An asymptotic formula for the voltage potential in a perturbed ε–periodic composite medium containing misplaced inclusions of size ε

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Abstract

We consider composite media made of homogeneous inclusions with $\mathcal{C}^{1,\alpha}$ boundaries. Our goal is to compare the potential u_ε in a perfectly periodic composite to the potential $u_{\varepsilon,d}$ of a perturbed periodic medium, where the periodicity defects consist of misplaced inclusions. We give an asymptotic expansion of the difference $u_{\varepsilon,d} - u_\varepsilon$ away from the defects and show that to first order, a misplaced inclusion manifests itself via a polarization tensor, which is characterized.

1 Introduction

In this work , we consider a composite medium made of an array of inclusions embedded in a homogeneous background material. We assume that the inclusions are centered on a ε–periodic lattice, except for a small number of them that might have been misplaced: the centers of these 'defects' are at a distance of order ε from the lattice points. Our goal is to compare, sufficiently far from the defects, the potential $u_{\varepsilon,d}$ of the perturbed medium with the potential u_ε of a perfectly periodic medium.

When the reference or background medium is homogeneous (or sufficiently smooth) D. Fengya, S. Moskow and M. Vogelius [10] (see also [3] and the references therein) studied the perturbations of the potential caused by the presence of small inhomogeneities, and derived an asymptotic expansion for the difference between the perturbed and background potentials. The first correction term in their asymptotic expansion is of the order of the volume ε^n of the inhomogeneities and has the following structure

$$\varepsilon^n \sum_{j=1}^{m} \nabla_x u(z_j) \cdot M_j \nabla_x G(z_j, z), \tag{1.1}$$

where u is the background potential and M_j is a polarization tensor which characterizes how the presence of the j-th inhomogeneity, centered at z_j, is felt in the far field.

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The above expression also involves the gradient of the background Green's function $G(x, z)$, which makes the expansion particularly interesting for numerical detection of inhomogeneities: linear sampling or MUSIC algorithms detect the singularities of the Green functions, and have proven quite efficient in both impedance imaging of inhomogeneities of low volume fraction [7] and inverse scattering by small inclusions [2].

In this paper we derive a similar asymptotic formula when the background medium is periodic. Our main result, Theorem 4.1, shows that inhomogeneities or defects of size comparable to the period affect the perturbed potential in a manner similar to the case of a homogeneous background. Indeed, the first term in our asymptotic formula (4.4) has the same structure as above. It involves the gradient of the *homogenized* potential at the center of the inclusions, the gradient of the *homogenized* Green's function, and a polarization tensor that combines the influence of the defect at infinity and the interaction of the defect with the periodic structure. Thus, numerical detection of such periodicity defects should be possible using MUSIC algorithms, provided that one has accurate knowledge of the background potential (which might be expensive in practice).

A possible application of our analysis concerns photonic cristals, periodic composite arrays of dielectric materials. In these structures, propagation of waves may be prohibited in certain intervals of frequencies, as a result of destructive interferences between the waves and the structure of the composite [13]. For a mathematical perspective, see the enthusiastic review of P. Kuchment [14].
Photonic crystals are an example of structures where periodicity or near–periodicity seems to play an important role. As the current manufacturing processes may not guarantee perfect periodicity, it is interesting to study the influence of periodicity defects in these structures, in the view of developing methods for non–destructive control.

Our analysis relies on fine regularity results on the potential gradients [16, 15], which require that the inclusions be somewhat smooth: their boundaries have regularity $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha \leq 1$. Under this hypothesis, $||\nabla u_{\varepsilon,d}||_\infty$ can be shown to be bounded independently of ε and independently of the distance between inclusions, misplaced or not.

The paper is organized as follows. In Section 2, we recall classical results about Green functions for uniformly elliptic operators in divergence form with merely bounded and measurable coefficients. We are particularly interested in their behavior at infinity, and throughout the paper we work in dimension 3 (although some of our results are valid and given in any dimension).
Section 3 only concerns the background potential u_ε. In 3.1, we recall a few classical results of periodic homogenization, while section 3.2, contains several estimates on the potential u_ε and on its gradient: We recall such interior estimates, that were derived by M. Avellaneda and F.H. Lin [4], when the coefficients of the medium have regularity $\mathcal{C}^{0,\mu}$. We then give interior estimates on the gradient ∇u_ε, in the case of a composite

medium made of a homogeneous background conductor containing homogeneous inclusions with $\mathcal{C}^{1,\alpha}$ boundaries (in which case the conductivity is only globally L^∞). These estimates are similar to those given in [15] (and so is their proof) but here we allow a nonzero source term.
We chose to study the case of Neumann boundary conditions for the periodic and perturbed media. The analysis also led us to compare the potentials $u_{\varepsilon,d}$ and u_ε to the potential u_0 of the homogenized medium. In particular, we give a L^2 error estimate on $u_\varepsilon - u_0$. This kind of estimate is well–known in the case of Dirichlet boundary conditions. In the case of Neumann boundary conditions, our result generalizes to the dimension 3, a 2D–estimate obtained by S. Moskow and M. Vogelius [20]. The proofs of all the estimates in this section are given in the Appendix.
Section 4, contains the main result. We derive there the asymptotic expansion of the potential in the domain with defects. We give the expression of the polarization tensor associated to a periodicity defect and compare it to the formula of [10] that describes the effect at infinity of an inhomogeneity embedded in a smooth matrix.
Throughout the paper, C denotes a generic positive constant, independent of ε.

2 Properties of the Green function

In this section, we present some known results and properties of the Green function for the elliptic operator

$$Lu = -\mathrm{div}(a(x)\nabla u), \tag{2.1}$$

when the conductivity $a(x)$ is merely a bounded measurable function in $\mathbf{R}^n$. The detailed proofs of the following results can be found in [18, 21, 22] in the symmetric case and are extended to the case of non–symmetric coefficients in [12].
Let Ω be a smooth bounded domain in $\mathbf{R}^n$. We consider a medium with conductivity $a \in L^\infty(\Omega)$ which is uniformly elliptic

$$0 \; < \; \lambda \; \le \; a(x) \; \le \; \Lambda \quad a.e. \; x \in \Omega.$$

Given a Radon measure μ defined on Ω, a function $u \in L^1(\Omega)$ is called a weak solution vanishing at the boundary $\partial\Omega$ of the equation $Lu = \mu$, if it satisfies

$$\int_\Omega u\, L\Phi \, dx \;\; = \;\; \int_\Omega \Phi \, d\mu,$$

for every $\Phi \in H^1_0(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ such that $L\Phi \in C^0(\overline{\Omega})$.
If $\mu = f$ with $f \in W^{-1,2}(\Omega)$, the Lax Milgram Lemma shows that there exists a unique solution $u \in H^1_0(\Omega)$ of

$$Lu \;\; = \;\; f. \tag{2.2}$$

One can thus define a continuous linear operator $\mathcal{G} \; : W^{-1,2}(\Omega) \to H^1_0(\Omega)$, called the Green operator, such that, for $f \in W^{-1,2}(\Omega)$, $u = \mathcal{G}(f)$ is the unique solution in $H^1_0(\Omega)$ of $Lu = f$.

A theorem of Stampacchia [22] extends the De Giorgi–Nash theorem on $\mathcal{C}^{0,\alpha}$ regularity of solutions to elliptic equations and shows that when Ω is sufficiently smooth and $f \in W^{-1,p}(\Omega)$ with $p > n$, the solution u to (2.2) lies in $\mathcal{C}^0(\overline{\Omega})$. Moreover, one has

$$\forall\, f \in \mathcal{C}^0(\overline{\Omega}), \qquad \max_{\overline{\Omega}} |\mathcal{G}(f)| \;\leq\; C\,\lambda\,|\Omega|^{1/n-1/p}\,\|f\|_{W^{-1,p}}, \tag{2.3}$$

where C only depends on p. Consequently, given a Radon measure μ, a function u is a weak solution vanishing on $\partial\Omega$ of the equation $Lu = \mu$ if and only if

$$\forall\, f \in \mathcal{C}^0(\overline{\Omega}), \qquad \int_\Omega u\, f\, dx \;=\; \int_\Omega \mathcal{G}(f)\, d\mu. \tag{2.4}$$

There is at most one solution to this problem. By (2.3), this solution satisfies

$$\forall\, f \in \mathcal{C}^0(\overline{\Omega}), \qquad \int_\Omega u\, f\, dx \;\leq\; C\,\lambda\,|\Omega|^{1/n-1/p} \int_\Omega |d\mu|\;\|f\|_{W^{-1,p}(\Omega)}.$$

Since, $\mathcal{C}^0(\overline{\Omega})$ is dense in $W^{-1,p}(\Omega)$, we see that $u \in W_0^{1,p'}(\Omega)$, $1/p + 1/p' = 1/n$, and

$$\|u\|_{W_0^{1,p'}(\Omega)} \;\leq\; C\,\lambda\,|\Omega|^{1/p'} \int_\Omega |d\mu|.$$

The transformation $\mu \to u$ is thus the adjoint operator $\mathcal{G}^*$ of $\mathcal{G}$: As $\mathcal{G}(W^{-1,p}(\Omega)) \subset \mathcal{C}^0(\Omega)$, the image by $\mathcal{G}^*$ of the space of Radon measures on Ω is contained in $W_0^{1,p'}(\Omega)$. This proves

Theorem 2.1 *[18]*
For every Radon measure μ, there exists a unique weak solution u to the equation $Lu = \mu$ vanishing on $\partial\Omega$, which lies in $W_0^{1,q}(\Omega)$ for every $q < n/(n-1)$.
Moreover, u belongs to $H_0^1(\Omega)$ if and only if $\mu \in W^{-1,2}(\Omega)$.

As a consequence, one can define a Green function for L in Ω

Definition 2.2 *The Green function $G(x, y)$, associated with the operator L on Ω, is defined as the weak solution vanishing on $\partial\Omega$ of the equation*

$$LG \;=\; \delta_y \qquad \text{in}\, \Omega,$$

where δ_y is the Dirac mass at y.

The Green function provides a representation formula (see theorem 6.1 in [18]): For every Radon measure μ, the integral

$$u(x) \;=\; \int G(x, y)\, d\mu(y) \tag{2.5}$$

is finite a.e. and is the weak solution vanishing on $\partial\Omega$ of the equation $Lu = \mu$.

The Green function has the following properties

Theorem 2.3 *[18, 12] For each $y \in \Omega$,*

$$
\begin{array}{lll}
G(.,y) \in L^*_{\frac{n}{n-2}}(\Omega) & \textit{and} & \|G\|_{L^*_{\frac{n}{n-2}}(\Omega)} \le C(n)\lambda^{-1}, \\
\nabla G(.,y) \in L^*_{\frac{n}{n-1}}(\Omega) & \textit{and} & \|\nabla G\|_{L^*_{\frac{n}{n-2}}(\Omega)} \le C(n,\lambda,\Lambda), \\
G(.,y) \in W^{1,s}_0(\Omega) & \textit{for} & s \in [1, \frac{n}{n-1}[.
\end{array}
\tag{2.6}
$$

Further, let G and $\overline{G}$ be the Green functions of two uniformly elliptic operators L and $\overline{L}$, with ellipticity constants λ, Λ and $\overline{\lambda}, \overline{\Lambda}$,respectively. Then, for any compact subset $K \subset\subset \Omega$, there exists positive constants c and C, which only depend on K, Ω, n and on the ellipticity constants, such that

$$
\forall\ (x,y) \in K \times K, \qquad c\,\overline{G}(x,y) \ \le\ G(x,y) \ \le\ C\,\overline{G}(x,y). \tag{2.7}
$$

The Lorentz spaces $L^*_p(\Omega)$ involved in these estimates are defined by

$$
L^*_p(\Omega) \ := \ \{f : \Omega \to \mathbf{R} \cup \{\infty\}, f \text{ measurable and } \|f\|_{L^*_p(\Omega)} < \infty\},
$$

where

$$
\|f\|_{L^*_p(\Omega)} \ := \ \sup_{t>0} t \left|\{x \in \Omega : |f(x)| > t\}\right|^{1/p},
$$

and are related to the classical L^p spaces via the estimates

$$
\left(\frac{p}{\beta}\right)^{\frac{1}{p-\beta}} |\Omega|^{\frac{-\beta}{p(p-\beta)}} \|f\|_{L^{p-\beta}(\Omega)} \ \le\ \|f\|_{L^*_p(\Omega)} \ \le\ \|f\|_{L^p(\Omega)},
$$

for $0 < \beta \le p-1$.
When $n \ge 3$, as the radius of Ω goes to infinity, the Green function converge to a function $G(.,y)$, Hölder continuous in $\mathbf{R}^n \setminus \{y\}$. Moreover, $G(.,y) \in W^{1,q}_{loc}(\mathbf{R}^n) \cap W^{1,2}_{loc}(\mathbf{R}^n \backslash y)$, $q < n/(n-1)$, and the representation formula (2.5) is valid. In particular, given $f \in W^{-1,2}(\mathbf{R}^n)$ the solution $u \in W^{1,2}(\mathbf{R}^n)$ of $Lu = f$ can be represented by

$$
u(x) \ = \ \int_{\mathbf{R}^n} G(x,y) f(y) dy.
$$

The estimates (2.7) hold uniformly in $\mathbf{R}^n$ with constants that only depend on the ellipticity constants and on n. Comparing the Green functions of L and of the Laplace operator in $\mathbf{R}^n$, we see from (2.7) that

$$
\forall\ (x,y) \in \mathbf{R}^n \times \mathbf{R}^n, \quad |G(x,y)| \ \le \ C|x-y|^{2-n} \tag{2.8}
$$

where C only depends on λ, Λ and n.
In the rest of this paper, we will be concerned with families of operators of the form $L_\varepsilon = \mathrm{div}(a(x/\varepsilon)\nabla)$ defined in a smooth domain $\Omega \subset \mathbf{R}^3$, where a is a $[0,1]^3$-periodic piecewise constant function.

