ASYMPTOTIC FORMULAS FOR THE VOLTAGE POTENTIAL IN A COMPOSITE MEDIUM CONTAINING CLOSE OR TOUCHING DISKS OF SMALL DIAMETER

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Abstract. We derive an expansion of the voltage potential in a composite medium, made of circular conducting inclusions of small diameter ε embedded in a homogeneous matrix phase, when the inhomogeneities are strongly interacting, i.e., when they are very close or even touching. The asymptotics of the voltage potential depend on the position of the inclusions and on the contrast between the inclusions and matrix conductivities via a polarization tensor. We are especially interested in determining an analytical expression of this tensor, in order to study how the terms in the expansion depend on the inter-inclusion distance, the inclusion size, and the conductivity contrast. We present numerical tests that compare the true voltage potential to our asymptotic formula when the inclusions are treated as a single inhomogeneity, and to the asymptotic formula when the inclusions are well-separated.

Key words. Asymptotic expansions, polarization tensor, conformal mapping

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1. Introduction. Let Ω be a a bounded smooth domain of $\mathbf{R}^2$ which represents a composite medium, made of conducting inclusions embedded in a conducting matrix phase. The voltage potential in Ω, denoted by u_ε, is the solution to

$$\left\{ \begin{array}{l} \mathrm{div}(\gamma_\varepsilon(x)\nabla u_\varepsilon) = 0 \quad \text{in } \Omega \\ \gamma_\varepsilon \dfrac{\partial u_\varepsilon}{\partial \nu}_{/\partial\Omega} = g \end{array} \right. . \tag{1.1}$$

For simplicity, the matrix phase is assumed to be homogeneous and $\gamma_\varepsilon(x) = 1$ in this phase. The j^{th} inclusion, of constant conductivity $\gamma_\varepsilon = k_j$, has the form

$$\omega_\varepsilon^j = \varepsilon B_j + z_j, \tag{1.2}$$

where B_j is a bounded smooth domain of $\mathbf{R}^2$ that contains the origin. We assume that the centers of the inclusions z_j are far from the boundary, i.e, that $dist(\omega_\varepsilon^j, \partial\Omega) > d_0 >> \varepsilon$ for some $d_0 > 0$. The applied boundary current g satisfies $\displaystyle\int_{\partial\Omega} g \, d\sigma_x = 0$, and ν denotes the unit outward normal to $\partial\Omega$.

When the number of inclusions is relatively small, u_ε is close to the solution u to the homogeneous PDE

$$\left\{ \begin{array}{ll} \Delta u = 0 & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu}_{/\partial\Omega} = g & \end{array} \right. . \tag{1.3}$$

To guarantee the uniqueness of the solutions to the problems (1.1) and (1.3), we choose the potentials u_ε and u such that $\displaystyle\int_{\partial\Omega} u_\varepsilon \, d\sigma_x = \int_{\partial\Omega} u \, d\sigma_x = 0$.

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An asymptotic expansion of u_ε, when the inhomogeneities are assumed to be well-separated from each other and well-separated from the boundary, has been derived by D.J. Fengya, S. Moskow and M. Vogelius [12]. In this case, the inclusions are not strongly interacting and the expansion takes the form

$$
\begin{aligned}
&u_\varepsilon(z) - u(z) + 2\int_{\partial\Omega} (u_\varepsilon(x) - u(x)) \frac{\partial G}{\partial \nu_x}(x, z)\, d\sigma_x \\
&= 2\varepsilon^2 \sum_{j=1}^{m} \frac{1-k_j}{k_j} \nabla_x G(z_j, z) \cdot A_j \nabla_x u(z_j) + O(\varepsilon^{5/2}), \quad z \in \partial\Omega
\end{aligned}, \tag{1.4}
$$

where G is the fundamental solution of the Laplacian in 2-D, m is the number of fibers. The polarization matrix A_j is a symmetric matrix associated with the j-th inclusion. It only depends on the shape and on the conductivity of the inclusion, and can be computed from the corrections to the voltage potential at infinity created by the rescaled j-th inclusion embedded in an infinite matrix phase.

This asymptotic formula is the basis of an efficient algorithm for conductivity imperfection identification [5, 12, 13, 3] and has been generalized to elasticity and to the Maxwell system [4, 21, 1].

When the inclusions are dispersed, well-separated and when their shape is regular, their effect on the potential u_ε cannot be too drastic: it is a piecewise smooth function, its gradient bounded. Such a picture could deteriorate when some inclusions are allowed to get close, forming narrow channels where currents could concentrate. The potential is still a piecewise smooth function (it globally has a $\mathcal{C}^{1,\alpha}$ regularity), but the values of its gradient could be much larger [18, 6]. These high gradients are the source of great difficulties in the numerical determination of the potential. An accurate computation of charge densities would require millions of Fourier coefficients in a multipole expansion, as noted by H. Cheng and L. Greengard [11], who propose a hybrid numerical method which combines multipole expansions and the method of images. The same situation arises in elasticity where nearly touching inclusions could create zones of large stresses which could potentially lead to fracture. Thus, it could be interesting to develop algorithms that detect clusters of inclusions from boundary measurements, which could be used to control the fabrication of certain composites.

The goal of this work is to study a model situation when 2 circular inclusions, of diameter ε and at a distance $\delta\varepsilon$ from each other, are interacting. We investigate how the asymptotic formula (1.4) is modified. We are especially interested in the resulting polarization tensor. In our particular geometry the solution to the auxiliary PDE, from which the tensor is computed, can be represented as a series [8, 18] (see also [16] where the so–called Added Mass tensor is computed as a series for two touching disks). In a recent work, H. Ammari, H. Kang, E. Kim and M. Lim [2] derived an asymptotic expansion for the voltage potential when Lipschitz inclusions are closely spaced and showed that the polarization tensor of such cluster could be represented by an equivalent ellipse. In this analysis, however, inclusions are not allowed to touch. In the particular case of disks, these authors used multiple reflections to derive a series representation of the polarization tensors.

In the cases of close or touching disks, we are able to analyze in a precise manner the influence of the different parameters : distance between inclusions, contrast between the conductivities of the inclusions and the homogeneous medium. In particular, the series that define the polarization tensor formally becomes singular as $\delta \to 0$. Using asymptotics of singular integrals [9, 10], we show however that the series can be expanded and converges to the polarization tensor of two touching inclusions (as it should). The expression of these series is similar to the expression of the gradient of the potential at the midpoint between the inclusions. Thus, we use the same kind of asymptotics to study how the gradient of the potential blows up with the conductivity contrast as the inclusions get nearer. We also show numerical tests concerning the accuracy of our asymptotic expansion that treats the 2 inclusions as a single inhomogeneity. In particular we investigate when the true solution u_ε is better approximated by our expansion, than by the expansion (1.4), when the inclusions are considered to be well-separated.

The paper is organized as follows: In Section 2, we compute the polarization tensors corresponding to two nearly touching and two touching disks. Section 3 is devoted to showing the asymptotic expansion of the potential u_ε in our particular geometry. The argument closely follows [12]. Section 4 presents the asymptotics of the series that define the polarization tensor and the potential gradient when $\delta \to 0$. Finally, in Section 5 we present numerical results about the accuracy of our asymptotic expansion.

2. Computation of the polarization tensor. We consider two conducting disks B_1, B_2 of radius 1, embedded in an infinite matrix phase. The conductivity of this medium is $\gamma(x) = 1$ in the matrix phase and $\gamma(x) = k \neq 1$ in the inclusions. Let w denote the solution to

$$\operatorname{div}(\gamma(x)\nabla w) = 0 \quad \text{in } \mathbf{R}^2, \tag{2.1}$$

which satisfies the far-field boundary conditions

$$\begin{cases} w(x_1, x_2) \sim \displaystyle\sum_{j=1}^{2} c_j x_j, & \text{as } |x| = |(x_1, x_2)| \to \infty, \\ w(x_1, x_2) - \displaystyle\sum_{i=1}^{2} c_j x_j \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{2.2}$$

Equivalently, w can be written

$$w(x_1, x_2) = \sum_{j=1}^{2} c_j (x_j + \varphi^{(j)}(x_1, x_2)), \tag{2.3}$$

where the functions $\varphi^{(j)}$ can be shown to decay at infinity as

$$\varphi^{(j)}(x) = -\frac{1}{2\pi} \sum_{l=1}^{2} m_{jl} \frac{x_l}{|x|^2} + O\left(\frac{1}{|x|^2}\right), \quad \text{as } |x| \to \infty. \tag{2.4}$$

The matrix $M = (m_{jl})$ is called the polarization tensor or the Pólya-Szego matrix [20]. Explicit formulas for M have been established in particular geometries (a cavity, a single rigid circular or elliptical inclusion) [19].

The function w may also be viewed as the x_3–component of the displacement in the transverse shear loading of an infinite elastic matrix that contains two cylindrical inclusions F_j, with axes parallel to x_3. The constants c_j in (2.2) then correspond to a constant stress–field applied at infinity $c_j = \tau_{x_j x_3}, j = 1, 2$. Because of this interpretation of the PDE (2.1) we sometimes refer to the inclusions as 'fibers'.

2.1. The case of two close inclusions . We consider the geometry illustrated in Figure (a). It consists of two circular inclusions, centered at $(\pm 1(\delta+1), 0)$, $\delta > 0$, of radii 1 and conductivity k. In order to simplify the computations, we restrict ourselves to the case when $c_1 \neq 0$ and $c_2 = 0$ at infinity (i.e. $\tau_{x_1 z} = \tau_0$ and $\tau_{x_2 z} = 0$ in the plane shear problem), i.e., we only present the calculation of the first line of the polarization tensor M_δ. The remaining coefficients can be computed in an analogous fashion.

To evaluate the matrix M_δ, we introduce the complex-valued function $f(z)$, $z = x_1 + i x_2$, such that

$$w = c_1 \, Re f. \tag{2.5}$$

This function is analytic in the matrix and in the fibers, and satisfies

$$f \sim z, \quad \text{when } |z| \to \infty. \tag{2.6}$$

The symmetries of the geometry and the uniqueness of w impose that

$$w(x_1, x_2) = -w(-x_1, -x_2) = w(x_1, -x_2),$$

from which it follows that $w(0, x_2) = \dfrac{\partial w}{\partial x_2}(x_1, 0) = 0$ and $Im\, f(0) = 0$, hence

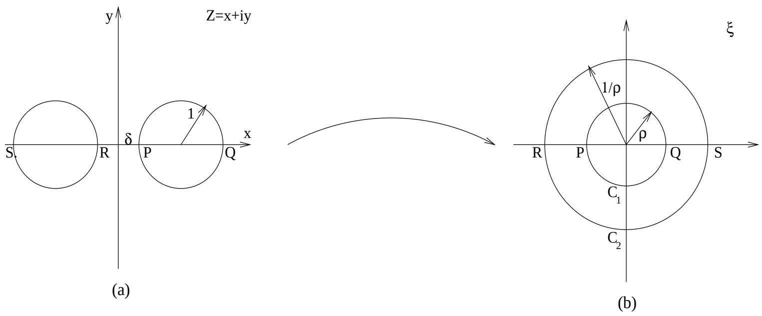
$$f(z) = -f(-z) = \overline{f}(z). \tag{2.7}$$

Let f_M and f_F denote the values of f in the matrix and in the right-hand fiber B_1, respectively. According to the interface conditions, $Re f$ and the conormal derivative $\gamma(z)\dfrac{\partial}{\partial \nu}(Re f)$ are continuous across ∂B_1. Due to the Cauchy Riemann equations, the jump conditions satisfied by w across ∂B_1 imply that $Re f$ and $\gamma(Z)\dfrac{\partial}{\partial \tau}(Im\, f)$ are continuous across ∂B_1, where $\partial/\partial\tau$ denotes the tangential derivative. We then have

$$Re\, f_M + i Im\, f_M = Re\, f_F + ik Im\, f_F, \quad \text{along } \partial B_1, \tag{2.8}$$

where $i^2 = -1$.

