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# PLANE STRESS ELASTO-PLASTIC CONSTITUTIVE EQUATIONS OBTAINED BY HOMOGENIZING ONE-DIMENSIONAL STRUCTURES (*)

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Abstract — *We consider 2-dimensional constitutive equations, for materials obtained as limits of periodic structures made of elasto-plastic rods The rods are attached together by hinges where the loads are applied The plasticity law involves internal parameters as for generalized standard materials The rod structure enables us to give a complete description of the homogenized limit. We show that the Poisson's ratio equals 1/3, for isotropic elastic constitutive equations, that can be obtained in the limit from simple rod-structures with forces acting on all the hinges We give an example of structures leading to isotropic elastic constitutive laws with* $\nu \neq 1/3$

Résumé — *Nous considérons des lois constitutives bi-dimensionnelles, pour des matériaux obtenus comme limites de structures périodiques composées de barres élasto-plastiques. Ces barres sont reliées entre elles par des charnières où sont appliquées des forces ponctuelles La loi de plasticité est du type matériau standard généralisé Le matériau limite est caractérisé explicitement Pour des structures simples, où les forces peuvent agir sur les extrémités de toutes les barres, nous montrons que les matériaux élastiques isotropes, obtenus à la limite, ont un coefficient de Poisson égal à 1/3 Nous donnons un exemple de structures, qui induisent des équations constitutives élastiques isotropes pour lesquelles* $\nu \neq 1/3$

## 1. INTRODUCTION

The aim of this work is to derive constitutive laws for plane stress plasticity, by homogenizing periodic structures made of small rods attached together with hinges. The behavior of each rod is governed by a one-dimensional constitutive equation which uses internal parameters to describe the state of the material and its hardening properties. The equilibrium equations for such a structure yield a system of nonlinear differential equations. When the length of the rods tends to 0, the solution of this system converges to the solution of a PDE, and yields, in the limit, the constitutive equations.

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We assume that, on each rod, the constitutive law is of the type of generalized standard materials [3]. In 1-d calculations, this law reproduces the experimental data reasonably well [1], and it has the advantage of being supported by mathematical results that guarantee existence of solutions to initial value problems [2], [6], [7], and convergence of numerical approximations [2], [6].

Since it is very easy to adjust 1-d models to reproduce uniaxial strain-stress experimental data, our construction should give us indications on how to choose a model of constitutive equations for 2-d (or 3-d) plasticity. Moreover, the results of convergence of approximations will also hold for the limiting 2-d structures.

This paper is organized as follows : in Section 2 we define constitutive equations for elasto-plastic rods and describe a periodic structure made of these rods. In Section 3 we state an initial value problem for this structure, and give a theorem for existence and uniqueness of a solution with a sketch of the proof. In Section 4, we let the length of the rods go to 0 and obtain, in the limit, the constitutive equations of a 2-d elasto-plastic material in plane stress. The associated yield criteria is of the Tresca type, i.e., polygonal surfaces in the stress space. When the rods are purely elastic, the constitutive law obtained has a Poisson's ratio $\nu$ equal to 1/3, if we require the limiting material to be isotropic. This property is well-known as Cauchy's relations [5]. By changing the geometry of the unit cell, the number of rods in the unit cell, the elastic moduli of rigidity and the yield functions of the rods, we can obtain a set of 2-d constitutive equations, through this homogenization process. In Section 5 we explain why Cauchy relation holds for all isotropic elastic constitutive laws of this family.

However, we would like to obtain a larger family of models of 2-d elasto-plasticity. This is the aim of the last section, in which we describe an example of rod construction that induces microscopic constraints in the cells. For this structure, when the rods are purely elastic and when isotropy is required, the limiting material has a Poisson ratio $\nu \neq 1/3$, i.e., does not satisfy Cauchy's relations.

## 2. PRELIMINARIES

### 2.1. Constitutive equations

Throughout this paper we consider structures made of elasto-plastic rods attached together at their extremities by hinges. Loads will be applied at the hinges only, so we assume the stress $\sigma$ is constant in each rod.

We assume that the state of a rod is defined by a yield function $\mathcal{F}$ which depends on $\sigma$ and on an internal variable $\alpha$ (which can be a vector), such that

$$\begin{aligned} \mathcal{F}(\sigma, \alpha) &\leqslant 0 \\ \mathcal{F}(0, 0) &\leqslant 0 . \end{aligned} \tag{1}$$

We assume that $\mathcal{F}$ is smooth, convex, and that its derivatives are uniformely bounded in the following sense : there exist positive constants $\gamma$, $\Gamma$, such that

$$\forall(\sigma, \alpha) / \mathcal{F}(\sigma, \alpha) = 0, \quad 0 < \gamma \leqslant |\partial_\alpha \mathcal{F}| (\sigma, \alpha), |\partial_\alpha \mathcal{F}| (\sigma, \alpha) \leqslant \Gamma . \tag{2}$$

The strain $w$, defined as the longitudinal elongation, is the sum of an elastic part $w^e$ and a plastic part $w^p$

$$w = w^e + w^p .$$

We assume that Hooke's law relates the stress and the elastic strain

$$\dot{\sigma} = E\dot{w}^e ,$$

where $E$ is the rod's elastic modulus of rigidity and the dot denotes the time derivative.

- If $\mathcal{F}(\sigma, \alpha) < 0$ or $\partial_\sigma \mathcal{F}(\sigma, \alpha) \cdot \dot{w} \leqslant 0$, the material is elastic and

$$\begin{cases} \dot{\sigma} = E\dot{w} \\ \dot{\alpha} = 0 . \end{cases} \tag{3}$$

- Otherwise, the bar is in a plastic state and the following normality condition holds

$$\exists \lambda \geqslant 0 \Bigg/ \begin{pmatrix} \dot{w}^p \\ -\dot{\alpha} \end{pmatrix} = \lambda \begin{pmatrix} \partial_\sigma \mathcal{F} \\ \partial_\alpha \mathcal{F} \end{pmatrix} . \tag{4}$$

The constitutive relations follow from (4)

$$\begin{cases} \dot{\sigma} = E_s(\sigma, \alpha)\, \dot{w} \\ \dot{\alpha} = \dfrac{\partial_\alpha \mathcal{F} \cdot \dot{\sigma}}{\partial_\alpha \mathcal{F}^T \partial_\alpha \mathcal{F}} \partial_\alpha \mathcal{F} , \end{cases} \tag{5}$$

$$\text{with} \quad E_s = E \frac{\partial_\alpha \mathcal{F}^T \partial_\alpha \mathcal{F}}{\partial_\alpha \mathcal{F}^T \partial_\alpha \mathcal{F} + E(\partial_\sigma \mathcal{F})^2} . \tag{6}$$

The assumption (2) implies that

$$E_s(\sigma, \alpha) \geqslant \frac{E\gamma^2}{\Gamma^2(1 + E)} > 0 .$$

This type of constitutive equations is a generalization of simple models of kinematic and isotropic hardening [6] which are extensively used in numerical calculations They can be rewritten as equations of generalized standard materials [3] To this effect, one can introduce the specific free energy

$$\phi(w^e, \alpha, T) = 1/2\,(E(w^e)^2 + \alpha^2)$$

The thermodynamic action $\mathcal{A} = \frac{\partial \phi}{\partial \alpha}$, associated with $\alpha$, reduces to $\alpha$ itself Relations (1, 4) and the convexity of $\mathcal{F}$ imply that the intrinsic dissipation

$$D = \sigma w^p - \mathcal{A}\alpha\,,$$

is positive

## 2.2. A rod structure

Let $\Omega$ be the unit square in $\mathbb{R}^2$ and $I = [0, T]$ denote a time interval Let $d = 1/N$ be the mesh size of a grid defined on $\Omega$ by the points

$$(x_i, y_j) = (i\ d, j\ d) \quad 1 \leqslant i, j \leqslant N$$

We subdivide $\bar{\Omega}$ into square cells of sidelength $d$ $e_{ij}$ denotes the cell whose upper right corner is at the point $(x_i, y_j)$ We further divide each cell into 2 triangles, along the axis $y = x$ We denote $e_{ij}^+$ the upper half cell and $e_{ij}^-$ the lower one, cf figure 1

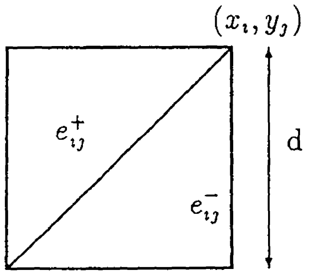


**Figure 1. — The cell $e_{ij}$**

The points $(x_i, y_j)$ represent hinges that link a set of rods in the pattern shown in figure 2