3 Asymptotic behavior of the background potential and of the associated Green function in periodic composite materials

Let Ω be a smooth bounded domain in $\mathbf{R}^3$ that contains a periodic composite medium composed of cells of size ε. These cells are deduced from the unit cell $Y = (0,1)^3$ by translation and rescaling, and are of the form $\varepsilon p + \varepsilon Y$, $p \in \mathbf{Z}^3$. The unit cell Y contains an inclusion $D_0 \subset\subset Y$ with boundary of class $\mathcal{C}^{1,\alpha}, 0 < \alpha \leq 1$. We assume that

$$dist(D_0, \partial Y) \geq d_0 > 0. \tag{3.1}$$

Let $0 < \lambda \leq \Lambda$ and $0 < \mu < 1$. $\mathcal{A}(\lambda, \Lambda, \mu, \alpha)$ denotes the class of Y-periodic functions a such that a is $\mathcal{C}^{0,\mu}$ in $\overline{D_0}$ and in $\overline{Y \setminus D_0}$ and such that $0 < \lambda \leq a(x) \leq \Lambda$ in Y. We also denote $\mathcal{L}(\lambda, \Lambda, \mu, \alpha)$ the class of elliptic operators with coefficients in $\mathcal{A}(\lambda, \Lambda, \mu, \alpha)$ of the form

$$L_\varepsilon \;=\; -\mathrm{div}(a(\frac{x}{\varepsilon})\nabla\cdot)) \;=\; -\mathrm{div}(a_\varepsilon(x)\nabla\cdot)), \quad 0 < \varepsilon < 1, \tag{3.2}$$

where $a_\varepsilon(x) = a(x/\varepsilon)$. We call these media ‘composites with sufficiently smooth inclusions’.

3.1 Homogenization

As ε tends to zero, we consider the sequence of elliptic problems

$$\left\{ \begin{array}{lcll} L_\varepsilon u_\varepsilon & = & f & \text{in } \Omega \\ & & & \\ a_\varepsilon \dfrac{\partial u_\varepsilon}{\partial \nu} & = & g & \text{on } \partial\Omega \end{array} \right. \tag{3.3}$$

with the normalization $\displaystyle\int_{\partial\Omega} g\, d\sigma_x = \int_\Omega f\, dx$ and $\displaystyle\int_\Omega u_\varepsilon\, dx = 0$.
The effective behavior of the composite and the asymptotic behavior of u_ε are described in terms of solutions $\chi \in (W^{1,2}_\#(Y))^3$ and $\Phi \in (W^{1,2}_\#(Y))^{3^2}$ to cell problems, defined by

$$\left\{ \begin{array}{l} -\mathrm{div}(a(y)\nabla(\chi(y) + y)) = 0 \;\text{ in } \mathbf{R}^3 \\ \displaystyle\int_Y \chi(y)\, dy = 0, \end{array} \right. \tag{3.4}$$

$$\left\{ \begin{array}{l} -\mathrm{div}(a(y)\nabla_y \Phi(y)) = B(y) - \displaystyle\int_Y B(y)\, dy \text{ in } Y; \\ \displaystyle\int_Y \Phi(y)\, dy = 0, \end{array} \right. \tag{3.5}$$

with

$$B(y) \;=\; a(y)I + a(y)\nabla_y \chi(y) + \mathrm{div}_y(a(y) I \otimes \chi(y)) \tag{3.6}$$

($W^{1,2}_{\#}$ denotes the subspace of periodic functions of $W^{1,2}(Y)$). The function u_ε can formally be sought with the ansatz [6]

$$u_\varepsilon(x) = u_0(x,\frac{x}{\varepsilon}) + \varepsilon u_1(x,\frac{x}{\varepsilon}) + \varepsilon^2 u_2(x,\frac{x}{\varepsilon}) + \varepsilon^3 u_3(x,\frac{x}{\varepsilon}) + \dots \tag{3.7}$$

where, each function $u_i(x,y)$ is Y-periodic with respect to the fast variable $y = \dfrac{x}{\varepsilon}$.

The function $u_0(x,y) = u_0(x)$ is independent from y and is the unique solution in $W^{1,2}(\Omega)$ to the homogenized equation

$$\begin{cases} L_0 u_0 = -\mathrm{div}(A\nabla u_0) = f & \text{in } \Omega \\ A\nabla u_0 \cdot \nu = g & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

with the normalization $\displaystyle\int_\Omega u_0\, d\sigma_x = 0$. The effective properties of the medium are expressed by the constant, symmetric, positive definite, homogenized matrix A defined by

$$A = \int_Y a(y)\,(I + \nabla\chi(y))\, dy.$$

The functions u_1 and u_2 can be written in terms of derivatives of u_0, up to arbitrary functions $\tilde{u}_1, \tilde{u}_2$ of the variable x only

$$u_1(x,\frac{x}{\varepsilon}) = \chi(\frac{x}{\varepsilon}) \cdot \nabla u_0(x) + \tilde{u}_1(x) \tag{3.9}$$

$$u_2(x,\frac{x}{\varepsilon}) = \Phi(\frac{x}{\varepsilon}) : \nabla u_0(x) + \chi(\frac{x}{\varepsilon}) \cdot \nabla\tilde{u}_1(x) + \tilde{u}_2(x). \tag{3.10}$$

If we approximate u_ε to first order by (3.7), we may choose the function $\tilde{u}_1$ to be 0. If we seek an approximation up to second order away from the boundary (neglecting boundary layers) $\tilde{u}_2$ may be chosen to be 0, but $\tilde{u}_1$ must satisfy

$$-\mathrm{div}(A\nabla\tilde{u}_1) = C_0 \nabla^3 u_0, \tag{3.11}$$

where

$$C_0 = \int_Y (a(y)\nabla\Phi(y) + a(y) I \otimes \chi(y))\, dy. \tag{3.12}$$

3.2 Error estimates

In this section, we give $W^{1,\infty}$–interior estimates for solutions u_ε to

$$L_\varepsilon u_\varepsilon = f \text{ in } \Omega,$$

i.e., we are concerned only with perfectly periodic media. We are particularly interested in pointwise estimates on the gradients of u_ε, which will be used in the proofs of Section 4.

When the conductivity a has global Hölder regularity on Y, $\|a\|_{C^{0,\mu}(Y)} \le M$, M. Avellaneda and F.H. Lin proved that the potentials u_ε are uniformly Lipschitz

Theorem 3.1 *(Theorem 2 in [4])*
Let u_ε satisfy

$$\begin{aligned} L_\varepsilon u_\varepsilon &= f \quad in\, \Omega \subset \mathbf{R}^n \\ u_\varepsilon &= g \quad on\, \partial\Omega, \end{aligned}$$

where, $f \in L^{n+\delta}$ for some $\delta > 0$ and $g \in \mathcal{C}^{1,\nu}(\partial\Omega)$, $0 < \nu \leq 1$. There exists a constant C that only depends on $\lambda, \Lambda, \mu, M, \Omega, \nu$ and δ, such that

$$\|u_\varepsilon\|_{\mathcal{C}^{0,1}(\Omega)} \leq C(\|g\|_{\mathcal{C}^{1,\nu}(\partial\Omega)} + \|f\|_{L^{n+\delta}(\Omega)}).$$

The regularity hypothesis on a can be relaxed to cover the case of composite media that contain inclusions with sufficiently smooth boundaries. For such media, one can show that the gradient of the potential is uniformly bounded, independently of the inter–inclusion distance. Results of this nature were first obtained by YanYan Li and M. Vogelius[16], then generalized to strongly elliptic systems by YanYan Li and L.Nirenberg [15]. We state here the version of [15] in the scalar case.
Let D be a bounded domain in $\mathbf{R}^3$ which contains L disjoint subdomains $D_1, \cdots, D_L$, of class $\mathcal{C}^{1,\alpha}$, $0 < \alpha \leq 1$, with $D = (\cup_{l=1}^{L} \overline{D}_l) \setminus \partial D$. We assume that any point $x \in D$ belongs to at most two of the boundaries of the D_l's. For $\eta > 0$, we set

$$D_\eta = \{x \in D; \mathrm{dist}(x, \partial D) > \eta\}.$$

Theorem 3.2 *(Theorem 0.1 in [15])*
Let $0 < \mu < 1$ and assume that the conductivity a is uniformly elliptic in D and belongs to $\mathcal{C}^\mu(\overline{D_l}), 1 \leq l \leq L$. Let $h \in \mathcal{C}^{0,\mu}(\overline{D}_l), 1 \leq l \leq L$, $f \in L^\infty(D)$ and let u be a solution in D to the equation

$$-div(a(x)\nabla u) \quad = \quad f + div(h).$$

For any $\eta > 0$, there exists a constant C such that for any $0 < \alpha' \leq min\{\mu, \frac{\alpha}{2(\alpha+1)}\}$, u satisfies

$$\sum_{l=1}^{L} \|u\|_{\mathcal{C}^{1,\alpha'}(\overline{D}_l \cap D_\eta)} \leq C\left(\|u\|_{L^2(D)} + \|f\|_{L^\infty(D)} + \sum_{l=1}^{L} \|h\|_{\mathcal{C}^{\alpha'}(\overline{D}_l)}\right). \tag{3.13}$$

Here C only depends on λ, Λ, μ, L, α, η, $\|a\|_{\mathcal{C}^{\alpha'}(\overline{D}_l)}$ and on the $\mathcal{C}^{1,\alpha'}$ norm of the D_l's. In particular,

$$\|\nabla u\|_{L^\infty(D_\eta)} \leq C\left(\|u\|_{L^2(D)} + \|f\|_{L^\infty(D)} + \sum_{l=1}^{L} \|h\|_{\mathcal{C}^{\alpha'}(\overline{D}_l)}\right). \tag{3.14}$$

In the sequel, for each r, $0 < r < 1$, and $x \in \mathbf{R}^3$, we set

$$B(x,r) \; = \; \{y \in \mathbf{R}^3 / |x-y| < r\} \quad \text{and} \quad B_r \; = \; B(0,r).$$

The constant C in the above theorem may however grow with the number of inclusions. However, in the case of *periodic* media, uniform pointwise estimates on the gradients do hold as in theorem 3.1. This is established in the following result, due to YanYan Li and L. Nirenberg (see also the remark in Section 5.3 of [4]). Its proof relies on Theorem 3.2 and on the 'three steps compactness method' of [4].

Theorem 3.3 *(Theorem 0.2 in [15])*
Assume that $a \in \mathcal{A}(\lambda, \Lambda, \mu, \alpha)$ and $L_\varepsilon \in \mathcal{L}(\lambda, \Lambda, \mu, \alpha)$. Let u_ε be a solution to

$$L_\varepsilon u_\varepsilon \quad = \quad 0 \quad \textit{in}\, B_1.$$

Then

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{1/2})} \quad \leq \quad C\|u_\varepsilon\|_{L^2(B_1)},$$

where C is independent of ε (thus, independent of the number of inclusions and of the distance between their boundaries).

We will need a slightly different version of Theorem 3.3, for solutions of elliptic equations in divergence form, with a source term of a particular form :

Theorem 3.4 *(Interior gradient estimates)*
Assume that $\mu \leq \frac{\alpha}{2(\alpha+1)}$. Let $a \in \mathcal{A}(\lambda, \Lambda, \mu, \alpha)$, $f \in L^\infty(B_1)$, $h \in \mathcal{C}^{0,\mu}(B_1)^3$ and $L_\varepsilon \in \mathcal{L}(\lambda, \Lambda, \mu, \alpha)$. Let b be a Y–periodic function such that b has regularity $\mathcal{C}^{0,\mu}$ in $\overline{D_0}$ and in $\overline{Y \setminus D_0}$. Assume that u_ε is a solution to

$$L_\varepsilon u_\varepsilon \quad = \quad f + \varepsilon\, div(b_\varepsilon h) \quad \textit{in}\, B_1.$$

Then,

$$\begin{aligned} &\|u_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{1/2})} + \|\nabla u_\varepsilon\|_{L^\infty(B_{1/2})} \\ &\quad \leq \quad C\left(\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}\right), \end{aligned} \tag{3.15}$$

where C is independent of ε (thus, of the number of inclusions and of the distances between their boundaries).

On the basis of theorem 3.4, one can proceed as in [5], and generalize to composite media with sufficiently smooth inclusions, error estimates between the potential u_ε and the homogenized potential u_0, and between the ε–periodic Green function and the Green function for the homogenized medium.
However, as we intend to apply such results in Neumann problems, we first state the following L^2 error estimate :

Lemma 3.5 *Assume that Ω is a smooth bounded domain in $\mathbf{R}^3$ and that $g \in \mathcal{C}^\infty(\partial\Omega)$ such that $\int_{\partial\Omega} g = 0$. Let u_ε denote the solution to*

$$\left\{ \begin{array}{rcll} div(a_\varepsilon \nabla u_\varepsilon) & = & 0 & in\, \Omega \\ a_\varepsilon \nabla u_\varepsilon \cdot \nu & = & g & on\, \partial\Omega, \end{array} \right. \tag{3.16}$$

normalized with the condition that $\int_\Omega u_\varepsilon = 0$. Let u_0 denote the solution to the corresponding homogenized problem

$$\left\{ \begin{array}{rcll} div(A \nabla u_0) & = & 0 & in\, \Omega \\ A \nabla u_0 \cdot \nu & = & g & on\, \partial\Omega, \end{array} \right. \tag{3.17}$$

also normalized by $\int_\Omega u_0 = 0$. Then, the following estimate holds

$$||u_\varepsilon - u_0||_{L^2(\Omega)} \leq C\varepsilon ||u_0||_{H^2(\Omega)}.$$

Estimates of this sort are well known for Dirichlet boundary conditions [6, 1, 19]. For Neumann boundary conditions, a similar error estimate was derived in 2D by S. Moskow and M. Vogelius [20] in the case of a convex polygon using harmonic conjugates. We show in section A.2 how this estimate generalizes to 3D.

We now state uniform error estimates between u_ε and u_0:

Theorem 3.6 *Let $\omega \subset\subset \Omega$. Assume that u_ε and u_0 solve $L_\varepsilon u_\varepsilon = 0$ and $L_0 u_0 = 0$ in Ω. Assume also that*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon^\sigma,$$

for some $0 < \sigma \leq 1$. Then there exists a constant C that only depends on $\lambda, \Lambda, \mu, \alpha, \Omega$ and ω such that,

$$\|u_\varepsilon - u_0\|_{L^\infty(\omega)} \leq C\,\varepsilon^\sigma \tag{3.18}$$

$$\|\nabla u_\varepsilon - (I + \nabla_y \chi(./\varepsilon))\nabla u_0\|_{L^\infty(\omega)} \leq C\,\varepsilon^\sigma. \tag{3.19}$$

We remark that by Lemma 3.5, this theorem applies to solutions of (3.16, 3.17).

Let G_ε and G_0 denote the respective Green functions, vanishing on $\partial\Omega$, of the operators L_ε and L_0. From theorem 3.6, we derive an estimate on the convergence rate of G_ε to G_0. This result is applied in section 4 when the source is far from the defect. For this reason, we consider below $G_\varepsilon(x, y)$ when $x \in \omega \subset\subset \Omega$ and $y \in \Omega \backslash \omega$ with $dist(y, \omega) > 0$. In [5], when the coefficients have Hölder regularity, similar estimates are established, which are valid on the whole of Ω (away from the source). Their derivation requires uniform boundary estimates on L_ε– harmonic functions. It would be interesting to study whether such estimates also hold in our context.