Let us consider the conformal mapping $\xi = \dfrac{z-a}{z+a}$, where $a = \sqrt{\delta(2+\delta)}$, which



sends the right-hand fiber into the interior of a circle C_1, centered at $\xi = 0$ and of radius $\rho = \dfrac{a-\delta}{a+\delta}$, the left-hand fiber into the exterior of a circle C_2, centered at $\xi = 0$

and of radius $1/\rho$, while the matrix phase is transformed into the annulus of radii $\rho < |\xi| < \rho^{-1}$, bounded by the circles C_1 and C_2 [8]. Therefore we can write

$$f_M = z + g_M(\xi) = a\frac{1+\xi}{1-\xi} + g_M(\xi), \tag{2.9}$$

where g_M is an analytic function in the annulus. Using the Cauchy integral formula, g_M can be decomposed as

$$g_M(\xi) = g_1(\xi) + g_2(\xi), \quad \rho < |\xi| < \rho^{-1},$$

where

$$g_1(\xi) = \frac{1}{2i\pi}\int_{C_2} \frac{g_M(\sigma)}{\sigma - \xi}\, d\sigma \quad \text{for } |\xi| < \rho^{-1}$$

and

$$g_2(\xi) = -\frac{1}{2i\pi}\int_{C_1} \frac{g_M(\sigma)}{\sigma - \xi}\, d\sigma \quad \text{for} \rho < |\xi|.$$

Using the symmetry properties (2.7), we have

$$g_M(\xi) = \overline{g}_M(\xi) = -g_M(1/\xi),$$

hence, $\overline{g_1}(\xi) = g_1(\xi)$, $\overline{g_2}(\xi) = g_2(\xi)$ and $g_1(\xi) = -g_2(1/\xi)$. Identity (2.9) becomes

$$\begin{aligned} f_M(\xi) &= a\frac{1+\xi}{1-\xi} + g_1(\xi) + g_2(\xi) \\ &= a\frac{1+\xi}{1-\xi} + g_1(\xi) - g_1(1/\xi), \end{aligned} \tag{2.10}$$

where $g_1(\xi)$ is analytic for $|\xi| < \rho^{-1}$.

In order to compute the coefficients of the tensor M_δ, we seek an expansion of $f_M(\xi)$ as a power series of ξ. To this end, the following Lemma gives us a representation of the analytic function g_1 in the ball $B(0, \rho^{-1})$.

Lemma 1. *[8] The function g_1 has the following expansion*

$$g_1(\xi) = g_1(0) + \sum_{n\geq 1} b_n \xi^n \qquad \textit{when } |\xi| \leq \rho^{-1} \tag{2.11}$$

where $b_n = 2a\rho^{2n}\left[\Lambda - \rho^{2n}\right]^{-1}$ and $\Lambda = (k+1)/(k-1)$.

We obtain then the form of M_δ :

Proposition 1. *The polarization tensor M_δ of two unit disks centered at $(\pm(\delta + 1), 0)$, $\delta > 0$, is given by*

$$M_\delta = \left(16a^2\pi \sum_{n\geq 1} n \frac{\rho^{2n}}{\Lambda + (-1)^j \rho^{2n}} \delta_{jl}\right)_{1\leq j,l\leq 2}, \tag{2.12}$$

where $\Lambda = (k+1)/(k-1)$ depends on the contrast, $a = \sqrt{\delta(2+\delta)}$ and $\rho = \dfrac{a-\delta}{a+\delta}$.

Proof : Using equation (2.11) and changing the variables back to the z-plane, we get

$$\begin{aligned}
g_M(\xi) &= \sum_{n\geq 1} b_n \xi^n - \sum_{n\geq 1} \frac{b_n}{\xi^n} \\
&= \sum_{n\geq 1} b_n \left(\sum_{p=0}^{n} C_n^p (2a)^p \left(\frac{(-1)^p}{(z+a)^p} - \frac{1}{(z-a)^p} \right) \right) \\
&= \left(\sum_{n\geq 1} n b_n \right) 2a \left(\frac{-2z}{z^2 - a^2} \right) + \sum_{n\geq 2} b_n \left(\sum_{p=2}^{n} C_n^p (2a)^p \left(\frac{(-1)^p}{(z+a)^p} - \frac{1}{(z-a)^p} \right) \right)
\end{aligned}$$

According to the d'Alembert criterion, the series $\sum n b_n$ converges absolutely when $|\rho| < 1$. We next prove that

$$\sum_{n\geq 2} b_n \left(\sum_{p=2}^{n} C_n^p (2a)^p \left(\frac{(-1)^p}{(z+a)^p} - \frac{1}{(z-a)^p} \right) \right) = O(\frac{1}{z^2}). \tag{2.13}$$

As $C_n^{p+2} = \frac{n(n-1)}{(p+2)(p+1)} C_{n-2}^p$, we get with $q = p - 2$,

$$\begin{aligned}
\sum_{n\geq 2} b_n \left(\sum_{p=2}^{n} C_n^p (2a)^p \frac{(-1)^p}{(z+a)^p} \right) &= \sum_{n\geq 2} b_n \left(\sum_{q=0}^{n-2} C_n^{q+2} (2a)^{q+2} \frac{(-1)^q}{(z+a)^{q+2}} \right) \\
&= \sum_{n\geq 2} b_n \left(\sum_{q=0}^{n-2} \frac{n(n-1)}{(q+2)(q+1)} C_{n-2}^q (2a)^{q+2} \frac{(-1)^q}{(z+a)^{q+2}} \right)
\end{aligned}$$

Hence,

$$\begin{aligned}
\left| \sum_{n\geq 2} b_n \left(\sum_{p=2}^{n} C_n^p (2a)^p \frac{(-1)^p}{(z+a)^p} \right) \right| &\leq \frac{2a^2}{|z+a|^2} \sum_{n\geq 2} n^2 b_n \left| 1 - \frac{2a}{z+a} \right|^{n-2} \\
&\leq \frac{2a^2}{|z+a|^2} \sum_{n\geq 2} n^2 b_n \left| \frac{z-a}{z+a} \right|^{n-2}.
\end{aligned}$$

When $|z|$ is large enough, $\left| \dfrac{z-a}{z+a} \right| < 1/\rho$ and the series above is bounded by the series

$$\sum_{n\geq 2} n^2 b_n \rho^{-n+2} \leq 2\alpha a \sum_{n\geq 2} n^2 \rho^{n+2},$$

which converges when $|\rho| < 1$. Thus, $\displaystyle\sum_{n\geq 2} b_n \left(\sum_{p=2}^{n} C_n^p (2a)^p \frac{(-1)^p}{(z+a)^p} \right) = O(\frac{1}{z^2})$ as $z \to \infty$. Similarly, we can show that $\displaystyle\sum_{n\geq 2} b_n \left(\sum_{p=2}^{n} C_n^p (2a)^p \frac{1}{(z-a)^p} \right) = O(\frac{1}{z^2})$, which proves (2.13). In conclusion,

$$f(z) = z - g_M(\xi) = z - 4a \left(\sum_{n\geq 1} n b_n \right) \frac{1}{z} + O(\frac{1}{z^2}).$$

We deduce that $\varphi^{(1)}(x_1,x_2) = Ref(z)-x_1 = -\frac{1}{2\pi}\sum_{j=1}^{2} m_{1j}\dfrac{x_j}{|x|^2}+O\left(\dfrac{1}{|x|^2}\right)$, from which we obtain m_{11} and m_{12}. The other components of the polarization tensor M can be computed in a similar fashion. The result is

$$m_{jl} = 16a^2\pi\sum_{n\geq 1} n\frac{\rho^{2n}}{\Lambda+(-1)^j\rho^{2n}}\delta_{jl} \quad j,l=1,2,$$

■

REMARK 1. *These computations can be generalized to the case of two inclusions B_1 and B_2, with radii $R_1 \leq R_2$, centered on the x_1–axis, and at a distance 2δ apart. We assume that the conductivity of the matrix phase is k_0, the conductivities of the inclusions are k_1 and k_2, and that the origin lies inside the right–hand inclusion B_1, we use the following conformal mapping to transform ∂B_1 and ∂B_2 into two concentric circles. The mapping has the form $z \to \xi = \frac{az}{a+z}$ where $a = R_2E$ and E is defined by*

$$E = \frac{\sqrt{(\beta-1)^2-2\alpha^2(\beta^2+1)+\alpha^4(\beta+1)^2}}{\alpha},$$

with $\alpha = 1+\frac{2\delta}{R_1+R_2}, \beta = \frac{R_1}{R_2}$. The boundaries of B_1 and B_2 are mapped onto the circles centered at 0 of radii $r_1 = R_2\frac{E\sqrt{E^2+4\beta^2}-E^2}{2\beta}$ and $r_2 = R_2\frac{E\sqrt{E^2+4}+E^2}{2}$.

The computation of the polarization tensor follows the same lines as above. One finds (see [14])

$$m_{ij} = 2\pi a^2 k_0\sum_{n=-\infty}^{\infty} na^n c_n^{(i)}\delta_{ij}.$$

The coefficients c_n are given by

$$\begin{aligned} c_n^{(i)} &= D_4(D_1a^nr_1^{-2n}+(-1)^{(i+1)}D_3a^{-n})L_n && \textit{if } n>0\\ c_n^{(i)} &= -D_3(D_2a^nr_2^{-2n}+(-1)^{(i+1)}D_4a^{-n})L_{-n} && \textit{if } n<0\end{aligned}$$

with $L_n = (D_1D_2r_2^{2n}r_1^{-2n}-D_3D_4)^{-1}$, $D_1 = k_1+k_0$, $D_2 = k_2+k_0$, $D_3 = k_1-k_0$ and $D_4 = k_2-k_0$.

2.2. The case of two touching inclusions . This case deserves special treatment as the series appearing in (2.12) diverges when $\delta = 0$ (i.e $\rho = 1$).

