \text{ l.s.c.}, \Omega = \{(x, y) \in Q : y < h(x)\}.$$

Note that if $\Omega \in \mathcal{G}$, $\partial\Omega \subseteq \overline{Q^+} = Q^+ \cup (\mathbf{S}^1 \times \{0\})$. Stating that $\Omega \in \mathcal{G}$ is equivalent to saying that $Q^- = Q \setminus \overline{Q^+} \subseteq \Omega$ and that for any $(x, y) \in \Omega$, $\{x\} \times (-\infty, y] \subset \Omega$. We also denote by $\mathcal{G}_L \subset \mathcal{G}$ the subgraphs of nonnegative Lipschitz functions.

If $(\Omega_n)_{n \geq 1}$ is a sequence of open sets, we say that it converges to Ω as n goes to infinity if it converges to Ω in the *Hausdorff-complement topology*, i.e., if $\Omega^c = Q \setminus \Omega$ is the Hausdorff limit of the complements Ω_n^c . We observe that \mathcal{G} is closed in the set of all open subsets of Q ; indeed, if Ω_n are the subgraphs of functions h_n , and Ω_n

converge to Ω as n goes to infinity, then Ω is the subgraph of the function

$$h(x) = \inf_{x_n \rightarrow x} \liminf_{n \rightarrow \infty} h_n(x_n).$$

In fact, we have $\mathcal{G} = \overline{\mathcal{G}_L}$.

2.1. The surface energy. Given $\sigma_c, \sigma_s > 0$, we define the surface energy of a regular domain $\Omega \in \mathcal{G}_L$ as

$$L^0(\Omega) = \sigma_c \mathcal{H}^1(\partial\Omega \cap Q^+) + \sigma_s \mathcal{H}^1(\partial\Omega \setminus Q^+).$$

(Notice that $\partial\Omega \setminus Q^+ = \partial\Omega \cap (\mathbf{S}^1 \times \{0\})$.) The idea is that the part $\partial\Omega \cap Q^+$ represents the free surface of the crystal, whose surface tension is σ_c , whereas $\partial\Omega \setminus Q^+$ is the surface of the substratum that is not recovered by the crystal and whose surface tension is σ_s .

We extend L^0 to \mathcal{G} by setting $L^0(\Omega) = +\infty$ if $\Omega \in \mathcal{G} \setminus \mathcal{G}_L$, and define the relaxed surface energy $L : \mathcal{G} \rightarrow [0, +\infty]$ as the l.s.c. envelope of L^0 . We have the following lemma.

LEMMA 2.1. *Let $\Omega \in \mathcal{G}$ and let h, \bar{h} be l.s.c. functions such that*

$$(2.1) \quad \Omega = \{(x, y) \in Q : y < h(x)\} \quad \text{and} \quad \overset{\circ}{\Omega} = \{(x, y) \in Q : y < \bar{h}(x)\}.$$

Then

$$(2.2) \quad L(\Omega) = \sigma_c \left\{ \mathcal{H}^1(\partial\bar{\Omega} \cap Q^+) + 2 \sum_{x \in \mathbf{S}^1} (\bar{h}(x) - h(x)) \right\} + (\sigma_c \wedge \sigma_s) \mathcal{H}^1(\partial\bar{\Omega} \setminus Q^+).$$

The proof of Lemma 2.1 is given in section 5. One consequence of this lemma, in particular, is the fact that if $\sigma_s > \sigma_c$, then it is better to recover all of the substratum with an infinitesimal layer of crystal atoms and pay the lower surface tension σ_c than to leave free any part of the surface of the substratum.

2.2. The global energy functional. We now introduce the energy

$$(2.3) \quad E(\Omega, u) = K \int_{\Omega \cap Q^+} \mathbf{A}e(u)(x, y) : e(u)(x, y) \, dx dy + L(\Omega),$$

defined for any $\Omega \in \mathcal{G}$ and $u \in X(\Omega)$, where $X(\Omega)$ denotes the set of functions $u \in L^2_{loc}(\pi^{-1}(\Omega); \mathbf{R}^2)$, $u(x, y) \equiv x$ for $y \leq 0$, with $u(x, y) - (x, 0)$ 1-periodic in x , and such that the linear deformation tensor (the symmetrized gradient) $e(u)$ is in $L^2(\Omega \cap Q^+; \mathbf{R}^4)$. The matrix $\mathbf{A} = (a_{ijkl})$ is a positive-definite symmetric tensor of order 4 (such that $a_{ijkl} = a_{jikl} = a_{klij}$ for any $i, j, k, l \in \{1, 2\}$). The scalar parameter K is related to the scalings of the physical constants and it balances the influence of each term in the energy. In the rest of the paper (except in section 4, where we illustrate the dependence on K) we set $K = 1$ for simplicity, without loss of generality.

2.3. The problem. We consider the following minimization problem:

$$(2.4) \quad \min_{\Omega \in \mathcal{G}, u \in X(\Omega)} E(\Omega, u) \quad \text{subject to } |\Omega \cap Q^+| = 1.$$

The volume constraint reflects conservation of mass: The model assumes that the relaxation of the film is much faster than the rate of deposition. We prove the following.

THEOREM 2.2. *Problem (2.4) has a solution.*

Proof. Consider (Ω_n, u_n) a minimizing sequence for (2.4). Since $|\Omega_n \cap Q^+| + L(\Omega_n)$ is bounded, $\Omega_n \cap Q^+$ is uniformly bounded. Up to a subsequence (not relabeled) we may thus assume that it converges to a domain $\Omega \in \mathcal{G}$ with $|\Omega \cap Q^+| = 1$ and, since L is l.s.c.,

$$(2.5) \quad L(\Omega) \leq \liminf_{n \rightarrow \infty} L(\Omega_n).$$

The function $e(u_n)$, that we extend to zero outside of Ω_n , is uniformly bounded in $L^2(Q^+; \mathbf{R}^4)$. We thus may assume that it converges weakly to some function $\mathcal{E} \in L^2(Q^+; \mathbf{R}^4)$. In particular, we have

$$(2.6) \quad \int_{\Omega \cap Q^+} \mathbf{A}\mathcal{E}(x, y): \mathcal{E}(x, y) \, dx dy \leq \liminf_{n \rightarrow \infty} \int_{\Omega_n \cap Q^+} \mathbf{A}e(u_n)(x, y): e(u_n)(x, y) \, dx dy.$$

Remark. It can be shown that $\mathcal{E} = 0$ a.e. in $Q \setminus \Omega$. Indeed, if K is the Hausdorff limit of some converging subsequence of $(\partial\Omega_n)_{n \geq 1}$, one can prove that $\mathcal{H}^1(K) < +\infty$ —thus $|K| = 0$ —and that $\Omega \cup K$ is the Hausdorff limit of $\bar{\Omega}_n$, or equivalently, that the domains $Q \setminus \bar{\Omega}_n$ converge to $Q \setminus (\Omega \cup K)$. (See section 5.1 for details.) Thus $\mathcal{E} = 0$ in $Q \setminus (\Omega \cup K)$ and, since $|K| = 0$, a.e. in $Q \setminus \Omega$.

Let $A \subset \Omega$ be a Lipschitz subgraph with $A \cap Q^+ \subset\subset \Omega$. (For simplicity we assume $A \supseteq Q^-$, although it is not essential.) For n large enough, $A \subset \Omega_n$ and u_n is defined on $\pi^{-1}(A)$. By Korn’s inequality, since $u_n(x, y) - (x, 0) \equiv 0$ for $y \leq 0$ and since ∂A is Lipschitz, there exist $c = c(A)$ and $c' = c'(A, \mathbf{A})$ such that

$$\int_{A \cap Q^+} |u_n(x, y)|^2 \, dx dy \leq c \int_{A \cap Q^+} e(u_n)(x, y): e(u_n)(x, y) \, dx dy \leq c' E(\Omega_n, u_n).$$

Thus u_n is uniformly bounded on $A \cap Q^+$, and thus some subsequence of (u_n) weakly converges (in fact, strongly) in $L^2(A; \mathbf{R}^2)$ to some function u . Since clearly for $y \leq 0$, $u(x, y) - (x, 0) \equiv 0$, and since $e(u) = \mathcal{E}$, the limit point u is unique, and the whole sequence (u_n) converges to u .

Since it holds for any Lipschitz subgraph $A \cap Q^+ \subset\subset \Omega$, this shows that there exists $u \in L^2_{loc}(\pi^{-1}(\Omega); \mathbf{R}^2)$, 1-periodic in x , with $u(x, y) - (x, 0) \equiv 0$ for $y \leq 0$, and such that $\mathcal{E} = e(u)$ in the distributional sense in Ω . With (2.5) and (2.6), we conclude that $u \in X(\Omega)$ and that (Ω, u) is a solution of problem (2.4).

3. An approximation scheme for problem (2.4). Given a (small) scale parameter $\varepsilon > 0$, we now introduce the following approximation of the energy E . We first choose $\eta_\varepsilon > 0$ such that $\eta_\varepsilon = o(\varepsilon)$ as ε goes to zero. Then we let

$$(3.1) \quad E_\varepsilon(v, u) = \int_{Q^+} (v(x, y) + \eta_\varepsilon) \mathbf{A}e(u)(x, y): e(u)(x, y) \, dx dy + L_\varepsilon(v),$$

where

$$(3.2) \quad L_\varepsilon(v) = 2\sigma_c \left(\frac{4\varepsilon}{\pi^2} \int_{Q^+} |\nabla v(x, y)|^2 \, dx dy + \frac{1}{\varepsilon} \int_{Q^+} v(x, y)(1 - v(x, y)) \, dx dy \right),$$

for $v \in H^1(Q^+)$ satisfying $0 \leq v(x, y) \leq 1$, $\partial_y v(x, y) \leq 0$ a.e. in Q^+ , and $v \geq v_s$ on $S^1 \times \{0\}$, and for $u \in H^1_{loc}(\mathbf{R} \times \mathbf{R}_+; \mathbf{R}^2)$ such that $u(x, y) - (x, 0)$ is 1-periodic in x and vanishes on $\mathbf{R} \times \{0\}$. The constant $v_s \in (0, 1]$ is given by

$$(3.3) \quad \int_0^{v_s} \sqrt{t(1-t)} \, dt = \frac{\sigma_c \wedge \sigma_s}{\sigma_c} \int_0^1 \sqrt{t(1-t)} \, dt = \left(\frac{\sigma_c \wedge \sigma_s}{\sigma_c} \right) \frac{\pi}{8}.$$

On the other hand, if v, u do not satisfy these properties, we set $E_\varepsilon(v, u) = +\infty$.

One can think of the Cahn–Hilliard energy L_ε introduced here as a mere mathematical artifact to approximate the perimeter of (rough) sets in the sense of Γ -convergence [23]. Minimizing the corresponding approximating energy E_ε is a much more regular problem. On physical grounds, though, E_ε may be interpreted as the energy of a diffuse interface model, where a smooth transition between two media is permitted over a length-scale ε . In this particular setting we show the following result, which strictly speaking is not a result of Γ -convergence but has the same practical consequences for the computation of minimizers of E .

THEOREM 3.1. *Let $(\varepsilon_n)_{n \geq 1}$ be an arbitrary sequence of positive numbers with $\varepsilon_n \downarrow 0$ as $n \rightarrow \infty$.*

(i) *Let (v_n, u_n) be functions such that*

$$(3.4) \quad \sup_{n \geq 1} E_\varepsilon(v_n, u_n) < +\infty$$

and $\sup_{n \geq 1} \int_{Q^+} v_n(\xi) \, d\xi < +\infty$. Then there exist $\Omega \in \mathcal{G}$, $u \in X(\Omega)$, and a subsequence of $(v_n, u_n)_{n \geq 1}$, still denoted by (v_n, u_n) , such that $v_n \rightarrow \chi_\Omega$ a. e. in Q^+ , $u_n \rightarrow u$ in $L^2_{loc}(\pi^{-1}(\Omega); \mathbf{R}^2)$ as n goes to infinity, and

$$(3.5) \quad E(\Omega, u) \leq \liminf_{n \rightarrow \infty} E_{\varepsilon_n}(v_n, u_n).$$

(ii) *Let $\Omega \in \mathcal{G}$, $u \in X(\Omega)$. Then there exists a sequence $(v_n, u_n)_{n \geq 1}$ such that $v_n \rightarrow \chi_\Omega$ a. e. in Q^+ , $u_n \rightarrow u$ in $L^2_{loc}(\Omega \cap Q^+; \mathbf{R}^2)$ as n goes to infinity, and*

$$(3.6) \quad \limsup_{n \rightarrow \infty} E_{\varepsilon_n}(v_n, u_n) \leq E(\Omega, u).$$

Moreover, we can assume that for all n , $\int_{Q^+} v_n(x, y) \, dx dy = |\Omega \cap Q^+|$.

In particular, this theorem shows that if $v_\varepsilon, u_\varepsilon$ are minimizers of E_ε , subject to the constraint $\int_{Q^+} v_\varepsilon(x, y) \, dx dy = 1$, then to each limit point u of $(u_\varepsilon)_{\varepsilon > 0}$ there corresponds a set Ω with $|\Omega \cap Q^+| = 1$ such that (Ω, u) is a solution of problem (2.4).

4. Numerical examples. The purpose of this section is purely illustrative: We present a few shapes obtained by minimizing approximate energies

$$(4.1) \quad E_\varepsilon(v, u) = K \int_{Q^+} (v(x, y) + \eta_\varepsilon) \mathbf{A}e(u)(x, y) : e(u)(x, y) \, dx dy + L_\varepsilon(v)$$

with L_ε given by (3.2). Expression (4.1) differs from (3.1) only by the presence of the parameter K in front of the elastic energy. It is related to the scaling from the physical dimensions to the model problem [5], and roughly measures the mean thickness of the specimen. One expects that when K is small, the term of surface energy is dominant, and therefore films with a flat free surface should minimize the energy. When K becomes large, minimizers should show a corrugated free surface.