Theorem 3.7 *Assume that $\omega \subset\subset \Omega$ is a smooth domain. Let G_ε and G_0 be the Green functions, vanishing on $\partial\Omega$, for the operators L_ε and L_0 (see section 2). There exists a positive constant C, independent of ε, such that for $y \in \Omega \setminus \omega$ with $dist(y, \omega) > 0$*

$$\begin{aligned} \|G_\varepsilon(.,y) - G_0(.,y)\|_{L^\infty(\omega)} &\leq C\varepsilon^{1/4}, && (3.20) \\ \|\nabla_x G_\varepsilon(.,y) - (I + \nabla_y \chi(./\varepsilon))\nabla_x G_0(.,y)\|_{L^\infty(\omega)} &\leq C\varepsilon^{1/4}. && (3.21) \end{aligned}$$

4 Main result: asymptotics of the perturbed potential

Let Y denotes the unit cell $(0,1)^3$ in $\mathbf{R}^3$. We assume that Y contains an inclusion D_0, the boundary of which has regularity $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha < 1$. We also assume that

$$dist(D_0, \partial Y) \geq d_0 > 0.$$

Let a be a measurable Y-periodic function equal to a constant k in D_0, $0 < k < \infty$, $k \neq 1$, and equal to 1 in $Y \setminus \overline{D}_0$.

As in section 3, we consider a bounded domain $\Omega \subset \mathbf{R}^3$ formed by the union of cells, translated and rescaled by ε from the elementary cell Y. The conductivity in Ω is denoted by $a_\varepsilon(x) = a(x/\varepsilon)$. We consider the elliptic operator $L_\varepsilon = -\mathrm{div}(a_\varepsilon(x)\nabla \cdot)$. We call background electrostatic potential in Ω, the solution u_ε to

$$\left\{ \begin{array}{l} L_\varepsilon u_\varepsilon = 0 \ \text{ in } \Omega \\ a_\varepsilon \dfrac{\partial u_\varepsilon}{\partial \nu}_{/\partial\Omega} = g \\ \displaystyle\int_{\partial\Omega} g \, d\sigma_x = \int_\Omega u_\varepsilon \, dx = 0 \end{array} \right. \tag{4.1}$$

We study the influence of a particular perturbation of such a medium which consists in misplacing one inclusion. More precisely, let $p \in \mathbf{Z}^3$ such that $Y_\varepsilon^p := \varepsilon(p + Y) \subset \Omega$, and so that $dist(Y_\varepsilon^p, \partial\Omega) >> \varepsilon$. If the medium were completely periodic, the inclusion contained in the cell Y_ε^p would occupy the subset $\omega_{\varepsilon,1} := \varepsilon(p + D_0)$. Instead, the inclusion lies in a subset $\omega_{\varepsilon,2} = \varepsilon(p + \delta + D_0)$, for some $0 < |\delta| < 1$. For simplicity, we asume that $\omega_{\varepsilon,2}$ does not intersect any of the remaining inclusions. Let ω_ε denote the symmetric difference of the sets $\omega_{\varepsilon,1}$ and $\omega_{\varepsilon,2}$.

The conductivity $a_{\varepsilon,d}$ of the perturbed medium is thus given by

$$\left\{ \begin{array}{lcll} a_{\varepsilon,d} &=& a_\varepsilon & \text{in } \Omega \setminus \overline{\omega}_\varepsilon \\ a_{\varepsilon,d} &=& 1 & \text{in } \omega_{\varepsilon,1} \setminus \omega_{\varepsilon,2} \\ a_{\varepsilon,d} &=& k & \text{in } \omega_{\varepsilon,2} \setminus \omega_{\varepsilon,1}. \end{array} \right.$$

The associated potential $u_{\varepsilon,d}$ solves

$$\left\{ \begin{array}{l} \mathrm{div}(a_{\varepsilon,d}(x)\nabla u_{\varepsilon,d}) = 0 \ \text{ in } \Omega \\ a_{\varepsilon,d} \dfrac{\partial u_{\varepsilon,d}}{\partial \nu}_{/\partial\Omega} = g \\ \displaystyle\int_{\partial\Omega} g \, d\sigma_x = \int_\Omega u_{\varepsilon,d} \, dx = 0. \end{array} \right. \tag{4.2}$$

Without loss of generality, we assume that $x_0 = 0$ belongs to the convex hull of ω_ε. Let

$$\tilde{\Omega}_\varepsilon \;=\; \{x/\varepsilon,\; x \in \Omega\} \qquad \tilde{\omega} \;=\; \{x/\varepsilon,\; x \in \omega_\varepsilon\},$$

(note that $\tilde{\omega}$ is independent of ε) and define the function a_d in $\mathbf{R}^3$ by $a_d(y) = a_{\varepsilon,d}(\varepsilon y)$). Throughout this Section, we denote by $a^+(x)$ and $a^-(x)$ the outward and inward limits of the discontinuous function a through an interface.
Let G_ε be the Green function associated to the operator L_ε, solution to

$$-\mathrm{div}(a_\varepsilon \nabla G_\varepsilon(x,z)) = \delta_z \quad \text{in } \Omega, \tag{4.3}$$

vanishing on $\partial\Omega$ (see Section 2)and G_0 the Green function of the homogenized operator L_0, defined by (3.8).
We now state the main result of this paper: an asymptotic expansion for $u_{\varepsilon,d} - u_\varepsilon$. This expansion has the same structure as that derived in the case of a homogeneous background medium [10], though it involves the homogenized potential u_0 and the homogenized Green's function G_0. As mentionned in the Introduction, the presence of the Green function (and its singularity) should make this expansion interesting for numerical detection purposes.
We note that our analysis easily extends to the case of several misplaced inclusions (or to the case of $O(\varepsilon)$ defects with different constant conductivities) provided that they are at distances larger than $O(\varepsilon)$ apart.

Theorem 4.1 *Assume that Ω and the Neumann data g are sufficiently regular so that the homogenized potential u_0 is smooth inside Ω. For any $z \in \Omega$ at a distance $d_0 > 0$ away from ω_ε, we have*

$$\begin{aligned} u_{\varepsilon,d}(z) - u_\varepsilon(z) + \int_{\partial\Omega} (u_{\varepsilon,d}(x) - u_\varepsilon(x)) \frac{\partial G_\varepsilon}{\partial \nu_x}\, d\sigma_x \\ = \;\; -\varepsilon^3 \nabla_x G_0(x_0, z) \cdot M \nabla_x u_0(x_0) + O(\varepsilon^{3+1/4}). \end{aligned} \tag{4.4}$$

The term $O(\varepsilon^{3+1/4})$ is uniformly bounded by $C\varepsilon^{3+1/4}$, where the constant C depends on d_0, k, α. The polarization tensor M is given by

$$M_{ij} = \int_{\partial\tilde{\omega}} (\frac{a^-}{a_d^-} - 1)(y_i + \chi^i(y)) \left(a^+(y) \frac{\partial \varphi^+_{j,d}}{\partial \nu_y} + a^-(y) \left(\nu_j + \frac{\partial \chi^j(y)}{\partial \nu_y} \right) \right) d\sigma_y \tag{4.5}$$

for $1 \le i, j \le 3$, where the cell function $\chi = (\chi^i)_{1 \le i \le 3}$ is defined by (3.4) and where the auxiliary functions $\varphi_{j,d}$ are defined by (4.14) below.

To prove the Theorem, we first establish three Lemmas. We introduce two auxiliary functions $v_{\varepsilon,d}$ and v_d, respective solutions to:

$$\left\{\begin{array}{l} \operatorname{div}(a(y)\nabla_y v_{\varepsilon,d}) \;=\; 0 \quad \text{in } \tilde\Omega_\varepsilon \setminus \overline{\tilde\omega} \qquad\quad \Delta v_{\varepsilon,d} \;=\; 0 \quad \text{in } \tilde\omega \\[2ex] v^+_{\varepsilon,d} - v^-_{\varepsilon,d} \;=\; 0 \quad \text{on } \partial\tilde\omega \\[2ex] a^+\dfrac{\partial v^+_{\varepsilon,d}}{\partial\nu} - a^-_d\dfrac{\partial v^-_{\varepsilon,d}}{\partial\nu} \;=\; -(a^- - a^-_d)(I + \nabla_y\chi^-)\nabla_x u_0(0)\cdot\nu_y \quad \text{on } \partial\tilde\omega \\[2ex] a_d\dfrac{\partial v_{\varepsilon,d}}{\partial\nu_y} \;=\; 0 \quad \text{on } \partial\tilde\Omega_\varepsilon \qquad\quad \displaystyle\int_{\tilde\Omega_\varepsilon} v_{\varepsilon,d} \;=\; 0, \end{array}\right. \tag{4.6}$$

and

$$\left\{\begin{array}{lcl} \operatorname{div}_y(a_d(y)\nabla_y v_d) & = & \mu \quad \text{in } \mathbf{R}^3 \\[2ex] v_d(y)\to 0 & & \text{when} \quad |y|\to\infty, \end{array}\right. \tag{4.7}$$

with $\mu = \operatorname{div}_y\left((a_d - a)(I + \nabla_y\chi)\nabla_x u_0(0)\right)$.

Lemma 4.2 *The function v_d, solution of (4.7), decays at infinity as*

$$\left\{\begin{array}{lcl} v_d(y) & = & O(|y|^{-1}) \\ \nabla_y v_d(y) & = & O(|y|^{-2}). \end{array}\right. \tag{4.8}$$

Proof: Since the support of μ is included in $\tilde\omega$, the function v_d can be represented in terms of the Green function G associated to $L = -\operatorname{div}_y(a_d(y)\nabla_y\cdot)$ in $\mathbf{R}^3$

$$v_d(y) = \int_{\mathbf{R}^3} G(y,z)\,d\mu(z) = \int_{\tilde\omega} G(y,z)\,d\mu(z). \tag{4.9}$$

(see theorem 6.1 in [18] where it is shown that the integral on the above right hand side exists a.e. as a consequence of Fubini's Theorem)
It follows from (2.8) that

$$v_d(y) = O(|y|^{-1}), \text{ as } |y|\to\infty.$$

Fix $z\in\mathbf{R}^3$ and $R > 4\,\mathrm{diam}(\tilde\omega)$. As a function of y, $G(y,z)$ satisfies

$$\operatorname{div}_y(a_d(y)\nabla_y G(y,z)) \;=\; 0 \quad \text{in } B(z,4R)\setminus B(z,R/4).$$

Thus, the rescaled function $g(x,z/R) = G(Rx,z)$ solves $\operatorname{div}_x(a_d(Rx)\nabla_x g(x,z/R)) = 0$ in the set $B(z/R,4)\setminus B(z/R,1/4)$. Applying theorem 3.4 and (2.8), we obtain

$$\begin{array}{rcl} \|\nabla_y G(y,z)\|_{L^\infty(B(z,2R)\setminus B(z,R/2))} & \leq & R^{-1}\|\nabla_x g(y/R,z/R)\|_{L^\infty(B(z/R,2)\setminus B(z/R,1/2))} \\ & \leq & CR^{-1}||g(.,z/R)||_{L^\infty(B(z/R,4)\setminus B(z/R,1/4))} \\ & \leq & CR^{-1}||G(.,z)||_{L^\infty(B(z,4R)\setminus B(z,R/4))} \\ & \leq & CR^{-1}\sup\{\dfrac{1}{|y-z|} \;\;/\;\; y\in B(z,4R)\setminus B(z,R/4)\}. \end{array}$$

It follows that

$$|\nabla_y G(y,z)| \quad = \quad O(|y-z|^{-2}) \quad \text{as } |y-z| \to \infty. \tag{4.10}$$

Noting that $G(.,z)$ is $\mathcal{C}^{1,\mu}$ away from $y=z$ as a consequence of theorem 3.2, we can differentiate (4.9) with respect to y, and we conclude from (4.10) that

$$|\nabla_y v_d(y)| = O(|y|^2),$$

which proves the Lemma. ■

Lemma 4.3 *There exists a constant $C > 0$, independent of ε, such that*

$$\|\nabla_y(u_{\varepsilon,d}(\varepsilon\cdot) - u_\varepsilon(\varepsilon\cdot) - \varepsilon v_{\varepsilon,d})\|_{L^2(\tilde{\Omega}_\varepsilon)} \quad \leq \quad C\varepsilon^2. \tag{4.11}$$

Proof: Let $z_{\varepsilon,d}(y) = u_{\varepsilon,d}(\varepsilon y) - u_\varepsilon(\varepsilon y) - \varepsilon v_{\varepsilon,d}(y)$. According to the equations (4.2, 4.1) and (4.6), $z_{\varepsilon,d}$ satisfies

$$\left\{ \begin{array}{rcl} \operatorname{div}(a(y)\nabla_y z_{\varepsilon,d}) & = & 0 \quad \text{in } \tilde{\Omega}_\varepsilon \setminus \overline{\tilde{\omega}} \quad \text{and in } \tilde{\omega} \\ z^+_{\varepsilon,d} - z^-_{\varepsilon,d} & = & 0 \quad \text{on } \partial\tilde{\omega} \\ a^+_d \dfrac{\partial z^+_{\varepsilon,d}}{\partial \nu} - a^-_d \dfrac{\partial z^-_{\varepsilon,d}}{\partial \nu} & = & -\varepsilon(a^- - a^-_d)\left(\nabla_x u^-_\varepsilon(\varepsilon y) - (I + \nabla_y \chi^-(y))\nabla_x u_0(0)\right) \cdot \nu_y \\ a_d \dfrac{\partial z_{\varepsilon,d}}{\partial \nu_y} & = & 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon \end{array} \right.$$

Thus, integrating by parts yields

$$\begin{aligned} & \int_{\tilde{\Omega}_\varepsilon} a_d(y)\nabla_y z_{\varepsilon,d}\nabla_y z_{\varepsilon,d}\,dy \\ & = \int_{\tilde{\Omega}_\varepsilon\setminus\tilde{\omega}} a_d(y)\nabla_y z_{\varepsilon,d}\nabla_y z_{\varepsilon,d}\,dy + \int_{\tilde{\omega}} a_d(y)\nabla_y z_{\varepsilon,d}\nabla_y z_{\varepsilon,d}\,dy \\ & = -\int_{\partial\tilde{\omega}} a^+_d \frac{\partial z^+_{\varepsilon,d}}{\partial \nu_y} z_{\varepsilon,d} + \int_{\partial\tilde{\omega}} a^-_d \frac{\partial z^-_{\varepsilon,d}}{\partial \nu_y} z_{\varepsilon,d} \\ & = -\varepsilon \int_{\partial\tilde{\omega}} (a^-(y) - a^-_d(y))\left(\nabla_x u^-_\varepsilon(\varepsilon y) - (I + \nabla_y\chi^-(y))\nabla_x u_0(0)\right) \cdot \nu_y z_{\varepsilon,d} \\ & = \varepsilon \int_{\tilde{\omega}} (a(y) - a_d(y))\left(\nabla_x u_\varepsilon(\varepsilon y) - (I + \nabla_y\chi(y))\nabla_x u_0(0)\right) \cdot \nabla_y z_{\varepsilon,d} \\ & \leq c\varepsilon \,\|\nabla_x u_\varepsilon(\varepsilon.) - (I + \nabla_y\chi)\nabla_x u_0(0)\|_{L^2(\tilde{\omega})} \,\|\nabla_y z_{\varepsilon,d}\|_{L^2(\tilde{\omega})} \end{aligned}$$