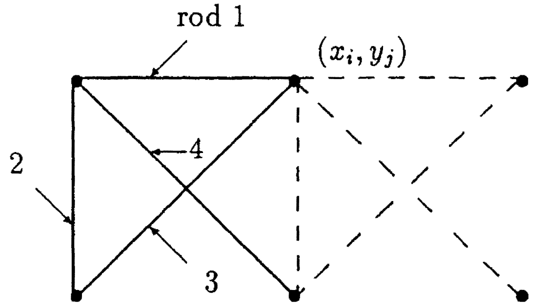


**Figure 2. — Positions of the rods in $e_{ij}$.**

Each cell $e_{ij}$ of the structure is formed of 4 rods whose moduli of elasticity are denoted $E^k$, $1 \leqslant k \leqslant 4$. Let $(\phi_{ij})$ be the set of piecewise linear basis functions associated with this triangulation such that

$$\phi_{ij}(x_k, y_l) = \delta_{ik}\,\delta_{jl}\,.$$

We consider the spaces

$$\mathcal{B}_d = \{\psi(x, y) = \sum_{ij} \psi_{ij}\,\phi_{ij}(x, y)\ /\ \psi_{/\partial\Omega} = 0\}$$

$$\mathcal{H}_d = \{u(x, y, t) = \sum_{ij} u_{ij}(t)\,\phi_{ij}(x, y)\ /\ u_{ij} \in C^1_+(I),\ u(x, y, t)_{/\partial\Omega} = 0 \text{ a.e. } t\}$$

$$\mathcal{G}_d = \{\sigma(x, y, t) = \sum_{ij} \sigma^+_{ij}(t)\,\chi_{e^+_{ij}} + \sigma^-_{ij}(t)\,\chi_{e^-_{ij}}\ /\ \sigma^\pm_{ij} \in C^1_+(I)\}\,,$$

where $C^1_+(I)$ denotes the space of continuous functions which have a uniformely bounded right derivative at each point, and where the characteristic function of a set $A$ is denoted by $\chi_A$. For an element $U_d = (u, v) \in (\mathcal{H}_d)^2$, let

$$u_{ij}(t) = u(x_i, y_j, t)$$

$$v_{ij}(t) = v(x_i, y_j, t)$$

represent the displacement of the extremities of the rods. The linear strain tensor $\varepsilon(U_d) = 1/2(\nabla U_d + \nabla U_d^T)$, is constant, with respect to the space variables, on each triangle $e^\pm_{ij}$. We define

$$w_d^1 = \varepsilon_{11}(U_d)$$

$$w_d^2 = \varepsilon_{22}(U_d)$$

$$w_d^3 = \left(\frac{\varepsilon_{11} + \varepsilon_{22}}{2} + \varepsilon_{12}\right)(U_d)$$

$$w_d^4 = \left(\frac{\varepsilon_{11} + \varepsilon_{22}}{2} - \varepsilon_{12}\right)(U_d) .$$

We notice that, for the rod $k$ of the cell $e_{ij}$, $w_d^k$ takes the same value on the two half cells adjacent to this rod, except for $k = 4$. Moreover, this value $w_{d,ij}^k$ (which depends on $t$) coincides with the elongation of the rod $k$ :

$$\begin{aligned}
(w_d^1)_{/e_{ij}^+} &= (w_d^1)_{/e_{i,j+1}^-} = w_{d,ij}^1 \\
&= \frac{1}{d}(u_{ij} - u_{i-1,j})(t) \\
(w_d^2)_{/e_{ij}^+} &= (w_d^2)_{/e_{i-1,j}^-} = w_{d,ij}^2 \\
&= \frac{1}{d}(v_{i-1,j} - v_{i-1,j-1})(t) \\
(w_d^3)_{/e_{ij}^+} &= (w_d^3)_{/e_{ij}^-} = w_{d,ij}^3 \\
&= \frac{1}{2d}(u_{ij} - u_{i-1,j-1} + v_{ij} - v_{i-1,j-1})(t).
\end{aligned}$$

We also define $w_{d,ij}^4 = 1/2\left((w_d^4)_{/e_{ij}^+} + (w_d^4)_{/e_{ij}^-}\right)$

$$= \frac{1}{2d}(u_{i,j-1} - u_{i-1,j} - v_{i,j-1} - v_{i-1,j})(t) .$$

Together with the strain $w_{d,ij}^k(t)$, we associate to each rod a triple $(\sigma_{d,ij}^k, \alpha_{d,ij}^k, E_{d,ij}^k)$ of stress, internal parameter and modulus of rigidity, satisfying the constitutive relations (3-6). Then, we define functions $(\sigma_d^k, \alpha_d^k, E_d^k)$ on the whole of $\Omega$, that take the same value on the two half cells adjacent to the rod $k$ (like the corresponding $w_d^k$) :

$$(\sigma^1_d, \alpha^1_d, E^1_d) = \sum_{ij} (\sigma^1_{d,ij}, \alpha^1_{d,ij}, E^1_{d,ij}) \chi_{e^+_{ij} \cup e^-_{i,j+1}}$$

$$(\sigma^2_d, \alpha^2_d, E^2_d) = \sum_{ij} (\sigma^2_{d,ij}, \alpha^2_{d,ij}, E^2_{d,ij}) \chi_{e^+_{ij} \cup e^-_{i-1,j}},$$

$$(\sigma^3_d, \alpha^3_d, E^3_d) = \sum_{ij} (\sigma^3_{d,ij}, \alpha^3_{d,ij}, E^3_{d,ij}) \chi_{e_{ij}}$$

$$(\sigma^4_d, \alpha^4_d, E^4_d) = \sum_{ij} (\sigma^4_{d,ij}, \alpha^4_{d,ij}, E^4_{d,ij}) \chi_{e_{ij}}.$$

### 2.3. An initial value problem

The structure is loaded by a system of force $F = (f_{ij}, g_{ij})(t)$, piecewise analytic, acting on the hinges. We seek a solution $(U_d, \sigma^k_d, \alpha^k_d)$ of the following equations (throughout the paper, repeated indices imply summation) :

$$\int_\Omega \sigma^k_d w^k_d = \int_\Omega F \cdot \Phi \quad \begin{array}{l} \text{a.e. } t \in I, \\ \forall \Phi \in (\mathcal{B}_d)^2 \end{array} \tag{7}$$

$$\mathcal{F}(\sigma^k_d, \alpha^k_d) \leq 0 \quad \text{a.e. } x \in \Omega\,, \tag{8}$$

with initial conditions

$$\begin{cases} U_d(x, y, 0) = 0 \\ \sigma^k_d(x, y, 0) = 0 \\ \alpha^k_d(x, y, 0) = 0 \end{cases} \quad \text{a.e. } x \in \Omega\,. \tag{9}$$

## 3. THE SEMI-DISCRETE PROBLEM

In this section we state an existence theorem for the initial value problem (7, 8, 9), and seek *a priori* estimates on $(U_d, \sigma^k_d, \alpha^k_d)$ that are independent of $d$.

THEOREM 3.1 : *Assume that the yield function $\mathcal{F}$ is smooth, convex and satisfies the bounds (1, 2), and that the load $F$ is piecewise analytic on $\Omega \times I$ and satisfies the compatibility condition*

$$F(x, y, 0) = 0\,.$$

*There exists a unique* $(U_d, \sigma_d^k, \alpha_d^k) \in (\mathcal{H}_d)^2 \times (\mathcal{G}_d)^4 \times (\mathcal{G}_d)^4$, *piecewise analytic in time, such that*

$$\int_\Omega \sigma_d^k w_d^k(\Phi) = \int_\Omega F \cdot \Phi \quad \begin{array}{l} \text{a.e. } in\ \Omega, \\ \forall \Phi \in (\mathcal{B}_d)^2 \end{array} \tag{10}$$

$$\mathcal{F}(\sigma_d^k, \alpha_d^k) \leqslant 0 \quad \text{a.e. } in\ \Omega \tag{11}$$

*with initial conditions (9).*

The proof, based on Miyoshi's ideas [6], [2], will only be sketched here. It consists of 3 steps.

**Step 1 :**

By a method of trial and error, the state of each rod can be predicted for a small interval of time $(t_0, t_0 + \delta)$, $\delta > 0$, from only the knowledge of $(U_d, \sigma_d^k, \alpha_d^k)$ and $F$ at time $t_0$. The equilibrium condition (10) yields a system of ODE's in the variables $(\dot{U}_d, \dot{\sigma}_d^k, \dot{\alpha}_d^k)$ which has a unique piecewise analytic solution, until one or several of the rods change state. This property is satisfied if incremental problems of the following type are well-posed : given the values $(\sigma_d^k, \alpha_d^k)$, $1 \leqslant k \leqslant 4$, find the minimum of the incremental energy functional

$$\mathcal{E}_{\sigma_d^k, \alpha_d^k}(V) = 1/2 \int_\Omega s_d^k w_d^k(V) - \int_\Omega \dot{F} \cdot V \quad V \in (\mathcal{B}_d)^2 \tag{12}$$

where the incremental stresses $s_d^k$ are defined by

$$s_d^k(V) = E^k w_d^k(V) \quad \begin{cases} \text{if} \quad \mathcal{F}(\sigma_d^k, \alpha_d^k) & \text{or} < 0 \\ \partial_\sigma \mathcal{F}(\sigma_d^k, \alpha_d^k)\, w_d^k(V) & \leqslant 0 \end{cases}$$
$$s_d^k(V) = E_s^k w_d^k(V) \quad \text{otherwise}$$

with $E_s^k$ as in (6).

Here we will only prove that for our particular structure the incremental energy functional (12) has a minimum. It is easily seen to be convex, so we just have to prove its boundedness.