This energy is minimized under the constraints

$$(4.2) \quad \begin{cases} u(x, 0) = (x, 0), & v(x, 0) \geq v_s, \\ u - (x, 0) \text{ and } v \text{ are 1-periodic in } x, \\ 0 \leq v \leq 1, \partial_y v \leq 0, \text{ and } \int_{Q^1} v = V. \end{cases}$$

For a fixed value of ε , we propose an iterative algorithm, based on a finite difference discretization, to minimize the energy (4.1). The computations are performed on a

fixed rectangle $Q^1 = \mathbf{S}^1 \times (0, 1)$, but the results are displayed on two periods. The domain Q^1 is discretized by a Cartesian mesh: a 200×200 mesh in the computations presented here.

At the n th step of the algorithm, the values of u_{n+1} are computed in a standard manner, as approximations of the solutions to the Euler equation

$$\operatorname{div}((v_n + \eta_\varepsilon)Ae(u_{n+1})) = 0,$$

with the boundary conditions and the periodicity given above.

For the density, our approach is inspired by algorithms for motion by mean curvature described in [28, 29], where in L_ε a nonregular potential of the same form is used. Minimizers of $E_\varepsilon(\cdot, u_{n+1})$ are computed as if they were stationary states of the parabolic problem

$$(4.3) \quad \begin{cases} \partial_t v = 2\sigma_c \left(\frac{8\varepsilon}{\pi^2} \Delta v - 1/\varepsilon(1 - 2v) \right) - Ae(u_{n+1}) : e(u_{n+1}) - \lambda, \\ 0 \leq v \leq 1, \quad \partial_y v \leq 0, \end{cases}$$

with the initial condition $v_{n+1}^{(0)} = v_n$, and where λ is a Lagrange multiplier for the volume constraint $\int_{Q^1} v = V$. More precisely, we compute functions $v_{n+1}^{(k+1)}$ 1-periodic in x and such that $v_{n+1}^{(k+1)}(x, 0) \geq v_s$, approximate solutions to the following variational problem:

$$(4.4) \quad \begin{cases} 1/\delta \int_{Q^1} (v_{n+1}^{(k+1)} - v_{n+1}^{(k)})\varphi \\ + 2\sigma_c \left(\frac{8\varepsilon}{\pi^2} \int_{Q^1} \nabla v_{n+1}^{(k+1)} \nabla \varphi + 1/\varepsilon \int_{Q^1} (1 - v_{n+1}^{(k+1)})\varphi \right) \\ + \int_{Q^1} (\lambda + Ae(u_{n+1}) : e(u_{n+1}))\varphi = 0, \end{cases}$$

where the test functions φ are 1-periodic in x and satisfy $\varphi(x, 0) = 0$. The constraints $0 \leq v_{n+1}^{(k+1)} \leq 1$ and $\partial_y v_{n+1}^{(k+1)} \leq 0$ are then enforced at each iteration by a simple truncation. The parameter δ corresponds to an artificial time step in view of (4.3) and is required to be smaller than $\varepsilon/2\sigma_c$. In this manner, the variational problem above is the Euler–Lagrange equation of a convex minimization problem, and therefore existence of a solution $v_{n+1}^{(k+1)}$ is guaranteed.

One of the virtues of this method is that the energy E_ε is decreasing in the course of the algorithm. Indeed, since (4.4) expresses that $v_{n+1}^{(k+1)}$ is the minimizer of the energy $E_\varepsilon(\cdot, u_{n+1}) + 1/2\delta \int_{Q^1} |\cdot - v_{n+1}^{(k)}|^2$, one can easily check that

$$E_\varepsilon(v_{n+1}^{(k+1)}, u_{n+1}) - E_\varepsilon(v_{n+1}^{(k)}, u_{n+1}) \leq -1/2\delta \int_{Q^1} |v_{n+1}^{(k+1)} - v_{n+1}^{(k)}|^2.$$

When $E_\varepsilon(v_{n+1}^{(k)}, u_{n+1})$ has become stationary, we assign $v_{n+1} = v_{n+1}^{(k)}$.

This method for computing equilibrium shapes thus depends on several parameters: $\varepsilon, \eta_\varepsilon, \delta$, and the mesh-size h . The transition zone should be described by at least a few mesh-points. In fact, during the computations, we let ε vary from 20 mesh sizes to 3. The parameter η_ε has been chosen equal to 0.001. The Hooke’s law of the film is that of an isotropic elastic material, with Lamé constants $\lambda = 1$ and $\mu = 2$.

We first illustrate the results obtained by this algorithm when considering a 1-d approximation for the elastic displacement, like in (1.1). The approximate energy functionals then reduce to

$$E_\varepsilon(v, u) = K \int_{Q^1} (v(x, y) + \eta_\varepsilon) |\partial_x u_1|^2 + L_\varepsilon(v),$$

where $u_1(x)$ denotes the average in y of the x -component of the displacement. In this context, it was shown in [5] that for K large the optimal configuration of the free surface consisted of half circles separated by cracks. Figure 1 shows that the algorithm does capture the optimal configuration.

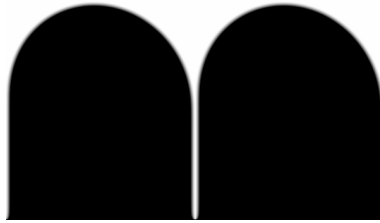


FIG. 1. *Equilibrium shape for the 1-d model.*

The rest of the figures are pertinent to the energy (4.1). The value of the density on the bottom of the film has been chosen to be $v \geq v_s = 0.99$ unless stated otherwise, and the volume of the film is constrained to be $V = 0.2$. In those computations the initial shape has the form $v_0(y) + a \sin(2\pi n x) f(y)$. The function v_0 is the optimal profile associated with minimizing, under the volume constraint, only the energy L_ε ; see (6.12). If the prescribed volume of the film is V , v_0 has the expression

$$v_0(y) = \begin{cases} 1 & \text{if } y \leq V - \varepsilon, \\ \frac{1}{2} \left(1 - \sin \frac{\pi(y-V)}{2\varepsilon} \right) & \text{if } V - \varepsilon \leq y \leq V + \varepsilon, \\ 0 & \text{if } y \geq V + \varepsilon. \end{cases}$$

The remaining term in the expression of the initial shape is a perturbation, consisting of a sinusoidal function in x , times a function f of y whose main attribute is to be concentrated in the transition zone of v_0 . The choice of the x -dependence of the perturbation stems from a linearized stability analysis around the shape v_0 , which can be shown to be a global minimizer for the energy E_ε , when K is small, corresponding to the flat free surface $\Omega = \mathbf{S}^1 \times (0, V)$, a solution to (2.4). These aspects will be discussed elsewhere, however.

Figure 2 shows the initial and final shapes for a computation with $K = 0.5$. The initial shape is $v_0(y)$ plus a perturbation of amplitude $a = 0.4$ and frequency $n = 2$. The resulting free surface is flat, which is consistent with the small value of K . Figure 3 shows the variations of the total energy, and of its elastic and surface energy components, during this computation

Figure 4 shows the initial shape (with amplitude $a = 0.7$ and frequency $n = 1$), and the final shapes resulting from the algorithm when $K = 6$ and for three different values of v_s (0.99, 0.6, and 0.4). In contrast with the 1-d model, no sharp cracks are observed; rather, wide zones with a nearly barren substrate are formed, while the film forms bumps in the shape of milestones (as expected with a low volume constraint).



FIG. 2. *Initial and final shapes, $K = 0.05$.*

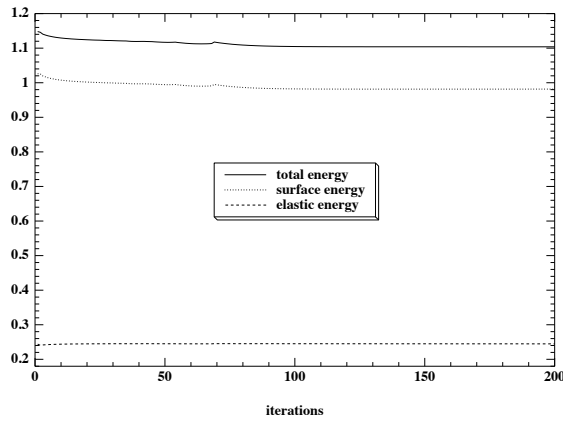


FIG. 3. *Energies, $K = 0.5$, initial shape $v_0(y) + 0.05 \sin(2\pi x)f(y)$.*

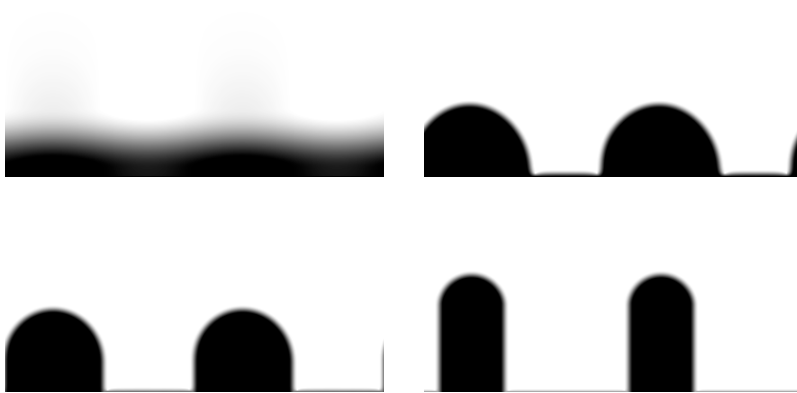
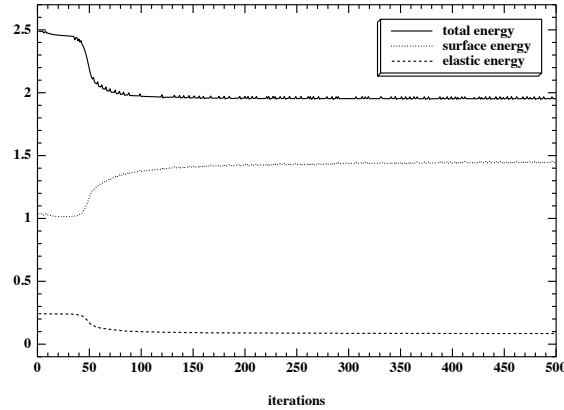


FIG. 4. *Initial and final shapes, $K = 6$.*

The width of the base of these milestones depends on the value of v_s ; this reflects the fact that v_s is itself a function of the ratio σ_c/σ_s , and determines the cost of leaving the substrate barren. The thin layers of film are exactly the wetting layers described in [8, 33, 31]. Figure 5 shows the history of energies, when $v_s = 0.99$. Finally, we present some results that demonstrate the unstable behavior of the system for large values of K . Figure 6 shows various images of the configurations obtained during a computation with $K = 20$, initiated with the profile $v_0(y)$, which is a local minimum. After a large number of iterations during which nothing seems to happen to the “flat surface,” instabilities are triggered by round-off errors. One observes corrugations

FIG. 5. *Energies*, $K = 6$, $v_s = 0.99$.FIG. 6. *Shapes at iterations* 1936, 1940, 1944, 1948, 2000, 2100; $K = 20$, *initial shape* $v_0(y)$.

that finally merge together to form a unique bigger milestone. These pictures represent the evolution of a free surface driven by the evolution equation (4.3). This model is a priori different from the model of motion by surface diffusion usually considered in the physical literature [14, 35, 32]. However, (4.3) governs the evolution of the tracer v but not that of the thickness of the film. Figure 7 shows the corresponding energies.

5. Proof of Lemma 2.1. In order to prove Lemma 2.1, we need to show that

(i) given any Ω_n converging to Ω in \mathcal{G} ,

$$(5.1) \quad L(\Omega) \leq \liminf_{n \rightarrow \infty} L^0(\Omega_n),$$

and that

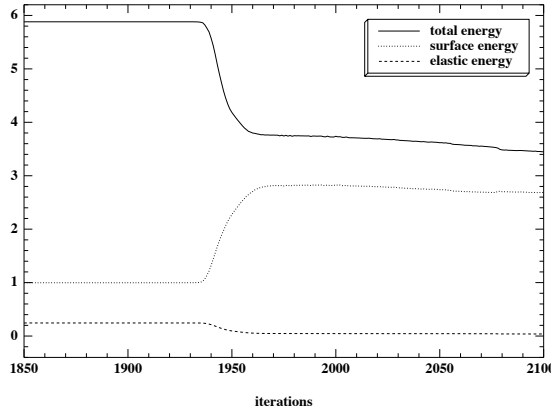


FIG. 7. *Energies, $K = 20$, initial shape $y(x) = 1$.*

(ii) for any $\Omega \in \mathcal{G}$, there exists a sequence Ω_n converging to Ω such that

$$(5.2) \quad \limsup_{n \rightarrow \infty} L^0(\Omega_n) \leq L(\Omega),$$

where L is defined by (2.2).

5.1. Proof of (i). To show (i), we consider Ω_n converging to Ω in \mathcal{G} . Without loss of generality we can assume that $a = \sup_n L^0(\Omega_n) + |\Omega_n \cap Q^+| < +\infty$ (since if $\liminf_{n \rightarrow \infty} L^0(\Omega_n) = +\infty$, then there is nothing to prove, and if $|\Omega_n \cap Q^+| \rightarrow \infty$ and $L^0(\Omega_n)$ is bounded, Ω_n converges to $Q \notin \mathcal{G}$). In particular, the Ω_n are subgraphs of Lipschitz functions $h_n : \mathbf{S}^1 \rightarrow [0, a]$.

Clearly, $\partial\Omega_n \subset \bar{Q}^a$, and thus we may assume (by extracting a subsequence, still denoted by (Ω_n)) that the boundaries $\partial\Omega_n$ converge in the Hausdorff metric to some compact set K . Notice that we easily deduce that $\bar{\Omega}_n$ converge to $\Omega \cup K$ in the Hausdorff sense; indeed, if $\xi \in \Omega \cup K$, either $\xi \in \Omega$ and therefore $\xi \in \Omega_n$ for large n , or $\xi \in K$ and there exists $\xi_n \in \partial\Omega_n \subset \bar{\Omega}_n$ such that $\xi = \lim_{n \rightarrow \infty} \xi_n$. Conversely, if $\xi_n \in \bar{\Omega}_n$ for all n and converge to some ξ as n goes to infinity, if $\xi \notin \Omega$, then there exists $\xi'_n \in Q \setminus \Omega_n$ with $\xi = \lim_{n \rightarrow \infty} \xi'_n$ (since $Q \setminus \Omega_n$ converge to $Q \setminus \Omega$ in the Hausdorff sense) and there exists $\xi''_n \in [\xi_n, \xi'_n] \cap \partial\Omega_n$. Since $\xi''_n \rightarrow \xi$, we deduce that $\xi \in K$. In a similar way we can show that $\partial\Omega \subset K$.