Lemma 3.5, theorem 3.6 and the smoothness of the homogenized potential u_0 show that

$$\begin{aligned} & \|\nabla_x u_\varepsilon - (I + \nabla_y\chi(./\varepsilon))\,\nabla_x u_0(0)\|_{L^\infty(\omega_\varepsilon)} \\ & \leq \|\nabla_x u_\varepsilon - (I + \nabla_y\chi(./\varepsilon))\,\nabla_x u_0(.)\|_{L^\infty(\omega_\varepsilon)} \\ & \quad + \|\,(I + \nabla_y\chi(./\varepsilon))\,(\nabla_x u_0(.) - \nabla_x u_0(0))\,\|_{L^\infty(\omega_\varepsilon)} \\ & \leq C\varepsilon. \end{aligned}$$

Since a_ε is bounded, we conclude that

$$\|\nabla_y z_{\varepsilon,d}\|^2_{L^2(\tilde\Omega_\varepsilon)} \le c\varepsilon^2 \|\nabla_y z_{\varepsilon,d}\|_{L^2(\tilde\omega)} \le c\varepsilon^2 \|\nabla_y z_{\varepsilon,d}\|_{L^2(\tilde\Omega_\varepsilon)}.$$

■

Lemma 4.4 *: There exists a constant C, independent of ε, such that*

$$\|\nabla_y \left(u_{\varepsilon,d}(\varepsilon.) - u_\varepsilon(\varepsilon.) - \varepsilon v_d\right)\|_{L^2(\tilde\Omega_\varepsilon)} \quad \le \quad C\varepsilon^{3/2}. \tag{4.12}$$

Proof: Lemma 4.3 shows that it is sufficient to prove that

$$\|\nabla_y (v_{\varepsilon,d} - v_d)\|_{L^2(\tilde\Omega_\varepsilon)} \quad \le \quad c\varepsilon^{1/2}.$$

To this end, let $\phi_{\varepsilon,d} = v_{\varepsilon,d}(y) - v_d(y) - c_\varepsilon$, where c_ε is chosen so that $\int_{\partial\tilde\Omega_\varepsilon} \phi_{\varepsilon,d} = 0.$ In view of (4.6) and (4.7), $\phi_{\varepsilon,d}$ solves :

$$\begin{cases} \operatorname{div}(a_d(y)\nabla\phi_{\varepsilon,d}) &= 0 \quad \text{in } \tilde\Omega_\varepsilon \\ a_d(y)\dfrac{\partial\phi_{\varepsilon,d}}{\partial\nu_y} &= -a_d(y)\dfrac{\partial v_d}{\partial \nu_y} \quad \text{on } \partial\tilde\Omega_\varepsilon \\ \displaystyle\int_{\partial\tilde\Omega_\varepsilon} \phi_{\varepsilon,d} &= 0. \end{cases}$$

Integrating by parts and changing variables back to the fixed domain Ω, we see that

$$\begin{aligned} \int_{\tilde\Omega_\varepsilon} a_d(y)\nabla_y\phi_{\varepsilon,d}\nabla_y\phi_{\varepsilon,d}\,dy &= \int_{\partial\tilde\Omega_\varepsilon} -a_d\frac{\partial v_d}{\partial\nu_y}\,\phi_{\varepsilon,d}d\sigma_y \\ &= \varepsilon^{-2}\int_{\partial\Omega} -a_d\frac{\partial v_d}{\partial\nu_y}(x/\varepsilon)\,\phi_{\varepsilon,d}(x/\varepsilon)d\sigma_x \\ &\le c\varepsilon^{-2}\|\frac{\partial v_d}{\partial\nu_y}(./\varepsilon)\|_{L^\infty(\partial\Omega)}\,\|\phi_{\varepsilon,d}(./\varepsilon)\|_{L^2(\partial\Omega)}. \end{aligned}$$

The trace Theorem and the Poincaré–Wirtinger inequality imply that

$$\|\phi_{\varepsilon,d}(./\varepsilon)\|_{L^2(\partial\Omega)} \le C\|\phi_{\varepsilon,d}(./\varepsilon)\|_{W^{1,2}(\Omega)} \le C\varepsilon^{1/2}\|\nabla_y\phi_{\varepsilon,d}(x/\varepsilon)\|_{L^2(\tilde\Omega)}.$$

Since v_d decays at infinity (see (4.8)), we have

$$\|\frac{\partial v_d}{\partial\nu_y}(./\varepsilon)\|_{L^\infty(\partial\Omega)} \quad = \quad O(\varepsilon^2),$$

and therefore

$$\begin{aligned} \|\nabla_y\phi_{\varepsilon,d}(x/\varepsilon)\|^2_{L^2(\tilde\Omega)} &\le C\int_{\tilde\Omega} a_\varepsilon(\varepsilon y)\nabla_y\phi_{\varepsilon,d}\nabla_y\phi_{\varepsilon,d}\,dy \\ &\le C\varepsilon^{1/2}\|\nabla_y\phi_{\varepsilon,d}\|_{L^2(\tilde\Omega)} \end{aligned}$$

and the Lemma is proved. ■

Proof of Theorem 4.1:
Let $u_{\varepsilon,d}$ and u_ε be the electrostatic potentials solutions to (4.2) and (4.1) respectively. Let z be a point in Ω, at a distance $d > 0$ away from ω_ε. We apply the Green formula in Ω to get

$$\begin{aligned}
u_{\varepsilon,d}(z) &= -\int_\Omega u_{\varepsilon,d}\,\mathrm{div}(a_\varepsilon(x)\nabla G_\varepsilon(x,z))\,dx \\
&= \int_{\Omega\setminus\omega_\varepsilon} a_\varepsilon(x)\nabla_x u_{\varepsilon,d}\nabla_x G_\varepsilon\,dx + \int_{\omega_\varepsilon} a_\varepsilon(x)\nabla_x u_{\varepsilon,d}\nabla_x G_\varepsilon\,dx \\
&\quad - \int_{\partial\Omega} u_{\varepsilon,d}a_\varepsilon(x)\frac{\partial G_\varepsilon}{\partial \nu_x}\,d\sigma_x \\
&= \int_{\partial\Omega} gG_\varepsilon\,d\sigma_x - \int_{\partial\Omega} u_{\varepsilon,d}a_\varepsilon\frac{\partial G_\varepsilon}{\partial \nu_x}\,d\sigma_x \\
&\quad - \int_{\partial\omega_\varepsilon}\left(a_\varepsilon^+\frac{\partial u_{\varepsilon,d}^+}{\partial \nu_x} - a_\varepsilon^-\frac{\partial u_{\varepsilon,d}^-}{\partial \nu_x}\right)G_\varepsilon\,d\sigma_x,
\end{aligned}$$

and

$$u_\varepsilon(z) = \int_{\partial\Omega} gG_\varepsilon\,d\sigma_x - \int_{\partial\Omega} u_\varepsilon a_\varepsilon\frac{\partial G_\varepsilon}{\partial \nu_x}\,d\sigma_x.$$

Using the continuity of $u_{\varepsilon,d}$ and the jump conditions satisfied by its normal derivative across $\partial\omega_\varepsilon$, the difference between these two equations yields

$$\begin{aligned}
&u_{\varepsilon,d}(z) - u_\varepsilon(z) - \int_{\partial\Omega}(u_\varepsilon - u_{\varepsilon,d})\,a_\varepsilon\frac{\partial G_\varepsilon}{\partial \nu_x}\,d\sigma_x \\
&= \int_{\partial\omega_\varepsilon}(a_\varepsilon^- - a_{\varepsilon,d}^-)\frac{\partial u_{\varepsilon,d}^-}{\partial \nu_x}G_\varepsilon\,d\sigma_x \\
&= \int_{\partial\omega_\varepsilon}(a_\varepsilon^- - a_{\varepsilon,d}^-)\left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y}\right)G_\varepsilon\,d\sigma_x \;+\; \int_{\partial\omega_\varepsilon}(a_\varepsilon^- - a_{\varepsilon,d}^-)\frac{\partial r_{\varepsilon,d}^-}{\partial \nu_x}G_\varepsilon\,d\sigma_x \\
&= I_1 + I_2,
\end{aligned} \tag{4.13}$$

where $r_{\varepsilon,d}(x) = u_{\varepsilon,d}(x) - u_\varepsilon(x) - \varepsilon v_d(x/\varepsilon)$.
Combining the $W^{1,\infty}$-error estimate (3.21) for the Green function G_ε and the fact that $\nabla_x u_\varepsilon$ is bounded on every compact subset of Ω that contains ω_ε (Theorem 3.4), shows that

$$\begin{aligned}
&\int_{\partial\omega_\varepsilon}(a_\varepsilon^- - a_{\varepsilon,d}^-)\left(\frac{\partial u_\varepsilon^-}{\partial \nu_x} + \frac{\partial v_d^-}{\partial \nu_y}\right)(G_\varepsilon(x,z) - G_0(x,z) - \varepsilon\nabla G_0(x,z)\chi(x/\varepsilon))\,d\sigma_x \\
&\leq C\|\nabla_x u_\varepsilon + \nabla_y v_d\|_{L^2(\omega_\varepsilon)}\,\|\nabla_x G_\varepsilon - (I + \nabla_y\chi(x/\varepsilon))\nabla G_0(x))\|_{L^2(\omega_\varepsilon)} \\
&\leq C\varepsilon^{3/2}(\|\nabla_x u_\varepsilon\|_{L^\infty(\omega_\varepsilon)} + \|\nabla_y v_d\|_{L^2(\tilde\omega)}) \\
&\qquad \varepsilon^{3/2}\|\nabla_x G_\varepsilon - (I + \nabla_y\chi(x/\varepsilon))\nabla G_0(x))\|_{L^\infty(\omega_\varepsilon)} \\
&\leq C\varepsilon^{3+1/4}.
\end{aligned}$$

Hence,

$$
\begin{aligned}
I_1 &= \int_{\partial\omega_\varepsilon} (a_\varepsilon^-(x) - a_{\varepsilon,d}^-(x)) \left(\frac{\partial u_\varepsilon^-}{\partial\nu_x} + \frac{\partial v_d^-}{\partial\nu_y} \right) G_\varepsilon \, d\sigma_x \\
&= \varepsilon^2 \int_{\partial\tilde{\omega}} (a^-(y) - a_d^-(y)) \left(\frac{\partial u_\varepsilon^-}{\partial\nu_x} + \frac{\partial v_d^-}{\partial\nu_y} \right) (G_0(\varepsilon y, z) + \varepsilon\chi(y)\cdot\nabla_x G_0(\varepsilon y, z)) \, d\sigma_y \\
&\quad + O(\varepsilon^{3+1/4}).
\end{aligned}
$$

Thus, by a Taylor expansion of G_ε about the origin,

$$
\begin{aligned}
I_1 &= \varepsilon^2 G_0(0,z) \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left(\frac{\partial u_\varepsilon^-}{\partial\nu_x} + \frac{\partial v_d^-}{\partial\nu_y} \right) d\sigma_y \\
&\quad + \varepsilon^3 \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left(\frac{\partial u_\varepsilon^-}{\partial\nu_x} + \frac{\partial v_d^-}{\partial\nu_y} \right) (\nabla_x G_0(0,z)\cdot(y+\chi(y))) \, d\sigma_y \\
&\quad + O(\varepsilon^{3+1/4}).
\end{aligned}
$$

Since $u_{\varepsilon,d}$ and v_d are harmonic in $\tilde{\omega}$, the first term in the right hand side of the above expression vanishes and I_1 reduces to

$$
\begin{aligned}
I_1 &= \varepsilon^3 \nabla_x G_0(0,z) \cdot \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left(\frac{\partial u_\varepsilon^-}{\partial\nu_x} + \frac{\partial v_d^-}{\partial\nu_y} \right) (y+\chi(y)) \, d\sigma_y \\
&\quad + O(\varepsilon^{3+1/4}).
\end{aligned}
$$

Invoking Theorem 3.6 in a fixed subset $\omega \subset\subset \Omega$ that contains ω_ε, we see that

$$
||\nabla_x u_\varepsilon(\varepsilon y) - (I + \nabla_y\chi(y))\nabla_x u_0(\varepsilon y)||_{L^\infty(\tilde{\omega})} \;\leq\; C\varepsilon,
$$

for some constant C independent of ε, and thus

$$
\begin{aligned}
&\nabla_x G_0(0,z) \cdot \int_{\partial\tilde{\omega}} (a^- - a_d^-) \frac{\partial u_\varepsilon^-}{\partial\nu_x} (y+\chi(y)) \, d\sigma_y \\
&= \int_{\tilde{\omega}} (a^- - a_d^-) \nabla_x u_\varepsilon \cdot (I + \nabla\chi(y)) \nabla_x G_0(0,z) \, dy \\
&= \int_{\tilde{\omega}} (a^- - a_d^-)(I + \nabla\chi(y))\nabla_x u_0(\varepsilon y) \cdot (I + \nabla\chi)\nabla_x G_0(0,z) \, dy + O(\varepsilon) \\
&= \nabla_x G_0(0,z) \int_{\partial\tilde{\omega}} (a^- - a_d^-)(I + \nabla\chi(y))\nabla u_0(0)\cdot\nu \, (y+\chi(y)) \, d\sigma_y + O(\varepsilon).
\end{aligned}
$$

Thus, up to $O(\varepsilon^{3+1/4})$, the term I_1 is equal to

$$
\varepsilon^3 \nabla_x G_0(0,z) \cdot \int_{\partial\tilde{\omega}} (a^- - a_d^-) \left((I + \nabla\chi(y))\nabla u_0(0)\cdot\nu + \frac{\partial v_d^-}{\partial\nu_y} \right) (y+\chi(y)) \; d\sigma_y
$$

Turning to I_2, integration by parts and the change of variables $y = x/\varepsilon$ give

$$I_2 \;=\; \int_{\partial\omega_\varepsilon} (a_\varepsilon^- - a_{\varepsilon,d}^-)\frac{\partial r_{\varepsilon,d}^-}{\partial\nu_x} G_\varepsilon \, d\sigma_x \;=\; \varepsilon^2 \int_{\tilde\omega} (a - a_d)\nabla_y r_{\varepsilon,d} \nabla G_\varepsilon \, dy.$$