PROPOSITION 3.2 : *The functional* $\mathcal{E}_{\sigma_d^k, \alpha_d^k}$ *defined on* $(\mathcal{B}_d)^2$ *with the norm of* $H^1(\Omega)$ *is bounded from below.*

*Proof:* The assumptions about $\mathcal{F}$ imply that the coefficients $E_{d,s}^k$ are uniformely bounded by

$$\kappa = \frac{\gamma^2 E}{\Gamma^2(1+E)} > 0 .$$

Hence

$$\forall 1 \leqslant k \leqslant 4 \quad \forall V \in (\mathcal{B}_d)^2$$

$$s_d^k(V)\, w_d^k(V) \geqslant E_{d,s}^k (w_d^k(V))^2 \geqslant \kappa (w_d^k(V))^2 .$$

We obtain (with $(\varepsilon = \varepsilon(V))$

$$\int_\Omega s_d^k w_d^k(V) \geqslant \kappa \int_\Omega \varepsilon_{11}^2 + \varepsilon_{22}^2 + \left(\frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \varepsilon_{12}\right)^2$$

$$\geqslant \frac{\kappa}{2} \int_\Omega \varepsilon_{11}^2 + \varepsilon_{22}^2 + \varepsilon_{12}^2 \tag{13}$$

$$\geqslant \frac{C\kappa}{4} \int_\Omega |\nabla V|^2$$

where $C$ is the positive constant figuring in Korn's inequality. □

**Step 2 :**

Inequality (13) and the constitutive relations (3-6) ensure boundedness of $(U_d, \sigma_d^k, \alpha_d^k)$ and enable us to continue the solution over the entire time interval

$$\|\nabla \dot{U}_d\|_{L^2}, \|\dot{w}_d^k\|_{L^2} \leqslant C \|\dot{F}\|_{L^2}$$

$$\text{a.e. } t \in I .$$

$$\|\dot{\sigma}_d^k\|_{L^2}, \|\dot{\alpha}_d^k\|_{L^2} \leqslant C \|\dot{w}_d^k\|_{L^2}$$

The positive constant $C$ depends only on $\Omega$ and on the $E^k$, $1 \leqslant k \leqslant 4$, which yields the following result.

PROPOSITION 3.3 : *The functions* $U_d$, $\dot{U}_d$, $\sigma_d^k$, $\dot{\sigma}_d^k$, $\alpha_d^k$, $\dot{\alpha}_d^k$ *are uniformely bounded with respect to* $d$ *in* $L^\infty(I, L^2(\Omega))$.

**Step 3 :**

Uniqueness is obtained using the convexity of the yield surface and writing the yield condition in the integral form.

PROPOSITION 3.4 : *Let*

$$\mathcal{K}_d = \left\{ (\tau, v) = \sum_{ij} (\tau_{ij}^{\pm}, v_{ij}^{\pm})\, 1_{e_{ij}^{\pm}} \;\Big/\; \begin{array}{ll} (\tau_{ij}^{\pm}, v_{ij}^{\pm}) & \in C_+^1(I) \\ \mathcal{F}(\tau_{ij}^{\pm}, v_{ij}^{\pm}) & \leqslant 0 \end{array} \right\}$$

*The problem (9, 10, 11) is equivalent to the following one : seek* $(U_d, \sigma_d^k, \alpha_d^k)$ *in* $(\mathcal{H}_d)^2 \times (\mathcal{K}_d)^4$ *such that*

$$\int_\Omega \sigma_d^k w_d^k(\phi) = \int_\Omega F \cdot \Phi \quad \begin{array}{l} \text{a.e. } t \in I, \\ \forall F \;\in (\mathcal{B}_d)^2 \end{array} \tag{14}$$

$$(\sigma_d^k, \alpha_d^k) \in \mathcal{K}_d \quad 1 \leqslant k \leqslant 4 \tag{15}$$

$$\int_\Omega (\dot{w}_d^k - (E^k)^{-1}\, \dot{\sigma}_d^k, \tau^k - \sigma_d^k) + \int_\Omega (-\dot{\alpha}_d^k, v^k - \alpha_d^k) \leqslant 0 \tag{16}$$

$$\forall (\tau^k, v^k) \text{ such that } \mathcal{F}(\tau^k, v^k) \leqslant 0\,,$$

*with initial conditions (9).*

*Proof:* Let $(U_d, \sigma_d^k, \alpha_d^k)$, be the solution of (9-10-11). Condition (4), expressing the normality rule, implies that the plastic increment $(\dot{w}_d^k - (E^k)^{-1}\, \dot{\sigma}_d^k,\ - \dot{\alpha}_d^k)$ is proportional to the outward normal to the yield surface, and since the surface is convex, it follows that

$$\forall (\tau^k, v^k) \in \mathcal{K}_d, \quad \forall 1 \leqslant k \leqslant 4$$

$$(\dot{w}_{d,ij}^k - (E^k)^{-1}\, \dot{\sigma}_{d,ij}^k, \tau_{ij}^k - \sigma_{d,ij}^k) - (\dot{\alpha}_{d,ij}^k, v_{ij}^k - \alpha_{d,ij}^k) \leqslant 0 \quad \text{on } e_{ij}^{\pm}$$

which yields (16). To complete the proof we will show that problem (14-16) has a unique solution : assuming that $(V, s^k, a^k)$ is another solution to (14-16), and choosing $(\tau^k, v^k) = (\sigma_d^k, \alpha_d^k)$ and $(\tau^k, v^k) = (s^k, a^k)$ in (16), yields

$$\int_\Omega (\dot{w}_d^k(U) - (E^k)^{-1}\, \dot{\sigma}_d^k, s^k - \sigma_d^k) - (\dot{\alpha}_d^k, a^k - \alpha_d^k) \leqslant 0$$

$$\int_\Omega (\dot{w}_d^k(V) - (E^k)^{-1}\, \dot{s}^k, \sigma_d^k - s^k) - (\dot{a}_d^k, \alpha_d^k - a^k) \leqslant 0\,.$$

Adding these two inequalities we obtain

$$1/2 \frac{d}{dt} \left( \| \sigma_d^k - s^k \|_{L^2}^2 + \| \alpha_d^k - a^k \|_{L^2}^2 \right) + \int_\Omega (\dot{w}_d^k(U) - w_d^k(V), \dot{\sigma}_d^k - \dot{s}^k) \leqslant 0\,. \tag{17}$$

Since both solutions satisfy the equilibrium condition (14),

$$\int_{\Omega} (\dot{w}_d^k(U) - \dot{w}^k(V), \dot{\sigma}_d^k - \dot{s}^k) = 0 .$$

And so, (17) and the initial conditions lead to

$$\sigma_d^k = s^k, \quad \alpha_d^k = a^k \quad \text{and} \quad U_d = V .$$

□

## 4. LIMITING SOLUTION

### 4.1. Convergence of the rod structures

From the above *a priori* estimates (proposition 3.3), we can extract a weakly convergent subsequence as $d$ tends to 0

$$\begin{aligned} &\dot{w}_d^k, \dot{\sigma}_d^k, \dot{\alpha}_d^k \rightharpoonup \dot{w}_d^k, \dot{\sigma}_d^k, \dot{\alpha}_d^k \\ &\qquad\qquad\qquad\qquad \text{weakly* in } L^\infty(I, L^2(\Omega)), \\ &w_d^k, \sigma_d^k, \alpha_d^k \rightharpoonup w^k, \sigma^k, \alpha^k \\ &U_d \rightharpoonup U \qquad\qquad \text{weakly* in } L^\infty(I, H_0^1(\Omega)) . \end{aligned}$$

We are now going to identify the initial value problem to which $(U, \sigma^k, \alpha^k)$ is a solution.

THEOREM 4.1 : $(U, \sigma^k, \alpha^k)$ $1 \leq k \leq 4$, *is the unique solution of*

$$U, \dot{U} \in L^\infty(I, H_0^1(\Omega))$$

$$\sigma, \dot{\sigma}, \alpha, \dot{\alpha} \in L^\infty(I, L^2(\Omega))$$

$$\int_{\Omega} \sigma^k w^k(\Phi) = \int_{\Omega} F \cdot \Phi \quad \forall \Phi \in (H_0^1(\Omega))^2 \tag{18}$$

$$\mathcal{F}(\sigma^k, \alpha^k) \leq 0 \quad 1 \leq k \leq 4 \tag{19}$$

$$\int_{\Omega} (\dot{w}^k - (E^k)^{-1} \dot{\sigma}^k, \tau^k - \sigma^k) - (\dot{\alpha}^k, v^k - \alpha^k) \leq 0 \quad \forall (\tau^k, v^k) \in \mathcal{K} \tag{20}$$

*which satisfies the initial condition* (9), *where*

$$\mathcal{K} = \{(\tau, v) \in (L^2(I \times \Omega))^2 / \mathcal{F}(\tau, v) \leq 0 \quad a.e.\}$$

*and for a function $\Phi$ in $(H_0^1(\Omega))^2$ we denote*

$$
\begin{aligned}
w^1(\Phi) &= \varepsilon_{11}(\Phi) \\
w^2(\Phi) &= \varepsilon_{22}(\Phi) \\
w^3(\Phi) &= \left(\frac{\varepsilon_{11}+\varepsilon_{22}}{2} + \varepsilon_{12}\right)(\Phi) \\
w^4(\Phi) &= \left(\frac{\varepsilon_{11}+\varepsilon_{22}}{2} - \varepsilon_{12}\right)(\Phi) .
\end{aligned}
\tag{21}
$$

*Proof:* We will proceed in 6 steps.

**Step 1 :**

First note that since the strains $w_d^k$ are linear functions of the displacements $U_d$, their limits $w^k$, $1 \leqslant k \leqslant 4$ are exactly $w^k(U)$ defined by (21). Also, uniqueness is proved as in Proposition 3.4.

**Step 2 :**

Since $\sigma_d^k, \dot{\sigma}_d^k, \alpha_d^k, \dot{\alpha}_d^k$ converge weakly* in $L^\infty(I, L^2(\Omega))$, one can define the traces $\sigma^k(x, y, 0)$, $\alpha^k(x, y, 0)$ [4], and moreover

$$
\left.
\begin{aligned}
\sigma_d^k(x, y, 0) &\rightharpoonup \sigma^k(x, y, 0) \\
\alpha_d^k(x, y, 0) &\rightharpoonup \alpha^k(x, y, 0)
\end{aligned}
\right\} \quad \text{weakly in } L^2(\Omega) .
$$

Hence $\sigma^k$, $\alpha^k$ (and similarly $U$) satisfy the initial conditions. In the same way, $(\sigma_d^k, \alpha_d^k)(x, y, T)$ converges to $(\sigma^k, \alpha^k)(x, y, T)$.