Then, invoking Golab's theorem (see [16, 24, 11]) and $\partial\Omega_n$ being a sequence of uniformly bounded 1-d compact connected sets, we get that

$$(5.3) \quad \mathcal{H}^1(\partial\Omega) \leq \mathcal{H}^1(K) \leq \liminf_{n \rightarrow \infty} \mathcal{H}^1(\partial\Omega_n) < +\infty.$$

Moreover, the proofs of Golab's theorem based on measure density arguments (see [24]) also show that the measures $\mu_n = \mathcal{H}^1 \llcorner \partial\Omega_n$, up to a subsequence (not relabeled), converge weakly-* to a measure μ supported by K such that

$$(5.4) \quad \mathcal{H}^1 \llcorner K \leq \mu.$$

We define the l.s.c. functions $h, \bar{h} : \mathbf{S} \rightarrow [0, a]$ as $h(x) = \min\{y : (x, y) \notin \Omega\}$ (resp., $\bar{h}(x) = \min\{y : (x, y) \notin \overset{\circ}{\Omega}\}$) so that (2.1) holds. Since $\Omega \subseteq \overset{\circ}{\Omega}$, $h \leq \bar{h}$.

We first observe that $\partial\Omega$ is the disjoint union of $\partial\bar{\Omega}$ and $(\bigcup_{x \in \mathbf{S}^1} \{x\} \times [h(x), \bar{h}(x)])$. Together with (5.3), this implies that $h(x) = \bar{h}(x)$ except for at most countably many $x \in \mathbf{S}^1$. As a matter of fact, $\partial\bar{\Omega} = \bar{\Omega} \setminus \overset{\circ}{\bar{\Omega}} \subseteq \bar{\Omega} \setminus \Omega = \partial\Omega$. Then $\partial\Omega \setminus \partial\bar{\Omega} = \overset{\circ}{\bar{\Omega}} \setminus \Omega$, so that by the definitions of h and \bar{h} ,

$$\partial\Omega \setminus \partial\bar{\Omega} = \{(x, y) \in Q : h(x) \leq y < \bar{h}(x)\} = \bigcup_{x \in \mathbf{S}^1} \{x\} \times [h(x), \bar{h}(x)),$$

which proves the claim.

We now define the measures $\nu_n = \sigma_c \mathcal{H}^1 \llcorner \partial\Omega_n \cap Q^+ + \sigma_s \mathcal{H}^1 \llcorner \partial\Omega_n \setminus Q^+$. It is not restrictive to assume that ν_n converges weakly-* to a positive measure ν as n goes to infinity, and we clearly have

$$(5.5) \quad (\sigma_s \wedge \sigma_c)\mu \leq \nu \leq (\sigma_s \vee \sigma_c)\mu.$$

Moreover, since Q^+ is open and $\nu_n \llcorner Q^+ = \sigma_c \mu_n \llcorner Q^+$ for all n , we also have

$$(5.6) \quad \nu \llcorner Q^+ = \sigma_c \mu \llcorner Q^+.$$

To deduce (5.1), it remains to show that the 1-d density of ν on the set $\partial\Omega \setminus \partial\bar{\Omega} = \bigcup_{x \in \mathbf{S}^1} \{x\} \times [h(x), \bar{h}(x))$ is at least $2\sigma_c$. Indeed, since

$$\begin{aligned} \liminf_{n \rightarrow \infty} L^0(\Omega_n) &= \liminf_{n \rightarrow \infty} \nu_n(Q) \geq \nu(Q) = \nu(K) \\ &\geq \nu(\partial\Omega) = \nu(\partial\bar{\Omega} \cap Q^+) + \nu(\partial\bar{\Omega} \setminus Q^+) + \nu(\partial\Omega \setminus \partial\bar{\Omega}), \end{aligned}$$

it will imply together with (5.4), (5.5), and (5.6) that

$$\begin{aligned} \liminf_{n \rightarrow \infty} L^0(\Omega_n) &\geq \sigma_c \mathcal{H}^1(\partial\bar{\Omega} \cap Q^+) + (\sigma_c \wedge \sigma_s) \mathcal{H}^1(\partial\bar{\Omega} \setminus Q^+) + 2\sigma_c \mathcal{H}^1(\partial\Omega \setminus \partial\bar{\Omega}) = L(\Omega). \end{aligned}$$

It remains therefore to show (see [10]) that for \mathcal{H}^1 -almost any $\xi = (x, y) \in \partial\Omega \setminus \partial\bar{\Omega}$,

$$(5.7) \quad \limsup_{\rho \downarrow 0} \frac{\nu(\overline{B(\xi, \rho)})}{2\rho} \geq 2\sigma_c.$$

Choose such a ξ , with $h(x) < y$, and choose $\rho > 0$ small enough, so that $h(x) < y - \rho$, $y + \rho < \bar{h}(x)$, and $B(\xi, 2\rho) \subset \overset{\circ}{\bar{\Omega}}$. (We exclude the case $h(x) = y$ since the set $\{(x, h(x)) : h(x) < \bar{h}(x)\}$ is at most countable and thus \mathcal{H}^1 -negligible.)

For some $\delta > 0$ small enough, $(x, y - \rho - \delta) \notin \Omega$, so that there exists $(x_n, y_n) \notin \Omega_n$ such that $(x_n, y_n) \rightarrow (x, y - \rho - \delta)$ as n goes to infinity. If n is large enough, $y_n < y - \rho$ and $h_n(x_n) < y - \rho$, where h_n is the Lipschitz function whose subgraph is Ω_n .

Fix $\varepsilon > 0$ small ($\varepsilon \ll \rho$) and choose x', x'' with $x - \varepsilon < x' < x < x'' < x + \varepsilon$. Choose also y', y'' such that $(x', y'), (x'', y'') \in B(\xi, 2\rho) \setminus \overline{B(\xi, \rho)}$ and $y' > y + \rho$, $y'' > y + \rho$. Since the $\bar{\Omega}_n$ converge in the Hausdorff sense to $\Omega \cup K \supset \bar{\Omega}$, and since $(x', y'), (x'', y'') \in \bar{\Omega}$, there are sequences (x'_n, y'_n) and (x''_n, y''_n) converging, respectively, to (x', y') and (x'', y'') such that $(x'_n, y'_n), (x''_n, y''_n) \in \bar{\Omega}_n$ for all n .

If n is large enough, $x - \varepsilon < x'_n < x_n < x''_n < x + \varepsilon$, and $h_n(x'_n), h_n(x''_n) > y + \rho$, $h_n(x_n) < y - \rho$, so that the length of the graph $\partial\Omega_n$ of h_n inside the ball $B(\xi, \rho)$ must be at least $2 \times 2\rho - O(\varepsilon^2)$. We deduce that for large n , $\nu_n(\overline{B(\xi, \rho)}) \geq 2\sigma_c 2\rho - O(\varepsilon^2)$,

and therefore $\nu(\overline{B(\xi, \rho)}) \geq \limsup_{n \rightarrow \infty} \nu_n(\overline{B(\xi, \rho)}) \geq 2\sigma_c 2\rho - O(\varepsilon^2)$. Sending ε to 0, we get that for any (small) $\rho > 0$

$$\frac{\nu(\overline{B(\xi, \rho)})}{2\rho} \geq 2\sigma_c,$$

so that (5.7) clearly holds.

5.2. Proof of (ii). We now must build, given $\Omega \in \mathcal{G}$, a sequence Ω_n converging to Ω satisfying (5.2). It is of course not restrictive to assume that $L(\Omega) < +\infty$.

We consider the l.s.c. functions h and \bar{h} as in (2.1). Since $L(\Omega) < +\infty$, h and \bar{h} are bounded and $h(x) = \bar{h}(x)$ except for at most a countable number of points $x \in \mathbf{S}^1$. Moreover, h and \bar{h} are functions of bounded variation, in the classical sense. (See the appendix, and in particular, section A.2.2 and inequality (A.3).) They thus have a right and left limit at each point. Now, since $h = \bar{h}$ a.e., we deduce that for any $x \in \mathbf{S}^1$, $\bar{h}(x-0) = h(x-0)$ and $\bar{h}(x+0) = h(x+0)$ (where $h(x \pm 0) = \lim_{\varepsilon \downarrow 0^+} h(x \pm \varepsilon)$).

Now, h being considered as a 1-periodic function defined on the whole real line (meaning that we still denote by h what should theoretically be $h \circ \pi$), for all $n \geq 1$ let h_n be the n -Lipschitz 1-periodic and nonnegative function defined by

$$(5.8) \quad h_n(x) = \inf_{x' \in \mathbf{R}} h(x') + n|x - x'|,$$

and define $\Omega_n \in \mathcal{G}$ as the subgraph of h_n . Notice that since h is l.s.c., the infimum in (5.8) is reached. Since $h_n \leq h$, $\Omega_n \subseteq \Omega$. It is also well known that for all x , $h_n(x) \uparrow h(x)$ as $n \rightarrow \infty$. Let us show that Ω is the limit of the sequence $(\Omega_n)_{n \geq 1}$. Let A be the limit of some converging subsequence $(\Omega_{n_k})_{k \geq 1}$. Since Ω_n is increasing, it is not difficult to show that A is the limit of the whole sequence $(\Omega_n)_{n \geq 1}$. Clearly $A \subseteq \Omega$, and we want to show the reverse inequality. Let $\xi = (x, y) \notin A$. There exist $\xi_n = (x_n, y_n) \notin \Omega_n$ such that $\xi_n \rightarrow \xi$ as $n \rightarrow \infty$. Since the infimum is reached in (5.8), there exist x'_n such that

$$y_n \geq h_n(x_n) = h(x'_n) + n|x_n - x'_n|$$

for every n . In particular, since $(y_n)_{n \geq 1}$ is bounded, $x'_n \rightarrow x$ as n goes to infinity. Since h is l.s.c., we deduce that

$$h(x) \leq \liminf_{n \rightarrow \infty} h(x'_n) \leq \liminf_{n \rightarrow \infty} y_n = y,$$

so that $\xi \notin \Omega$; therefore $\Omega \subseteq A$, and we have proved that Ω is the limit of Ω_n .

Now we will show that (5.2) holds for the sequence Ω_n . We have $\partial\Omega_n = \partial\bar{\Omega}_n = \{(x, h_n(x)) : x \in \mathbf{S}^1\}$.

We split $\partial\Omega_n$ into two parts, $\partial\Omega_n \cap \partial\Omega$ and $\partial\Omega_n \setminus \partial\Omega$. First notice that $\partial\Omega \cap \partial\Omega_n$ is essentially equal to $\partial\bar{\Omega} \cap \partial\Omega_n$. Indeed, let $(x, y) \in \partial\Omega_n \cap (\partial\Omega \setminus \partial\bar{\Omega})$. We have $y = h_n(x) \leq h(x)$ and $h(x) \leq y < \bar{h}(x)$, thus $y = h(x)$ and $h(x) < \bar{h}(x)$, but we know that this happens for at most a countable number of points x . Thus, $\partial\Omega_n \cap (\partial\Omega \setminus \partial\bar{\Omega})$ is at most countable.

Now, suppose that $h_n(x) = 0$; then there exists $x' \in \mathbf{R}$ such that $h(x') + n|x - x'| = 0$. Since h is nonnegative, it implies that $x' = x$ and $h(x) = 0$. On the other hand, if $h(x) = 0$, since $0 \leq h_n \leq h$, $h_n(x) = 0$. We deduce that $\partial\Omega_n \setminus Q^+ = \partial\Omega \setminus Q^+$ for any $n \geq 1$; moreover, this set is essentially equal to $\partial\bar{\Omega} \setminus Q^+$.

We thus deduce that, for every n ,

$$(5.9) \quad \begin{aligned} L^0(\Omega_n) &= \sigma_s \mathcal{H}^1(\partial\Omega_n \setminus Q^+) + \sigma_c \mathcal{H}^1(\partial\Omega_n \cap Q^+) \\ &= \sigma_s \mathcal{H}^1(\partial\bar{\Omega} \setminus Q^+) + \sigma_c \mathcal{H}^1(\partial\bar{\Omega} \cap \partial\Omega_n \cap Q^+) + \sigma_c \mathcal{H}^1(\partial\Omega_n \setminus \partial\Omega). \end{aligned}$$

We now need to estimate $\mathcal{H}^1(\partial\Omega_n \setminus \partial\Omega)$. Notice that clearly, $\partial\Omega_n \setminus \partial\Omega = \{(x, h_n(x)) : h_n(x) < h(x)\}$. Since $\partial\Omega_n \setminus \partial\Omega$ is open, it can be written as a disjoint union of open connected arcs:

$$\partial\Omega_n \setminus \partial\Omega = \bigcup_{k \in K} \{(x, h_n(x)) : x \in I_k\},$$

where the set K is finite or countable, and for each k , $I_k = (a_k, b_k) \subset \mathbf{S}^1$ is an open interval.

Fix $k \in K$ and consider such an I_k . We claim that for all $x \in I_k$

$$(5.10) \quad h_n(x) = \min\{h(a_k) + n|x - a_k|, h(b_k) + n|x - b_k|\}.$$

In order to simplify the notations, we temporarily drop the subscript k , and let therefore $I = (a, b) = I_k$. We can consider (a, b) as an interval in \mathbf{R} , with $a < b$. Since h_n is n -Lipschitz, $h_n(x) - h_n(a) \leq n|x - a|$, so that for any x , $h_n(x) \leq h_n(a) + n|x - a| \leq h(a) + n|x - a|$. Thus, $h_n(x) \leq \min\{h(a) + n|x - a|, h(b) + n|x - b|\}$.