Lemma 4.4 shows that $\|\nabla_y r_{\varepsilon,d}(\varepsilon y)\|_{L^2(\tilde\Omega_\varepsilon)} \le c\varepsilon^{3/2}$. Moreover, Theorem 3.4 implies that $\nabla_x G_\varepsilon(x,z)$ is uniformly bounded in ω_ε. Consequently,

$$I_2 \;=\; O(\varepsilon^{7/2}).$$

Thus, (4.13) yields

$$\begin{aligned}
&u_{\varepsilon,d}(z) - u_\varepsilon(z) - \int_{\partial\Omega} (u_\varepsilon - u_{\varepsilon,d}) a_\varepsilon \frac{\partial G_\varepsilon}{\partial\nu_x}\, d\sigma_x \\
=\;& \varepsilon^3 \nabla_x G_0(0,z)\cdot \int_{\partial\tilde\omega} (a^- - a_d^-)\left((I + \nabla\chi(y))\nabla u_0(0)\cdot\nu + \frac{\partial v_d^-}{\partial\nu_y}\right)(y + \chi(y))\, d\sigma_y \\
&\qquad + O(\varepsilon^{3+1/4}).
\end{aligned}$$

To enlight the structure of this expression, we introduce the functions $\varphi_{j,d}$, $1 \le j \le 3$, solutions to

$$\left\{\begin{array}{l}
\mathrm{div}_y(a_d(y)\nabla\varphi_{j,d}) = 0 \text{ in } \tilde\omega \qquad \mathrm{div}(a_d(y)\nabla\varphi_{j,d}) = 0 \text{ in } \mathcal{R}^3\backslash\tilde\omega \\[2ex]
\varphi_{j,d} \text{ is continuous across } \partial\tilde\omega \\[2ex]
a_d^+(y)\dfrac{\partial\varphi_{j,d}^+}{\partial\nu_y} - a_d^-(y)\dfrac{\partial\varphi_{j,d}^-}{\partial\nu_y} = (a_d^- - a^-)(\nu_j + \dfrac{\partial\chi^j(y)}{\partial\nu_y}) \quad \text{on } \partial\tilde\omega \\[2ex]
\varphi_{j,d}(y) \to 0 \;\text{ when } |y| \to \infty
\end{array}\right. \tag{4.14}$$

Noticing that $v_d(y) = \displaystyle\sum_{j=1}^{3} \varphi_{j,d}(y)\frac{\partial u_0}{\partial x_j}(0)$ allows us to rewrite

$$\begin{aligned}
u_{\varepsilon,d}(z) - u_\varepsilon(z) \;=\;& \int_{\partial\Omega} (u_\varepsilon - u_{\varepsilon,d}) a_\varepsilon \frac{\partial G_\varepsilon}{\partial\nu_x}\, d\sigma_x \\
&+ \varepsilon^3 \nabla_x G_0(0,z) M \nabla_x u_0(0) \;+\; O(\varepsilon^{3+1/4})
\end{aligned}$$

where the tensor M is defined by

$$M_{ij} \;=\; \int_{\partial\tilde\omega} (a^-(y) - a_d^-(y))(y_i + \chi^i(y))\left(\frac{\partial\varphi_{j,d}^-}{\partial\nu_y} + \left(\nu_j + \frac{\partial\chi^j(y)}{\partial\nu_y}\right)\right) d\sigma_y.$$

Using the jump condition satisfied by $\varphi_{j,d}$ across $\partial\tilde\omega$,

$$\frac{\partial\varphi_{j,d}^-}{\partial\nu_y} \;=\; \frac{1}{a_d^-}\left(a_d^+\frac{\partial\varphi_{j,d}^+}{\partial\nu_y} + (a^- - a_d^-)(\nu_j + \frac{\partial\chi^j}{\partial\nu_y})(y)\right),$$

one sees that M_{ij} can be expressed as

$$M_{ij} = \int_{\partial\tilde{\omega}} (\frac{a^-}{a_d^-} - 1)(y_i + \chi^i(y)) \left(a^+(y) \frac{\partial \varphi^+_{j,d}}{\partial \nu_y} + a^-(y) \left(\nu_j + \frac{\partial \chi^j(y)}{\partial \nu_y} \right) \right) d\sigma_y,$$

which proves Theorem 4.1. ■

This formula defines a polarization tensor in the same spirit as in [10, 9]. It describes the influence on the far field of a localized defect within the periodic medium. One easily checks that the expression of M_{ij} reduce to that given in [10], when instead of a misplaced inclusion, one considers a defect ($a_d \neq 1$ in $\omega_{\varepsilon,2}$) in a homogeneous background medium (a constant and $\chi = 0$). Also, adapting the proof of Lemma 5 in [10] shows that M is symmetric.

A Proofs of the estimates

The proofs of theorems 3.4–3.7 and of Lemma 3.5 are collected in this Appendix.

A.1 Proof of Theorem 3.4

The proof of this result is based on two main ingredients: The first is the 'three-step compactness method' of M. Avellaneda and F.H. Lin [4, 5], who proved Hölder and Lipschitz estimates on u_ε, when the coefficients of L_ε are smooth (Hölder continuous). The second is the Hölder regularity results for the gradients in composite media containing inclusions with $\mathcal{C}^{1,\alpha}$–regular boundaries [16, 15]. Theorem 3.4 generalizes Theorem 3.3 to nonzero right–hand side. Its proof closely follows the proof of Theorem 3.3 (Theorem 0.2 in [15]), which itself is based on the arguments of [4, 5].

In the sequel, for each r, $0 < r < 1$, and $x \in \mathbf{R}^3$, we set

$$B(x,r) = \{y \in \mathbf{R}^3 / |x-y| < r\}, \qquad B_r = B(0,r),$$
$$\fint_D f = \frac{1}{|D|} \int_D f, \qquad (\overline{u}_\varepsilon)_{x,r} = \fint_{B(x,r)} u_\varepsilon.$$

We recall the classical characterization of Hölder spaces [8] in terms of the semi-norm

$$[u]_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} = \sup_{x,x' \in \Omega} \frac{|u(x) - u(x')|}{|x - x'|^\alpha}.$$

For each $0 < \alpha < 1$, there exist positive constants c_1, c_2, which only depend on Ω and α, such that for all $u \in \mathcal{C}^{0,\alpha}(\overline{\Omega})$,

$$c_1 [u]_{\mathcal{C}^{0,\alpha}(\overline{\Omega})} \leq \sup_{x \in \Omega} \sup_{r > 0} \left[\frac{1}{r^{2\alpha}} \fint_{\Omega \cap B(x,r)} (u - (\overline{u})_{x,r})^2 \right]^{1/2} \leq c_2 [u]_{\mathcal{C}^{0,\alpha}(\overline{\Omega})}. \qquad \text{(A.1)}$$

We assume that the coefficient a is piecewise smooth and that the boundaries of the inclusions have regularity $\mathcal{C}^{1,\alpha}$ for some $0<\alpha<1$. Let $\mu = \dfrac{\alpha}{2(\alpha+1)}$. We begin with proving interior Hölder estimates on u_ε (see Theorem 5.1 in [15]).

Theorem A.1 *(interior Hölder estimates)*
Let $f \in L^\infty(B_1)$ and $h \in \mathcal{C}^{0,\mu}(B_1)^3$. Assume that u_ε satisfies

$$L_\varepsilon u_\varepsilon \;=\; f + \varepsilon\, div(b_\varepsilon h) \quad in\ B_1.$$

There exists a constant C, which only depends on μ, λ, Λ and α, but which is independent of ε and of the distances between the inclusions, such that

$$\|u_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{\frac{1}{2}})} \;\le\; C(\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}). \tag{A.2}$$

The Theorem results from the three following Lemmas. The difference with [4] mainly lies in the proof of the third Lemma, where the regularity hypotheses on the conductivity are determinant.

Lemma A.2 *(One–step improvement) There exist $\theta>0$ and $0<\varepsilon_0<1$, which only depend on μ,α,λ and Λ, such that, if u_ε, f and h satisfy*

$$\begin{cases} L_\varepsilon u_\varepsilon &= f + \varepsilon\, div(b_\varepsilon h) \quad in\ B_1 \\ ⨍_{B_1} |u_\varepsilon|^2 &\le 1 \\ ||f||_{L^\infty(B_1)} &\le \varepsilon_0 \\ ||h||_{\mathcal{C}^{0,\mu}(B_1)} &\le \varepsilon_0, \end{cases}$$

then, for $0<\varepsilon\le\varepsilon_0$,

$$⨍_{B_\theta} |u_\varepsilon - (\overline{u_\varepsilon})_{0,\theta}|^2 \;\le\; \theta^{2\mu}. \tag{A.3}$$

Proof: Let $\mu < \mu' < 1$. As the homogenized operator L_0 is elliptic with constant coefficients, solutions to $-\text{div}(A\nabla u_0) = 0$ in B_1 are smooth. In particular, there exists $0<\theta<1$ such that

$$⨍_{B_\theta} (u_0 - (\overline{u_0})_{0,\theta})^2 \;\le\; \theta^{2\mu'} ⨍_{B_1} u_0^2. \tag{A.4}$$

We fix a value of θ for which (A.4) holds. We prove (A.3) by contradiction: Assume that there is a sequence $L^j_{\varepsilon_j}, u_{\varepsilon_j}, f_{\varepsilon_j}, h_{\varepsilon_j}$ which satisfies

$$L^j_{\varepsilon_j} u_{\varepsilon_j} \;=\; f_{\varepsilon_j} + \varepsilon_j \text{div}(b_{\varepsilon_j} h_{\varepsilon_j}) \quad \text{in } B_1,$$

with $⨍_{B_1} u^2_{\varepsilon_j} \le 1$ and $\lim \|f_{\varepsilon_j}\|_{L^\infty(B_1)} = \lim \|h_{\varepsilon_j}\|_{\mathcal{C}^{0,\mu}(B_1)} = \lim \varepsilon_j = 0$, and such that

$$⨍_{B_\theta} |u_{\varepsilon_j} - (\overline{u_{\varepsilon_j}})_{0,\theta}|^2 \;>\; \theta^{2\mu}. \tag{A.5}$$

Extracting a subsequence, we find an operator L_0, limit of the operators $L^j_{\varepsilon_j}$ in the sense of homogenization, and a function $u_0 \in H^1_{\text{loc}}(B_1)$, such that

$$\left\{ \begin{array}{lll} u_{\varepsilon_j} & \rightharpoonup & u_0 \quad \text{weakly in } L^2(B_1), \\ u_{\varepsilon_j} & \rightharpoonup & u_0 \quad \text{weakly in } H^1(B_\theta). \end{array} \right.$$

As $f_{\varepsilon_j} + \varepsilon_j \text{div}(b_{\varepsilon_j} h_{\varepsilon_j})$ converges to 0 strongly in $H^{-1}(B_1)$, we see that $L_0(u_0) = 0$ in B_1. Taking limits in (A.5) we get

$$\theta^{2\mu} \ \leq \ \fint_{B_\theta} |u_0 - (\overline{u_0})_{0,\theta}|^2 \ \leq \ \theta^{2\mu'},$$

a contradiction. Hence, (A.3) holds for some $\varepsilon_0 > 0$. ■

Lemma A.3 *(Iteration)*
Let θ and ε_0 be as in Lemma A.2. Then, for all $u_\varepsilon \in L^2(B_1), f \in L^\infty(B_1)$ and $h \in \mathcal{C}^{0,\mu}(B_1)$ which satisfy

$$L_\varepsilon u_\varepsilon \ = \ f + \varepsilon\, div(b_\varepsilon h) \quad in\ B_1,$$

and for all $k \geq 1$ such that $\varepsilon/\theta^k \leq \varepsilon_0$,

$$\fint_{B_{\theta^k}} |u_\varepsilon - (\overline{u_\varepsilon})_{0,\theta^k}|^2 \ \leq \ \theta^{2k\mu} \left[\left(\fint_{B_1} |u_\varepsilon|^2 \right)^{1/2} + \frac{1}{\varepsilon_0} (\|f_\varepsilon\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\alpha}(B_1)}) \right]^2 \quad \text{(A.6)}$$

Proof: The proof is by induction on k. Lemma A.2 shows that (A.6) holds for $k = 1$. Let

$$J \ = \ \left[\left(\fint_{B_1} |u_\varepsilon|^2 \right)^{1/2} + \frac{1}{\varepsilon_0} (\|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}) \right]^2. \qquad \text{(A.7)}$$

and, for k satisfying $\varepsilon/\theta^k \leq \varepsilon_0$ and $x \in B_1$, let

$$w_\varepsilon(x) \ = \ J^{-1} \theta^{-k\mu} \left[u_\varepsilon(\theta^k x) - (\overline{u_\varepsilon})_{0,\theta^k} \right].$$

Then w_ε solves

$$L_{\varepsilon/\theta^k} w_\varepsilon \ = \ \hat{f}_\varepsilon + \frac{\varepsilon}{\theta^k} \text{div}(b_{\varepsilon/\theta^k} \hat{h}_\varepsilon),$$

where for $x \in B_1$, $\hat{f}_\varepsilon(x) = J^{-1}\theta^{k(2-\mu)} f(\theta^k x)$, and $\hat{h}_\varepsilon(x) = J^{-1}\theta^{k(2-\mu)} h(\theta^k x)$. One easily checks that

$$\left\{ \begin{array}{lllll} \|\hat{f}_\varepsilon\|_{L^\infty(B_1)} & \leq & J^{-1}\theta^{k(2-\mu)} \|f\|_{L^\infty(B_1)} & \leq & \varepsilon_0, \\ \|\hat{h}_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_1)} & \leq & J^{-1}\theta^{k(2-\mu)} \|h\|_{\mathcal{C}^{0,\mu}(B_1)} & \leq & \varepsilon_0. \end{array} \right.$$

By the induction hypothesis, we see that

$$\fint_{B_1} |w_\varepsilon(x)|^2 \ \leq \ 1.$$

Thus, we can apply lemma A.2: w_ε satisfies (A.3) which, expressed in terms of u_ε, yields (A.6). ■

Lemma A.4 *(Blow up)*
Assume that $u_\varepsilon \in L^2(B_1), f \in L^\infty(B_1)$ and $h \in \mathcal{C}^{0,\mu}(B_1)$ satisfy

$$L_\varepsilon u_\varepsilon = f + \varepsilon\, div(b_\varepsilon h) \quad in\ B_1.$$

Then there exists a constant C, that only depends on μ, λ, Λ and the regularity of the dividing interfaces, such that

$$\|u_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{1/2})} \;\leq\; C\left(\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}\right). \qquad (A.8)$$