The polarization tensor is again defined via the solution w to (2.1), where B_j, $j = 1,2$, now denote two circular inclusions of radius 1 with centers $(\pm 1, 0)$. The conductivity $\gamma(x)$ is equal to $k \neq 1$ inside B_j, and to 1 in the outside matrix phase. Again, we only detail the calculation when $c_1 \neq 0$ and $c_2 = 0$, in which case w has the following asymptotic behavior :

$$w(x_1,x_2) = x_1+\varphi^{(1)}(x_1,x_2), \qquad \|x\| \to \infty, \tag{2.14}$$

with $\varphi^{(1)}(x) \to 0$ at infinity.

A generic representation of solutions to such problems can be obtained as follows (see [18]). By symmetry, w is even with respect to the x_2–axis and odd with respect to the x_1–axis. Setting $z = x_1+ix_2$, the conformal mapping $z \to \xi = 1/z$ transforms the complex plane containing the two unit circles centered at ± 1 onto the complex plane with the vertical lines $Re\xi = -1/2$ and $Re\xi = 1/2$. The interior of the right-hand circle is mapped onto $Re\xi > 1/2$ and that of the left one to $Re\xi < -1/2$, while the exterior of the fibers is mapped onto the layer $-1/2 < Re\xi < 1/2$.

Let ϕ be an odd function of ξ, analytic in $\mathbf{C} \setminus 0$, which satisfies also :

$$\phi(\overline{\xi}) = \overline{\phi(\xi)} \tag{2.15}$$

$$|\phi(\xi)| \leq C\beta^{|Re\xi|}, \quad 1/2 < |Re\xi|, \tag{2.16}$$

for some $0 < \beta < |\Lambda|$. We define

$$\Phi(\xi) = -\frac{2}{k+1}\sum_{n=0}^{\infty}\Lambda^{-n}\phi(n-\xi), \quad \text{if } Re\xi < -1/2$$

$$\Phi(\xi) = \phi(\xi) + \sum_{n=1}^{\infty}\Lambda^{-n}\left(\phi(n+\xi) - \phi(n-\xi)\right), \quad \text{if } -1/2 < Re\xi < 1/2 \tag{2.17}$$

$$\Phi(\xi) = \frac{2}{k+1}\sum_{n=0}^{\infty}\Lambda^{-n}\phi(n+\xi), \quad \text{if } 1/2 < Re\xi.$$

PROPOSITION 2. *The function* $w(x_1, x_2) = Re\,\Phi(1/z)$ *solves*

$$div(\gamma(x)\nabla w) = 0. \tag{2.18}$$

This function is even with respect to the x_2*-axis and odd with respect to the* x_1*–axis.*

We choose $\phi(\xi) = 1/\xi$ in Proposition 2, which satisfies hypotheses (2.15) and (2.16) for $\beta = 1$. We next show that this choice guarantees that the corresponding w grows linearly as $z \to \infty$. Indeed, in the layer $-1/2 < Re\xi < 1/2$, we have

$$\begin{aligned}
\Phi(\xi) &= \frac{1}{\xi} + \sum_{n=1}^{\infty}\Lambda^{-n}\left(\frac{1}{n+\xi} - \frac{1}{n-\xi}\right) \\
&= \frac{1}{\xi} + \sum_{n=1}^{\infty}\Lambda^{-n}\frac{-2\xi}{n^2}\frac{1}{1-(\xi/n)^2} \\
&= \frac{1}{\xi} - 2\xi\sum_{n=1}^{\infty}\frac{1}{n^2\Lambda^n}\left(\sum_{p=0}^{\infty}(\frac{\xi}{n})^{2p}\right) \\
&= z - 2\left(\sum_{n=1}^{\infty}\frac{1}{n^2\Lambda^n}\right)\frac{1}{z} - 2\left(\sum_{n=1}^{\infty}\frac{1}{n^4\Lambda^n}\sum_{p=0}^{\infty}\frac{1}{(nz)^{2p}}\right)\frac{1}{z^3}
\end{aligned}$$

When $|z|$ is sufficiently large, the series $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^4\Lambda^n}\sum_{p=0}^{\infty}\frac{1}{(nz)^{2p}} = \sum_{n\geq 1}\frac{1}{n^2\Lambda^n}\frac{z^2}{(nz)^2-1}$ converges and is bounded by $\displaystyle\sum_{n\geq 1}\frac{1}{n^2\Lambda^n}$. Therefore,

$$\Phi(\xi) = z - 2\left(\sum_{n=1}^{\infty}\frac{1}{n^2\Lambda^n}\right)\frac{1}{z} + O(1/z^3).$$

Hence it follows that

$$\varphi^{(1)}(x_1,x_2) = Re\left(-2\left(\sum_{n=1}^{\infty}\frac{1}{n^2\Lambda^n}\right)\frac{1}{z} + O(1/z^3)\right)$$
$$= -\frac{1}{2\pi}\sum_{j=1}^{2} m_{1j}\frac{x_j}{|x|^2} + O(1/|x|^2),$$

and the coefficients m_{1j} of the first line of the polarization tensor M_0 prove to be

$$m_{11} = 4\pi\sum_{n=1}^{\infty}\frac{1}{\Lambda^n n^2} \quad \text{and} \quad m_{12} = 0.$$

The remaining coefficients of the polarization tensor are determined by seeking solutions to (2.1) which are odd with respect to the x_2–axis, and the adequate choice of ϕ yields

$$w(x_1,x_2) = \; Re(-i/\xi + \sum_{n=1}^{\infty}(-1)^n\Lambda^{-n}\left(\frac{i}{n-\xi} - \frac{i}{n+\xi}\right)),$$

which grows linearly as $z \to \infty$. It follows that

$$m_{22} = -4\pi\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2\Lambda^n} \quad \text{and} \quad m_{21} = 0.$$

We have thus proved the following

PROPOSITION 3. *The polarization tensor M_0 of two touching inclusions centered at the points $(\pm 1, 0)$ is given by*

$$M_{0,ij} = (-1)^{(j+1)}4\pi\sum_{n=1}^{\infty}\frac{(-1)^{(j+1)n}}{\Lambda^n n^2}. \tag{2.19}$$

3. Derivation of the asymptotic formula. This Section follows closely the work of D.G. Fengya, S. Moskow and M. Vogelius. Merely, we show that the proof [12] still holds when two close fibers are considered a single inhomogeneity.

When the inclusions are not degenerate (i.e. their conductivity is $k_j > 0, k_j \neq 1$) the first term in the expansion of u_ε is the background potential u, solution to the homogeneous problem (1.3). In fact, u_ε converges strongly in $H^1(\Omega)$ to u when ε tends to 0. This is a consequence of the following estimate of the $H^1(\Omega)$ norm of $u_\varepsilon - u$:

LEMMA 2. *[12] There exists a constant C, independent of ε, such that*

$$\int_\Omega \left(|\nabla(u-u_\varepsilon)|^2 + |u-u_\varepsilon|^2\right)\, dx \leq C\varepsilon^2.$$

Henceforth, we focus on our particular geometry, i.e., that of two inclusions of the form

$$\omega_\varepsilon^j = \varepsilon RB + z_j, \quad j = 1,2, \tag{3.1}$$

where B is the unit ball centered at the origin, R a dilatation parameter, and where the points z_j are the centers of the touching or close disks. Without loss of generality, we assume that $R = 1$ and $z_j = ((-1)^j\varepsilon(\delta+1), 0)$, $\delta \geq 0$ small. Let us consider the sets

$$\tilde{\Omega}_\varepsilon = \{x/\varepsilon : x \in \Omega\}$$

$$\text{and } \tilde{\omega} = \{x/\varepsilon : x \in \omega_\varepsilon\}$$

deduced from Ω and from $\omega_\varepsilon = \omega_\varepsilon^1 \cup \omega_\varepsilon^2$ by the rescaling $y = x/\varepsilon$. Let ν_y be the outward unit normal to both $\partial\tilde{\Omega}_\varepsilon$ and $\partial\tilde{\omega}$. We introduce the solutions v and v_ε to the two following PDE's:

$$\left\{\begin{array}{l} \Delta_y v = 0 \text{ in } \tilde{\omega} \qquad \Delta_y v = 0 \text{ in } \mathbf{R}^2\backslash\tilde{\omega} \\[2mm] v \text{ is continuous across } \partial\tilde{\omega} \\[2mm] \dfrac{\partial v^+}{\partial\nu_y} - k\dfrac{\partial v^-}{\partial\nu_y} = (k-1)\nabla_x u(0)\cdot\nu_y \quad \text{on } \partial\tilde{\omega} \\[2mm] v = -\dfrac{1}{2\pi}\nabla_x u(0)\cdot M\dfrac{y}{|y|^2} + O(\dfrac{1}{|y|^2}) \text{ when } |y| \to \infty \end{array}\right. \tag{3.2}$$

and

$$\left\{\begin{array}{l} \Delta_y v_\varepsilon = 0 \text{ in } \tilde{\omega} \qquad \Delta_y v_\varepsilon = 0 \text{ in } \tilde{\Omega}_\varepsilon\backslash\tilde{\omega} \\[2mm] v_\varepsilon \text{ is continuous across } \partial\tilde{\omega} \\[2mm] \dfrac{\partial v_\varepsilon^+}{\partial\nu_y} - k\dfrac{\partial v_\varepsilon^-}{\partial\nu_y} = (k-1)\nabla_x u(0)\cdot\nu_y \quad \text{on } \partial\tilde{\omega} \\[2mm] \dfrac{\partial v_\varepsilon}{\partial\nu_y} = 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon, \qquad \displaystyle\int_{\partial\tilde{\Omega}_\varepsilon} v_\varepsilon = 0. \end{array}\right. \tag{3.3}$$

REMARK 2. *The function v is connected to the background potential u and to the functions $\varphi^{(1)}$ and $\varphi^{(2)}$, introduced in the previous section and satisfying (2.3)-(2.4), by the relation*

$$v(y) = \sum_{j=1}^{2}\frac{\partial u}{\partial x_j}(0)\varphi^{(j)}(y). \tag{3.4}$$

The next two lemmas were proved in [12] for a domain containing a single inclusion. We give the proof of these two results for our particular geometry.

LEMMA 3. *There exists a constant C, independent of ε, such that*

$$||\nabla_y(u_\varepsilon(\varepsilon y) - u(\varepsilon y) - \varepsilon v_\varepsilon(y))||_{L^2(\tilde{\Omega}_\varepsilon)} \leq C\varepsilon^2. \tag{3.5}$$

Proof: We define $z_\varepsilon(y) = u_\varepsilon(\varepsilon y) - u(\varepsilon y) - \varepsilon v_\varepsilon(y) - c_\varepsilon$, the constant c_ε being chosen so that $\displaystyle\int_{\partial\tilde{\omega}} z_\varepsilon = 0$.