Assume now that there exists $x \in I$ such that the inequality is strict, and let $x' \in \mathbf{R}$ be a point where the infimum is reached in (5.8). We have $h(x') + n|x - x'| = h_n(x) < h(a) + n|x - a|$, so that if $x' \leq a$, $h(x') + n|a - x'| < h(a)$, but this is in contradiction with the fact that $h_n(a) = h(a)$; thus $x' > a$. In the same way, we show that $x' < b$, so that $x' \in I$.

But if x'' is such that $h_n(x') = h(x'') + n|x' - x''|$, then

$$\begin{aligned} h_n(x) &\leq h_n(x'') + n|x'' - x| && \text{(since } h_n \text{ is } n\text{-Lipschitz)} \\ &\leq h(x'') + n|x'' - x'| + n|x' - x| \\ &= h_n(x') + n|x' - x| \\ &\leq h(x') + n|x' - x| = h_n(x), \end{aligned}$$

so that $h_n(x') = h(x')$ and $x' \notin I$. Therefore x cannot exist, and (5.10) holds for every point in $I = I_k$.

We deduce an estimate for the contribution of the interval $I = I_k$ to the length of $\partial\Omega_n \setminus \partial\Omega$. We consider the two cases

- (a) for all $x \in I$, $h_n(x) = h(a) + n(x - a)$ (or $h_n(x) = h(b) + n(b - x)$);
- (b) there exists $c \in I$ such that $h_n(x) = h(a) + n(x - a)$ for all $x \in (a, c]$, and $h_n(x) = h(b) + n(b - x)$ for all $x \in [c, b)$.

In the first case, the graph of h_n in $\bar{I} \times \mathbf{R}$ is a straight line going from $(a, h(a))$ to $(b, h(b))$, while the boundary $\partial\Omega \cap (\bar{I} \times \mathbf{R})$ contains a curve connecting these two points. This curve is made of a possible piece of straight line going from $(a, h(a))$ to $(a, h(a+0))$, then a curve going from $(a, h(a+0))$ to $(b, h(b-0))$, essentially contained in $\partial\bar{\Omega} \cap (I \times \mathbf{R})$, and then another possible piece of straight line going from $(b, h(b-0))$ to $(b, h(b))$, so that

$$(5.11) \quad \begin{aligned} \mathcal{H}^1(\partial\Omega_n \cap (\bar{I}_k \times \mathbf{R})) &\leq h(a_k + 0) - h(a_k) + h(b_k - 0) - h(b_k) \\ &\quad + \mathcal{H}^1(\partial\bar{\Omega} \cap (I_k \times \mathbf{R})). \end{aligned}$$

In case (b), the graph of h_n in $\bar{I} \times \mathbf{R}$ is made of two straight lines, one going from $(a, h(a))$ to $(c, h_n(c))$ and the other from $(c, h_n(c))$ to $(b, h(b))$. The boundary $\partial\Omega \cap (\bar{I} \times \mathbf{R})$ contains a curve connecting $(a, h(a))$ to $(b, h(b))$ and passing through the point $(c, \bar{h}(c))$, and since $\max\{h(a), h(b)\} < h_n(c) < \bar{h}(c)$, (5.11) still holds.

Summing over $k \in K$, we deduce that

$$(5.12) \quad \begin{aligned} \mathcal{H}^1(\partial\Omega_n \setminus \partial\Omega) &\leq \mathcal{H}^1\left(\partial\bar{\Omega} \cap \bigcup_{k \in K} (I_k \times \mathbf{R})\right) \\ &+ \sum_{k \in K} h(a_k + 0) - h(a_k) + h(b_k - 0) - h(b_k). \end{aligned}$$

Now, it is possible to show that for any x , if, for instance, $h(x + 0) \geq h(x - 0)$, then (recalling that h and \bar{h} are l.s.c.) $\bar{h}(x) = h(x - 0)$ and

$$(5.13) \quad \begin{aligned} h(x + 0) + h(x - 0) - 2h(x) &= \bar{h}(x + 0) - \bar{h}(x) + 2\bar{h}(x) - 2h(x) \\ &= \mathcal{H}^1(\partial\bar{\Omega} \cap (\{x\} \times \mathbf{R})) + 2(\bar{h}(x) - h(x)). \end{aligned}$$

From (5.12) and (5.13), we deduce that

$$(5.14) \quad \mathcal{H}^1(\partial\Omega_n \setminus \partial\Omega) \leq \mathcal{H}^1\left(\partial\bar{\Omega} \cap \bigcup_{k \in K} (\bar{I}_k \times \mathbf{R})\right) + 2\mathcal{H}^1(\partial\Omega \setminus \partial\bar{\Omega}).$$

Now it is clear that $\partial\bar{\Omega} \cap \bigcup_{k \in K} (\bar{I}_k \times \mathbf{R})$ is contained in Q^+ , and that, up to at most a countable number of points, it is disjoint from $\partial\bar{\Omega} \cap \partial\Omega_n \cap Q^+$. Thus we can deduce from (5.9) and (5.14) that

$$L^0(\Omega_n) \leq \sigma_s \mathcal{H}^1(\partial\bar{\Omega} \setminus Q^+) + \sigma_c (\mathcal{H}^1(\partial\bar{\Omega} \cap Q^+) + 2\mathcal{H}^1(\partial\Omega \setminus \partial\bar{\Omega})) = L(\Omega);$$

thus (5.2) holds and (ii) is proved. This achieves the proof of Lemma 2.1. \square

6. Proof of Theorem 3.1.

6.1. Proof of point (i) of the theorem. Given a sequence $(\varepsilon_n)_{n \geq 1}$ with $\varepsilon_n \downarrow 0$ as n goes to infinity, we first consider a sequence $(v_n, u_n)_{n \geq 1}$ that satisfies (3.4). Then, by standard results [23, 1] on the Cahn–Hilliard energy L_ε , we know that up to a subsequence (still denoted by v_n), there exists a Caccioppoli set $F \subseteq Q^+$ such that $v_n \rightarrow \chi_F$ a.e. in Q^+ ; moreover,

$$(6.1) \quad \sigma_c \mathcal{H}^1(\partial^* F) \leq \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n),$$

where $\partial^* F$ is the reduced boundary of F inside Q^+ (see [10] or section A.2.3). Since $\sup_{n \geq 1} \int_{Q^+} v_n(\xi) d\xi < +\infty$, the set F is bounded, and since $\partial_y v_n \leq 0$, $F \cup \bar{Q}^-$ is Lebesgue—essentially equivalent to the subgraph of a (nonnegative) bounded variation function $g : \mathbf{S}^1 \rightarrow [0, +\infty)$.

Now fix an integer $k \geq 2$ and write for every n , using the coarea formula (see (A.4), appendix)

$$\begin{aligned} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left| D\chi_{\{v_n > s\}} \right| (Q^+) ds &= k(k+1) \int_{\{\xi \in Q^+ : \frac{1}{k+1} < v_n(\xi) < \frac{1}{k}\}} |\nabla v_n(\xi)| d\xi \\ &\leq (k(k+1))^{\frac{3}{2}} \int_{Q^+} |\nabla v_n(\xi)| \sqrt{v_n(\xi)(1-v_n(\xi))} d\xi \\ &\leq (k(k+1))^{\frac{3}{2}} \frac{\pi}{8\sigma_c} L_{\varepsilon_n}(v_n), \end{aligned}$$

which by (3.4) is uniformly bounded by some constant c_k . We deduce that there exists a level $s_n^k \in (\frac{1}{k+1}, \frac{1}{k})$ such that $|D\chi_{\{v_n > s_n^k\}}|(Q^+) \leq c_k$.

Moreover, since $\partial_y v_n \leq 0$, $\{v_n > s_n^k\}$ is the subgraph of some function of bounded variation. Thus, if we define $h_n^k : \mathbf{S}^1 \rightarrow [0, +\infty)$ to be the largest l.s.c. representative of this function, the open set $\Omega_n^k = \{(x, y) \in Q : y < h_n^k(x)\} \in \mathcal{G}$ is such that $\overline{\Omega_n^k} = \Omega_n^k$ and is (Lebesgue-) essentially equal in Q^+ to $\{v_n > s_n^k\}$. In particular, $v_n > s_n^k$ a.e. in Ω_n^k , $v_n \leq s_n^k$ a.e. out of Ω_n^k , and $\mathcal{H}^1(\partial\Omega_n^k \cap Q^+) = |D\chi_{\{v_n > s_n^k\}}|(Q^+) \leq c_k$ so that $\mathcal{H}^1(\partial\Omega_n^k) \leq 1 + c_k$.

Define the sequence $(n_p^1)_{p \geq 1}$ by $n_p^1 = p$ for every p . For each $k = 2, 3, \dots$, we can extract (by induction) from the sequence (n_p^{k-1}) a sequence (n_p^k) such that $\Omega_{n_p^k}^k$ converges to some open set $\Omega^k \in \mathcal{G}$ as p goes to infinity. We build in this way a family of sets $(\Omega^k)_{k \geq 2}$ such that, for every k , the sequence $(\Omega_{n_p^k}^k)$ converges to Ω^k in \mathcal{G} .

In what follows we will relabel this subsequence and denote again by n what should be n_p^p . We let $\Omega = \bigcup_{k \geq 2} \Omega^k$ and

$$N = \{\xi \in Q^+ : v_n(\xi) \not\rightarrow \chi_F(\xi) \text{ as } n \rightarrow \infty\} \\ \cup \bigcup_{k,n} (\{v_n > s_n^k\} \setminus \Omega_n^k) \cup (\{v_n \leq s_n^k\} \cap \Omega_n^k)$$

and observe that $|N| = 0$. Notice that Ω has the following characterization:

$$(6.2) \quad Q \setminus \Omega = \{\xi \in Q : \forall \rho > 0, \lim_{n \rightarrow \infty} \text{infess}_{B(\xi, \rho)} v_n = 0\}.$$

Indeed, consider $\xi \notin \Omega$ and fix $\rho > 0$. For any fixed $k \geq 2$ there exists $\xi_n \notin \Omega_n^k$ such that $\xi_n \rightarrow \xi$. But since $|N| = 0$ and $\Omega_n^k = \overline{\Omega_n^k}$, we have that $|B(\xi_n, \rho/2) \setminus \Omega_n^k \setminus N| > 0$. If n is large enough, $|\xi_n - \xi| < \rho/2$; therefore $B(\xi, \rho) \supset B(\xi_n, \rho/2)$ and $\text{infess}_{B(\xi, \rho)} v_n \leq s_n^k < 1/k$. This shows that for every $k \geq 2$, $\limsup_{n \rightarrow \infty} \text{infess}_{B(\xi, \rho)} v_n \leq 1/k$; thus $\lim_{n \rightarrow \infty} \text{infess}_{B(\xi, \rho)} v_n = 0$. We will not need it in the following, but it is also easy to show that if $\xi \in \Omega$, then $\limsup_{n \rightarrow \infty} \text{infess}_{B(\xi, \rho)} v_n > 0$.

We will now prove that the set Ω satisfies the thesis of point (i) of Theorem 3.1. We will first show in section 6.1.1 that

$$(6.3) \quad L(\Omega) \leq \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n),$$

and then, in section 6.1.2, that (up to a subsequence) u_n converges to a function u such that

$$(6.4) \quad \int_{\Omega \cap Q^+} \mathbf{A}e(u)(x, y) : e(u)(x, y) \, dx dy \\ \leq \liminf_{n \rightarrow \infty} \int_{Q^+} (v_n(x, y) + \eta_{\varepsilon_n}) \mathbf{A}e(u_n)(x, y) : e(u_n)(x, y) \, dx dy.$$

6.1.1. Estimate of $L(\Omega)$. In order to show (6.3) we need more information on the structure of $\partial\Omega$ and on the relationship of Ω with F .

Notice that if $\xi \in \Omega \setminus N \cap Q^+$, then $\xi \in \Omega^k$ for some k and therefore $\xi \in \Omega_n^k$ for large n . In particular, $v_n(\xi) > s_n^k > 1/(k+1)$ and, since it tends either to 0 or to 1, $\lim_{n \rightarrow \infty} v_n(\xi) = 1$ and $\xi \in F$. Therefore $\Omega \cap Q^+ \setminus N \subseteq F$.

On the other hand, we claim that $|F \setminus \Omega| = 0$.

Indeed, we first notice that, given any $k \geq 2$, $\chi_{\Omega_n^k} \rightarrow \chi_{\Omega^k}$ a.e. in Q^+ as $n \rightarrow \infty$ (up to a subsequence) and thus, for instance, in $L^2(Q^a)$ for every $a > 0$. This is true because, up to a subsequence, we can assume that $\partial\Omega_n^k$ tends to some compact subset $K \subset \overline{Q^+}$ in the Hausdorff sense; then, by Gołab's theorem (see [16, 11, 24]), $\mathcal{H}^1(K) \leq \liminf_{n \rightarrow \infty} \mathcal{H}^1(\partial\Omega_n^k) < +\infty$ (since $\mathcal{H}^1(\partial\Omega_n^k) \leq 1 + c_k$ for every n) and thus $|K| = 0$. Eventually, observe that if $\xi \in \Omega^k$, then $\xi \in \Omega_n^k$ for large n and $\chi_{\Omega_n^k}(\xi) = 1$, and that if $\xi \notin \Omega^k \cup K$, then we can easily show that $\xi \notin \Omega_n^k$ for large n and $\chi_{\Omega_n^k}(\xi) = 0$. (See section 5.1 for details of a similar proof.)

Then we have $v_n \rightarrow 1$ a.e. in F , and in particular, $v_n \chi_F \rightarrow \chi_F$ in $L^2(Q^+)$. Thus,

$$\begin{aligned} |Q^a \cap F \setminus \Omega^k| &= \int_{Q^a} \chi_F(\xi)(1 - \chi_{\Omega^k})(\xi) \, d\xi \\ &= \lim_{n \rightarrow \infty} \int_{Q^a} v_n(\xi)\chi_F(\xi)(1 - \chi_{\Omega_n^k})(\xi) \, d\xi \\ &= \lim_{n \rightarrow \infty} \int_{Q^a} v_n(\xi)\chi_F(\xi)\chi_{\{v_n \leq s_n^k\}}(\xi) \, d\xi \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{k} \int_{Q^a} \chi_F(\xi)(1 - \chi_{\Omega_n^k})(\xi) \, d\xi = \frac{1}{k} |Q^a \cap F \setminus \Omega^k|, \end{aligned}$$

so that $|Q^a \cap F \setminus \Omega^k|$ must be zero for any $a > 0$. This shows that $|F \setminus \Omega^k| = 0$, and since $\Omega^k \subset \Omega$, it proves the claim. We conclude that $\Omega \cap Q^+ = F$ up to a Lebesgue-negligible set.