Proof: In view of (A.1), we need only prove that

$$\fint_{B(x,r)} |u_\varepsilon - (\overline{u_\varepsilon})_{x,r}|^2 \;\leq\; Cr^{2\mu}\left(\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)}\right)^2, \quad (A.9)$$

for all $0 < r \leq 1/4$ and $|x| < 1/2$. We establish (A.9) for $x = 0$. By Lemma A.3, (A.9) with $x = 0$ holds for $r \geq \varepsilon/\varepsilon_0$. For $y \in B_{2/\varepsilon_0}$, let

$$w_\varepsilon(y) \;=\; \varepsilon^{-\mu}(u_\varepsilon(\varepsilon y) - (\overline{u_\varepsilon})_{0,2\varepsilon/\varepsilon_0}).$$

Applying (A.9) with $r = 2\varepsilon/\varepsilon_0$, shows that

$$\begin{aligned}
\|w_\varepsilon\|^2_{L^2(B_{2/\varepsilon_0})} &= \frac{\varepsilon^{-2\mu}}{\varepsilon_0^3}\fint_{B_{2\varepsilon/\varepsilon_0}} (u_\varepsilon(x) - (\overline{u_\varepsilon})_{0,2\varepsilon/\varepsilon_0})^2\, dx \\
&\leq \frac{\varepsilon^{-2\mu}}{\varepsilon_0^3}\,(\frac{2\varepsilon}{\varepsilon_0})^{2\mu} J^2 \\
&\leq CJ^2, \qquad (A.10)
\end{aligned}$$

with J as in (A.7), so that w_ε is uniformly bounded in $L^2(B_{2/\varepsilon_0})$. Moreover, w_ε solves an equation where the operator and the domain (and in particular the number of inclusions) are independent of ε: indeed,

$$L_1 w_\varepsilon \;=\; \hat{f}_\varepsilon + \mathrm{div}(b_1 \hat{h}_\varepsilon) \quad \text{in } B_{2/\varepsilon_0}, \qquad (A.11)$$

with, for $x \in B_{2/\varepsilon_0}$, $\hat{f}_\varepsilon(x) = \varepsilon^{2-\mu} f(\varepsilon x)$ and $\hat{h}_\varepsilon(x) = \varepsilon^{2-\mu} h(\varepsilon x)$. We notice that

$$\left\{\begin{array}{lll}
\|\hat{f}_\varepsilon\|_{L^\infty(B_{2/\varepsilon_0})} & \leq & \varepsilon^{2-\mu}\|f\|_{L^\infty(B_1)}, \\
\|\hat{h}_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{2/\varepsilon_0})} & \leq & \varepsilon^{2-\nu}\|h\|_{\mathcal{C}^{0,\mu}(B_1)},
\end{array}\right. \qquad (A.12)$$

and that $b(y)\hat{h}_\varepsilon(y)$ has regularity $\mathcal{C}^{0,\mu}$ on each of the inclusions $\varepsilon l \overline{D}_0$, $l \in \mathbf{Z}^3$, contained in B_{2/ε_0} and has the same regularity on their complementary in B_{2/ε_0}.
Therefore, we can apply the interior Hölder gradient estimates (3.13) to w_ε, to obtain

$$\|w_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{1/\varepsilon_0})} \;\leq\; C\left(\|w_\varepsilon\|_{L^2(B_{2/\varepsilon_0})} + \|\hat{f}_\varepsilon\|_{L^\infty(B_{2/\varepsilon_0})} + \|\hat{h}_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{2/\varepsilon_0})}\right).$$

Thus, for all $s \leq 1/\varepsilon_0$, we have

$$\begin{aligned}\varepsilon^{-2\mu} \fint_{B_{\varepsilon s}} |u_\varepsilon(x) - (\overline{u_\varepsilon})_{0,\varepsilon s}|^2 \, dx &= \fint_{B_s} |w_\varepsilon - (\overline{w_\varepsilon})_{0,s}|^2 \\ &\leq Cs^{2\mu} \left(\|w_\varepsilon\|_{L^2(B_{2/\varepsilon_0})} + \|\hat{f}_\varepsilon\|_{L^\infty(B_{2/\varepsilon_0})} + \|\hat{h}_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{2/\varepsilon_0})} \right)^2. \qquad \text{(A.13)}\end{aligned}$$

Setting $r = s\varepsilon$ and combining this last identity with (A.12), (A.10) and (A.13), we finally obtain

$$\fint_{B_r} |u_\varepsilon - (\overline{u_\varepsilon})_{0,r}|^2 \quad \leq \quad Cr^{2\nu} (\|u_\varepsilon\|_{L^2(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)})^2,$$

which is (A.9) at $x = 0$. By translation, this estimate remains true for all $x \in B_{1/2}$. The Lemma (and Theorem A.1) is proved. ∎

Let χ be the cell function defined in (3.4). To prove Theorem 3.4, we apply again the three-steps method, this time to estimate the quantity

$$\|u_\varepsilon(x) - u_\varepsilon(0) - (x + \varepsilon\chi(x/\varepsilon))(\overline{\nabla u_\varepsilon})_{0,\theta}\|_{L^\infty(B_\theta)}.$$

Lemma A.5 *(One-step improvement) There exist $0 < \theta, \varepsilon_0 < 1$ which only depends on $\lambda, \Lambda, \mu, \alpha$, such that, if u_ε, f and h satisfy*

$$L_\varepsilon u_\varepsilon \quad = \quad f + \varepsilon \, div(b_\varepsilon h) \quad in \; B_1,$$

with $\|u_\varepsilon\|_{L^\infty(B_1)} \leq 1, \|f\|L^\infty(B_1) \leq 1$ and $\|h\|_{\mathcal{C}^{0,\mu}(B_1)} \leq 1$ then, for $0 < \varepsilon \leq \varepsilon_0$,

$$\sup_{|x|<\theta} |u_\varepsilon(x) - u_\varepsilon(0) - (x + \varepsilon\chi(x/\varepsilon))(\overline{\nabla u_\varepsilon})_{0,\theta}| \quad \leq \quad \theta^{1+\mu/2}. \qquad \text{(A.14)}$$

Proof: Let $\mu < \mu' < 1$. Recalling (3.8), let u_0 and f_0 satisfy

$$L_0 u_0 \quad = \quad f_0 \text{ in } B_1.$$

Classical regularity estimates [11] show that $u_0 \in \mathcal{C}^{1,\mu}(\overline{\Omega})$. Thus, there exists $0 < \theta < 1$, which only depends on λ and Λ, such that

$$\sup_{|x|<\theta} |u_0(x) - u_0(0) - x(\overline{\nabla u_0})_{0,\theta}| \quad \leq \quad \theta^{1+\mu'/2} (\|u_0\|_{L^\infty(B_1)} + \|f_0\|_{L^\infty(B_1)}). \text{ (A.15)}$$

Fixing this value of θ, we prove (A.14) by contradiction.

Suppose on the contrary that there is a sequence $\varepsilon_j \to 0$ and sequences $L^j_{\varepsilon_j}$, $u_{\varepsilon_j}, f_{\varepsilon_j}$ and h_{ε_j}, such that

$$L^j_{\varepsilon_j} u_{\varepsilon_j} \quad = \quad f_{\varepsilon_j} + \varepsilon_j \text{div}(b_{\varepsilon_j} h_{\varepsilon_j}) \quad \text{in } B_1,$$

and

$$\left\{\begin{array}{lcl}\|u_{\varepsilon_j}\|_{L^\infty(B_1)} & \leq & 1,\\ \|f_{\varepsilon_j}\|_{L^\infty(B_1)} & \leq & 1,\\ \|h_{\varepsilon_j}\|_{\mathcal{C}^{0,\mu}(B_1)} & \leq & 1,\end{array}\right.$$

and for which

$$\sup_{|x|<\theta} |u_{\varepsilon_j}(x) - u_{\varepsilon_j}(0) - (x + \varepsilon_j \chi(x/\varepsilon_j))(\overline{\nabla u_{\varepsilon_j}})_{0,\theta}| \quad > \quad \theta^{1+\mu/2}. \tag{A.16}$$

Passing to a subsequence (not renamed) and using Theorem A.1, we find an operator L_0 and functions $u_0 \in H^1_{loc}(B_1)$ and $f_0 \in L^\infty(B_1)$, such that

$$\begin{array}{lcllcl} f_{\varepsilon_j} & \rightharpoonup & f_0 \text{ weakly in } L^\infty(B_1), & u_{\varepsilon_j} & \to & u_0 \text{ uniformly in } B_\theta,\\ u_{\varepsilon_j} & \rightharpoonup & u_0 \text{ weakly * in } L^\infty(B_1), & \nabla u_{\varepsilon_j} & \rightharpoonup & \nabla u_0 \text{ weakly in } L^2(B_\theta).\end{array}$$

We also notice that $\varepsilon_j \mathrm{div}(b_{\varepsilon_j} h_{\varepsilon_j}) \to 0$ strongly in $H^{-1}(B_1)$ so that $L_0 u_0 = f_0$ in B_1.

Before passing to the limit in (A.16), we show that $|(\overline{\nabla u_{\varepsilon_j}})_{0,\theta}|$ is uniformly bounded by a constant that only depends on θ. Indeed, let $v \in \mathcal{C}^\infty_0(B_{(1+\theta)/2})$ satisfy $0 \leq v \leq 1$ and $v \equiv 1$ on B_θ. We have

$$\begin{aligned} |(\overline{\nabla u_{\varepsilon_j}})_{0,\theta}| &= |\frac{1}{|B_\theta|}\int_{B_\theta} \nabla u_{\varepsilon_j}| \\ &\leq \frac{1}{|B_\theta|^{1/2}}\left(\int_{B_\theta} |\nabla (u_{\varepsilon_j})|^2\right)^{1/2} \\ &\leq \frac{1}{|B_\theta|^{1/2}}\left(\int_{B_1} |\nabla (v u_{\varepsilon_j})|^2\right)^{1/2}. \end{aligned}$$

One easily checks that

$$\begin{aligned} \int_{B_1} |\nabla (v u_{\varepsilon_j})|^2 &= \int_{B_1} \nabla u_{\varepsilon_j} \cdot \nabla (v^2 u_{\varepsilon_j}) + \int_{B_1} u^2_{\varepsilon_j} |\nabla v|^2 \\ &\leq \int_{B_1} f_{\varepsilon_j} v^2 u_{\varepsilon_j} - \varepsilon_j \int_{B_1} b_{\varepsilon_j} h_{\varepsilon_j} \cdot \nabla(v^2 u_{\varepsilon_j}) \\ &\quad + \int_{B_1} u^2_{\varepsilon_j} |\nabla v|^2. \end{aligned}$$

Therefore, given the uniform bounds on u_{ε_j} and f_{ε_j}, we conclude that

$$|(\overline{\nabla u_{\varepsilon_j}})_{0,\theta}| \quad \leq \quad C(\theta)$$

and

$$\sup_{|x|<\theta} |\varepsilon_j \chi(\frac{x}{\varepsilon_j})(\overline{\nabla u_{\varepsilon_j}})_{0,\theta}| \quad \leq \quad \varepsilon_j C(\theta) \to 0$$

Returning to (A.16), and passing to the limit $\varepsilon_j \to 0$ yields

$$\theta^{1+\mu/2} \;\leq\; \sup_{|x|<\theta} |u_0(x) - u_0(0) - x(\overline{\nabla u_0})_{0,\theta}| \;\leq\; \theta^{1+\mu'/2},$$

which contradicts the fact that $\theta < 1$. ■

Lemma A.6 *(Iteration)*
Let θ and ε_0 be as in Lemma A.5. Suppose that $u_\varepsilon \in L^\infty(B_1)$, $f \in L^\infty(B_1)$ and $h \in \mathcal{C}^{0,\mu}(B_1)$ satisfy

$$L_\varepsilon u_\varepsilon \quad = \quad f + \varepsilon\, div(b_\varepsilon h) \quad \text{in } B_1.$$

Then for all $k \geq 1$ with $\varepsilon/\theta^k \leq \varepsilon_0$, there exist $a_k^\varepsilon \in \mathbf{R}$ and $B_k^\varepsilon \in \mathbf{R}^3$ such that

$$|a_k^\varepsilon| \quad \leq \quad C_1 \left(\|u_\varepsilon\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)} \right), \tag{A.17}$$

$$|B_k^\varepsilon| \quad \leq \quad C_2(1 + \sum_{j=1}^k \theta^{j\mu/2}) \left(\|u_\varepsilon\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)} \right), \tag{A.18}$$

and such that

$$\begin{aligned} &\sup_{|x|<\theta^k} |u_\varepsilon(x) - u_\varepsilon(0) - \varepsilon a_k^\varepsilon - [x + \varepsilon\chi(x/\varepsilon)] B_k^\varepsilon| \\ &\quad \leq \quad \theta^{k(1+\mu/2)} \left(\|u_\varepsilon\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)} \right). \end{aligned} \tag{A.19}$$

Here C_1 and C_2 are generic constants, which only depend on $\theta, \varepsilon_0, \lambda, \Lambda$.

Proof: We set

$$J \quad = \quad \left(\|u_\varepsilon\|_{L^\infty(B_1)} + \|f\|_{L^\infty(B_1)} + \|h\|_{\mathcal{C}^{0,\mu}(B_1)} \right). \tag{A.20}$$

The proof is by induction on k. By lemma A.5, estimate (A.19) holds for $k = 1$, with $a_1^\varepsilon = 0$ and $B_1^\varepsilon = (\overline{\nabla u_\varepsilon})_{0,\theta}$.