According to equations (1.1), (1.3) and (3.3), z_ε solves the problem

$$
\left\{
\begin{array}{l}
\Delta_y z_\varepsilon = 0 \text{ in } \tilde{\omega} \qquad \Delta_y z_\varepsilon = 0 \text{ in } \tilde{\Omega}_\varepsilon \backslash \tilde{\omega} \\
z_\varepsilon \text{ continuous across } \partial\tilde{\omega} \\
\dfrac{\partial z_\varepsilon^+}{\partial \nu_y} - k\dfrac{\partial z_\varepsilon^-}{\partial \nu_y} = \varepsilon(1-k)(\nabla_x u(0) - \nabla_x u(\varepsilon y)) \cdot \nu_y \quad \text{on } \partial\tilde{\omega} \\
\dfrac{\partial z_\varepsilon}{\partial \nu_y} = 0 \quad \text{on } \partial\tilde{\Omega}_\varepsilon
\end{array}
\right. \tag{3.6}
$$

We consider the quantity

$$\int_{\tilde{\Omega}_\varepsilon} \gamma_\varepsilon(\varepsilon y) \nabla_y z_\varepsilon \nabla_y z_\varepsilon \, dy = \int_{\tilde{\Omega}_\varepsilon \backslash \tilde{\omega}} \nabla_y z_\varepsilon \nabla_y z_\varepsilon \, dy + \int_{\tilde{\omega}} k \nabla_y z_\varepsilon \nabla_y z_\varepsilon \, dy$$

If we integrate by parts the integrals on the right-hand side and use the transmission conditions satisfied by z_ε across $\partial\tilde{\omega}$, we get

$$
\begin{aligned}
\int_{\tilde{\Omega}_\varepsilon} \gamma_\varepsilon(\varepsilon y) \nabla_y z_\varepsilon \nabla_y z_\varepsilon \, dy &= -\int_{\partial\tilde{\omega}} \frac{\partial z_\varepsilon^+}{\partial \nu_y} z_\varepsilon + \int_{\partial\tilde{\omega}} k \frac{\partial z_\varepsilon^-}{\partial \nu_y} z_\varepsilon \\
&= \varepsilon(k-1) \int_{\partial\tilde{\omega}} (\nabla_x u(0) - \nabla_x u(\varepsilon y)) \cdot \nu_y z_\varepsilon \\
&\leq \varepsilon |k-1| \|(\nabla_x u(0) - \nabla_x u(\varepsilon y)) \cdot \nu_y\|_{L^2(\partial\tilde{\omega})} \|z_\varepsilon\|_{L^2(\partial\tilde{\omega})}
\end{aligned}
$$

The Taylor expansion and the fact that u and all its first and second derivatives are uniformly bounded in ω_ε imply that

$$\|(\nabla_x u(0) - \nabla_x u(\varepsilon y)) \cdot \nu_y\|_{L^\infty(\partial\tilde{\omega})} = O(\varepsilon).$$

Using the Trace Theorem, the Poincaré Wirtinger inequality, and the fact that γ_ε is bounded we get

$$
\begin{aligned}
\|\nabla_y z_\varepsilon\|^2_{L^2(\tilde{\Omega}_\varepsilon)} &\leq c\varepsilon^2 \|z_\varepsilon\|_{L^2(\partial\tilde{\omega})} &&\leq c\varepsilon^2 \|z_\varepsilon\|_{H^1(\tilde{\omega})} \\
&\leq c\varepsilon^2 \|\nabla_y z_\varepsilon\|_{L^2(\tilde{\omega})} &&\leq c\varepsilon^2 \|\nabla_y z_\varepsilon\|_{L^2(\tilde{\Omega}_\varepsilon)},
\end{aligned}
$$

which proves the Lemma. ■

LEMMA 4. *There exists a constant C, independent of ε, such that*

$$\|\nabla_y \left(u_\varepsilon(\varepsilon y) - u(\varepsilon y) - \varepsilon v(y)\right)\|_{L^2(\tilde{\Omega}_\varepsilon)} \leq C\varepsilon^2. \tag{3.7}$$

Proof: According to Lemma 3, it suffices to prove that $\|\nabla_y(v_\varepsilon(y) - v(y))\|_{L^2(\tilde{\Omega}_\varepsilon)} \leq c\varepsilon$. To this end, we set $\phi_\varepsilon = v_\varepsilon(y) - v(y) - c_\varepsilon$, where c_ε is chosen so that $\int_{\partial\tilde{\Omega}_\varepsilon} \phi_\varepsilon = 0$. Using equations (3.2) and (3.3), we deduce that ϕ_ε is the solution to

$$
\left\{
\begin{array}{l}
div(\gamma_\varepsilon(\varepsilon y) \nabla \phi_\varepsilon) = 0 \text{ in } \tilde{\Omega}_\varepsilon \\
\dfrac{\partial \phi_\varepsilon}{\partial \nu_y} = -\dfrac{\partial v}{\partial \nu_y} \quad \text{on } \partial\tilde{\Omega}_\varepsilon \quad \textit{and} \int_{\partial\tilde{\Omega}_\varepsilon} \phi_\varepsilon = 0.
\end{array}
\right. \tag{3.8}
$$

Integrating by parts and changing variables back to the fixed domain Ω, yields

$$\begin{aligned}\int_{\tilde{\Omega}_\varepsilon} \gamma_\varepsilon(\varepsilon y)\nabla_y\phi_\varepsilon(y)\nabla_y\phi_\varepsilon(y)\,dy &= \int_{\partial\tilde{\Omega}_\varepsilon} -\frac{\partial v}{\partial\nu_y}(y)\phi_\varepsilon(y)\\ &= \varepsilon^{-1}\int_{\partial\Omega} -\frac{\partial v}{\partial\nu_y}(x/\varepsilon)\phi_\varepsilon(x/\varepsilon)\\ &\le \varepsilon^{-1}\|\frac{\partial v}{\partial\nu_y}(x/\varepsilon)\|_{L^2(\partial\Omega)}\,\|\phi_\varepsilon(x/\varepsilon)\|_{L^2(\partial\Omega)}.\end{aligned}$$

According to the Trace Theorem and to the Poincaré Wirtinger inequality, we have

$$\|\phi_\varepsilon(x/\varepsilon)\|_{L^2(\partial\Omega)} \le c\|\phi_\varepsilon(x/\varepsilon)\|_{H^1(\Omega)} \le c\|\nabla_y\phi_\varepsilon(y)\|_{L^2(\tilde{\Omega}_\varepsilon)},$$

while the asymptotic behavior (2.4) of $\varphi^{(j)}$ implies that

$$\frac{\partial v}{\partial\nu_y}(x/\varepsilon) = O(\varepsilon^2).$$

We conclude that

$$\begin{aligned}\|\nabla_y\phi_\varepsilon(x/\varepsilon)\|^2_{L^2(\tilde{\Omega}_\varepsilon)} &\le c\int_{\tilde{\Omega}_\varepsilon}\gamma_\varepsilon(\varepsilon y)\nabla_y\phi_\varepsilon\nabla_y\phi_\varepsilon\,dy\\ &\le c\varepsilon\|\nabla_y\phi_\varepsilon(x/\varepsilon)\|_{L^2(\tilde{\Omega}_\varepsilon)},\end{aligned}$$

and the lemma is proved. ∎

Let G denote the fundamental solution of the Laplacian in $\mathbf{R}^2$

$$G(x,y) = -\frac{1}{2\pi}\log|x-y|, \quad \text{for } x,y\in\Omega',\ \Omega\subset\subset\Omega'.$$

The asymptotic behavior of the potential u_ε can now be deduced from (3.7)

Theorem 1. *Let Ω be an open bounded smooth domain in $\mathbf{R}^2$, containing two circular inclusions with centers $z_i\in\Omega$, corresponding to the geometry (3.1). Denote by z_0 the middle of the segment $[z_1,z_2]$. Let u_ε and u be the potentials, solutions of the problems (1.1) and (1.3) respectively. Then, for all $z\in\partial\Omega$ and for ε small enough, we have*

$$\begin{aligned}&u_\varepsilon(z)-u(z)+2\int_{\partial\Omega}(u_\varepsilon(x)-u(x))\frac{\partial G}{\partial\nu_x}\,d\sigma_x\\ &= -2\varepsilon^2\nabla_x G(z_0,z)\cdot M_\delta\nabla_x u(z_0)+O(\varepsilon^3).\end{aligned}\tag{3.9}$$

When the distance between inclusions is equal to $2\delta\varepsilon>0$, the polarization tensor M_δ is given by (2.12), while in the case of touching fibers it is given by (2.19).

Proof : We only consider the case of two inclusions centered at $((-1)^j\varepsilon(\delta+1),0)$, $\delta\ge 0$, of radii ε with $R=1$. Let z be a point in Ω which lies at a fixed distance d away from the fibers. From Green's formula we have

$$\begin{aligned}u_\varepsilon(z) &= -\int_\Omega u_\varepsilon\Delta_x G(x,z)\,dx\\ &= \int_{\Omega\setminus\omega_\varepsilon}\nabla_x u_\varepsilon\nabla_x G\,dx+\int_{\omega_\varepsilon}\nabla_x u_\varepsilon\nabla_x G\,dx-\int_{\partial\Omega}u_\varepsilon\frac{\partial G}{\partial\nu_x}\,d\sigma_x\end{aligned}$$

$$= \int_{\partial\Omega} gG\,d\sigma_x - \int_{\partial\omega_\varepsilon} \frac{\partial u_\varepsilon^+}{\partial\nu_x} G\,d\sigma_x + \int_{\partial\omega_\varepsilon} \frac{\partial u_\varepsilon^-}{\partial\nu_x} G\,d\sigma_x - \int_{\partial\Omega} u_\varepsilon \frac{\partial G}{\partial\nu_x}\,d\sigma_x$$

$$= \int_{\partial\Omega} gG(x,z)\,d\sigma_x + (1-k)\int_{\partial\omega_\varepsilon} \frac{\partial u_\varepsilon^-}{\partial\nu_x} G(x,z)\,d\sigma_x - \int_{\partial\Omega} u_\varepsilon \frac{\partial G}{\partial\nu_x}(x,z)\,d\sigma_x,$$

where in the last equation we have used the transmission condition $\partial u_\varepsilon^+/\partial\nu_x = k(\partial u_\varepsilon^-/\partial\nu_x)$ on $\partial\omega_\varepsilon$. Introducing $r_\varepsilon(x) = u_\varepsilon(x) - u(x) - \varepsilon v(x/\varepsilon)$, where v denotes the solution to (3.2), we have

$$u_\varepsilon(z) = \int_{\partial\Omega} (gG - u_\varepsilon \frac{\partial G}{\partial\nu_x})\,d\sigma_x$$

$$+(1-k)\int_{\partial\omega_\varepsilon} \left(\frac{\partial u}{\partial\nu_x} + \frac{\partial v^-}{\partial\nu_y}\right) G \;+\; \frac{\partial r_\varepsilon^-}{\partial\nu_x} G\;d\sigma_x \tag{3.10}$$

Integrating by parts the last term in (3.10) and changing the variables, we obtain

$$\int_{\partial\omega_\varepsilon} \frac{\partial r_\varepsilon^-}{\partial\nu_x}(x)G(x,z)\,d\sigma_x = \varepsilon\int_{\tilde\omega} \nabla_y r_\varepsilon(\varepsilon y)\nabla_x G(\varepsilon y,z)\,dy.$$