**Step 3 :**

Consider a function $\Phi$ in $(C_c^\infty(\Omega))^2$. Let $\Phi_d$ be its projection on the space $(\mathcal{B}_d)^2$ of piecewise linear functions. The functions $w_d^k(\Phi_d)$ converge strongly towards $w^k(\Phi)$ defined by (21). Writing the discrete equilibrium equation (14) with $\Phi_d$ as test function and letting $d$ tend to 0, yields (18) for all $\Phi$ in $(C_c^\infty(\Omega))^2$ and, by density, for all $\Phi$ in $(H_0^1(\Omega))^2$.

**Step 4 :**

For $(\tau, v) \in \mathcal{K}$, there exists an approximating sequence $(\tau_d, v_d)_d$ of elements of $\mathcal{K}_d$ that converges strongly in $L^2(I \times \Omega)$. Indeed, consider a sequence of mollifiers $\rho_\eta$ defined on $I \times \Omega$. For $z \in I \times \Omega$, we let

$$\tau_\eta(z) = \int_{I \times \Omega} \rho_\eta \, \tau(z - y)\, dy$$

and we define a similar function $v_\eta(z)$. For simplicity we will assume that $I \times \Omega$ has measure 1 (recall that the yield function $\mathcal{F}$ is assumed to be convex and that $(\mathcal{F}(0,0) < 0)$. Jensen's inequality yields

$$\begin{aligned}\mathcal{F}(\tau_\eta(z), v_\eta(z)) &\leqslant \int_{I \times \Omega} \mathcal{F}(\rho_\eta(y)\, \tau(z-y), \rho_\eta(y)\, v(z-y))\, dy \\ &\leqslant \int_{I \times \Omega} [\rho_\eta(z-y)\, \mathcal{F}(\tau(y), v(y)) + \\ &\quad + (1-\rho_\eta(z-y))\, \mathcal{F}(0,0)]\, dy \\ &\leqslant 0 ,\end{aligned}$$

and so $(\tau_\eta, v_\eta)$ belongs to $\mathcal{K}$. Moreover, these functions are elements of $C^\infty(I \times \Omega)$ : by interpolation on the set of piecewise linear functions $(\mathcal{B}_d)^2$ we obtain a sequence $(\tau_d, v_d)_d$ that converges strongly towards $(\tau, v)$ in $L^2(I \times \Omega)$.

**Step 5 :**

For $(\tau^k, v^k) \in \mathcal{K}$, consider an approximating sequence $(\tau_d^k, v_d^k)$ for the strong $L^2$ norm. Relation (16) yields

$$\int_{I \times \Omega} (w_d^k - (E^k)^{-1} \dot{\sigma}_d^k, \tau_d^k - \sigma_d^k) - (\dot{\alpha}_d^k, v_d^k - \alpha_d^k) \leqslant 0$$

or

$$\int_{I \times \Omega} (\dot{w}_d^k - (E^k)^{-1} \dot{\sigma}_d^k, \tau_d^k) - (\dot{\alpha}_d^k, v_d^k) - \int_{I \times \Omega} (\dot{w}_d^k, \sigma_d^k) + $$

$$+ 1/2 \frac{d}{dt} \int_{I \times \Omega} (\sigma_d^k, \sigma_d^k) + (\alpha_d^k, \alpha_d^k) \leqslant 0 . \quad (22)$$

As $d$ tends to 0, the first term on the left-hand side of (22) tends to

$$\int_{I \times \Omega} (\dot{w}^k - (E^k)^{-1} \dot{\sigma}^k, \tau^k) - (\dot{\alpha}^k, v^k) .$$

The second term is equal to the right hand side of (14), and tends to

$$\int_{I \times \Omega} F \cdot \dot{U} .$$

Finally, using the initial conditions, the last term reduces to

$$1/2 \sum_k ( \|D^k \sigma_d^k\|_{L^2}^2 + \|\alpha_d^k\|_{L^2}^2 ) (T) ,$$

with $D^k = (E^k)^{-1/2}$. Since the traces $\sigma_d^k(x, y, T)$, $\alpha_d^k(x, y, T)$ converge to $\sigma^k(x, y, T)$, $\alpha^k(x, y, T)$ weakly in $L^2$, we have by weak lower semi-continuity of the norm that

$$\liminf_d \|D^k \sigma_d^k\| (T) \geqslant \|D^k \sigma^k\| (T)$$

$$\liminf_d \| \alpha_d^k\| (T) \geqslant \| \alpha^k\| (T) .$$

Hence, as $d$ tends to 0, (22) yields

$$\int_{I \times \Omega} ( \dot{w}^k - (E^k)^{-1} \dot{\sigma}^k, \tau^k ) - (\dot{\sigma}^k, v^k) - (\dot{w}^k, \sigma^k) + \\ + 1/2 \sum_k ( \|D^k \sigma^k\|^2 + \|\alpha^k\|^2 ) (T) \leqslant 0 ,$$

which reduces to (20).

**Step 6 :**

Let $P$ denote the projection on the convex set $\mathcal{K}$. We have

$$\int_{I \times \Omega} ( (\sigma^k, \alpha^k) - P(\sigma^k, \alpha^k), (\tau, v) - P(\sigma^k, \alpha^k) ) \leqslant 0 \quad \forall (\tau, v) \in \mathcal{K} .$$

Since $\mathcal{K}_d \subset \mathcal{K}$, we can choose $(\tau, v) = (\sigma_d^k, \alpha_d^k)$ in the inequality above, and we get as $d$ tends to 0,

$$\int_{I \times \Omega} \| (\sigma^k, \alpha^k) - P(\sigma^k, \alpha^k) \|_{L^2(\Omega)}^2 \leqslant 0 ,$$

and from the initial condition,

$$(\sigma^k, \alpha^k) = P(\sigma^k, \alpha^k) \quad \text{a.e.} \quad \text{in } I \times \Omega .$$

That is, (19) is fulfilled and this completes the proof of Theorem 4.1. □

From here on, we can proceed as [6], [2], to show that if the solution $(U, \sigma^k, \alpha^k)$ is absolutely continuous, it satisfies the following relations :

$$\begin{cases} \dot{\sigma}^k(t) = E^k w^k(\dot{U}(t)) \\ \dot{\alpha}^k(t) = 0 \end{cases} \quad \text{a.e.} \quad \text{in } \mathscr{C}^k \tag{23}$$

$$\begin{cases} \sigma^k(t) = E_s^k\, w^k(\dot{U}(t)) \\ \\ \alpha^k(t) = \dfrac{-\partial_\sigma \mathcal{F} \quad \sigma^k}{\partial_\sigma \mathcal{F}^T \partial_\alpha \mathcal{F}} \end{cases} \quad \text{a e} \quad \text{in}\ \mathcal{P}^k \tag{24}$$

where

$$E_s^k = \frac{(E^k \partial_\sigma \mathcal{F})^2}{\partial_\alpha \mathcal{F}^T \partial_\alpha \mathcal{F} + E^k(\partial_\sigma \mathcal{F})^2}(\sigma^k, \alpha^k)$$

$$\mathcal{E}^k = \{(x, y, t) \,/\, \mathcal{F}(\sigma^k, \alpha^k) < 0 \quad \text{or} \quad \partial_\sigma \mathcal{F} \quad \sigma^k \leqslant 0\}$$

$$\mathcal{P}^k = \{(x, y, t) \,/\, \mathcal{F}(\sigma^k, \alpha^k) = 0 \quad \text{and} \quad \partial_\sigma \mathcal{F} \quad \sigma^k > 0\}$$

### 4.2. Limiting constitutive equations

We are now able to write the constitutive equations of a two-dimensional homogenized material which can be obtained as the limit of the rod structures described in Section 2

THEOREM 4 2 *Let* $(U, \sigma^k, \alpha^k)$ *be the limiting solution given by Theorem* 4 1 *U is the displacement field of a 2-d plate under the load F, whose stress tensor* $\Sigma$ *satisfies*

$$-\operatorname{div}(\Sigma) = F \quad \text{in}\ I \times \Omega \tag{25}$$

$$\begin{cases} \Sigma_{11} = \sigma_1 + (\sigma_3 + \sigma_4)/2 \\ \Sigma_{12} = (\sigma_3 - \sigma_4)/2 \\ \Sigma_{22} = \sigma_2 + (\sigma_3 + \sigma_4)/2 \end{cases} \tag{26}$$

$$U_{/\partial\Omega} = 0, \quad \dot{\Sigma} = M\varepsilon \quad \textit{with}$$

$$M = \begin{bmatrix} \eta_1 + (\eta_3 + \eta_4)/4 & (\eta_3 + \eta_4)/4 & (\eta_3 - \eta_4)/2 \\ (\eta_3 + \eta_4)/4 & \eta_2 + (\eta_3 + \eta_4)/4 & (\eta_3 - \eta_4)/2 \\ (\eta_3 - \eta_4)/4 & (\eta_3 - \eta_4)/4 & (\eta_3 + \eta_4)/2 \end{bmatrix} \tag{27}$$

*where the materials coefficients are given by*

$$\begin{cases} \eta_k = E^k \\ \qquad\qquad \text{a.e. } \mathscr{E}^k \\ \alpha^k = 0 \end{cases}$$

$$\begin{cases} \eta_k = \dfrac{(E^k \partial_\sigma \mathscr{F})^2}{\partial_\alpha \mathscr{F}^T \partial_\alpha \mathscr{F} + E^k(\partial_\sigma \mathscr{F})^2}(\sigma^k, \alpha^k) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } \mathscr{P}^k \\ \dot{\alpha}^k = \dfrac{\partial_\sigma \mathscr{F} \cdot \dot{\sigma}^k}{\partial_\alpha \mathscr{F}^T \partial_\alpha \mathscr{F}} \end{cases}$$

*together with initial conditions* (9).