Recall that F is the subgraph of a function of bounded variation $g \in BV(\mathbf{S}^1)$. Let \bar{h} be the greatest l.s.c. representative of g . (For instance, \bar{h} can be built as the l.s.c. envelope of any u.s.c. representative of g .) Since $\Omega \cap Q^+ = F$ up to a Lebesgue-negligible set, we easily show that

$$\overset{\circ}{\Omega} = \{(x, y) \in Q : y < \bar{h}(x)\}$$

and that $\partial\bar{\Omega} \cap Q^+$ is \mathcal{H}^1 -essentially equal to ∂^*F .

For any Borel set $B \subseteq Q$ we introduce the following localization of L_ε :

$$L_\varepsilon(v, B) = 2\sigma_c \left(\frac{4\varepsilon}{\pi^2} \int_{Q^+ \cap B} |\nabla v(x, y)|^2 \, dx dy + \frac{1}{\varepsilon} \int_{Q^+ \cap B} v(x, y)(1 - v(x, y)) \, dx dy \right). \tag{6.5}$$

For any open set A , we also have $\sigma_c \mathcal{H}^1(\partial^*F \cap A) \leq \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n, A)$. In particular, if we choose $\delta > 0$ and define $A^\delta = \{\xi \in Q : \text{dist}(\xi, \partial\bar{\Omega} \cap Q^+) < \delta\}$, then we have that

$$\sigma_c \mathcal{H}^1(\partial\bar{\Omega} \cap Q^+) \leq \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n, A^\delta). \tag{6.6}$$

Next define $B^\delta = \{\xi \in Q \setminus \bar{A}^\delta : \text{dist}(\xi, \partial\bar{\Omega} \setminus Q^+) < \delta\}$; $B^\delta \cap \partial\bar{\Omega}$ is a finite union of segments in \mathbf{S}^1 . We want to estimate $\mathcal{H}^1(B^\delta \cap \partial\bar{\Omega})$. Write $\partial\bar{\Omega} \cap B^\delta = \bigcup_{k=1}^N (a_k, b_k) \times \{0\}$. We have for any $\delta' \leq \delta$

$$\begin{aligned} L_{\varepsilon_n}(v_n, B^\delta) &\geq \sum_{k=1}^N \int_{a_k}^{b_k} \left\{ 2\sigma_c \int_0^{\delta'} \frac{4\varepsilon_n}{\pi^2} |\partial_y v_n(x, y)|^2 + \frac{v_n(x, y)(1 - v_n(x, y))}{\varepsilon_n} \, dy \right\} dx \\ &\geq \sum_{k=1}^N \int_{a_k}^{b_k} \left\{ \frac{8\sigma_c}{\pi} \int_0^{\delta'} \sqrt{v_n(x, y)(1 - v_n(x, y))} |\partial_y v_n(x, y)| \, dy \right\} dx. \end{aligned}$$

Recalling that $v_n(x, 0) = v_s$, where v_s is defined by (3.3), we deduce

$$\begin{aligned} L_{\varepsilon_n}(v_n, B^\delta) &\geq \sum_{k=1}^N \int_{a_k}^{b_k} \left\{ \frac{8\sigma_c}{\pi} \int_{v_n(x, \delta')}^{v_s} \sqrt{t(1-t)} dt \right\} dx \\ &= \sum_{k=1}^N \int_{a_k}^{b_k} \left\{ \sigma_c \wedge \sigma_s - \frac{8\sigma_c}{\pi} \int_0^{v_n(x, \delta')} \sqrt{t(1-t)} dt \right\} dx. \end{aligned}$$

Since $v_n \rightarrow 0$ a.e. in $B^\delta \cap Q^+$, for a.e. $\delta' \in (0, \delta]$, we must have that for a.e. x in $\bigcup_{k=1}^N (a_k, b_k)$, $v_n(x, \delta') \rightarrow 0$ as $n \rightarrow \infty$. Therefore, choosing such a δ' , we have for a.e. $x \in \bigcup_{k=1}^N (a_k, b_k)$

$$\lim_{n \rightarrow \infty} \int_0^{v_n(x, \delta')} \sqrt{t(1-t)} dt = 0,$$

so that

$$(6.7) \quad \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n, B^\delta) \geq (\sigma_c \wedge \sigma_s) \sum_{k=1}^N |b_k - a_k| = (\sigma_c \wedge \sigma_s) \mathcal{H}^1(\partial\bar{\Omega} \cap B^\delta).$$

Now, we want to estimate the length of $\partial\Omega \setminus \partial\bar{\Omega}$. As usual $h : \mathbf{S}^1 \rightarrow [0, +\infty)$ will denote the l.s.c. function such that $\Omega = \{(x, y) \in Q : y < h(x)\}$, and it has been shown in section 5.1 that

$$\partial\Omega \setminus \partial\bar{\Omega} = \bigcup_{x \in \mathbf{S}^1} \{x\} \times [h(x), \bar{h}(x)).$$

Given $\delta > 0$, choose $x_0 \in \mathbf{S}^1$ such that $\bar{h}(x_0) - h(x_0) > \delta$ and $\eta > 0$, small enough to have (since $[h(x_0), \bar{h}(x_0) - \delta] \subset \overset{\circ}{\bar{\Omega}}$)

$$C^{\delta, \eta} = [x_0 - \eta, x_0 + \eta] \times [h(x_0), \bar{h}(x_0) - \delta] \subset \bar{\Omega}.$$

Let $\rho > 0$, $\rho < \min\{\delta, \eta\}/2$. Since $(x_0, h(x_0)) \notin \Omega$ and Ω is characterized by (6.2), $\lim_{n \rightarrow \infty} \inf_{\text{ess } B((x_0, h(x_0)), \rho)} v_n = 0$. As v_n is nonincreasing in y , there exists $x_n \in (x_0 - \rho, x_0 + \rho)$ such that $v_n(x_n, y) \rightarrow 0$ as $n \rightarrow \infty$ for a.e. $y \in (h(x_0) + \rho, +\infty)$. On the other hand, $v_n \rightarrow 1$ a.e. in $\bar{\Omega}$, thus a.e. in $(x_0 - \eta, x_0 - \rho) \times (\bar{h}(x_0) - \delta - \rho, \bar{h}(x_0) - \delta)$ and in $(x_0 + \rho, x_0 + \eta) \times (\bar{h}(x_0) - \delta - \rho, \bar{h}(x_0) - \delta)$, and there exist $x'_n \in (x_0 - \eta, x_0 - \rho)$, $x''_n \in (x_0 + \rho, x_0 + \eta)$ such that $v_n(x'_n, y)$ and $v_n(x''_n, y)$ converge to 1 as $n \rightarrow \infty$ for a.e. $y \in (0, h(x_0) - \delta - \rho)$.

Now, we have

$$\begin{aligned} L_{\varepsilon_n}(v_n, C^{\delta, \eta}) &\geq 2\sigma_c \int_{h(x_0)+\rho}^{\bar{h}(x_0)-\delta-\rho} \left\{ \int_{x'_n}^{x''_n} \frac{4\varepsilon_n}{\pi^2} |\partial_x v_n(x, y)|^2 + \frac{v_n(x, y)(1-v_n(x, y))}{\varepsilon_n} dx \right\} dy \\ &\geq \frac{8\sigma_c}{\pi} \int_{h(x_0)+\rho}^{\bar{h}(x_0)-\delta-\rho} \left\{ - \int_{x'_n}^{x_n} \sqrt{v_n(1-v_n)} \partial_x v_n dx + \int_{x_n}^{x''_n} \sqrt{v_n(1-v_n)} \partial_x v_n dx \right\} dy \\ &= \frac{8\sigma_c}{\pi} \int_{h(x_0)+\rho}^{\bar{h}(x_0)-\delta-\rho} \left\{ \int_{v_n(x_n, y)}^{v_n(x'_n, y)} \sqrt{t(1-t)} dt + \int_{v_n(x_n, y)}^{v_n(x''_n, y)} \sqrt{t(1-t)} dt \right\} dy. \end{aligned}$$

Since for a.e. $y \in (h(x_0) + \rho, \bar{h}(x_0) - \delta - \rho)$ the terms $\int_{v_n(x_n,y)}^{v_n(x'_n,y)} \sqrt{t(1-t)} dt$ and $\int_{v_n(x_n,y)}^{v_n(x'_n,y)} \sqrt{t(1-t)} dt$ converge to $\int_0^1 \sqrt{t(1-t)} dt = \pi/8$ as $n \rightarrow \infty$, we get

$$\liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n, C^{\delta,\eta}) \geq 2\sigma_c(\bar{h}(x_0) - h(x_0) - \delta - 2\rho),$$

and sending ρ to zero,

$$(6.8) \quad 2\sigma_c(\bar{h}(x_0) - h(x_0) - \delta) \leq \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n, C^{\delta,\eta}).$$

We are now able to show (6.3). Choose x_1, \dots, x_N such that $h(x_k) < \bar{h}(x_k)$ and $\delta < \min_{k=1, \dots, N}(\bar{h}(x_k) - h(x_k))$, and choose $\eta > 0$ such that the sets $C_k^{\delta,\eta} = [x_k - \eta, x_k + \eta] \times [h(x_k), \bar{h}(x_k) - \delta]$ are disjoint and all included in $\overset{\circ}{\Omega}$. Choose also $\delta' > 0$ such that the (disjoint) sets $A^{\delta'}, B^{\delta'}$ defined as before do not touch any of the $C_k^{\delta,\eta}$. From (6.6), (6.7), and (6.8), we get that

$$\begin{aligned} \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n) &\geq \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n, A^{\delta'}) + \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n, B^{\delta'}) \\ &\quad + \sum_{k=1}^N \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n, C_k^{\delta,\eta}) \\ &\geq \sigma_c \mathcal{H}^1(\partial \bar{\Omega} \cap Q^+) + (\sigma_c \wedge \sigma_s) \mathcal{H}^1(\partial \bar{\Omega} \cap B^{\delta'}) \\ &\quad + 2\sigma_c \sum_{k=1}^N (\bar{h}(x_k) - h(x_k) - \delta). \end{aligned}$$

Sending first δ' to zero, and then δ , we get

$$\sigma_c \mathcal{H}^1(\partial \bar{\Omega} \cap Q^+) + (\sigma_c \wedge \sigma_s) \mathcal{H}^1(\partial \bar{\Omega}) + 2\sigma_c \sum_{k=1}^N (\bar{h}(x_k) - h(x_k)) \leq \liminf_{n \rightarrow \infty} L_{\varepsilon_n}(v_n).$$

Since this is true for any $\{x_1, \dots, x_N\} \subseteq \{x \in \mathbf{S}^1 : h(x) < \bar{h}(x)\}$, we deduce (6.3).

6.1.2. Convergence of the sequence $(u_n)_{n \geq 1}$. First of all (since (3.4) holds), we may extract a subsequence (still denoted by (u_n, v_n)) such that $\sqrt{v_n}e(u_n)$ weakly converges in $L^2(Q^+; \mathbf{R}^4)$ to some function \mathcal{E} .

Consider now a Lipschitz subgraph $A \subset \Omega$, with $A \cap Q^+ \subset\subset \Omega$. Since $\overline{A \cap Q^+}$ is compact and included in $\bigcup_{k \geq 2} \Omega^k$, there exists $k \geq 2$ such that $A \cap Q^+ \subset\subset \Omega^k$; in particular, $A \subset \Omega_n^k$ for large n . Since $v_n > s_n^k > 1/(k+1)$ a.e. in Ω_n^k , $1/\sqrt{v_n}$ is uniformly bounded by $\sqrt{k+1}$ in A , and since it converges to 1 a.e., it follows that $e(u_n) = \sqrt{v_n}e(u_n) \times 1/\sqrt{v_n} \rightharpoonup \mathcal{E}$ weakly in $L^2(A \cap Q^+; \mathbf{R}^4)$ as $n \rightarrow \infty$. Using Korn's inequality and the fact that $e(u_n)$ is uniformly bounded in $L^2(A \cap Q^+; \mathbf{R}^4)$, we deduce (since $u_n - (x, 0) \equiv 0$ on $\{y = 0\}$) that $(u_n)_{n \geq 1}$ is also compact in $L^2_{loc}(\pi^{-1}(A \cap Q^+); \mathbf{R}^2)$, and since, if u is a limit point of the sequence, we must have $u - (x, 0) \equiv 0$ on $\{y = 0\}$ and $e(u) = \mathcal{E}$, the possible limit point is unique; therefore u_n converges in $L^2(A \cap Q^+; \mathbf{R}^2)$.

Since this holds for every Lipschitz subgraph A with $A \cap Q^+ \subset\subset \Omega$, u_n converges in $L^2_{loc}(\pi^{-1}(\Omega \cap Q^+); \mathbf{R}^2)$ to some function $u \in X(\Omega)$, with $e(u) = \mathcal{E}$ in the distributional sense. As \mathcal{E} is the weak limit in $L^2(Q^+; \mathbf{R}^4)$ of $\sqrt{v_n}e(u_n)$, (6.4) follows and point (i) of Theorem 3.1 is proved.

It remains to prove (ii).