According to Lemma 4, $\|\nabla_y r_\varepsilon(\varepsilon y)\|_{L^2(\tilde\Omega_\varepsilon)} \le c\varepsilon^2$, and since $\nabla_x G(\varepsilon y,z)$ is uniformly bounded on $\tilde\omega$, we deduce that

$$\int_{\partial\omega_\varepsilon} \frac{\partial r_\varepsilon^-}{\partial\nu_x}(x)G(x,z)\,d\sigma_x = O(\varepsilon^3). \tag{3.11}$$

Expanding the kernel G in a Taylor series about the origin, in the second term in the right–hand side of (3.10), yields

$$\begin{aligned}\int_{\partial\omega_\varepsilon} \left(\frac{\partial u}{\partial\nu_x} + \frac{\partial v^-}{\partial\nu_y}\right) G\,d\sigma_x &= \varepsilon\int_{\partial\tilde\omega} \left(\frac{\partial u}{\partial\nu_x}(\varepsilon y) + \frac{\partial v^-}{\partial\nu_y}(y)\right) G(\varepsilon y,z)\,d\sigma_y \\ &= \varepsilon G(0,z)\int_{\partial\tilde\omega} \left(\frac{\partial u}{\partial\nu_x}(\varepsilon y) + \frac{\partial v^-}{\partial\nu_y}(y)\right)\,d\sigma_y \\ &\quad +\varepsilon^2\nabla_x G(0,z)\int_{\partial\tilde\omega} \left(\frac{\partial u}{\partial\nu_x}(\varepsilon y) + \frac{\partial v^-}{\partial\nu_y}(y)\right) y\,d\sigma_y + O(\varepsilon^3).\end{aligned}$$

Since $\displaystyle\int_{\partial\tilde\omega} \frac{\partial u}{\partial\nu_x}(\varepsilon y)\,d\sigma_y = \int_{\partial\tilde\omega} \frac{\partial v^-}{\partial\nu_y}(\varepsilon y)\,d\sigma_y = 0$, we obtain

$$\int_{\partial\omega_\varepsilon} \left(\frac{\partial u}{\partial\nu_x} + \frac{\partial v^-}{\partial\nu_y}\right) G\,d\sigma_x = \varepsilon^2\nabla_x G(0,z)\int_{\partial\tilde\omega} \left(\frac{\partial u}{\partial\nu_x}(\varepsilon y) + \frac{\partial v^-}{\partial\nu_y}(y)\right) y\,d\sigma_y + O(\varepsilon^3).$$

Using the transmission conditions of v across $\partial\omega_\varepsilon$, and inserting the above identity and (3.11) in (3.10), leads to

$$u_\varepsilon(z) = \int_{\partial\Omega} (gG - u_\varepsilon\frac{\partial G}{\partial\nu_x}) + \varepsilon^2(1-k)\nabla_x G(0,z)\int_{\partial\tilde\omega} \left(\nabla_x u(0)\cdot\nu_y + \frac{\partial v^-}{\partial\nu_y}(y)\right) y + O(\varepsilon^3).$$

Combining the last equation and (3.4), the representation formula for u_ε becomes

$$u_\varepsilon(z) = \int_{\partial\Omega} (gG - u_\varepsilon\frac{\partial G}{\partial\nu_x})\,d\sigma_x + \varepsilon^2\frac{1-k}{k}\nabla_x G(0,z)\cdot A\nabla_x u(0) + O(\varepsilon^3), \tag{3.12}$$

where the coefficients of the matrix A are given by

$$a_{lj} = k \int_{\partial\tilde{\omega}} \left(\nu_j + \frac{\partial \varphi^{(j)^-}}{\partial \nu_y} \right) y_l \, d\sigma_y. \tag{3.13}$$

Next, we determine A explicitly from the expression of the polarization tensor M and from the asymptotic behavior of $\varphi^{(j)}, j = 1, 2$. The jump condition $\partial\varphi^{(j)^+}/\partial\nu - k\partial\varphi^{(j)^-}/\partial\nu = (k-1)\nu_j$, satisfied by $\varphi^{(j)}$ across $\partial\tilde{\omega}$, shows that

$$\int_{\partial\tilde{\omega}} \frac{\partial \varphi^{(j)^-}}{\partial \nu_y} y_l \, d\sigma_y = \frac{1}{k} \int_{\partial\tilde{\omega}} \frac{\partial \varphi^{(j)^+}}{\partial \nu_y} y_l \, d\sigma_y + \frac{1-k}{k} \int_{\partial\tilde{\omega}} \nu_j y_l \, d\sigma_y, \tag{3.14}$$

and as $\varphi^{(j)}$ is harmonic in $\tilde{\omega}$ and in $\mathbf{R}^2 \setminus \tilde{\omega}$, it follows that

$$\begin{aligned} \int_{\partial\tilde{\omega}} \frac{\partial \varphi^{(j)^+}}{\partial \nu_y} y_l \, d\sigma_y &= \lim_{R\to\infty} \left(\int_{\partial B_R} \frac{\partial \varphi^{(j)}}{\partial \nu_y} y_l \, d\sigma_y - \int_{\partial B_R} \varphi^{(j)} \nu_l \, d\sigma_y \right) + \int_{\partial\tilde{\omega}} \varphi^{(j)} \nu_l \, d\sigma_y \\ &= \lim_{R\to\infty} \left(\int_{\partial B_R} \frac{\partial \varphi^{(j)}}{\partial \nu_y} y_l - \int_{\partial B_R} \varphi^{(j)} \nu_l \right) + \int_{\partial\tilde{\omega}} \frac{\partial \varphi^{(j)^-}}{\partial \nu_y} y_l \end{aligned} \tag{3.15}$$

Consequently, inserting identity (3.15) in (3.14),

$$\int_{\partial\tilde{\omega}} \frac{\partial \varphi^{(j)^-}}{\partial \nu_y} y_l \, d\sigma_y = \frac{1}{k-1} \lim_{R\to\infty} \left(\int_{\partial B_R} \frac{\partial \varphi^{(j)}}{\partial \nu_y} y_l \, d\sigma_y - \int_{\partial B_R} \varphi^{(j)} \nu_l \, d\sigma_y \right) - \int_{\partial\tilde{\omega}} \nu_j y_l \, d\sigma_y, \tag{3.16}$$

and substituting (3.16) in (3.13) yields

$$a_{lj} = \frac{k}{k-1} \lim_{R\to\infty} \left(\int_{\partial B_R} \frac{\partial \varphi^{(j)}}{\partial \nu_y} y_l \, d\sigma_y - \int_{\partial B_R} \varphi^{(j)} \nu_l \, d\sigma_y \right).$$

The right-hand side can be computed from the asymptotic behavior of the functions $\varphi^{(j)}, j = 1, 2$, and it follows that

$$a_{lj} = \frac{k}{k-1} m_{jj} \delta_{lj} = \frac{k}{k-1} m_{lj} \quad 1 \le i, j \le 2,$$

with m_{jj} as in (2.12) or (2.19).

Finally, Green's formula applied to the background potential shows that

$$u_\varepsilon(z) = \int_{\partial\Omega} (gG - u_\varepsilon \frac{\partial G}{\partial \nu_x}) \, d\sigma_x,$$

so that for all $z \in \Omega \cap \{dist(z, \tilde{\Omega}_\varepsilon) \ge d_0\}$, (3.12) becomes

$$u_\varepsilon(z) - u(z) = \int_{\partial\Omega} (u(x) - u_\varepsilon(x)) \frac{\partial G}{\partial \nu_x} \, d\sigma_x - \varepsilon^2 \nabla_x G(0, z) \cdot M \nabla_x u(0) + O(\varepsilon^3).$$

When z converges to $\partial\Omega$, the double layer potential $\displaystyle\int_{\partial\Omega} (u(x) - u_\varepsilon(x)) \frac{\partial G}{\partial \nu_x} \, d\sigma_x$ converges uniformly on $\partial\Omega$ to

$$-\frac{1}{2}(u(z) - u_\varepsilon(z)) + \int_{\partial\Omega} (u(x) - u_\varepsilon(x)) \frac{\partial G}{\partial \nu_x} \, d\sigma_x.$$

By continuity, we obtain the desired formula (3.9). ∎

4. Convergence properties of the polarization tensor and $\nabla\omega$ as $\delta \to 0$.

4.1. Preliminaries. We again consider the configurations of paragraphs 2.1 and 2.2 : two circular inclusions of radius 1, centered at $(\pm(1+\delta), 0)$, with $\delta \geq 0$. The inclusions are filled with a material of conductivity k and the rest of the plane has conductivity 1. Let w_δ denote the potential solution to

$$\left\{ \begin{array}{lcl} \operatorname{div}(a(x)\nabla w_\delta) & = & 0 \quad \text{in } \mathbf{R}^2 \\ w & \sim & x_2 \quad \text{as } |x| \to \infty. \end{array} \right. \qquad \delta > 0. \tag{4.1}$$

When $\delta \to 0$, w_δ converges to w_0 and thus we expect that the polarization tensors M_δ converge to M_0. This is however not obvious given the expressions of the series (2.12) and (2.19). We recall that the polarization tensor $M_\delta = (m_{jl})_{1 \leq j,l \leq 2}$ is equal to

$$m_{jl} = 16a^2\pi \sum_{n\geq 1} n \frac{\rho^{2n}}{\Lambda + (-1)^j \rho^{2n}} \delta_{jl} \tag{4.2}$$

where $a = \sqrt{\delta(2+\delta)}$, $\rho = \dfrac{a-\delta}{a+\delta}$ and $\Lambda = \dfrac{k+1}{k-1}$. This series converges uniformly for fixed Λ and for $0 \leq \rho \leq \rho_0 < 1$. When $\delta \to 0$, $\rho \to 1$ and summing the series formally amounts to summing the divergent series $\dfrac{1}{\Lambda+1} \displaystyle\sum_{n \geq 1} (-1)^n n$.

The series that give the coefficients of the polarization tensors are very similar to the series that express the values of the gradient of w_δ at the midpoint between the inclusions. The same asymptotics may then be performed on the series for $\nabla w_\delta(0)$.

In the context of antiplane shear, for instance, $a\nabla w$ represents the stresses. Most linear fracture models suppose that the fracturing will occur at points with extreme stress concentrations. The symmetries of our configuration imply that ∇w is extremal at the origin, which explains the interest of computing $\nabla w(0)$ for close-to-touching inclusions. In fact, in this case, the current concentrates in a narrow channel. The gradient of the potential could be very large even if the potential is still smooth.

The behavior of the potential gradient, near points where two circular fibers are close or touch, was studied in [6], and shown to remain bounded independently of the distance between the inclusions. This regularity result was then generalized to the case of arbitrary $C^{1,\alpha}$ inclusions by YanYan Li and M. Vogelius [18] and to strongly elliptic systems by Yan Yan Li and L. Nirenberg [17]. The bounds on the gradient may degenerate as the conductivity contrast becomes large [6]. Our calculations provide an example where we can study precisely how the gradient blows up with the contrast.