*Proof:* Let $\Phi = (\phi, \psi) \in (H_0^1(\Omega))^2$. Integrating by parts in (18) yields for almost every *t*,

$$\int_\Omega -\phi\left[\frac{\partial}{\partial x}\left(\sigma^1 + \frac{\sigma^3 + \sigma^4}{2}\right) + \frac{\partial}{\partial y}\left(\frac{\sigma^3 - \sigma^4}{2}\right)\right] - \psi\left[\frac{\partial}{\partial x}\left(\frac{\sigma^3 - \sigma^4}{2}\right) + \right.$$

$$\left. + \frac{\partial}{\partial y}\left(\sigma^2 + \frac{\sigma^3 + \sigma^4}{2}\right)\right] = \int_\Omega F \quad \Phi .$$

This relation is the weak formulation of the equilibrium equation (25) where the symmetric tensor $\Sigma$, defined by (26), represents the stress tensor of the homogenized material. □

Thus, we have constructed a 2-d elasto-plastic material which has the following properties :

- If no plastic deformation has occured, necessary conditions for isotropy yield

$$E^1 = E^2 = E^3 = E^4 ,$$

which corresponds to Lamé's coefficients

$$\lambda = \mu = \frac{E^1}{2}$$

and Poisson's ratio

$$\nu = \frac{\lambda}{\lambda + 2\mu} = \frac{1}{3} .$$

- If plasticity occurs, the yield surface is given by 4 conditions,

$$\mathcal{F}(\sigma^k, \alpha^k) \leq 0 ,$$

in terms of linear combinations of the components of the stress tensor In the stress space the yield surface is a polygon, like Tresca's criterion
- Plasticity does not only affect the stress deviator in this construction, as is usually assumed in 2-d or 3-d plasticity Indeed,

$$\operatorname{tr}(\Sigma) = \left(\eta_1 + \frac{\eta_3 + \eta_4}{4}\right)\varepsilon_{11} + \left(\eta_2 + \frac{\eta_3 + \eta_4}{4}\right)\varepsilon_{22}$$

and an elastic relation between $\operatorname{tr}(\Sigma)$ and $\operatorname{tr}(\varepsilon)$ would restrict the moduli $\eta_i$ to stay elastic

## 5. ELASTIC LIMITS OF SIMILAR ROD STRUCTURES

For the particular rod pattern considered in Section 2, we have seen that, when no plastic deformation has occured, the requirement of isotropy implies equality of the elastic moduli of rigidity of the rods This restriction is called Cauchy's relations [5] Thus, in the family of constitutive equations for 2-d isotropic elastic materials, we can only achieve those whose Poisson ratio equals 1/3, as limits of rod structures as those considered in Section 2

We are going to show that, for periodic rod-structures, this property is independent of the geometry of the basic cell, i e of the pattern of the coverage of $\Omega$ by the rods, provided that there are as many cells as hinges where the load is applied In other words, there is no internal mechanism within each cell and the equilibrium equations at each hinge uniquely determine the displacement

Let us consider a periodic coverage of $\Omega$ by cells of area $d^2$, which contain elastic rods As in the previous section, forces act on the hinges that attach together the extremities of the rods We assume that there are as many hinges as cells We define the strain in the rod joining 2 points $M_n^k$, $M_n^l$ of a cell $e_n$ by

$$w_{dn}^{kl} = \frac{1}{d^{kl}}\left[(u_n^k - u_n^l)\cos\theta^{kl} + (v_n^k - v_n^l)\sin\theta^{kl}\right]$$

where $U_d = (u, v)$ denotes the displacement, $d_{kl}$, the length of the rod $(M^k, M^l)$, and $\theta^{kl}$, the angle it makes with the horizontal The stress is defined in each rod by the elastic law

$$\sigma_{dn}^{kl} = E^{kl}\, w_{dn}^{kl}$$

Let $N$ be the number of hinges which are not on the boundary of $\Omega$. Let $\mathscr{C}_d$, be the set of piecewise linear functions $\Phi$ defined on the triangulation made by these $N$ points, such that

$$\Phi_{/\partial\Omega} = 0 .$$

When a system of loads $(F_{dn}) = (f_{dn}, g_{dn})$ $1 \leqslant n \leqslant N$ is applied to the structure, the equilibrium equations write

$$\begin{cases} \displaystyle\sum_{M_n \in (M_p^k, M_p^l)} \sigma_{dp}^{kl} \cos(\theta^{kl}) = df_{dn} \\ \\ \displaystyle\sum_{M_n \in (M_p^k, M_p^l)} \sigma_{dp}^{kl} \sin(\theta^{kl}) = dg_{dn} \end{cases} \qquad 1 \leqslant n \leqslant N . \tag{28}$$

Let $\Phi \in (C_c^\infty(\Omega))^2$, and $\Phi_d$ denote its projection on the space $\mathscr{C}_d$. Equations (28) yield

$$\sum_n \sum_{k,l} \left( \frac{d^{kl}}{d} \sigma_{dn}^{kl} w_{dn}^{kl}(\Phi) \right) = \sum_n F_{dn} \cdot \Phi_{dn} .$$

We extend the stresses and strains $(\sigma_{dn}^{kl}, w_{dn}^{kl})$ as piecewise constant functions defined on $\Omega$

$$(\sigma_d^{kl}, w_d^{kl}) = \sum_n (\sigma_{dn}^{kl}, w_{dn}^{kl}) \chi_{e_n} ,$$

and rewrite the equilibrium equations in the form

$$\int_\Omega \frac{d^{kl}}{d} \sigma_d^{kl}, w_d^{kl}(\Phi_d) = \int_\Omega F_d \cdot \Phi_d .$$

Let us assume that these equations have solutions $U_d$ which are uniformely bounded in $H_0^1(\Omega)$ (this is the case if the associated discrete energy functional is bounded from below).

We obtain the limiting equilibrium equation (note that the ratio $\frac{d^{kl}}{d}$ is independent of $d$),

$$\int_\Omega \left( \frac{d^{kl}}{d} \cos^2(\theta^{kl}) \sigma^{kl} \right) \varepsilon_{11}(\Phi) + \left( \frac{d^{kl}}{d} \sin^2(\theta^{kl}) \sigma^{kl} \right) \varepsilon_{22}(\Phi) +$$

$$+ 2\left( \frac{d^{kl}}{d} \cos(\theta^{kl}) \sin(\theta^{kl}) \sigma^{kl} \right) \varepsilon_{12}(\Phi) = \int_\Omega F \cdot \Phi \tag{29}$$

since as $d$ tends to 0,

$$w_d^{kl}(\Phi) \to \cos^2(\theta^{kl})\,\varepsilon_{11}(\Phi) + \sin^2(\theta^{kl})\,\varepsilon_{22}(\Phi) + \sin(2\,\theta^{kl})\,\varepsilon_{12}(\Phi)$$

strongly in $L^2(\Omega)$, and since

$$w_d^{kl}(U_d) \rightharpoonup \cos^2(\theta^{kl})\,\varepsilon_{11}(U) + \sin^2(\theta^{kl})\,\varepsilon_{22}(U) + \sin(2\,\theta^{kl})\,\varepsilon_{12}(U)$$

$$\sigma_d^{kl} \rightharpoonup \sigma^{kl}$$

weakly in $L^2(\Omega)$ ; with

$$\sigma^{kl} = E^{kl}[\cos^2(\theta^{kl})\,\varepsilon_{11}(U) + \sin^2(\theta^{kl})\,\varepsilon_{22}(U) + \sin(2\,\theta^{kl})\,\varepsilon_{12}(U)]\,. \tag{30}$$

The macroscopic stresses can thus be defined by

$$\Sigma_{11} = \frac{d^{kl}}{d}\cos^2(\theta^{kl})\,\sigma^{kl}$$

$$\Sigma_{22} = \frac{d^{kl}}{d}\sin^2(\theta^{kl})\,\sigma^{kl}$$

$$\Sigma_{12} = \frac{d^{kl}}{d}\cos(\theta^{kl})\sin(\theta^{kl})\,\sigma^{kl}\,.$$

Replacing the expressions (30) in these formulae yields the matrix $M$ of elastic coefficients of the limiting material :

$$M = \begin{bmatrix} G^{kl}\cos^4(\theta^{kl}) & G^{kl}\cos^2(\theta^{kl})\sin^2(\theta^{kl}) & 2\,G^{kl}\cos^3(\theta^{kl})\sin(\theta^{kl}) \\ G^{kl}\cos^2(\theta^{kl})\sin^2(\theta^{kl}) & G^{kl}\sin^4(\theta^{kl}) & 2\,G^{kl}\cos(\theta^{kl})\sin^3(\theta^{kl}) \\ G^{kl}\cos^3(\theta^{kl})\sin^3(\theta^{kl}) & G^{kl}\cos(\theta^{kl})\sin^3(\theta^{kl}) & 2\,G^{kl}\cos^2(\theta^{kl})\sin^2(\theta^{kl}) \end{bmatrix},$$

where $G^{kl} = \frac{d^{kl}}{d}E^{kl}$ (no summation with respect to $k$, $l$ here). Isotropy requirements force the relations

$$\frac{d^{kl}}{d}E^{kl}\cos^3(\theta^{kl})\sin(\theta^{kl}) = 0$$

$$\frac{d^{kl}}{d} E^{kl} \cos(\theta^{kl}) \sin^3(\theta^{kl}) = 0$$

$$\lambda + 2\mu = \frac{d^{kl}}{d} E^{kl} \cos^4(\theta^{kl})$$

$$= \frac{d^{kl}}{d} E^{kl} \sin^4(\theta^{kl})$$

$$= \frac{3\, d^{kl}}{d} E^{kl} \cos^2(\theta^{kl}) \sin^2(\theta^{kl})$$

$$\lambda = \mu = \frac{d^{kl}}{d} E^{kl} \cos^2(\theta^{kl}) \sin^2(\theta^{kl})$$

and thus Poisson's ratio equals 1/3.