6.2. Proof of point (ii) of Theorem 3.1.

6.2.1. The optimal profile for L_ε . Consider the following 1-d version of L_ε :

$$(6.9) \quad l_\varepsilon(v, I) = 2\sigma_c \left(\frac{4\varepsilon}{\pi^2} \int_I |v'(t)|^2 dt + \frac{1}{\varepsilon} \int_I v(t)(1 - v(t)) dt \right),$$

defined for an interval $I \subseteq \mathbf{R}$ and $v : I \rightarrow [0, 1]$. Consider also the two problems

$$(6.10) \quad \min \left\{ l_\varepsilon(v, \mathbf{R}) : v : \mathbf{R} \rightarrow [0, 1], v' \leq 0, \lim_{-\infty} v = 1, \lim_{+\infty} v = 0 \right\}$$

and

$$(6.11) \quad \min \left\{ l_\varepsilon(v, [0, +\infty)) : v : [0, +\infty) \rightarrow [0, 1], v' \leq 0, v(0) = v_s, \lim_{+\infty} v = 0 \right\}.$$

It is known (and easy to prove) that problem (6.10) is solved by $v_\varepsilon(t) = \gamma(\frac{t}{\varepsilon})$, where the *optimal profile* γ is

$$(6.12) \quad \gamma(t) = \begin{cases} 1 & \text{if } t \leq -1, \\ \frac{1}{2}(1 - \sin \frac{\pi t}{2}) & \text{if } -1 \leq t \leq 1, \\ 0 & \text{if } t \geq 1. \end{cases}$$

The value of the minimum (6.10) is σ_c . Similarly, problem (6.11) is solved by the function $\tilde{v}_\varepsilon(t) = \tilde{\gamma}(\frac{t}{\varepsilon})$, where for any $t \geq 0$, $\tilde{\gamma}(t) = \gamma(t+t_s)$ and $t_s = \frac{2}{\pi} \arcsin(1-2v_s) \in [-1, 1]$, so that $\gamma(t_s) = v_s$. In this case the value of the minimum (6.11) is $\sigma_c \wedge \sigma_s$.

6.2.2. Construction of a sequence $(v_\varepsilon, u_\varepsilon)$ for a regular (Ω, u) . We will first consider the case where $\partial\Omega$ is regular: We assume that Ω is the subgraph of a Lipschitz function h , that $\partial\Omega \cap Q^+$ is a finite union of C^2 arcs, and that $\partial\Omega \setminus Q^+$ is a finite union of segments $(a_k, b_k) \times \{0\} \subset \mathbf{S}^1 \times \{0\}$, $k = 1, \dots, N$. We also assume that $u \in L^\infty(\Omega, \mathbf{R}^2)$.

We define the signed distance d_Ω to $\partial\Omega$ as

$$d_\Omega(\xi) = \text{dist}(\xi, \Omega) - \text{dist}(\xi, Q \setminus \Omega).$$

In particular, $d_\Omega(\xi) = 0 \Leftrightarrow \xi \in \partial\Omega$ and $d_\Omega(x, y) \leq y < 0$ if $y < 0$.

For every $\varepsilon > 0$ (small) and $\xi \in Q$ we define

$$v_\varepsilon(\xi) = \gamma \left(\frac{d_\Omega(\xi)}{\varepsilon} + t_s \right),$$

so that $v_\varepsilon \equiv v_s$ on $\partial\Omega$, and, in particular, $v_\varepsilon \geq v_s$ on $\partial Q^+ = \mathbf{S}^1 \times \{0\}$. Notice that $|\nabla v_\varepsilon(\xi)| = |\gamma'(d_\Omega(\xi)/\varepsilon + t_s) \nabla d_\Omega(\xi)/\varepsilon| = -\gamma'(d_\Omega(\xi)/\varepsilon + t_s)/\varepsilon$ a.e. in Q , and that $|\nabla v_\varepsilon(\xi)| = v_\varepsilon(\xi)(1 - v_\varepsilon(\xi)) = 0$ if $|d_\Omega(\xi)| > 2\varepsilon$.

If $A \subseteq Q$ is open, $L_\varepsilon(v_\varepsilon, A)$ defined by (6.5) is bounded by

$$(6.13) \quad \begin{aligned} L_\varepsilon(v_\varepsilon, A) &\leq \frac{2\sigma_c}{\varepsilon} \int_{Q^+ \cap A} \frac{4}{\pi^2} \gamma' \left(\frac{d_\Omega(\xi)}{\varepsilon} + t_s \right)^2 + \gamma \left(\frac{d_\Omega(\xi)}{\varepsilon} + t_s \right) \left(1 - \gamma \left(\frac{d_\Omega(\xi)}{\varepsilon} + t_s \right) \right) d\xi \\ &\leq 4\sigma_c \frac{1}{4\varepsilon} |\{\xi \in A : |d_\Omega(\xi)| < 2\varepsilon\}| \\ &\leq 4\sigma_c \mathcal{H}^1(\partial\Omega \cap A) + c_\varepsilon, \end{aligned}$$

with $c_\varepsilon = c_\varepsilon(A, \delta)$ converging to 0 as $\varepsilon \rightarrow 0$, since, as $\partial\Omega$ is closed, the Minkowski content $\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} |\{\xi \in A : |d_\Omega(\xi)| < \varepsilon\}|$ is equal to the length $\mathcal{H}^1(\partial\Omega \cap A)$. (See, for instance, [13].)

Now, given $\delta > 0$ small, we let $\partial\Omega_\pm^\delta = \{\xi \in \partial\Omega \cap Q^\pm : \text{dist}(\xi, Q^\mp) > \delta\}$, $\partial\Omega_0^\delta = \{\xi \in \partial\Omega \setminus Q^+ : \text{dist}(\xi, \partial\Omega \cap Q^+) > \delta\}$, and $A_+^\delta = \{\xi + \zeta\nu(\xi) : \xi \in \partial\Omega_\pm^\delta, -\delta < \zeta < \delta\}$, where $\nu(\xi)$ is the normal to $\partial\Omega$ at ξ and $A_0^\delta = \{\xi + (0, \zeta) : \xi \in \partial\Omega_0^\delta, 0 < \zeta < \delta\}$. Notice that $A_+^\delta \cup A_0^\delta \subset Q^+$. We let $B^\delta = \{\xi \in Q : \text{dist}(\xi, \partial\Omega \setminus (\partial\Omega_\pm^\delta \cup \partial\Omega_0^\delta)) < 2\delta\}$.

If δ is small enough, $\{\xi \in Q^+ : \text{dist}(\xi, \partial\Omega) < \delta\} \subset A_0^\delta \cup A_+^\delta \cup B^\delta$, and if $2\varepsilon < \delta$,

$$L_\varepsilon(v_\varepsilon) \leq L_\varepsilon(v_\varepsilon, A_0^\delta) + L_\varepsilon(v_\varepsilon, A_+^\delta) + L_\varepsilon(v_\varepsilon, B^\delta).$$

The last quantity is

$$L_\varepsilon(v_\varepsilon, B^\delta) \leq 4\sigma_c \mathcal{H}^1(\partial\Omega \cap B^\delta) + c_\varepsilon$$

by (6.13). It is easy to check that

$$\partial\Omega \cap B^\delta \subset C^\delta = (\partial\Omega \cap Q^{3\delta}) \cup ((\partial\Omega \setminus Q^+) \setminus \partial\Omega_0^{3\delta}),$$

so that the previous inequality yields

$$(6.14) \quad L_\varepsilon(v_\varepsilon, B^\delta) \leq 4\sigma_c \mathcal{H}^1(C^\delta) + c_\varepsilon.$$

We estimate the two other integrals:

$$\begin{aligned} L_\varepsilon(v_\varepsilon, A_0^\delta) &= \sum_{k=1}^N \int_{a_k+\delta}^{b_k-\delta} \left(\frac{2\sigma_c}{\varepsilon} \int_0^\delta \frac{4}{\pi^2} \gamma' \left(\frac{\zeta}{\varepsilon} + t_s \right)^2 \right. \\ &\quad \left. + \gamma \left(\frac{\zeta}{\varepsilon} + t_s \right) \left(1 - \gamma \left(\frac{\zeta}{\varepsilon} + t_s \right) \right) d\zeta \right) dx \\ &= \left(\sum_{k=1}^N |b_k - a_k| - 2N\delta \right) 2\sigma_c \int_{t_s}^{\delta/\varepsilon} \frac{4}{\pi^2} \gamma'(t)^2 + \gamma(t)(1 - \gamma(t)) dt \\ (6.15) \quad &\leq (\sigma_c \wedge \sigma_s) (\mathcal{H}^1(\partial\Omega \setminus Q^+) - 2N\delta) \end{aligned}$$

by definition of t_s . On the other hand,

$$\begin{aligned} L_\varepsilon(v_\varepsilon, A_+^\delta) &= \int_{\partial\Omega_+^\delta} d\mathcal{H}^1(z) \left(\frac{2\sigma_c}{\varepsilon} \int_{-\delta}^\delta J(z, \zeta) \frac{4}{\pi^2} \gamma' \left(\frac{\zeta}{\varepsilon} + t_s \right)^2 + \gamma \left(\frac{\zeta}{\varepsilon} + t_s \right) \left(1 - \gamma \left(\frac{\zeta}{\varepsilon} + t_s \right) \right) d\zeta \right) \\ &= \int_{\partial\Omega_+^\delta} d\mathcal{H}^1(z) \left(2\sigma_c \int_{-1}^1 J(z, \varepsilon(t - t_s)) \frac{1}{2} \cos^2 \left(\frac{\pi t}{2} \right) dt \right), \end{aligned}$$

where $J(z, \zeta)$ is the Jacobian of the transformation $\xi \in A_+^\delta \mapsto (z, \zeta) : z \in \partial\Omega_+^\delta, \xi = z + \zeta\nu(z)$, which is well defined if δ is small since $\partial\Omega \cap Q^+$ is C^2 . It can be shown that J is continuous and $J(z, 0) \equiv 1$; therefore

$$(6.16) \quad \lim_{\varepsilon \downarrow 0} L_\varepsilon(v_\varepsilon, A_+^\delta) = \int_{\partial\Omega_+^\delta} d\mathcal{H}^1(z) \left(\sigma_c \int_{-1}^1 \cos^2 \left(\frac{\pi t}{2} \right) dt \right) = \sigma_c \mathcal{H}^1(\partial\Omega_+^\delta).$$

Equations (6.14), (6.15), and (6.16) yield

$$\begin{aligned} \limsup_{\varepsilon \downarrow 0} L(v_\varepsilon) &\leq (\sigma_c \wedge \sigma_s) (\mathcal{H}^1(\partial\Omega \setminus Q^+) - 2N\delta) + \sigma_c \mathcal{H}^1(\partial\Omega_+^\delta) + 4\sigma_c \mathcal{H}^1(C^\delta) \\ &\leq L(\Omega) - 2N(\sigma_c \wedge \sigma_s)\delta + 4\sigma_c \mathcal{H}^1(C^\delta), \end{aligned}$$

so that, sending δ to zero,

$$(6.17) \quad \limsup_{\varepsilon \downarrow 0} L(v_\varepsilon) \leq L(\Omega).$$

Indeed, $\cap_{\delta>0} C^\delta = \bigcup_{k=1}^N \{a_k, b_k\}$, and the latter set has length zero; thus $\mathcal{H}^1(C^\delta)$ tends to zero.

We now build the associated functions u_ε . Let $c > 0$ be the Lipschitz constant of h ; we assume $c \geq 1$. If $\xi = (x, y) \in Q \setminus \Omega$ and $\xi' = (x', h(x')) \in \partial\Omega$, then $|h(x) - h(x')| \leq c|x - x'|$, so that $y - h(x) \leq c|x - x'| + |h(x') - y|$, and we deduce that $y - h(x) \leq c \operatorname{dist}(\xi, \Omega)$. Thus, if in addition $v_\varepsilon(\xi) > 0$, we have $\operatorname{dist}(\xi, \Omega) < 2\varepsilon$, so that $y - h(x) < 2c\varepsilon$, and $(x, y - 2c\varepsilon) \in \Omega$. We define for all $(x, y) \in Q$ the function $w_\varepsilon(x, y) = 1 \wedge (v_\varepsilon(x, y - 2c\varepsilon)/v_s) \in H^1_{loc}(Q)$. Now $v_\varepsilon(x, y) > 0$ implies $w_\varepsilon(x, y) = 1$ (since $(x, y - 2c\varepsilon) \in \Omega$ and $v_\varepsilon \geq v_s$ in Ω), and $w_\varepsilon(x, y) > 0$ implies $(x, y - 4c\varepsilon) \in \Omega$.

We can define, for all $(x, y) \in Q^+$,

$$u_\varepsilon(x, y) = \begin{cases} u(x, y - 4c\varepsilon)w_\varepsilon(x, y) & \text{if } w_\varepsilon(x, y) > 0 \ (\Rightarrow (x, y - 4c\varepsilon) \in \Omega), \\ 0 & \text{if } w_\varepsilon(x, y) = 0. \end{cases}$$

The function $u_\varepsilon(x, y) - (x, 0)$ is 1-periodic in x and vanishes for $y = 0$, and $u_\varepsilon(x, y) = u(x, y - 4c\varepsilon)$ as soon as $v_\varepsilon(x, y) > 0$. We have

$$e(u_\varepsilon)(x, y) = \begin{cases} e(u)(x, y - 4c\varepsilon)w_\varepsilon(x, y) & \text{if } w_\varepsilon(x, y) > 0, \\ 0 & \text{if } w_\varepsilon(x, y) = 0, \end{cases} + u(x, y - 4c\varepsilon) \odot \nabla w_\varepsilon(x, y)$$

where $a \odot b$ denotes the symmetrized tensor product, and $e(u_\varepsilon)(x, y) = e(u)(x, y - 4c\varepsilon)$ if $v_\varepsilon(x, y) > 0$. Thus, for some constant $C > 0$,

$$\begin{aligned} &\int_{Q^+} (v_\varepsilon(\xi) + \eta_\varepsilon) \mathbf{A}e(u_\varepsilon)(\xi): e(u_\varepsilon)(\xi) \, d\xi \\ &\leq (1 + \eta_\varepsilon) \int_{\{v_\varepsilon>0\}} \mathbf{A}e(u_\varepsilon)(x, y - 4c\varepsilon): e(u_\varepsilon)(x, y - 4c\varepsilon) \, dx dy \\ &\quad + C\eta_\varepsilon \int_{\{0 < w_\varepsilon < 1\}} |e(u)(x, y - 4c\varepsilon)|^2 + |u(x, y - 4c\varepsilon)|^2 |\nabla w_\varepsilon(x, y)|^2 \, dx dy \\ &\leq (1 + \eta_\varepsilon) \int_{\Omega} \mathbf{A}e(u)(x, y): e(u)(x, y) \, dx dy \\ &\quad + C\eta_\varepsilon \int_{\{0 < v_\varepsilon < v_s\}} |e(u)(x, y - 2c\varepsilon)|^2 + |u(x, y - 2c\varepsilon)|^2 |\nabla v_\varepsilon(x, y)/v_s|^2 \, dx dy. \end{aligned}$$

We claim that this last integral goes to zero as $\varepsilon \downarrow 0$, so that

$$\limsup_{\varepsilon \downarrow 0} \int_{Q^+} (v_\varepsilon(\xi) + \eta_\varepsilon) \mathbf{A}e(u_\varepsilon)(\xi): e(u_\varepsilon)(\xi) \, d\xi \leq \int_{\Omega} \mathbf{A}e(u)(\xi): e(u)(\xi) \, d\xi,$$

and in view of (6.17), inequality (3.6) holds in this case.