Suppose that (A.19) holds for some k such that $\varepsilon/\theta^k \leq \varepsilon_0$. For $|x| < 1$ let

$$w_\varepsilon(x) \quad = \quad \theta^{-k(1+\mu/2)} J^{-1} \Big(u_\varepsilon(\theta^k x) - u_\varepsilon(0) - \varepsilon a_k^\varepsilon - [\theta^k x + \varepsilon\chi(\theta^k x/\varepsilon)] B_k^\varepsilon \Big).$$

This function solves

$$L_{\varepsilon/\theta^k} w_\varepsilon \quad = \quad \hat{f}_\varepsilon + \frac{\varepsilon}{\theta^k} \mathrm{div}(b_{\varepsilon/\theta^k} \hat{h}_\varepsilon) \quad \text{in } B(0,1),$$

where for $x \in B_1$, $\hat{f}_\varepsilon(x) = \theta^{k(1-\mu/2)} J^{-1} f(\theta^k x)$ and $\hat{h}_\varepsilon(x) = \theta^{k(1-\mu/2)} J^{-1} h(\theta^k x)$. One easily sees that $\|\hat{f}_\varepsilon\|_{L^\infty(B_1)} \leq 1$ and $\|\hat{h}_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_1)} \leq 1$. By the induction hypotheses (A.19), $\|w_\varepsilon\|_{L^\infty(B_1)} \leq 1$, so that applying Lemma A.5 to w_ε, we get

$$\sup_{|x|<\theta} |w_\varepsilon(x) - w_\varepsilon(0) - [x + \varepsilon/\theta^k \chi(\theta^k x/\varepsilon)](\overline{\nabla w_\varepsilon})_{0,\theta}| \quad \leq \quad \theta^{1+\mu/2}.$$

Rewriting this inequality in terms of u_ε, we obtain

$$\begin{aligned} &\sup_{|x|<\theta} \Big| u_\varepsilon(\theta^k x) - u_\varepsilon(0) + \varepsilon\chi(0) B_k^\varepsilon - \Big[\theta^k x + \varepsilon\chi(\theta^k x/\varepsilon)\Big] \left(B_k^\varepsilon + J\theta^{k\mu/2} (\overline{\nabla w_\varepsilon})_{0,\theta}\right) \Big| \\ &\leq \quad J\, \theta^{(k+1)(1+\mu/2)}. \end{aligned} \tag{A.21}$$

If we set $a^\varepsilon_{k+1} = -\chi(0)B^\varepsilon_k$, $B^\varepsilon_{k+1} = B^\varepsilon_k + J\theta^{k\mu/2}(\overline{\nabla w_\varepsilon})_{0,\theta}$, substitute these expressions in (A.21), and make the change of variables $\theta^k x \to x$, we obtain (A.19) with $k+1$ instead of k. Moreover, as in the proof of Lemma (A.5)

$$|(\overline{\nabla w_\varepsilon})_{0,\theta}| \quad \leq \quad C,$$

where C depends on θ but not on ε, k, and given the initial choice of a^ε_1 and B^ε_1, it is easy to check that the sequences $\{a^\varepsilon_k\}$ and $\{B^\varepsilon_k\}$ satisfy the estimates (A.17) and (A.18). Lemma A.6 is thus proved. ∎

Proof of Theorem 3.4:
Let $k \in \mathbb{N}$ be such that $\varepsilon/\theta^k \leq \varepsilon_0 \leq \varepsilon/\theta^{k+1}$. By lemma A.6 and recalling (A.20),

$$\sup_{|x|<\varepsilon/\varepsilon_0} |u_\varepsilon(x) - u_\varepsilon(0) - \varepsilon a^\varepsilon_k - (x + \varepsilon\chi(x/\varepsilon))B^\varepsilon_k| \quad \leq \quad \theta^{k(1+\mu/2)} J.$$

Invoking the estimates (A.17) and (A.18) and rescaling the above inequality, we get

$$\sup_{|x|<1/\varepsilon_0} |\frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0)}{\varepsilon}| \quad \leq \quad |a^\varepsilon_k - [x + \chi(x)]B^\varepsilon_k| + \varepsilon_0 \frac{\theta^{k(1+\mu/2)}}{\theta^{k+1}} J \ \leq \ CJ.$$

For $|x| < 1/\varepsilon$, the function

$$v_\varepsilon(x) \quad = \quad \frac{u_\varepsilon(\varepsilon x) - u_\varepsilon(0)}{\varepsilon},$$

satisfies

$$L_1 v_\varepsilon \quad = \quad \hat{f}_\varepsilon(x) + \operatorname{div}(b_1(y)\hat{h}_\varepsilon) \quad \text{in } B_{1/\varepsilon_0},$$

where $\hat{f}_\varepsilon(x) = \varepsilon f(\varepsilon x)$ and $\hat{h}_\varepsilon(x) = \varepsilon h(\varepsilon x), x \in B_{1/\varepsilon_0}$ and

$$\begin{cases} \|v_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} & \leq \quad CJ, \\ \|\hat{f}_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} & \leq \quad \varepsilon J, \\ \|\hat{h}_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{1/\varepsilon_0})} & \leq \quad \varepsilon J. \end{cases}$$

In the above equation, neither the operator nor the domain depend on ε. Also we remark again that the function $b\hat{h}_\varepsilon$ has regularity $\mathcal{C}^{0,\mu}$ on each of the inclusions contained in B_{1/ε_0}, and on the complementary of the inclusions in B_{1/ε_0}. We can therefore apply the interior gradient estimate (3.14) to v_ε, to obtain

$$\|\nabla v_\varepsilon\|_{L^\infty(B_{1/2\varepsilon_0})} \quad \leq \quad C\left(\|v_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} + \|\hat{f}_\varepsilon\|_{L^\infty(B_{1/\varepsilon_0})} + \|\hat{h}_\varepsilon\|_{\mathcal{C}^{0,\mu}(B_{1/\varepsilon_0})}\right),$$

which shows that

$$\|\nabla u_\varepsilon\|_{L^\infty(B_{\varepsilon/2\varepsilon_0})} \leq CJ.$$

The same estimate can be established in $B(x, \varepsilon/2\varepsilon_0)$ for any $x \in B_{1/2}$. The proof of theorem 3.4 follows by combining this estimate with (A.2). ∎

A.2 Proof of Lemma 3.5

Error estimates in L^2 between u_ε and u_0 are well–known for Dirichlet problems. For Neumann boundary conditions, S. Moskow and M. Vogelius [20] derived 2D estimates, using the fact that the harmonic conjugate of the potential satisfies Dirichlet boundary conditions. Our proof does not use this property, although we follow the structure of S. Moskow and M. Vogelius' argument.

step 1: We transform the equation into a first order system

$$\left\{\begin{array}{lcl} a_\varepsilon \nabla u_\varepsilon - v_\varepsilon & = & 0 \\ -\mathrm{div}(v_\varepsilon) & = & 0. \end{array}\right.$$

and seek an asymptotic expansion for both u_ε and v_ε. Such an expansion is given explicitely in 3D, in [6], pp. 58–65. Recalling the notations of Section 3.1, one easily checks that the first term in the expansion of u_ε must be the potential u_0 of (3.8) and that

$$\left\{\begin{array}{ccl} -\mathrm{div}_y v_0 & = & 0 \\ a(y)\nabla_y u_1 + a(y)\nabla_x u_0 - v_0 & = & 0 \\ -\mathrm{div}_y v_1 - \mathrm{div}_x v_0 & = & 0. \end{array}\right. \tag{A.22}$$

Denoting by $e_p, 1 \le p \le 3$ the canonical basis of $\mathbf{R}^3$, we set

$$u_1(x,y) \quad = \quad -\chi_j(y)\frac{\partial u_0}{\partial x_j}$$

and we define functions $\tilde{\chi}_p \in H^1_\#(Y)^3$ by

$$\left\{\begin{array}{ccl} \mathrm{curl}_y\left(a^{-1}(y)\mathrm{curl}_y(\tilde{\chi}_p)\right) & = & \mathrm{curl}_y(a^{-1}(y)e_p) \\ \mathrm{div}_y(\tilde{\chi}_p) & = & 0 \\ \fint_Y \tilde{\chi}_p & = & 0. \end{array}\right. \tag{A.23}$$

These functions are related to the usual correctors χ_j defined in (3.4) by

$$a^{-1}(y)(I - \mathrm{curl}_y\tilde{\chi}) \quad = \quad (I - \nabla_y \chi)A^{-1},$$

where $\mathrm{curl}_y\tilde{\chi}$ and $\nabla_y\chi$ denote the matrices the columns of which are the vectors $\mathrm{curl}_y\tilde{\chi}_p$ and $\nabla\chi_j$ respectively (see [6]). In particular, according to Theorem 3.2, the above relation shows that, under our hypothesis on the conductivity, $\tilde{\chi} \in W^{1,\infty}(Y)^3$. It also shows that $\tilde{\chi}(x/\varepsilon)$ has a trace on $\partial\Omega$, which is uniformly bounded in $L^\infty(\partial\Omega)$. Following [6], we set

$$\left\{\begin{array}{lcl} v_0(x,y) & = & \rho(x) - \mathrm{curl}_y(\tilde{\chi}_p(y))\,\rho_p(x) \\ & = & (I - \mathrm{curl}_y\tilde{\chi})A\nabla u_0(x) \\ v_1(x,y) & = & -\mathrm{curl}_x(\tilde{\chi}_p(y)\rho(x)). \end{array}\right. \tag{A.24}$$

which satisfy (A.22) (note that $\rho(x) = \displaystyle\fint_Y v_0(x,y)dy = A\nabla u_0(x)$).

We then form

$$\begin{aligned} z_\varepsilon(x) &= u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, x/\varepsilon) \\ \eta_\varepsilon(x) &= a_\varepsilon(x)\nabla u_\varepsilon(x) - v_0(x, x/\varepsilon) - \varepsilon v_1(x, x/\varepsilon) \end{aligned}$$

and note that z_ε and η_ε are uniformly bounded in $H^1(\Omega)$ and $L^2(\Omega)$ respectively. Since u_0, u_1, v_0 and v_1 satisfy (A.22), the proof of Proposition 1 in [19] shows that

$$\begin{aligned} \operatorname{div}(\eta_\varepsilon) &= 0 \quad \text{in } \Omega && \text{(A.25)} \\ ||a_\varepsilon(x)\nabla z_\varepsilon - \eta_\varepsilon||_{L^2(\Omega)} &\leq C\varepsilon ||u_0||_{H^2(\Omega)}. && \text{(A.26)} \end{aligned}$$

As in [20], we introduce a boundary corrector defined by

$$\left\{ \begin{aligned} -\operatorname{div}(a_\varepsilon \nabla B_\varepsilon) &= 0 \quad \text{in } \Omega \\ a_\varepsilon(x)\nabla B_\varepsilon \cdot \nu &= \varepsilon^{-1}\eta_\varepsilon \cdot \nu \quad \text{on } \partial\Omega. \end{aligned} \right. \tag{A.27}$$

We note that as $\eta_\varepsilon \in L^2(\Omega)$ and $\operatorname{div}(\eta_\varepsilon) = 0$, the normal trace $\eta_\varepsilon \cdot \nu$ is well defined in $H^{-1/2}(\partial\Omega)$.

We normalize B_ε by requiring that $\int_\Omega \varepsilon B_\varepsilon d\sigma = \int_\Omega z_\varepsilon d\sigma = \int_\Omega \varepsilon u_1 d\sigma$ so that

$$\left| \int_\Omega \varepsilon B_\varepsilon dx \right| \leq C\varepsilon ||u_0||_{H^1(\Omega)}. \tag{A.28}$$

step 2: It is easy to see that Proposition 1 in [20] also holds in 3D, which states that

$$||u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, x/\varepsilon) - \varepsilon B_\varepsilon(x)||_{H^1(\Omega)} \leq C\varepsilon ||u_0||_{H^2(\Omega)}. \tag{A.29}$$

step 3: The regularity of u_0, the boundedness of $\tilde{\chi}$, and the definition of η_ε show that

$$||\eta_\varepsilon \cdot \nu||_{L^2(\partial\Omega)} \leq C||u_0||_{H^2(\Omega)}. \tag{A.30}$$

We next show an estimate on $||\eta_\varepsilon \cdot \nu||_{H^{-1}(\partial\Omega)}$. Let $\phi \in H^1(\partial\Omega)$ and consider the boundary value problem

$$\left\{ \begin{aligned} -\operatorname{div}(A\nabla v) &= 0 \quad \text{in } \Omega \\ v &= \phi \quad \text{on } \partial\Omega. \end{aligned} \right. \tag{A.31}$$

From the definition of η_ε and from (A.24), it follows that

$$\begin{aligned} \int_{\partial\Omega} \eta_\varepsilon \cdot \nu \, \phi d\sigma &= \int_\Omega \eta_\varepsilon \cdot \nabla v && \text{(A.32)} \\ &= \int_\Omega a_\varepsilon \nabla u_\varepsilon \cdot \nabla v - \int_\Omega (v_0 + \varepsilon v_1) \cdot \nabla v \\ &= \int_\Omega (a_\varepsilon \nabla u_\varepsilon - A\nabla u_0) \cdot \nabla v + \int_\Omega \sum_{p=1}^3 \operatorname{curl}_y(\tilde{\chi}_p)(x/\varepsilon)\rho_p(x) \cdot \nabla v \\ &\quad + \varepsilon \int_\Omega \operatorname{curl}_x(\tilde{\chi}\rho)(x, x/\varepsilon) \cdot \nabla v \\ &= \varepsilon \int_\Omega \sum_{p=1}^3 \operatorname{curl}_x(\tilde{\chi}_p(x/\varepsilon))\rho_p(x) \cdot \nabla v + \varepsilon \int_\Omega \operatorname{curl}_x(\tilde{\chi}\rho)(x, x/\varepsilon) \cdot \nabla v \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}\int_{\partial\Omega} \eta_\varepsilon \cdot \nu \, \phi d\sigma &= \varepsilon \Bigg\{ \int_\Omega \sum_{p=1}^{3} \tilde{\chi}(x/\varepsilon) \cdot (\nabla \rho_p(x) \times \nabla v(x)) \\ &\quad - \int_{\partial\Omega} \sum_{p=1}^{3} (\tilde{\chi}(x/\varepsilon) \times \nu) \cdot \rho_p(x) \nabla v(x) \, d\sigma \\ &\quad - \int_{\partial\Omega} (\tilde{\chi}_p(x/\varepsilon)\rho_p(x) \times \nu) \cdot \nabla v(x) \, d\sigma \Bigg\} \\ &= \varepsilon \Bigg\{ \int_\Omega \sum_{p=1}^{3} \tilde{\chi}(x/\varepsilon) \cdot (\nabla \rho_p(x) \times \nabla v(x)) \\ &\quad - \int_{\partial\Omega} \sum_{p=1}^{3} \tilde{\chi}(x/\varepsilon) \cdot \rho_p(x)(\nu \times \nabla v(x)) \, d\sigma \\ &\quad - \int_{\partial\Omega} \tilde{\chi}_p(x/\varepsilon)\rho_p(x) \cdot (\nu \times \nabla v(x)) \, d\sigma \Bigg\}.\end{aligned}$$

We remark that $||\nabla v||_{L^2(\Omega)} \leq C||\phi||_{H^1(\partial\Omega)}$ and that $\nu \times \nabla v$ only involves tangential derivatives of v, hence $\nu \times \nabla v \in L^2(\partial\Omega)$ and $||\nu \times \nabla v||_{L^2(\partial\Omega)} \leq C||\phi||_{H^1(\partial\Omega)}$. Since $\tilde{\chi}(x/\varepsilon)$ is bounded in $L^\infty(\partial\Omega)$ and since $\rho(x) = A\nabla u_0(x)$ is regular, it follows that

$$\int_{\partial\Omega} \eta_\varepsilon \cdot \nu \, \phi d\sigma \leq C\varepsilon ||u_0||_{H^2(\Omega)} ||\phi||_{H^1(\partial\Omega)},$$

in other words, $\eta_\varepsilon \cdot \nu_{/\partial\Omega} \in H^{-1}(\partial\Omega)$ and

$$||\eta_\varepsilon \cdot \nu||_{H^{-1}(\partial\Omega)} \leq C\varepsilon ||u_0||_{H^2(\Omega)}. \qquad \text{(A.33)}$$