## 6. A HOMOGENIZED ROD STRUCTURE WITH $\nu \neq 1/3$

In this section, we design a rod structure with « internal constraints ». Theorem 3.1 and Theorem 4.2 apply to this structure, but the constraints induce limiting elastoplastic constitutive equations different from those obtained in the previous sections. In particular, if the rods are purely elastic, we obtain a limiting isotropic material with $\nu \neq 1/3$.

### 6.1. Description of the structure

We consider a triangulation of $\Omega$ made by equilateral triangles of sidelength $d$. We assume that the mesh points are numbered $(M_n)$. Each unit cell is a rhombus that consists of 2 triangles, as shown in figure 3. The cell, whose most western point is $M_n$, is denoted by $e_n$. The upper triangle is denoted by $e_n^+$, the lower triangle by $e_n^-$. In the cell $e_n$, we associate a local number for the vertices (*cf. fig.* 3) : we define

$$M_n^1 = M_n \qquad M_n^3 = M_n + d(1/2, \sqrt{3/2})$$
$$M_n^2 = M_n + d(1, 0) \qquad M_n^5 = M_n + d(1/2, -\sqrt{3/2}) .$$

We denote by $M_n^4$ (resp. $M_n^6$) the center of gravity of the triangle $e_n^+$ (resp. $e_n^-$), i.e.,

$$M_n^4 = M_n + d(1/2, \sqrt{3/6})$$

$$M_n^6 = M_n + d(1/2, -\sqrt{3/6}) .$$

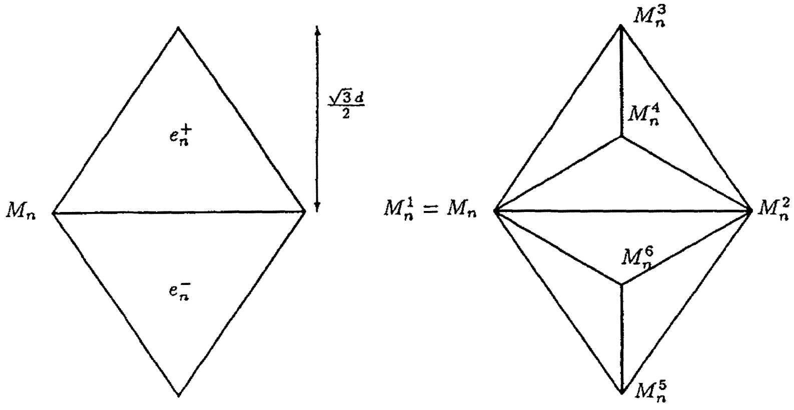


**Figure 3. — The cell $e_n$ and the position of the rods in $e_n$ .**

As in section 2, we assume that the vertices are connected by rods. We also assume that 'internal rods' connect the center of gravity of each triangle $e_n^{\pm}$ to the vertices of the triangle

Let $N$ be the number of vertices in $\Omega$, $M$ be the number of centers of gravity. Thus, the number of hinges is $N+M$. Let $\mathcal{D}_d$ be the set of continuous piecewise linear functions $\Phi$ defined on the triangulation made by the $M+N$ hinges, such that

$$\Phi_{/\partial\Omega} = 0 .$$

Let $\Phi = (\phi, \psi) \in (C_c^\infty(\Omega))^2$, and let $\Phi_d$ be its projection on the space $\mathcal{D}_d$. We write $\Phi_{dm} = \Phi_d(M_m) = \Phi(M_m)$, for a hinge located at $M_m$.

With the notation of the previous section, we define approximate strains

$$w_{dn}^{kl}(\Phi) = \frac{1}{d^{kl}} [(\phi_n^k - \phi_n^l) \cos(\theta^{kl}) + \\ + (\psi_n^k - \psi_n^l) \sin(\theta^{kl})] \quad 1 \leqslant n \leqslant N \quad (31)$$

for $k < l$, such that a rod connects the points $M_n^k$ and $M_n^l$. Let $U_d(t)$ be a $\mathcal{C}_+^1$ function of $t$, with values in $\mathcal{D}_d$, that represents the displacements of the endpoints of the rods. We denote $w_{dn}^{kl} = w_{dn}^{kl}(U_d(t))$ the strains on the rod $M_n^k$ and $M_n^l$ defined as in (31). The stress, internal parameters and modulus of rigidity are defined in each rod by an elasto-plastic law of the type (3-6)

$$\dot{\sigma}_{dn}^{kl} = E_{dn}^{kl}(\sigma_{dn}^{kl}, \alpha_{dn}^{kl})\, \dot{w}_{dn}^{kl} . \quad (32)$$

As in Section 2, some linear combination of the strain tensor $\varepsilon(U_d)$ coincides with the value of $w_{dn}^{kl}$ on the triangles adjacent to the rod $M_n^k M_n^l$ (for the triangulation defined by the $N+M$ hinges).

We define

$$w_d^1 = \Sigma_n w_{dn}^{21} \chi_{e_n^+ \cup e_n^-}$$

$$w_d^2 = \Sigma_n w_{dn}^{23} \chi_{e_n^+} + w_{dn}^{51} \chi_{e_n^-}$$

$$w_d^3 = \Sigma_n w_{dn}^{31} \chi_{e_n^+} + w_{dn}^{52} \chi_{e_n^-}$$

$$w_d^4 = \Sigma_n w_{dn}^{41} \chi_{e_n^+} + w_{dn}^{26} \chi_{e_n^-}$$

$$w_d^5 = \Sigma_n w_{dn}^{24} \chi_{e_n^+} + w_{dn}^{61} \chi_{e_n^-}$$

$$w_d^6 = \Sigma_n w_{dn}^{34} \chi_{e_n^+} + w_{dn}^{65} \chi_{e_n^-} \,. \tag{33}$$

We define the functions $\sigma_d^k$, $\alpha_d^k$, $E_{d,s}^k$, $1 \leqslant k \leqslant 6$, in the same way. We assume that a system of force $F_{dn}$, $1 \leqslant n \leqslant N+M$, is loading the structure, which is fixed at its boundary. The equilibrium equations (28)

$$\sum_{m=1}^{N+M} \sum_{M_n^k = M_m} \frac{d^{kl}}{d} \sigma_{dn}^{kl} w_{dn}^{kl}(\Phi) = \sum_m \dot{F}_{dm} \cdot \Phi_{dm}$$

can be rewritten in terms of the definitions (33) as

$$\int_\Omega \sum_{k \leqslant 3} \dot{\sigma}_d^k \dot{w}_d^k(\Phi) + \sum_{k>3} \frac{\sqrt{3}}{3} \dot{\sigma}_d^k \dot{w}_d^k(\Phi) = \sum_m \frac{\sqrt{3}\, d^2}{2} \dot{F}_{dm} \cdot \Phi_{dm} \,. \tag{34}$$

We seek a solution $(U_d, \sigma_d^k, \alpha_d^k)$ of the initial value problem (34, 8, 9). The following proposition is the analog of Theorem 3.1.

PROPOSITION 6.1 : *With the hypothesis of Theorem* 3.1, *the initial value problem* (34, 8, 9) *has a unique solution* $(U_d, \sigma_d^k, \alpha_d^k)$ *such that* $\sigma_d^k$, $\alpha_d^k$ *are piecewise constant functions on the triangles* $e_n^\pm$, *and are* $C_+^1$ *functions in time ; moreover,*

$$\mathcal{F}(\sigma_d^k, \alpha_d^k) \leqslant 0 \quad \text{a.e. in } I \times \Omega \,.$$

*Proof:* To adapt the proof of Theorem 3.1 to this new rod structure, it suffices to show that the incremental energy functional

$$\mathscr{E}_{\sigma_d^k, \alpha_d^k} = \left( \int_\Omega \sum_{k \leqslant 3} \dot{\sigma}_d^k \dot{w}_d^k(U_d) + \sum_{k > 3} \frac{\sqrt{3}}{3} \dot{\sigma}_d^k \dot{w}_d^k(U_d) \right) - \\ - \sum_n \frac{\sqrt{3}\, d^2}{2} \dot{F}_{dn} \cdot U_{dn} ,$$

defined on $\mathscr{D}_d$ has a minimum ; i.e. that it is bounded from below, since the convexity properties are still guaranteed by the constitutive relations (3-5).