The function w_δ is the real part of a piecewise analytic function f given by

$$f(z) = -iz + g_1(\xi) - g_1(1/\xi), \tag{4.3}$$

for a point z outside the inclusions where $\xi = \dfrac{z-a}{z+a}$. From Lemma 1, the function g_1 has the following expansion

$$g_1(\xi) = g_1(0) + \sum_{n\geq 1} b_n \xi^n \qquad \text{when } |\xi| \leq \rho^{-1}$$

where $b_n = 2a\dfrac{\rho^{2n}}{\Lambda + \rho^{2n}}$.

To compute the gradient of the potential, we differentiate (4.3) at $z=0$

$$\begin{aligned} f'(0) &= -i + \frac{2a}{(z+a)^2}_{/z=0} \left(g'(\xi) + 1/\xi^2 g'(1/\xi)\right)_{/\xi=-1} \\ &= -i\left(1 + 4/a \sum_{n\geq 1} (-1)^n n b_n\right) \\ &= -i\left(1 + 8\sum_{n\geq 1}(-1)^n n \frac{\rho^{2n}}{\Lambda+\rho^{2n}}\right). \end{aligned} \tag{4.4}$$

We study the behavior of M_δ with the techniques developed for singular asymptotics by C. Callias and X. Markenscoff [9, 10], and expand the series (4.2) and (4.4) in terms of $s = -2Log(\rho) \sim 2\sqrt{2\delta}$.

Let us fix $\varepsilon > 0$. Our first step consists in rewriting the series $S_0 = \sum_{n\geq 1} n \frac{\rho^{2n}}{\Lambda+\rho^{2n}}$ and $S_1 = \sum_{n\geq 1} (-1)^n n \frac{\rho^{2n}}{\Lambda+\rho^{2n}}$, when $\rho < 1$, in the form

$$\begin{aligned} S_p &= \sum_{n\geq 1} \frac{ne^{-sn}}{\Lambda + e^{-sn}} \cos(\pi n p) \\ &= \frac{1}{2i}\int_C H(sz, 1/z)\cos(\pi p z)\cot(\pi z)dz \qquad p = 0, 1, \end{aligned} \tag{4.5}$$

where C is the contour $\{Im(z) = \pm\varepsilon, 1/2 \leq Re(z)\} \cup \{Re(z) = 1/2, -\varepsilon < Im(z) < \varepsilon\}$ and $H(sz, 1/z) = \frac{z\, e^{-sz}}{\Lambda + e^{-sz}}$. This follows from the analycity of $z \to H(sz, 1/z)$ in each rectangle $R_n = [n-1/2, n+1/2] \times [-\varepsilon, \varepsilon]$ and from the Residue Theorem :

$$\begin{aligned} \frac{1}{2i\pi}\int_{\partial R_n} H(sz,1/z)\cos(\pi p z)\cot(\pi z)dz &= \mathrm{Ind}(\partial R_n)\mathrm{Res}\left(H(sz,1/z)\cos(\pi p z)\cot(\pi z), R_n\right) \\ &= \frac{(-1)^{np}}{\pi} H(sn, 1/n). \end{aligned}$$

Moreover, since

$$\begin{aligned} |\cos(\pi p(x \pm i\varepsilon))\cot(\pi(x\pm i\varepsilon))| &\leq \cosh(\pi\varepsilon p)\ \left|\frac{1+e^{-2i\pi x}e^{\pm 2\pi\varepsilon}}{1-e^{-2i\pi x}e^{\pm 2\pi\varepsilon}}\right| \\ &\leq \cosh(\pi\varepsilon p)\ \frac{1+e^{2\pi\varepsilon}}{|1-e^{\pm 2\pi\varepsilon}|}, \end{aligned}$$

one easily checks that the integral in (4.5) is well defined.

To expand the integral with respect to s, we introduce a smooth cut-off function $\chi(z)$, which is equal to 1 for $|z| > 2$ and which vanishes for $|z| < 1$ and we split the integral into

$$\begin{aligned} 2iS_p &= \int_C (1-\chi(z))\, H(sz,1/z)\cos(\pi p z)\cot(\pi z)dz \\ &+ \int_{C\cap\{Im(z)>0\}} \chi(z)\, H(sz,1/z)\cos(\pi p z)\cot(\pi z)dz \end{aligned}$$

$$+\int_{C\cap\{Im(z)<0\}} \chi(z)\, H(sz,1/z)\cos(\pi p z)\cot(\pi z)dz$$
$$= I_0 + I^+ + I^-. \tag{4.6}$$

Clearly, the first term can be expanded in s as the integrand is a smooth function and integration is performed on a compact set. The result is

$$\begin{aligned} I_0 &= \int_C (1-\chi(z))\, H(0,1/z)\cos(\pi p z)\cot(\pi z)dz + O(s) \\ &= \frac{1}{\Lambda+1}\int_C (1-\chi(z))\, z\cos(\pi p z)\cot(\pi z)dz + O(s). \end{aligned} \tag{4.7}$$

The difficulties lie in the remaining terms, which, after the change of variables $z = 1/x \pm i\varepsilon$, can be rewritten in the form

$$2is^3\, I^{\pm} = \int_0^\infty \chi(1/x)\, h_\pm(s/x,s)\, \phi_\pm(x)dx, \tag{4.8}$$

where we have set

$$\begin{aligned} h_\pm(y,s) &= y^2(y\pm is\varepsilon)\frac{e^{-y\mp is\varepsilon}}{\Lambda+e^{-y\mp is\varepsilon}} \\ \phi_\pm(x) &= \cos\left(\pi p(1/x\pm i\varepsilon)\right)\cot\left(\pi(1/x\pm i\varepsilon)\right),\ p=0,1. \end{aligned}$$

We focus on I^- (omitting the subscript for simplicity). Our goal is to expand I^- with respect to s. The results on singular integrals of C. Callias and X. Markenscoff show that such an integral can be expanded up to order n in s, provided the integrand is in $\mathcal{C}^{n+1}$ and satisfies appropriate decay conditions.

In the expression of I^-, the function $\chi(1/x)h(s,s/x)$ is smooth up to $x=0$, but ϕ is smooth only for $x>0$. To cast I^- in a suitable form, we regularize ϕ in the following way: we define $\phi_{(1)}$ by

$$\phi_{(1)}(x) = \frac{1}{x}\int_0^x \phi(t)dt,$$

and, for $n\geq 1$, we denote by $\phi_{(n+1)}$ the regularization of $\phi_{(n)}$. We can then transform I^- by integration by parts

$$\begin{aligned} (2is^3)\, I^- &= \int_0^\infty \chi(1/x)\, h(s/x,s)\,\partial_x\left[x\phi_{(1)}(x)\right] dx \\ &= \left[\chi(1/x)h(s/x,x)\, x\phi_{(1)}(x)\right]_0^\infty - \int_0^\infty \partial_x\left[\chi(1/x)h(s/x,s)\right] x\phi_{(1)}(x)dx \\ &= \int_0^\infty (y\partial_y - x\partial_x)\left[\chi(1/x)h(y,s)\right]_{/y=s/x}\phi_{(1)}(x)dx. \end{aligned}$$

Notice that the boundary terms in the above computation vanish, because $\chi(1/x)$ has compact support in x and because of the exponential factor in h. Integrating by parts again, we obtain, for $n\geq 1$

$$(2is^3)\, I^- = \int_0^\infty (y\partial_y - x\partial_x)^n\left[\chi(1/x)h(y,s)\right]_{/y=s/x}\phi_{(n)}(x)dx. \tag{4.9}$$

Proposition 5 below, shows that $\phi_{(n)}$ is indeed of class $\mathcal{C}^{n-1}$. The integrand has the form $\mathcal{H}(x,y,s) = (y\partial_y - x\partial_x)^n \left[\chi(1/x)h(y,s)\right]\phi_{(n)}(x)$.

THEOREM 2. *[9] Let $\mathcal{H}$ satisfies:*

i/ $\mathcal{H} \in \mathcal{C}^n\left([0,\infty)\times[0,\infty)\right)$,

ii/ $\mathcal{H}$ has a compact support in x uniformly in y,

iii/ for every triple (β,q,r) of non-negative integers, for every $y \geq 0$ and for $0 \leq s \leq 1$,

$$|\partial_x^\beta \partial_y^q \partial_s^r \mathcal{H}(y,s)| \leq H_{\beta,q,r}(y)\, y^{\beta-q},$$

with $\int_0^1 H_{\beta,q,r}(1/t)dt < \infty$,

then, the following asymptotic expansion holds

$$\int_0^\infty \mathcal{H}(x,s/x,s)dx = \sum_{j=0}^{l} \frac{s^j}{j!}\left\{A_0^j + \sum_{m=1}^{l-j} A_m^j s^m + ln(s)\sum_{m=1}^{l-j} B_m^j s^m\right\} + \mathcal{R}_{l+1} \tag{4.10}$$

where $\mathcal{R}_{l+1} = O(s^{l+1})$,

$$\begin{aligned}
A_0^j &= \int_0^\infty \partial_s^j \mathcal{H}(x,0,0)dx \\
A_m^j &= U_m^j(\mathcal{H}) + L_m^j(\mathcal{H}) + B_m^j \sum_{\lambda=1}^{m-1} 1/\lambda \\
B_m^j &= \frac{1}{m!(m-1)!}\partial_x^{m-1}\partial_y^m \partial_s^j \mathcal{H}(0,0,0) \\
U_m^j(\mathcal{H}) &= -\frac{1}{m!(m-1)!}\int_0^\infty ln(x)\partial_x^m \partial_y^m \partial_s^j \mathcal{H}(x,0,0)dx \\
L_m^j(\mathcal{H}) &= -\frac{1}{(m-1)!}\int_0^\infty ln(\xi)\partial_\xi\left[\xi^m \partial_x^{m-1} R_{m+1}^j(0,1/\xi)\right] d\xi,
\end{aligned}$$

and where R_{m+1}^j is the remainder in the Taylor series of $\partial_s^j\mathcal{H}(x,y,0)$ about $y=0$ at order m, i.e.,

$$R_{m+1}^j(x,y) = \partial_s^j \mathcal{H}(x,y,0) - \sum_{\lambda=0}^{m} \frac{y^\lambda}{\lambda!}\partial_y^\lambda \partial_s^j \mathcal{H}(x,0,0).$$

The next two propositions show that the above Theorem applies in our context.

PROPOSITION 4. *The function $F(x,y,s) = \chi(1/x)h(y,s)$, satisfies*

i/ $F \in \mathcal{C}^\infty(\mathbf{R}^+ \times \mathbf{R}^+ \times \mathbf{R}^+)$,

ii/ F has compact support in x,

iii/ For every triple (β,q,r) of non-negative integers, there exists a function $H_{\beta,q,r}(y)$ such that for $x,y \geq 0$,

$$|\partial_x^\beta \partial_y^q \partial_s^r F(x,y,s)| \leq H_{\beta,q,r}(y) y^{p-q},$$

and such that $\int_0^1 H_{\beta,q,r}(1/t)dt < \infty$.