Clearly, $\lim_{\varepsilon \downarrow 0} \eta_\varepsilon \int_{\{0 < v_\varepsilon < v_s\}} |e(u)(x, y - 2c\varepsilon)|^2 dx dy = 0$. Then,

$$\begin{aligned} & \eta_\varepsilon \int_{\{0 < v_\varepsilon < v_s\}} |u(x, y - 2c\varepsilon)|^2 |\nabla v_\varepsilon(x, y)/v_s|^2 dx dy \\ & \leq \|u\|_\infty^2 \frac{\eta_\varepsilon}{v_s^2 \varepsilon^2} \|\gamma'\|_\infty^2 |\{\xi \in Q^+ : 0 < v_\varepsilon(\xi) < v_s\}| \\ & \leq \left(\frac{\eta_\varepsilon}{\varepsilon}\right) \frac{\|u\|_\infty^2 \|\gamma'\|_\infty^2}{v_s^2} \frac{1}{\varepsilon} |\{\xi \in Q^+ : |d_\Omega(\xi)| < 2\varepsilon\}| = o(1), \end{aligned}$$

since $\eta_\varepsilon = o(\varepsilon)$ and $\frac{1}{\varepsilon} |\{ |d_\Omega| < 2\varepsilon \}|$ tends to $4\mathcal{H}^1(\partial\Omega)$.

Therefore point (ii) of Theorem 3.1 is proved in the case where $\partial\Omega$ and u have some regularity.

6.2.3. Construction of $(v_\varepsilon, u_\varepsilon)$ in the general case. Now, for an arbitrary $\Omega \in \mathcal{G}$ and u in $X(\Omega)$, suppose that we are able to build a sequence $(\Omega_n)_{n \geq 1}$ converging to Ω and $u_n \in X(\Omega_n) \cap L^\infty(\Omega_n)$ such that $u_n \rightarrow u$ in $L^2(A; \mathbf{R}^2)$ for every $A \subset\subset \Omega$, each Ω_n is the subgraph of a Lipschitz function, $\partial\Omega_n \cap Q^+$ is C^2 , $\partial\Omega_n \setminus Q^+$ is a finite union of segments, and such that (Ω_n, u_n) satisfies

$$\limsup_{n \rightarrow \infty} E(\Omega_n, u_n) \leq E(\Omega, u).$$

Then a simple diagonalization argument will lead to the result. Indeed, if we consider $(v_{n,\varepsilon}, u_{n,\varepsilon})_{\varepsilon > 0}$ obtained for each n as described in section 6.2.2, we first build by induction a sequence $\varepsilon(n)$ such that for every n , $\varepsilon(n) < \min\{\varepsilon(n-1), 1/n\}$, and such that

$$\|u_{n,\varepsilon} - u_n\|_{L^2(\{\text{dist}(\xi, Q \setminus \Omega_n) > 1/n\}; \mathbf{R}^2)} < \frac{1}{n},$$

$v_{n,\varepsilon} \equiv 1$ on $\{\xi : \text{dist}(\xi, Q \setminus \Omega_n) > 1/n\}$ as soon as $\varepsilon < \varepsilon(n)$, and

$$\sup_{\varepsilon < \varepsilon(n)} E_\varepsilon(v_{n,\varepsilon}, u_{n,\varepsilon}) \leq E(u_n, \Omega_n) + \frac{1}{n}.$$

Then we let $u_\varepsilon = u_{n,\varepsilon}$, $v_\varepsilon = v_{n,\varepsilon}$ whenever $\varepsilon(n+1) \leq \varepsilon < \varepsilon(n)$. We have $v_\varepsilon \rightarrow \chi_\Omega$, $\limsup_{\varepsilon \downarrow 0} E(v_\varepsilon, u_\varepsilon) \leq E(u, \Omega)$, and if $A \subset\subset \Omega$, for large n we have $A \subset \{\xi \in Q : \text{dist}(\xi, Q \setminus \Omega_n) > 1/n\}$, so that for small ε ,

$$\begin{aligned} \|u_\varepsilon - u\|_{L^2(A; \mathbf{R}^2)} & \leq \|u_\varepsilon - u_{n(\varepsilon)}\|_{L^2(A; \mathbf{R}^2)} + \|u_{n(\varepsilon)} - u\|_{L^2(A; \mathbf{R}^2)} \\ & \leq 1/n(\varepsilon) + \|u_{n(\varepsilon)} - u\|_{L^2(A; \mathbf{R}^2)} \rightarrow 0 \end{aligned}$$

as $\varepsilon \rightarrow 0$, where $n(\varepsilon)$ is defined by $\varepsilon(n(\varepsilon)+1) \leq \varepsilon < \varepsilon(n(\varepsilon))$ and goes to $+\infty$ as $\varepsilon \rightarrow 0$.

In order to build the sequence (Ω_n, u_n) , we first assume we can find for every n an n -Lipschitz function $h_n \leq h$, where h satisfies $\Omega = \{(x, y) \in Q : y < h(x)\}$. This is proved in section 5.2 (h_n is given by (5.8)), and it is established that the sets $\Omega'_n = \{(x, y) \in Q : y < h_n(x)\}$ converge to Ω and that $L(\Omega'_n) \rightarrow L(\Omega)$. Now, if ρ is a 1-d smoothing kernel ($\rho \in C^\infty(\mathbf{R})$, $\int_{\mathbf{R}} \rho = 1$, $\rho \geq 1$, and $\text{supp } \rho \subset [-1/2, 1/2]$) and $\rho_n(t) = n^2 \rho(n^2 t)$ for every $n \geq 1$, we let $g_n = h_n * \rho_n$ and $\Omega_n = \{(x, y) \in Q : y < g_n(x)\}$. It is not difficult to see that $\Omega_n \rightarrow \Omega$, that $\partial\Omega_n$ is smooth (in fact, C^∞), and that $\partial\Omega_n \setminus Q^+$ is a finite union of segments. We also have that $L(\Omega_n) \simeq L(\Omega'_n) \rightarrow L(\Omega)$. Moreover, g_n is Lipschitz ($g_n \in C^\infty(\mathbf{S}^1)$), so that Ω_n is a Lipschitz subgraph. Eventually it is easy to check that $g_n \leq h_n + c/n$ (where the constant $c = \int_{\mathbf{R}} |t| \rho(t) dt$), so that $\{(x, y - 2c/n) : (x, y) \in \Omega_n\} \subset\subset \Omega'_n \subseteq \Omega$. Therefore, we can build $u_n \in X(\Omega_n) \cap L^\infty(\Omega_n)$ as a suitable regularization of the function $u(x, y - 2c/n)$, which is defined in a neighborhood of Ω_n .

6.2.4. A remark on the integral of v_ε . We now know how to build a family $(v_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ for an arbitrary domain $\Omega \in \mathcal{G}$ and $u \in X(\Omega)$. In order to achieve the proof of Theorem 3.1 it remains to show that we can also impose that, for every ε , $\int_{Q^+} v_\varepsilon(\xi) d\xi = |\Omega|$.

We have that $\int_{Q^+} v_\varepsilon(\xi) d\xi \rightarrow |\Omega|$ as $\varepsilon \downarrow 0$. Now we let $\alpha_\varepsilon = \int_{Q^+} v_\varepsilon(\xi) d\xi / |\Omega|$, which converges to 1. We define a new family $(v'_\varepsilon, u'_\varepsilon)_{\varepsilon>0}$ by

$$v'_\varepsilon(x, y) = v_\varepsilon(x, \alpha_\varepsilon y),$$

$$(u'^x_\varepsilon(x, y), u'^y_\varepsilon(x, y)) = (u^x_\varepsilon(x, \alpha_\varepsilon y), \alpha_\varepsilon u^y_\varepsilon(x, \alpha_\varepsilon y))$$

for every $(x, y) \in Q^+$. It is not difficult to show that this family still satisfies point (ii) of Theorem 3.1, and that $\int_{Q^+} v'_\varepsilon(\xi) d\xi = |\Omega|$ for every $\varepsilon > 0$.

6.2.5. A link between (i) and (ii) in Theorem 3.1. The $(v_\varepsilon, u_\varepsilon)_{\varepsilon>0}$ constructed in section 6.2 have the property that the set Ω would be the set obtained from any subsequence $(v_{\varepsilon_n})_{n \geq 1}$ by the method described in section 6.1. In particular, we deduce that, for this family $(v_\varepsilon, u_\varepsilon)$,

$$(6.18) \quad \lim_{\varepsilon \downarrow 0} E_\varepsilon(v_\varepsilon, u_\varepsilon) = E(\Omega, u).$$

Appendix. Functions of bounded variation, Caccioppoli sets. We review in this appendix some standard definitions and properties of the functions of bounded variation. For more details, the reader should consult [15, 10, 13, 2].

A.1. Definitions.

A.1.1. Classical and measure theoretical total variation. In the classical sense, the variation of a real-valued function $f : I \rightarrow \mathbf{R}$, defined on an interval $I \subseteq \mathbf{R}$, is given by

$$(A.1) \quad \text{Var}(f, I) = \sup \left\{ \sum_{i=1}^m |f(t_i) - f(t_{i-1})| : m \in \mathbf{N}, t_0, \dots, t_m \in I, t_0 < t_1 < \dots < t_m \right\}.$$

The definition for a function $f : \mathbf{S}^1 \rightarrow \mathbf{R}$ is similar. The function f has bounded variation on I in the classical sense if $\text{Var}(f, I) < +\infty$.

In the measure theoretical sense, on the other hand, a function f is said to belong to $BV(\Omega)$, the space of functions of bounded variation on the open domain $\Omega \subseteq \mathbf{R}^N$, $N \geq 1$, if and only if $f \in L^1(\mathbf{R}^N)$ and its distributional gradient Df is a bounded vector measure on Ω . In this case, the total variation of f is the mass $|Df|(\Omega)$ of the measure Df . The definition is similar if $\Omega \subseteq \mathbf{S}^1 \times \mathbf{R}$, which is usually the case in this paper.

A.1.2. Sets with finite perimeter. A Caccioppoli set, or set with finite perimeter in Ω , is a set $E \subseteq \Omega$ such that the distributional gradient $D\chi_E$ of its characteristic function is a bounded vector measure on Ω . (If E is bounded, we thus have $\chi_E \in BV(\Omega)$.) In this definition, a set E is of course identified with all sets E' such that $|E' \Delta E| = 0$. The perimeter of E in Ω is the total variation $|D\chi_E|(\Omega) < +\infty$.

A.2. Properties of functions with bounded variation.

A.2.1. Links between both definitions. If $I \subseteq \mathbf{R}$ (or $I \subseteq \mathbf{S}^1$) is an open interval, then

$$\int_I |f(t)| dt < +\infty \quad \text{and} \quad \text{Var}(f, I) < +\infty \Rightarrow f \in BV(I).$$

More precisely, the equivalence class of functions a.e. equal to f is an element of $BV(I)$.

On the other hand, every $f \in BV(I)$ has a representative g ($g = f$ a.e.) such that $\text{Var}(g, I) = |Df|(I) < +\infty$. In fact, one can prove that

$$|Df|(I) = \min \{ \text{Var}(g, I) : g = f \text{ a.e.} \}.$$

Moreover, each f always has a representative g' with $\text{Var}(g', I) = +\infty$; for instance, the characteristic function of \mathbf{Q} belongs to $BV(\mathbf{R})$, but in the BV sense $\chi_{\mathbf{Q}} = 0$, whereas in the classical sense $\text{Var}(\chi_{\mathbf{Q}}, \mathbf{R}) = 2\#\mathbf{Q} = +\infty$.

A.2.2. Continuity properties. If $f \in BV_{loc}(\Omega)$, we can define at each point $x \in \Omega$ the approximate lower limit of f by

$$f^-(x) = \sup \left\{ t \in \mathbf{R} : \limsup_{\rho \downarrow 0} \frac{|\{f < t\} \cap B(x, \rho)|}{\rho^N} = 0 \right\}$$

and the approximate upper limit $f^+(x)$ in the same way ($f^+(x) = -(-f)^-(x)$). If $f^+(x) = f^-(x) \neq \pm\infty$ (this is true almost everywhere in Ω), we set $\tilde{f}(x) = f^+(x) = f^-(x)$ and say that the precise representative \tilde{f} is approximately continuous at x .

If $N = 1$, $\Omega = I \subseteq \mathbf{R}$ (or $I \subseteq \mathbf{S}^1$), and $f \in BV(I)$ for any g representative of f , we have

$$\text{Var}(g, I) = |Df|(I) \Leftrightarrow g(x) \in [f^-(x), f^+(x)] \quad \text{for every } x \in I.$$

Moreover, in the 1-d case, f also has an approximate left limit at each point

$$\begin{aligned} f(x-0) &= \sup \left\{ t \in \mathbf{R} : \limsup_{\rho \downarrow 0} \frac{|\{f < t\} \cap (x-\rho, x)|}{\rho} = 0 \right\} \\ &= \inf \left\{ t \in \mathbf{R} : \limsup_{\rho \downarrow 0} \frac{|\{f > t\} \cap (x-\rho, x)|}{\rho} = 0 \right\} \end{aligned}$$

and a right limit $f(x+0)$ similarly defined. For every $x \in I$, $\{f(x-0), f(x+0)\} = \{f^-(x), f^+(x)\}$. Notice that if $I = (a, b)$, then up to a constant $f(x-0) = |Df|((a, x))$ and $f(x+0) = |Df|((a, x])$. In particular, $f(x \pm 0)$ is continuous at each point of $I \setminus S_f$, where $S_f = \{f(x-0) \neq f(x+0)\} = \{f^-(x) < f^+(x)\}$ is the (at most countable) set of essential discontinuities of f . The representative f^- of f is l.s.c.; it is the largest l.s.c. representative of f . Similarly, f^+ is the lowest u.s.c. representative of f .