Let us consider one of the triangles $e_n^+$ of the triangulation. Since the moduli of rigidity are uniformly bounded away from 0, there exists a positive constant $\kappa$ such that the contribution of $e_n^+$ to the quadratic term of the energy is greater than

$$\kappa \int_{e_n^+} \sum_{1 \leqslant k \leqslant 6} (\dot{w}_d^k)^2 .$$

The integral term in this expression is greater than

$$\int_{T_n^1} (\dot{w}_d^1)^2 + \frac{(\dot{w}_d^4)^2 + (\dot{w}_d^5)^2}{2} + \int_{T_n^2} (\dot{w}_d^2)^2 + \frac{(\dot{w}_d^5)^2 + (\dot{w}_d^6)^2}{2}$$

$$\int_{T_n^3} (\dot{w}_d^3)^2 + \frac{(\dot{w}_d^4)^2 + (\dot{w}_d^6)^2}{2}$$

where $T_n^1$ (respectively $T_n^2, T_n^3$) denotes the triangle $(M_n^1, M_n^2, M_n^4)$, (respectively $(M_n^2, M_n^3, M_n^4)$, $(M_n^1, M_n^4, M_n^3)$).

Since $U_d$ is piecewise linear on the triangulation defined by the hinges, using the definition (33) of the functions $w_d^k$, the last expression can also be written as

$$\int_{T_n^1} (\varepsilon_{11})^2 + \frac{(\varepsilon_{11} + 3\,\varepsilon_{22} + \sqrt{3}\,\varepsilon_{12})^2 + (\varepsilon_{11} + 3\,\varepsilon_{22} - \sqrt{3}\,\varepsilon_{12})^2}{32} + \\ + \int_{T_n^2} \frac{(3\,\varepsilon_{11} + \varepsilon_{22} + \sqrt{3}\,\varepsilon_{12})^2}{4} + \frac{(\varepsilon_{11} + 3\,\varepsilon_{22} - \sqrt{3}\,\varepsilon_{12})^2}{32} + \frac{(\varepsilon_{22})^2}{2} + \\ + \int_{T_n^3} \frac{(3\,\varepsilon_{11} + \varepsilon_{22} - \sqrt{3}\,\varepsilon_{12})^2}{4} + \frac{(\varepsilon_{11} + 3\,\varepsilon_{22} + \sqrt{3}\,\varepsilon_{12})^2}{32} + \frac{(\varepsilon_{22})^2}{2} . \qquad (35)$$

Let us consider the first term of (35) :

$$\int_{T_n^1} (\varepsilon_{11})^2 + \frac{(\varepsilon_{11} + 3\,\varepsilon_{22} + \sqrt{3}\,\varepsilon_{12})^2 + (\varepsilon_{11} + 3\,\varepsilon_{22} - \sqrt{3}\,\varepsilon_{12})^2}{32} \geq$$

$$\geq \int_{T_n^1} (\varepsilon_{11})^2 + \frac{1}{16}\left[(\varepsilon_{11})^2 + 9(\varepsilon_{22})^2 + 3(\varepsilon_{12})^2 - 6\,\varepsilon_{11}\,\varepsilon_{22}\right]$$

$$\geq \int_{T_n^1} \frac{7}{8}(\varepsilon_{11})^2 + \frac{3}{8}(\varepsilon_{22})^2 + \frac{3}{16}(\varepsilon_{12})^2$$

$$\geq C \int_{T_n^1} |\nabla \dot{U}_d|^2$$

for some positive constant $C$, independent of $d$. We can treat the 2 other terms similarly which yields the boundedness of $\mathscr{C}_d$. □

We only consider systems of forces that vanish on the interior hinges of each cell : this will constrain the displacement of the vertices and affect the limiting constitutive equations. To also be able to go to the limit if plasticity occurs, we have to impose appropriate conditions on the behavior of the internal rods. More precisely, we make the following assumptions concerning our rod structure :

- The elastic coefficients for the internal rods are equal : according to (33), this condition implies that

$$E^4 = E^5 = E^6 . \tag{36}$$

- At $t = 0$, the internal rods are all in a virgin state, i.e.,

$$\begin{aligned} \sigma_d^4 = \sigma_d^5 = \sigma_d^6 = 0 \\ \alpha_d^4 = \alpha_d^5 = \alpha_d^6 = 0 . \end{aligned} \tag{37}$$

- The forces do not act on the centers of gravity of the triangles $e_n^{\pm}$, i.e., $F_{dm} = F_d(M_m) = 0$, for $N + 1 \leq m \leq M + N$. A $d$ tends to 0, we assume that the piecewise constant functions $F_d$, that takes the value $F_d(M_n)$ on the cell $e_n$, converge to $F$.

### 6.2. The effect of internal constraints

We will now let the size of the rods $d$ tend to 0. We will verify that, if only elastic deformation takes place, limiting isotropic materials satisfy $\nu \neq 1/3$.

Similar uniform estimates as in Section 2 hold for the semi-discrete solutions $(U_d, \sigma_d^k, \alpha_d^k)$. So as $d$ tends to 0, we can extract subsequences that converge weakly

$$\dot{U}_d, U_d \rightharpoonup \dot{U}, U$$

$$\dot{\sigma}_d^k, \sigma_d^k, \dot{\alpha}_d^k, \alpha_d^k \rightharpoonup \dot{\sigma}^k, \sigma^k, \dot{\alpha}^k, \alpha^k .$$

On the other hand, the strains $w_d^k(\Phi)$ converge strongly in $L^2(I \times \Omega)$ as follows :

$$\begin{aligned}
&w_d^1(\Phi) \to \varepsilon_{11}(\Phi) \\
&w_d^2(\Phi) \to \frac{\varepsilon_{11} + 3\,\varepsilon_{22} - 2\sqrt{3}\,\varepsilon_{12}}{4}(\Phi) \\
&w_d^3(\Phi) \to \frac{\varepsilon_{11} + 3\,\varepsilon_{22} + 2\sqrt{3}\,\varepsilon_{12}}{4}(\Phi) \\
&w_d^4(\Phi) \to \frac{3\,\varepsilon_{11} + \varepsilon_{22} + 2\sqrt{3}\,\varepsilon_{12}}{4}(\Phi) \\
&w_d^5(\Phi) \to \frac{3\,\varepsilon_{11} + \varepsilon_{22} - 2\sqrt{3}\,\varepsilon_{12}}{4}(\Phi) \\
&w_d^6(\Phi) \to \varepsilon_{22}(\Phi) .
\end{aligned} \tag{38}$$

Using the hypothesis on $F_d$, we can rewrite the right hand side of (35) as

$$\begin{aligned}
\sum_{n=1}^{N+M} \frac{\sqrt{3}\,d^2}{2} F_{dn} \cdot \Phi_{dn} &= \sum_{n=1}^{N} |e_n|\, F_{dn} \cdot \Phi_{dn} \\
&= \int_\Omega F_d \cdot \Phi_d .
\end{aligned} \tag{39}$$

As $d$ tends to 0, this expression converges to

$$\int_\Omega F \cdot \Phi .$$

Relations (38, 39) yield the equilibrium equation for the limiting material

$$\forall \Phi \in (C_c^\infty(\Omega))^2, \quad \int_\Omega \varepsilon_{11}(\Phi)\left[\sigma^1 + \frac{1}{4}(\sigma^2+\sigma^3) + \frac{\sqrt{3}}{4}(\sigma^4+\sigma^5)\right] +$$

$$+ \varepsilon_{22}(\Phi)\left[\frac{3}{4}(\sigma^2+\sigma^3) + \frac{\sqrt{3}}{12}(\sigma^4+\sigma^5) + \frac{\sqrt{3}}{3}\sigma^6\right] +$$

$$+ 2\,\varepsilon_{12}(\Phi)\left[\frac{\sqrt{3}}{4}(\sigma^3-\sigma^2) + \frac{1}{4}(\sigma^4-\sigma^5)\right] = \int_\Omega F\ \Phi$$

and we define the macroscopic stress tensor by

$$\dot{\Sigma}_{11} = \sigma^1 + \frac{1}{4}(\sigma^2+\sigma^3) + \frac{\sqrt{3}}{4}(\sigma^4+\sigma^5)$$

$$\dot{\Sigma}_{22} = \frac{3}{4}(\sigma^2+\sigma^3) + \frac{\sqrt{3}}{12}(\sigma^4+\sigma^5) + \frac{\sqrt{3}}{3}\sigma^6$$

$$\dot{\Sigma}_{12} = \frac{\sqrt{3}}{4}(\sigma^3-\sigma^2) + \frac{1}{4}(\sigma^4-\sigma^5) \tag{40}$$

We can deduce the macroscopic constitutive equations from $\dot{\Sigma}$, provided we can express the functions $\sigma_k$, $1 \leqslant k \leqslant 6$, in terms of the values of $\dot{U}_d$ at the $N$ vertices only