By interpolation ([17], Theorem 12.3), we obtain from (A.30,A.33)

$$||a_\varepsilon \nabla(\varepsilon B_\varepsilon) \cdot \nu||_{H^{-1/2}(\partial\Omega)} = ||\eta_\varepsilon \cdot \nu||_{H^{-1/2}(\partial\Omega)} \leq C\varepsilon^{1/2} ||u_0||_{H^2(\Omega)}, \qquad \text{(A.34)}$$

and thus from standard elliptic theory we get

$$||\varepsilon B_\varepsilon||_{H^1(\Omega)} \leq C\varepsilon^{1/2} ||u_0||_{H^2(\Omega)}. \qquad \text{(A.35)}$$

step 4: Let $h \in L^2(\Omega)$, with $\int_\Omega h = 0$, and let w_ε denote the solution to

$$\begin{cases} -\mathrm{div}(a_\varepsilon \nabla w_\varepsilon) = h & \text{in } \Omega \\ a_\varepsilon \nabla w_\varepsilon \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases}$$

We define the first terms in the expansion of w_ε by w_0, w_1 and C_ε analogously to u_0, u_1 and B_ε. Similarly to (A.29), they satisfy

$$\begin{aligned}||w_\varepsilon(x) - w_0(x) - \varepsilon w_1(x, x/\varepsilon) - \varepsilon C_\varepsilon(x)||_{H^1(\Omega)} &\leq C\varepsilon ||w_0||_{H^2(\Omega)} \\ &\leq C\varepsilon ||h||_{L^2(\Omega)}. \qquad \text{(A.36)}\end{aligned}$$

From (A.35), we have

$$\begin{aligned}\left|\int_\Omega \varepsilon B_\varepsilon h\right| &\leq \left|\int_\Omega a_\varepsilon \nabla(\varepsilon B_\varepsilon)\cdot\nabla(w_0+\varepsilon w_1+\varepsilon C_\varepsilon)\right| \\ &\quad + C\varepsilon^{3/2}||u_0||_{H^2(\Omega)}||w_0||_{H^2(\Omega)}.\end{aligned} \tag{A.37}$$

We split the first term in the above right hand side in three parts:

$$I_1 = \int_\Omega a_\varepsilon \nabla(\varepsilon B_\varepsilon)\cdot\nabla w_0 = \int_\Omega \eta_\varepsilon\cdot\nabla w_0,$$

has the same form as the right hand side of (A.32). Proceeding as in step 3, we see that this term can be rewritten as

$$\begin{aligned} I_1 &= \varepsilon\left\{\int_\Omega \sum_{p=1}^3 \tilde\chi(x/\varepsilon)\cdot(\nabla\rho_p(x)\times\nabla w_0) - \int_{\partial\Omega}\sum_{p=1}^3(\tilde\chi(x/\varepsilon)\times\nu)\cdot\rho_p(x)\nabla w_0\, d\sigma\right. \\ &\quad \left. - \int_{\partial\Omega}(\tilde\chi\rho(x)\times\nu)\cdot\nabla w_0\, d\sigma\right\}\end{aligned}$$

and thus, as $\tilde\chi(x/\varepsilon)$ is uniformly bounded in $L^\infty(\partial\Omega)$, as $\rho = A\nabla u_0$ and ∇w_0 have at least regularity $H^1(\Omega)$ and $H^{1/2}(\partial\Omega)$, it follows that

$$|I_1| \leq C\varepsilon||u_0||_{H^2(\Omega)}||w_0||_{H^2(\Omega)}. \tag{A.38}$$

For the second term

$$\begin{aligned}|I_2| &= \left|\int_\Omega a_\varepsilon\nabla(\varepsilon B_\varepsilon)\cdot\nabla(\varepsilon w_1)\right| = \left|\int_{\partial\Omega} a_\varepsilon\nabla(\varepsilon B_\varepsilon)\cdot\nu\,\varepsilon w_1\, d\sigma\right| \\ &\leq \varepsilon||a_\varepsilon\nabla(\varepsilon B_\varepsilon)\cdot\nu||_{H^{-1/2}(\partial\Omega)}\,||\chi_j(x/\varepsilon)\frac{\partial w_0}{\partial x_j}||_{H^{1/2}(\partial\Omega)}.\end{aligned}$$

As $\chi_j\in W^{1,\infty}(Y)\cap\mathcal{C}^{1,\mu}(Y), 1\leq j\leq 3$, one easily checks that $||\chi_j(x/\varepsilon)\dfrac{\partial w_0}{\partial x_j}||_{L^2(\partial\Omega)}$ and $\varepsilon||\chi_j(x/\varepsilon)\dfrac{\partial w_0}{\partial x_j}||_{H^1(\partial\Omega)}$ are uniformly bounded. By interpolation, it follows that

$$||\chi_j(x/\varepsilon)\frac{\partial w_0}{\partial x_j}||_{H^{1/2}(\partial\Omega)} \leq C\varepsilon^{-1/2},$$

so that, using (A.34), we obtain

$$I_2 \leq C\varepsilon||u_0||_{H^2(\Omega)}||w_0||_{H^2(\Omega)}. \tag{A.39}$$

Finally, the third term is easily controlled using (A.35) and its equivalent for C_ε

$$\begin{aligned}|I_3| &= \left|\int_\Omega a_\varepsilon\nabla(\varepsilon B_\varepsilon)\cdot\nabla(\varepsilon C_\varepsilon)\right| \\ &\leq C||\varepsilon B_\varepsilon||_{H^1(\Omega)}||\varepsilon C_\varepsilon||_{H^1(\Omega)} \\ &\leq C\varepsilon||u_0||_{H^2(\Omega)}||w_0||_{H^2(\Omega)}.\end{aligned} \tag{A.40}$$

We thus conclude from (A.37–A.40) that

$$\left|\int_\Omega \varepsilon B_\varepsilon h\right| \leq C\varepsilon ||u_0||_{H^2(\Omega)} ||h||_{L^2(\Omega)}$$

and thus

$$||\varepsilon B_\varepsilon||_{L^2(\Omega)} \leq C\varepsilon ||u_0||_{H^2(\Omega)}. \tag{A.41}$$

step 5: Finally, the uniform boundedness of $||u_1(x, x/\varepsilon)||_{L^2(\Omega)}$, (A.29) and (A.41) yield

$$\begin{aligned} ||u_\varepsilon(x) - u_0(x)||_{L^2(\Omega)} &\leq ||u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, x/\varepsilon) - \varepsilon B_\varepsilon(x)||_{H^1(\Omega)} \\ &\quad + \varepsilon ||u_1(x, x/\varepsilon)||_{L^2(\Omega)} + ||\varepsilon B_\varepsilon(x)||_{L^2(\Omega)} \\ &\leq C\varepsilon ||u_0||_{H^2(\Omega)}. \end{aligned}$$

■

A.3 Proof of Theorem 3.6

Let χ and Φ be the first and the second-order matrix of correctors, solutions of the cell problems (3.4) and (3.5)-(3.6) respectively.
Consider the auxiliary function

$$z_\varepsilon = u_\varepsilon - u_0 - \varepsilon\chi(x/\varepsilon) \cdot \nabla u_0 - \varepsilon^2 \Phi(x/\varepsilon) : \nabla^2 u_0. \tag{A.42}$$

Applying L_ε to z_ε, we obtain (with $\chi_\varepsilon(x) = \chi(x/\varepsilon)$ and $\Phi_\varepsilon(x) = \Phi(x/\varepsilon)$)

$$\begin{aligned} L_\varepsilon z_\varepsilon &= \mathrm{div}((a_\varepsilon I - A)\nabla u_0) + \varepsilon \mathrm{div}(a_\varepsilon \nabla(\chi_\varepsilon \nabla u_0)) + \varepsilon^2 \mathrm{div}(a_\varepsilon \nabla(\Phi_\varepsilon \nabla^2 u_0)) \\ &= \mathrm{div}\left((a_\varepsilon(I + \nabla_y \chi) - A)\nabla u_0\right) + \varepsilon \mathrm{div}(a_\varepsilon \chi_\varepsilon \nabla^2 u_0) \\ &\quad + \varepsilon \mathrm{div}(a_\varepsilon \nabla_y \Phi \nabla^2 u_0) + \varepsilon^2 \mathrm{div}(a_\varepsilon \Phi_\varepsilon \nabla^3 u_0) \\ &= (a(I + \nabla_y \chi) - A)\nabla^2 u_0 + \varepsilon a \chi_\varepsilon \nabla^3 u_0 + \mathrm{div}_y(a\chi)\nabla^2 u_0 \\ &\quad + \mathrm{div}_y(a\nabla_y \Phi)\nabla^2 u_0 + \varepsilon a \nabla_y \Phi \nabla^3 u_0 + \varepsilon^2 \mathrm{div}(a_\varepsilon \Phi_\varepsilon \nabla^3 u_0) \end{aligned}$$

Since Φ is the solution of (3.5)-(3.6) and $\displaystyle\int_Y B(y)\, dy = A(y)$, we get

$$L_\varepsilon z_\varepsilon = \varepsilon \left[F_\varepsilon + \varepsilon \mathrm{div}(b_\varepsilon H_\varepsilon)\right]$$

with

$$\begin{aligned} F_\varepsilon(x) &= a(x/\varepsilon)\chi(x/\varepsilon)\nabla^3 u_0(x) + a(x/\varepsilon)\nabla_y \Phi(x/\varepsilon)\nabla^3 u_0(x) \\ b(y) &= a(y)\Phi(y), \\ H_\varepsilon(x) &= \nabla^3 u_0(x), \end{aligned}$$

Noting that Φ has regularity $\mathcal{C}^{0,\mu}$ in $\overline{D_0}$ and in $\overline{Y \setminus D_0}$ by (3.5) and (3.13), we can apply theorem 3.4: for a suitable constant C

$$\|F_\varepsilon\|_{L^\infty(\Omega)} + \|H_\varepsilon\|_{\mathcal{C}^{0,\mu}(\Omega)} \leq C.$$

The interior gradient estimates (3.15) applied to z_ε and lemma 3.5 show then that

$$||z_\varepsilon||_{L^\infty(\omega)} + \|\nabla z_\varepsilon\|_{L^\infty(\omega)} \leq (\|z_\varepsilon\|_{L^2(\Omega)} + C\varepsilon).$$

Since by hypothesis $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon^\sigma$, we conclude that

$$|u_\varepsilon(x) - u_0(x)| + |\nabla u_\varepsilon(x) - (I + \nabla_y \chi(x/\varepsilon))\nabla u_0(x)| \leq C\varepsilon^\sigma \quad \text{a.e. } x \in \omega.$$

■

A.4 Proof of Theorem 3.7

Proof: From the results of section 2, we see that G_ε is Hölder continuous away from its singular point and that there exist a constant C that only depends on λ such that

$$\|G_\varepsilon(.,y)\|_{L^2(\Omega)} \leq C, \quad \text{for } y \in \Omega,$$

since $L^2(\Omega) \subset L^*_{\frac{n}{n-2}}(\Omega)$ when $n = 3$.
Let ω_1, ω_2 be smooth domains such that $\omega \subset\subset \omega_1 \subset\subset \omega_2 \subset\subset \Omega$ and let $y \in \Omega \setminus \omega_2$.
Let

$$\begin{aligned} \sigma_\varepsilon &= \sup\{|G_\varepsilon(x,y) - G_0(x,y)| \;:\; x \in \overline{\omega_1}\} \\ &=: |G_\varepsilon(x_\varepsilon, y) - G_0(x_\varepsilon, y)|. \end{aligned}$$

As $L_\varepsilon G_\varepsilon(.,y) = 0$ in ω_2, theorem 3.4 shows that there is a positive constant C independent of ε and y, such that

$$\|\nabla G_\varepsilon(.,y)\|_{L^\infty(\omega_2)} + \|\nabla G_0(.,y)\|_{L^\infty(\omega_2)} \leq C.$$

Setting $\rho = \min(\sigma_\varepsilon/2C, dist(\omega_1, \omega_2))$, we see that

$$|G_\varepsilon(x,y) - G_0(x,y)| \geq \sigma_\varepsilon/2, \quad \text{for } x \in B_\rho(x_\varepsilon) \cap \omega_1. \tag{A.43}$$

In particular, $G_\varepsilon(.,y) - G_0(.,y)$ keeps a constant sign in $B_\rho(x_\varepsilon)$. Let $f \in \mathcal{C}^\infty_0(B_\rho(x_\varepsilon))$ such that $0 \leq f \leq 1$ and such that $f \equiv 1$ on $B_{\rho/2}(x_\varepsilon)$. We consider the solutions w_ε and w_0, vanishing on $\partial\Omega$ of

$$L_\varepsilon w_\varepsilon = f \quad \text{and} \quad L_0 w_0 = f \qquad \text{in } \Omega.$$

Theorem 2.3 in [1] gives us an interior L^2 estimate for the convergence rate of w_ε to w_0: for some constant C independent of ε

$$\|w_\varepsilon - w_0\|_{L^2(\omega_2)} \leq C\varepsilon.$$

Applying Theorem 3.6, we see that

$$\|w_\varepsilon - w_0\|_{L^\infty(\omega_1)} \leq C\varepsilon. \tag{A.44}$$

Thus we obtain for $x \in B_{\rho/2}(x_\varepsilon) \cap \omega_1$

$$\begin{aligned} C\varepsilon \geq |w_\varepsilon(x) - w_0(x)| &= \int_{B_\rho(x_\varepsilon)} |G_\varepsilon(x,y) - G_0(x,y)| f(y)\, dy. \\ &\geq \int_{B_{\rho/2}(x_\varepsilon)\cap\omega_1} \frac{\sigma_\varepsilon}{2}\, dy. \end{aligned}$$

As the Lebesgue measure of $B_{\rho/2}(x_\varepsilon) \cap \omega_1$ is proportional to ρ^3, we conclude from its definition that $\sigma_\varepsilon = O(\varepsilon^{1/4})$ and that

$$\|G_\varepsilon(.,y) - G_0(.,y)\|_{L^\infty(\omega_1)} \leq C\varepsilon^{1/4}.$$

Moreover, as this estimate also holds in $L^2(\omega_1)$, another application of Theorem 3.6 yields (3.20, 3.21). ■

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