If g is a representative of $f \in BV(I)$ such that $\text{Var}(g, I) < +\infty$, at each $x_0 \in I$ the (classical) left and right limits exist and we have

$$\lim_{\substack{x \rightarrow x_0 \\ x < x_0}} g(x) = g(x_0 - 0) = f(x_0 - 0) \quad \text{and}$$

$$\lim_{\substack{x \rightarrow x_0 \\ x > x_0}} g(x) = g(x_0 + 0) = f(x_0 + 0).$$

A.2.3. Sets and subgraphs. In the case where $f = \chi_E$, we let $E^0 = \{\tilde{f} = 0\}$, $E^1 = \{\tilde{f} = 1\}$, and define a reduced boundary by $\partial^*E = \Omega \setminus (E^0 \cup E^1)$. It is known that the measure $D\chi_E$ is supported by the set ∂^*E , and that $|D\chi_E| = \mathcal{H}^{N-1} \llcorner \partial^*E$; thus $|D\chi_E|(\Omega) = \mathcal{H}^{N-1}(\partial^*E)$ is the perimeter of E .

If $I \subseteq \mathbf{R}$ (or $I \subseteq \mathbf{S}^1$), let $f \in BV(I)$. Then the set $E = \{(x, y) \in I \times \mathbf{R} : y < f(x)\} \subset I \times \mathbf{R}$ has a finite perimeter. This can be shown, for instance, by approximating f by a sequence of regular functions f_n such that $\int_I |f'_n(t)| dt \rightarrow |Df|(I)$, and invoking the lower semicontinuity of the total variation. We get

$$|D\chi_E|(\Omega) \leq \liminf_{n \rightarrow \infty} |D\chi_{\{y < f_n(x)\}}|(\Omega) = \liminf_{n \rightarrow \infty} \int_I \sqrt{1 + |f'_n(t)|^2} dt \leq |I| + |Df|(I).$$

In fact, one can show that $|D_x\chi_E|(I) = |Df|(I)$ and $|D_y\chi_E|(I) = |I|$, so the fact that the subgraph E has a finite perimeter yields that $f \in BV(I)$, and

$$(A.2) \quad |Df|(I) \leq \mathcal{H}^1(\partial^*E) \leq |I| + |Df|(I).$$

We define the open sets $A = \{y < f^-(x)\}$ and $B = \{y > f^+(x)\}$. It is not hard to see that $\bar{A} = B^c$ (and, equivalently, $\bar{B} = A^c$), so that $\overset{\circ}{A} = \bar{B}^c = (A^c)^c = A$ and $\partial A = \partial \bar{A}$. We can check easily from the definitions that $A \subset E^1$ and $B \subset E^0$, and in particular, $\partial^*E \subseteq \partial A = \partial B = \{(x, y) \in I \times \mathbf{R} : f^-(x) \leq y \leq f^+(x)\}$. This last set can be shown to have finite length; in fact, $\mathcal{H}^1(\partial A) = \mathcal{H}^1(\partial^*E)$. In particular, $A = B^c = E$ Lebesgue-essentially, and $\partial A = \partial^*E$ \mathcal{H}^1 -essentially.

One last result we want to state is a “classical” equivalent of (A.2). We claim that, if g is an l.s.c. function and if we consider the open set $\Omega = \{(x, y) \in I \times \mathbf{R} : y < g(x)\}$, then

$$(A.3) \quad \frac{1}{2} \text{Var}(g, I) \leq \mathcal{H}^1(\partial\Omega) \leq |I| + \text{Var}(g, I).$$

Indeed, if we consider $m \in \mathbf{N}$ and $t_0, \dots, t_m \in I$, $t_0 < t_1 < \dots < t_m$, then $|g(t_i) - g(t_{i-1})| \leq \sqrt{|t_i - t_{i-1}|^2 + |g(t_i) - g(t_{i-1})|^2}$, which is the length of the segment joining $(t_{i-1}, g(t_{i-1}))$ to $(t_i, g(t_i))$. But $\partial\Omega \cap ([t_{i-1}, t_i] \times \mathbf{R})$ contains at least a path connecting these two points; therefore this length is smaller than $\mathcal{H}^1(\partial\Omega \cap ([t_{i-1}, t_i] \times \mathbf{R}))$. Summing over i , we get

$$\sum_{i=1}^m |g(t_i) - g(t_{i-1})| \leq \sum_{i=1}^m \mathcal{H}^1(\partial\Omega \cap ([t_{i-1}, t_i] \times \mathbf{R})) \leq 2\mathcal{H}^1(\partial\Omega);$$

thus the left-hand inequality in (A.3) holds.

To show the other inequality, we refer the reader to the techniques used in section 5. Basically, one considers \bar{g} such that $\overset{\circ}{\Omega} = \{(x, y) \in I \times \mathbf{R}, y < \bar{g}(x)\}$. One then can split $\mathcal{H}^1(\partial\Omega) = \mathcal{H}^1(\partial\bar{\Omega}) + \mathcal{H}^1(\partial\Omega \setminus \partial\bar{\Omega})$. We get $\mathcal{H}^1(\partial\bar{\Omega}) \leq |I| + |D\bar{g}|(I) = |I| + \text{Var}(\bar{g}, I)$ by (A.3), and we then show that $\partial\Omega \setminus \partial\bar{\Omega} = \bigcup_{\{g < \bar{g}\}} \{x\} \times [g(x), \bar{g}(x))$. We deduce that $2\mathcal{H}^1(\partial\Omega \setminus \partial\bar{\Omega}) = 2 \sum_{\{g < \bar{g}\}} (\bar{g}(x) - g(x)) = \text{Var}(g, I) - \text{Var}(\bar{g}, I)$; hence the result.

A.2.4. Coarea formula. We eventually state the coarea formula: For every $f \in BV(\Omega)$ ($\Omega \subseteq \mathbf{R}^N$, or, as is common in this paper, $\Omega \subseteq \mathbf{S}^1 \times \mathbf{R}$),

$$(A.4) \quad |Df|(\Omega) = \int_{-\infty}^{+\infty} |D\chi_{\{f > t\}}|(\Omega) dt.$$

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REFERENCES

- [1] G. ALBERTI, *Variational models for phase transitions, an approach via Γ -convergence*, in *Calculus of Variations and Differential Equations*, G. Buttazzo, A. Marino, and M. K. V. Murthy, eds., Springer, Berlin, 2000, pp. 327–337. Also available online at <http://cvgmt.sns.it>.
- [2] L. AMBROSIO, N. FUSCO, AND D. PALLARA, *Functions of Bounded Variation and Free Discontinuity Problems*, Clarendon Press, Oxford, 2000.
- [3] L. AMBROSIO AND V. M. TORTORELLI, *Approximation of functionals depending on jumps by elliptic functionals via Γ -convergence*, *Comm. Pure Appl. Math.*, 43 (1990), pp. 999–1036.
- [4] R. J. ASARO AND W. A. TILLER, *Interface morphology development during stress corrosion cracking: Part I: Via surface diffusion*, *Metall. Trans.*, 3 (1972), pp. 1789–1796.
- [5] E. BONNETIER, R. S. FALK, AND M. A. GRINFELD, *Analysis of a one-dimensional variational model of the equilibrium shape of a deformable crystal*, *M2AN Math. Model. Numer. Anal.*, 33 (1999), pp. 573–591.
- [6] J. W. CAHN AND J. E. HILLIARD, *Free energy of a nonuniform system I—interfacial free energy*, *J. Chem. Phys.*, 28 (1958), pp. 258–267.
- [7] C. H. CHIU AND H. GAO, *Numerical simulation of diffusion controlled surface evolution*, in *Mechanisms of Thin Film Evolution*, *Mater. Res. Soc. Sympos. Proc.* 317, S. M. Yalisove, C. V. Thomspon, and D. J. Eaglesham, eds., MRS, Warrendale, PA, 1994, pp. 369–374.
- [8] C. H. CHIU AND H. GAO, *A numerical study of stress controlled surface diffusion during epitaxial film growth*, in *Thin Films: Stresses and Mechanical Properties V*, *Mater. Res. Soc. Sympos. Proc.* 356, S. P. Baker, P. Borgesen, P. H. Townsend, C. A. Ross, and C. A. Volkert, eds., MRS, Warrendale, PA, 1995, pp. 33–44.
- [9] G. DAL MASO, *An Introduction to Γ -Convergence*, *Progr. Nonlinear Differential Equations Appl.* 8, Birkhäuser Boston, Boston, 1993.
- [10] L. C. EVANS AND R. F. GARIEPY, *Measure Theory and Fine Properties of Functions*, CRC Press, Boca Raton, FL, 1992.
- [11] K. J. FALCONER, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, UK, 1985.
- [12] R. FALK, *private communication*.
- [13] H. FEDERER, *Geometric Measure Theory*, Springer, Berlin, 1969.
- [14] H. J. GAO AND W. D. NIX, *Surface roughening of heteroepitaxial thin films*, *Annu. Rev. Mater. Sci.*, 29 (1999), pp. 173–209.
- [15] E. GIUSTI, *Minimal Surfaces and Functions of Bounded Variation*, Birkhäuser Boston, Cambridge, MA, 1984.
- [16] S. GOLAB, *Sur quelques points de la théorie de la longueur*, *Annales de la Société Polonaise de Mathématiques*, VII (1928), pp. 227–241.
- [17] M. A. GRINFELD, *Instability of the separation boundary between a non-hydrostatically stressed elastic body and a melt*, *Soviet Physics Doklady*, 31 (1986), pp. 831–834.
- [18] M. A. GRINFELD, *Stress driven instabilities in crystals: mathematical models and physical manifestations*, *J. Nonlinear Sci.*, 3 (1993), pp. 35–83.
- [19] J. E. GUYER AND P. W. VOORHEES, *Morphological stability and compositional uniformity of alloy thin films*, *J. Cryst. Growth*, 187 (1998), pp. 150–165.
- [20] K. KASSNER AND C. MISBAH, *A phase-field approach for stress-induced instabilities*, *Europhys. Lett.*, 46 (1999), pp. 217–223.
- [21] P. H. LEO, J. S. LOWENGRUB, AND H. J. JOU, *A diffusive interface model for micro-structural evolution in elastically stressed solids*, *Acta Mater.*, 46 (1998), pp. 2113–2130.
- [22] P. H. LEO AND R. F. SEKERKA, *The effects of surface stress on crystal–melt and crystal–crystal equilibrium*, *Acta Metall.*, 37 (1989), p. 3119.
- [23] L. MODICA AND S. MORTOLA, *Il limite nella Γ -convergenza di una famiglia di funzionali ellittici*, *Boll. Unione. Mat. Itale. Sez. A Mat. Soc. Cult.* (5), 14 (1977), pp. 526–529.
- [24] J.-M. MOREL AND S. SOLIMINI, *Variational Methods in Image Segmentation*, Birkhäuser Boston, Cambridge, MA, 1995.
- [25] *MRS Bulletin*, 21 (1996).
- [26] J. MULLER AND M. GRANT, *Model of surface instabilities induced by stress*, *Phys. Rev. Lett.*, 82 (1999), pp. 1736–1739.
- [27] W. W. MULLINS, *Theory of thermal grooving*, *J. Appl. Phys.*, 28 (1957), pp. 333–339.
- [28] R. H. NOCHETTO, S. ROVIDA, M. PAOLINI, AND C. VERDI, *Variational approximation of the*

- geometric motion of fronts*, in Motion by Mean Curvature and Related Topics (Trento, 1992), de Gruyter, Berlin, 1994, pp. 124–149.
- [29] M. PAOLINI AND C. VERDI, *Asymptotic and numerical analyses of the mean curvature flow with a space-dependent relaxation parameter*, *Asympt. Anal.*, 5 (1992), pp. 553–574.
 - [30] C. D. RUDIN AND B. J. SPENCER, *Equilibrium island ridge arrays in strained solid films*, *J. Appl. Phys.*, 86 (1999), pp. 5530–5536.
 - [31] B. J. SPENCER, *Asymptotic derivation of the glued-wetting-layer model and contact-angle condition for Stranski-Krastanow islands*, *Phys. Rev. B*, 59 (1999), pp. 2011–2017.
 - [32] B. J. SPENCER AND D. L. MEIRON, *Nonlinear evolution of stress-driven morphological instability in a two-dimensional semi-infinite solid*, *Acta Metall. Mater.*, 42 (1994), pp. 3629–3641.
 - [33] B. J. SPENCER AND J. TERSOFF, *Equilibrium shapes of islands in epitaxially strained solid films*, in *Mathematics of Multiscale Materials*, K. M. Golden, G. R. Grimmett, R. D. James, G. W. Milton, and P. N. Sen, eds., IMA Vol. Math. Appl., Springer, 1998, pp. 255–269.
 - [34] B. J. SPENCER, P. W. VOORHEES, AND S. H. DAVIS, *Morphological instability in epitaxially strained dislocation-free solid films*, *Phys. Rev. Lett.*, 67 (1991), pp. 3696–3699.
 - [35] B. J. SPENCER, P. W. VOORHEES, AND S. H. DAVIS, *Morphological instability in epitaxially strained dislocation-free solid films: Linear stability theory*, *J. Appl. Phys.*, 73 (1993), pp. 4955–4970.
 - [36] B. J. SPENCER, P. W. VOORHEES, AND J. TERSOFF, *Enhanced instability of strained alloy film due to compositional stresses*, *Appl. Phys. Lett.*, 84 (2000), pp. 2449–2452.
 - [37] B. J. SPENCER, P. W. VOORHEES, AND J. TERSOFF, *Stabilization of strained alloy film growth by a difference in atomic mobilities*, *Appl. Phys. Lett.*, 76 (2000), pp. 3022–3024.
 - [38] D. J. SROLOVITZ, *On the stability of stressed solids*, *Acta Metall.*, 37 (1989), pp. 621–624.