To this effect, notice that, under the hypothesis on the load, the equilibrium conditions (28) at the centers of gravity of the triangles $e_n^+$ yield the following constraints on the rates of stresses of the internal bars

$$\sigma_{dn}^{41} = \sigma_{dn}^{42} = \sigma_{dn}^{43}, \tag{41}$$

or, in terms of the restrictions on $e_n^+$ of the functions $\sigma_d^k$

$$\sigma_d^4 = \sigma_d^5 = \sigma_d^6$$

We solve this system for the rate of displacement, $\dot{U}_d^4 = (u_d^4, v_d^4)$, of the center of gravity of the triangle $e_n^+$

$$\dot{u}_d^4 = \frac{\sqrt{3}}{6 \cdot \Delta_d} [\dot{u}_d^1(\sqrt{3}\, E_d^4 E_d^5 + 2\sqrt{3}\, E_d^4 E_d^6) + \dot{v}_d^1(E_d^4 E_d^5 + 2\, E_d^4 E_d^6)$$

$$+ \dot{u}_d^2(\sqrt{3}\, E_d^4 E_d^5 + 2\sqrt{3}\, E_d^5 E_d^6) - \dot{v}_d^2(E_d^4 E_d^5 + 2\, E_d^5 E_d^6)$$

$$+ \dot{v}_d^3 \cdot 2\, E_d^6(E_d^5 - E_d^4)$$

$$\dot{v}_d^4 = \frac{1}{2 \cdot \Delta_d} [\dot{u}_d^1 \sqrt{3}\, E_d^4 E_d^5 + \dot{v}_d^1 E_d^4 E_d^5 - \dot{u}_d^2 \sqrt{3}\, E_d^4 E_d^5$$

$$+ \dot{v}_d^2\, E_d^4 E_d^5 + \dot{v}_d^3\, 2\, E_d^6(E_d^5 + E_d^4)]$$

where

$$\Delta_d = E_d^4 E_d^5 + E_d^4 E_d^6 + E_d^5 E_d^6 .$$

We replace these expressions in the formulae for the strain rates :

$$\begin{cases} \dot{w}_d^4 = \dfrac{E_d^5 E_d^6}{\Delta_d} \dot{w}_d \\ \dot{w}_d^5 = \dfrac{E_d^4 E_d^6}{\Delta_d} \dot{w}_d \\ \dot{w}_d^6 = \dfrac{E_d^4 E_d^5}{\Delta_d} \dot{w}_d \end{cases}$$

with the effective strain $\dot{w}_d$ defined on $e_n^+$ by

$$\dot{w}_d = \frac{\sqrt{3}}{2\, d} (\sqrt{3}(\dot{u}_d^2 - \dot{u}_d^1) - \dot{v}_d^1 - \dot{v}_d^2 + 2 \cdot \dot{v}_d^3) .$$

The yield condition for internal bars can thus be written only in terms of $\dot{w}_d$,

$$\mathcal{F}(\sigma_d^k, \alpha_d^k) = 0 \quad and \quad \partial_\sigma \mathcal{F}(\sigma_d^k, \alpha_d^k)\dot{w}_d > 0 \quad 4 \leq k \leq 6 .$$

Assumptions (36, 37) and relations (41) imply that the internal bars in $e_n^+$ are in the same state,

$$\begin{cases} E_d^4 = E_d^5 = E_d^6 \\ \sigma_d^4 = \sigma_d^5 = \sigma_d^6 \\ \alpha_d^4 = \alpha_d^5 = \alpha_d^6 \end{cases} \qquad \text{a.e. } t, \text{ on } e_n^+ ,$$

and the constraints on the internal strain rates on $e_n^+$ are thus

$$\dot{w}_d^4 = \dot{w}_d^5 = \dot{w}_d^6 = \frac{1}{3}\dot{w}_d .$$

The same relations hold on the triangles $e_n^-$ .

Let $V_d$ be the projection of $U_d$ on the triangulation defined by only the $N$ vertices, i.e., $V_d$ is the set of continuous piecewise linear functions such that $V_d(M_n) = U_d(M_n)$, $1 \leqslant n \leqslant N$. As $\|\nabla \dot{V}_d\|_{L^2}$ is uniformely bounded with respect to $d$, $V_d$ has the same weak limit as $U_d$. Thus, we can express the limits of the functions $\dot{w}_d^k$, $1 \leqslant k \leqslant 3$ in terms of $U$, since

$$\dot{w}_d^1 = \varepsilon_{11}(\dot{V}_d) \rightharpoonup \varepsilon_{11}(\dot{U})$$

$$\dot{w}_d^2 = \frac{\varepsilon_{11} + 3\,\varepsilon_{22} - 2\sqrt{3}\,\varepsilon_{12}}{4}(\dot{V}_d) \rightharpoonup \frac{\varepsilon_{11} + 3\,\varepsilon_{22} - 2\sqrt{3}\,\varepsilon_{12}}{4}(\dot{U})$$

$$w_d^3 = \frac{\varepsilon_{11} + 3\,\varepsilon_{22} + 2\sqrt{3}\,\varepsilon_{12}}{4}(\dot{V}_d) \rightharpoonup \frac{\varepsilon_{11} + 3\,\varepsilon_{22} + 2\sqrt{3}\,\varepsilon_{12}}{4}(\dot{U}) .$$

Also note that, as $d$ tends to 0, the effective strain rate $\dot{w}_d$ converges to

$$\dot{w}_d \rightharpoonup \frac{3}{2}(\varepsilon_{11} + \varepsilon_{22})(\dot{U}) .$$

The analog of Theorem 4.2 enables us to let $d$ tend to 0 in the weak form of the constitutive equations. We obtain macroscopic relations of the form :

$$\dot{\sigma}^1 = \eta_1\,\varepsilon_{11}$$

$$\dot{\sigma}^2 = \eta_2\,\frac{3\,\varepsilon_{11} + \varepsilon_{22} - 2\sqrt{3}\,\varepsilon_{12}}{4}$$

$$\dot{\sigma}^3 = \eta_3\,\frac{3\,\varepsilon_{11} + \varepsilon_{22} + 2\sqrt{3}\,\varepsilon_{12}}{4}$$

$$\dot{\sigma}^k = \eta_k \lim w_d^k = \frac{\eta_k}{2} (\varepsilon_{11} + \varepsilon_{22}) (\dot{u}) \quad 4 \leqslant k \leqslant 6 ,$$

$$\begin{cases} \eta_k = E^4 & \text{if} \quad \mathcal{F}(\sigma^k, \alpha^k) \leqslant 0, \\ \dot{\alpha}^k = 0 & \text{or} \quad \partial_\sigma \mathcal{F} \cdot \dot{\sigma}^k \leqslant 0 \end{cases}$$

$$\begin{cases} \eta_k = \dfrac{(E^4 \, \partial_\sigma \mathcal{F})^2}{\partial_\alpha \mathcal{F}^T \partial_\alpha \mathcal{F} + E^4 \cdot (\partial_\sigma \mathcal{F})^2} (\sigma^k, \alpha^k) & \\ & \text{otherwise.} \\ \dot{\alpha}^k = \dfrac{-\partial_\sigma \mathcal{F} \cdot \sigma^k}{\partial_\alpha \mathcal{F}^T \partial_\alpha \mathcal{F}} & \end{cases}$$

Replacing $\dot{\sigma}^k$, $4 \leqslant k \leqslant 6$, in the expression of macroscopic stresses (40), yields the constitutive relations

$$\dot{\Sigma}_{11} = \left[ \eta_1 + \frac{\eta_2 + \eta_3}{16} + \frac{\sqrt{3}\,\eta_4}{4} \right] \varepsilon_{11}(\dot{U})$$

$$+ \left[ \frac{3}{16} (\eta_2 + \eta_3) + \frac{\sqrt{3}\,\eta_4}{4} \right] \varepsilon_{22}(\dot{U}) + \left[ \frac{\sqrt{3}}{8} (\eta_3 - \eta_2) \right] \varepsilon_{12}(\dot{U})$$

$$\dot{\Sigma}_{22} = \left[ \frac{3}{16} (\eta_2 + \eta_3) + \frac{\sqrt{3}\,\eta_4}{4} \right] \varepsilon_{11}(\dot{U})$$

$$+ \left[ \frac{9}{16} (\eta_2 + \eta_3) + \frac{\sqrt{3}\,\eta_4}{4} \right] \varepsilon_{22}(\dot{U}) + \left[ \frac{\sqrt{3}}{8} (\eta_3 - \eta_2) \right] \varepsilon_{12}(\dot{U})$$

$$\dot{\Sigma}_{12} = + \left[ \frac{\sqrt{3}}{16} (\eta_3 + \eta_2) \right] \varepsilon_{11}(\dot{U}) + \left[ \frac{\sqrt{3}}{16} (\eta_3 - \eta_2) \right] \varepsilon_{22}(\dot{U})$$

$$+ \left[ \frac{\sqrt{3}}{8} (\eta_3 + \eta_2) \right] \varepsilon_{12}(\dot{U}) .$$

As in Section 3, we obtain an elasto-plastic material with a constitutive equation of the same type as for the 1-d case, whose yield surface is a polygon in the stress space. When only elastic deformation takes place, we have the following proposition.

PROPOSITION 6.2 : *Isotropic elastic materials with Poisson ratio* $\nu \neq 1/3$ *can be modelled following this process.*

*Proof :* From the formulae above it is clear that isotropy implies the equality of $E_2$ and $E_3$. The matrix of elasticity coefficients reduces then to

$$\begin{bmatrix} E_1 + \frac{E_2}{8} + E & \frac{3 \cdot E_2}{8} + E & 0 \\ \frac{3 \cdot E_2}{8} + E & \frac{9 \cdot E_2}{8} + E & 0 \\ 0 & 0 & \frac{3 \cdot E_2}{8} \end{bmatrix},$$

with $E = \frac{\sqrt{3}\, E_4}{4}$. We must also take $E_1$ equal to $E_2$, and we recognize the usual form of the elasticity matrix with

$$\lambda \cdot \frac{3 \cdot E_1}{8} + E \quad \textit{and} \quad \mu = \frac{3 \cdot E_1}{8},$$

and Poisson's ratio

$$\nu = \frac{\lambda}{\lambda + 2\mu} = \left( 3 - \frac{16\, E}{3\, E_1 + 8\, E} \right)^{-1}.$$

□

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