In the terminology of [10], F is of extended class B.
Proof: The first and second point result from the exponential in the numerator of h and from the compact support of $\chi(1/x)$ in x. Since $\chi(1/x)$ is smooth and is identically equal to 1 when $0 \leq x \leq 1$, it suffices to show that for every couple (q,r) of non-negative integers and for every $y \geq 0$,

$$|(y\partial_y)^q \partial_s^r h(y,s)| \leq H_{q,r}(y),$$

with $\int_0^1 H_{q,r}(1/t)dt < \infty$. One easily checks that $(y\partial_y)^q \partial_s^r h(y,s)$ has the form

$$e^{-y}\ P\left(y, e^{-y}, (\Lambda + e^{-(y+is\varepsilon)})^{-1}\right),$$

where P is a polynomial, so that one can choose $H_{q,r}(y) = e^{-y}|P|$. ∎

Proposition 5. *The regularizations of the functions* $\phi(x) = \cos(p\pi(1/x - i\varepsilon))\cot(\pi(1/x - i\varepsilon))$*, where* $p = 0,1$*, satisfy*

(i) $\phi_{(n)} \in C^{n-1}([0,\infty))$*,*

(ii) $\forall\ 0 < m < n, \quad \dfrac{d^m}{dx^m}\phi_{(n)}(0) \ = \ 0.$

(iii) $\forall\ 0 < n, \quad \phi_{(n)}(0) \ = \ \begin{cases} -i & \text{if } p = 0 \\ 0 & \text{if } p = 1. \end{cases}$

Proof: (See also [10]). The function ϕ has the form

$$\begin{aligned}\phi(x) &= i/2\ \left(e^{(i\pi/x+\pi\varepsilon)p} + e^{(-i\pi/x-\pi\varepsilon)p}\right) \frac{e^{i\pi/x+\pi\varepsilon} + e^{-i\pi/x-\pi\varepsilon}}{e^{i\pi/x+\pi\varepsilon} - e^{-i\pi/x-\pi\varepsilon}} \\ &= i/2\ \left(e^{(i\pi/x+\pi\varepsilon)p} + e^{(-i\pi/x-\pi\varepsilon)p}\right) \frac{e^{2\pi\varepsilon} + e^{-2i\pi/x}}{e^{2\pi\varepsilon} - e^{-2i\pi/x}} \\ &= e^{ip\pi/x} f_+\left((e^{-i\pi/x})^2\right) + e^{-ip\pi/x} f_-\left((e^{-i\pi/x})^2\right),\end{aligned}$$

where $f_+(Z)$ and $f_-(Z)$ are analytic functions of Z, for $|Z| < 1+\eta$ for some $\eta > 0$ and $p = 0,1$. These functions have a power series expansion that converges absolutely in Z, for $|Z| < 1+\eta$, so that ϕ can be rewritten

$$\phi(x) = \sum_{q=0}^{\infty} f_{+,q} e^{-i\pi(2q-p)/x} + f_{-,q} e^{-i\pi(2q+p)/x}. \tag{4.11}$$

Due to the absolute convergence of the series, ϕ can be regularized term by term. It is thus sufficient to show that the Proposition applies to each term $e^{-i\pi(2q\pm p)/x}$, $p = 0,1$, of the expansion.

One easily checks that if $\varphi \in \mathcal{C}^l([0,\infty))$, and $\varphi^{(j)}(0) = 0$ for $0 \leq j \leq l$, then $\varphi_{(1)}$ has the same properties. Expanding a function $\varphi \in \mathcal{C}^l([0,\infty))$ as $\sum_{j=0}^{l} \varphi^{(j)}(0)\frac{x^j}{j!} + R(x)$ shows then that $\varphi_{(1)}^{(j)}(0) = \dfrac{1}{j+1}\varphi^{(j)}(0)$.

Further, if $\alpha \neq 0, l \geq 0$ and $\varphi_l(x) = x^l e^{i\alpha/x}$, integration by parts shows that

$$(\varphi_l)_{(1)}(x) = i/\alpha\left[\varphi_{l+1}(x) - (l+2)(\varphi_{l+1})_{(1)}(x)\right],$$

from which it follows that

$$(\varphi_l)_{(n)} \in \mathcal{C}^{l+n-1}([0,\infty))\text{and}(\varphi_l)^{(j)}_{(n)}(0) \;=\; 0 \quad \text{for } j \leq l+n-1. \tag{4.12}$$

This shows (i) and (ii).
Additionally, if $p = 0$, we deduce from (4.11) and (4.12) that

$$\forall\, 0 < n, \quad \phi_{(n)}(0) \;=\; (f_{+,0} + f_{-,0}) \;=\; -i,$$

while if $p = 1$, (4.12) implies that $\forall\, 0 < n, \quad \phi_{(n)}(0) \;= 0.$ ■

4.2. Asymptotics of the polarization tensor. First, we focus on the computation of the terms of the series S_0. We seek the term of order s in (4.8), when $s \to 0$. After regularizing $\phi = \cot(\pi(1/x - i\varepsilon))$ three times, I^- becomes

$$I^- \;=\; \int_0^\infty \mathcal{H}(x,y,s)dx \;=\; \int_0^\infty (y\partial_y - x\partial_x)^3 \left[\chi(1/x)h(y,s)\right] \phi_{(3)}(x)dx.$$

where $h_- = y^2(y - is\varepsilon)\dfrac{e^{-y+is\varepsilon}}{\Lambda + e^{-y+is\varepsilon}}$.

4.2.1. Computation of the terms of the series. For simplicity, we denote by $\sigma(x)$ the function $\chi(1/x)$ and we notice that all its derivatives vanish at $x = 0$. Applying Theorem 2, the integral rewrites as

$$\begin{aligned}(2is^3)I^- &= \int_0^\infty (y\partial_y - x\partial_x)^3 \left[\sigma(x)h(y,s)\right]_{y=s/x} \phi_{(3)}(x)dx \\ &= \sum_{j=0}^{1} \frac{s^j}{j!}\left\{ A_0^j + \sum_{m=1}^{1-j} A_m^j\, s^m \;+\; \ln(s) \sum_{m=1}^{1-j} B_m^j\, s^m \right\} + O(s^2) \\ &= A_0^0 + \left(A_0^1 + A_1^0 + \ln(s)B_1^0\right) s + O(s^2).\end{aligned}$$

• The terms $A_0^k, k = 0, 1$: Their expression is

$$A_0^j = \int_0^\infty (y\partial_y - x\partial_x)^3 \left(\sigma(x)\partial_s^j h(y,0)\right)_{/y=0} \phi_{(3)}(x)dx.$$

The integrand can be rewritten in the form

$$\sum_{q=0}^{3} \binom{3}{q} (y\partial_y)^q \partial_s^j h(y,0)_{/y=0}\, (-1)^{3-q}(x\partial_x)^{3-q}\sigma\, \phi_{(3)}(x). \tag{4.13}$$

The expression of h shows that $\partial_s^j h(y,0) = O(y^2)$ for $0 \leq j \leq 1$ and thus

$$(y\partial_y)^q \partial_s^j h(y,0) = O(y^2), \tag{4.14}$$

so that all the terms A_0^j are equal to 0.

• The term B_1^0 : (4.13) and (4.14) show that

$$B_1^0 \;=\; \partial_y \left[(y\partial_y - x\partial_x)^3 \left(\sigma(x)h(y,0)\right)_{/y=s/x} \phi_{(3)}(x)\right]_{(x,y)=(0,0)} \;=\; 0.$$

• The term A_1^0, which reduces to $U_1^0 + L_1^0$: U_1^0 is an integral of $ln(x)\, f_1^0(x,0)$, where

$$f_1^0(x,y) = \partial_x \partial_y \left[(y\partial_y - x\partial_x)^3 (\sigma(x) h)\, \phi_{(3)}(x)\right]_{(x,y)=(x,0)}$$
$$= \sum_{q=0}^{3} \binom{3}{q} \partial_y (y\partial_y)^q h(y,0)\, (-1)^{3-q} \partial_x \left[(x\partial_x)^{3-q} \sigma(x)\, \phi_{(3)}(x)\right].$$

Again from (4.14), $f_1^0(x,0) = 0$, which implies that $U_1^0 = 0$. As for L_1^0, we have

$$L_1^0 = \lim_{d\to 0} \left\{ -\int_d^{\infty} R_2^0(0,1/\xi) d\xi \right.$$
$$\left. + ln(d) \partial_y \left[(y\partial_y - x\partial_x)^3 \left(\sigma(x) h(y,0)\right)_{/y=s/x} \phi_{(3)}(x) \right]_{(x,y)=(0,0)} \right\}.$$

Computing $\partial_y h$ shows that the last term vanishes. The remainder R_2^0 in the Taylor expansion of $\mathcal{H}(x,y,0)$ about $y=0$ up to order y^2 is actually equal to $\mathcal{H}(x,y,0)$. Thus, $R_2(0,1/\xi) = (y\partial_y - x\partial_x)^3 \left(\sigma(x) \frac{y^3 e^{-y}}{\Lambda + e^{-y}}\right)_{x=0, y=1/\xi} \phi_{(3)}(0)$. From Proposition 5, $\phi_{(3)}(0) = -i$. and since all the derivatives of $\sigma(x)$ vanish at $x=0$, and since $\sigma(0) = 1$, we finally get

$$A_1 \;=\; L_1^0 \;=\; -i \int_0^{\frac{\Lambda+1}{\Lambda}} \frac{\ln(t)}{1-t}\, dt \;=\; i \int_0^{\frac{-1}{\Lambda}} \frac{\ln(1-v)}{v}\, dv.$$

Since $\Lambda = \dfrac{k+1}{k-1} > 1$, $\ln(1-v)$, for $|v| < 1/\Lambda < 1$, can be expanded as a power series in v to obtain

$$A_1 \;=\; -i \sum_{n=1}^{\infty} \frac{1}{n} \int_0^{-\frac{1}{\Lambda}} v^{n-1} d\,v \;=\; -i \sum_{n=1}^{\infty} \frac{(-1)^n}{\Lambda^n n^2},$$

4.2.2. The leading term in (4.2) . The above calculations show that the integral I^- can be expanded with respect to s as

$$I^- = -\frac{1}{2s^2} \sum_1^{\infty} \frac{(-1)^n}{\Lambda^n n^2} + O(\frac{1}{s}).$$

Similar calculations yield the asymptotics of I^+, and (4.7), show that

$$S_0 = -\frac{1}{s^2} \sum_1^{\infty} \frac{(-1)^n}{\Lambda^n n^2} + O(\frac{1}{s}).$$

from which we deduce the term m_{22} of the polarization tensor

$$m_{22} = -16\pi \frac{a^2}{s^2} \sum_1^{\infty} \frac{(-1)^n}{\Lambda^n n^2} + O(\frac{a^2}{s}).$$

The same kind of computations can be carried out for m_{11}. The results are given in the following (recalling that $a^2 \sim 2\delta$ and $s \sim 2\sqrt{2\delta}$)

PROPOSITION 6. *The polarization tensor $M_\delta = (m_{jl})_{1\le j,l\le 2}$, of a configuration with two circular inclusions at a distance 2δ apart, satisfies*

$$m_{jl} = (-1)^{(j+1)} 4\pi \sum_{n=1}^{\infty} \frac{(-1)^{(j+1)n}}{\Lambda^n n^2}\, \delta_{jl} + O(\sqrt{\delta}). \tag{4.15}$$

In other words, M_δ converges to M_0, the polarization tensor of two touching discs, as $\delta \to 0$.

4.3. Computation of $\nabla w(0)$. In this case, we are interested in the term of order s^0 in the series S_1 *i.e.* in the term of order s^3 in (4.8) when $p = 1$. Therefore, we regularize ϕ five times and apply (4.10) with

$$\mathcal{H}(x, y, s) = (y\partial_y - x\partial_x)^5 \left[\sigma(x) h(y, s)\right] \phi_{(5)}(x),$$

where again all the derivatives at $x = 0$ of $\sigma(x) = \chi(1/x)$ vanish.

4.3.1. Computation of the terms of the series. Applying Theorem 2, the integral I^- rewrites as

$$\begin{aligned}(2is^3) I^- &= \int_0^\infty (y\partial_y - x\partial_x)^5 \left[\sigma(x) h(y, s)\right]_{y=s/x} \phi_{(5)}(x) dx \\ &= \sum_{j=0}^{3} \frac{s^j}{j!} \left\{ A_0^j + \sum_{m=1}^{3-j} A_m^j\, s^m \;+\; \ln(s) \sum_{m=1}^{3-j} B_m^j\, s^m \right\} + O(s^4).\end{aligned}$$

• The terms B_m^j : We recall that

$$B_m^j = \frac{1}{m!(m-1)!} \partial_x^{m-1} \partial_y^m \left[(y\partial_y - x\partial_x)^5 \left(\sigma(x) \partial_s^j h(y, 0)\right)_{/y=s/x} \phi_{(5)}(x) \right]_{(x,y)=(0,0)}.$$

By Proposition 5, $\partial_x^{m-1}\phi_{(5)}(0) = 0$, for $0 \le m-1 < 5$. Since $0 \le j \le 3$ and $1 \le m \le 3 - j$, one easily checks that $B_m^j = 0$.

• The terms L_m^j: These terms are interpreted in the following sense

$$\begin{aligned}L_m^j = \lim_{d\to 0} \{ &- \int_d^\infty \xi^{m-1} \partial_x^{m-1} R_{m+1}^j(0, 1/\xi) d\xi \\ &+ \frac{ln(d)}{m!} \partial_x^{m-1} \partial_y^m \left[(y\partial_y - x\partial_x)^5 \left(\sigma(x) \partial_s^j h(y, 0)\right)_{/y=s/x} \phi_{(5)}(x) \right]_{(x,y)=(0,0)} \}.\end{aligned}$$

Again, Proposition 5 shows that the last term in the above expression vanishes. Using Leibniz's rule, we rewrite

$$\begin{aligned}R_{m+1}^j(x, y) = \sum_{q=0}^{5} \binom{5}{q} &\left[(y\partial_y)^q \frac{\partial^j h}{\partial s^j}(y, 0) - \sum_{l=0}^{m} \frac{y^l}{l!} \frac{\partial^l}{\partial y^l} (y\partial_y)^q \frac{\partial^j h}{\partial s^j}(y, 0) \right] \\ &(-1)^{5-q} \partial_x^{m-1} \left[(x\partial_x)^{5-q} \sigma\, \phi_{(5)}(x) \right].\end{aligned}$$

We conclude by Proposition 5, that since $\partial_x^{m-1}\phi_{(5)}(0) = 0$, for $0 \le m-1 < 5$,

$$\left(\partial_x^{m-1} R_{m+1}^j(x, y)\right)_{/x=0} = 0,$$

and that all the terms L_m^j are equal to 0.

• The terms A_0^j: Their expression is

$$A_0^j = \int_0^\infty (y\partial_y - x\partial_x)^5 \left(\sigma(x)\partial_s^j h(y,0)\right)_{/y=0} \phi_{(5)}(x)dx,$$

where the integrand can be rewritten in the form

$$\sum_{q=0}^{5} \begin{pmatrix} 5 \\ q \end{pmatrix} (y\partial_y)^q \partial_s^j h(y,0)_{/y=0} (-1)^{5-q}(x\partial_x)^{5-q}\sigma\, \phi_{(5)}(x).$$

The expression of h shows that $\partial_s^j h(y,0) = O(y^2)$ for $0 \le j \le 3$ and thus

$$(y\partial_y)^q \partial_s^j h(y,0) = O(y^2),$$

so that all the terms A_0^j are equal to 0.

• The terms U_m^j: Their expression is $\dfrac{-1}{m!\,(m-1)!}\displaystyle\int_0^\infty ln(x)\; f_m^j(x,0)\,dx$, where

$$\begin{aligned} f_m^j(x,y) &= \partial_x^m \partial_y^m \left[(y\partial_y - x\partial_x)^5(\sigma(x)\partial_s^j h)\,\phi_{(5)}(x)\right]_{(x,y)=(x,0)} \\ &= \sum_{q=0}^{5} \begin{pmatrix} 5 \\ q \end{pmatrix} \partial_y^m (y\partial_y)^q \partial_s^j h(y,0)\,(-1)^{5-q}\partial_x^m \left[(x\partial_x)^{5-q}\sigma(x)\,\phi_{(5)}(x)\right]. \end{aligned}$$

It is easily checked that

$$\partial_y^m (y\partial_y)^q \partial_s^j h(y,0)_{/y=0} = m^q \partial_y^m \partial_s^j h(0,0),$$

so that the term f_m^j simplifies to

$$f_m^j(x,0) = \partial_y^m \partial_s^j h(0,0)\; \partial_x^m \left[(m - x\partial_x)^5\sigma(x)\,\phi_{(5)}(x)\right].$$

From the expression of h we compute

$$\begin{aligned} \partial_y h(0,0) &= 0, \quad \partial_y^2 h(0,0) \;=\; 0, \quad \partial_y^3 h(0,0) \;=\; \frac{6}{\Lambda+1}, \\ \partial_y\partial_s h(0,0) &= 0, \partial_y^2\partial_s h(0,0) \;=\; \frac{-2i\varepsilon}{\Lambda+1}, \partial_y\partial_s^2 h(0,0) \;=\; 0, \end{aligned}$$

thus all the U_m^j's vanish but U_2^1 and U_3^0.

4.3.2. The leading term in the series S_1 . The above calculations show that the integral I^- can be expanded with respect to s as

$$\begin{aligned} I^- &= \frac{1}{2is^3}\left(U_3^0 s^3 + U_2^2 s^3\right) + O(s) \\ &= \frac{1}{\Lambda+1}\left(-3i\int_0^\infty ln(x)\partial_x \left[(1-x\partial_x)^5\sigma(x)\,\phi_{(5)}(x)\right]\, dx \right. \\ &\quad \left. +i\varepsilon\int_0^\infty ln(x)\partial_x^2 \left[(2-x\partial_x)^5\sigma(x)\,\phi_{(5)}(x)\right]\, dx\right) + O(s). \end{aligned}$$

Similar calculations yield the asymptotics of I^+:

$$I^+ = \frac{1}{\Lambda+1}\left(-3i\int_0^\infty ln(x)\partial_x\left[(1-x\partial_x)^5\sigma(x)\,\phi_{(5)}(x)\right]\,dx\right.$$
$$\left.-i\varepsilon\int_0^\infty ln(x)\partial_x^2\left[(2-x\partial_x)^5\sigma(x)\,\phi_{(5)}(x)\right]\,dx\right)+O(s).$$

Recalling (4.7), we finally arrive at

$$S_1 = \frac{1}{\Lambda+1}\left(\int_C(1-\chi(z))\,z\cos(\pi z)\cot(\pi z)dz\right.$$
$$\left.-6i\int_0^\infty ln(x)\partial_x\left[(1-x\partial_x)^5\chi(1/x)\,\phi_{(5)}(x)\right]\,dx\right)+O(s).$$

To check that the leading term in the expansion does not vanish, notice that Theorem 2 can also be used to compute the series

$$S' = \sum_{n\geq 1}(-1)^n ne^{-sn}$$
$$= \frac{1}{2i}\int_C H'(sz,1/z)\cos(\pi z)\cot(\pi z)dz,$$

where C is the same contour and where $H'(sz,1/z) = z\,e^{-sz}$. Comparing the terms in the expansion for S_1 and S' shows that

$$S_1 - \frac{S'}{\Lambda+1} = O(s).$$

On the other hand, S' can be computed explicitly as the derivative of a geometric series, and is equal to

$$S' = \frac{-e^{-s}}{(1+e^{-s})^2} \quad = \quad -1/4+O(s^2).$$

Thus $S_1 = \dfrac{-1}{4(\Lambda+1)}+O(s)$. Recalling (4.4) and the relationship between the complex potential and the function w, we conclude that

Proposition 7. *Consider two circular inclusions of conductivity k, of radius 1, at distance 2δ apart. Assume that $k<1$, that w solves (4.1) (weakly conducting inclusions, transverse current). Then the gradient ∇w at the midpoint between the inclusions satisfies*

$$\frac{\partial w}{\partial x_1}(0,0) \;=\; O(\sqrt{\delta}) \qquad \frac{\partial w}{\partial x_2}(0,0) \;=\; 1/k+O(\sqrt{\delta}).$$

In particular, the gradient blows up linearly like k^{-1} when $k\to 0$.

A similar results holds (considering harmonic conjugates) for strongly conducting inclusions ($k>1$) when $w\sim x_1$ at infinity : in this setting the x_1 component of ∇w blows up like k as $k\to\infty$.

5. Numerical results. In this part, we describe some computational experiments that attempt to quantify the error

$$e_1(z) = u_\varepsilon(z) - u(z) + 2\int_{\partial\Omega} (u_\varepsilon(x) - u(x))\frac{\partial G}{\partial \nu_x}\, d\sigma_x + 2\varepsilon^2 \nabla_x G(z_0, z) \cdot M_\delta \nabla_x u(z_0),$$

on $\partial\Omega$, where, u_ε is the voltage potential in presence of the imperfections, u is the background potential, and the polarization tensor M_δ is equal to

$$M_\delta = Q \begin{pmatrix} m_{11} & 0 \\ 0 & m_{22} \end{pmatrix} Q^t,$$

where Q is the rotation matrix of angle θ between the x_1–axis and the line $(z_1 z_2)$ and m_{ij}'s are given by (2.12) or by (2.19). We define also the remainder

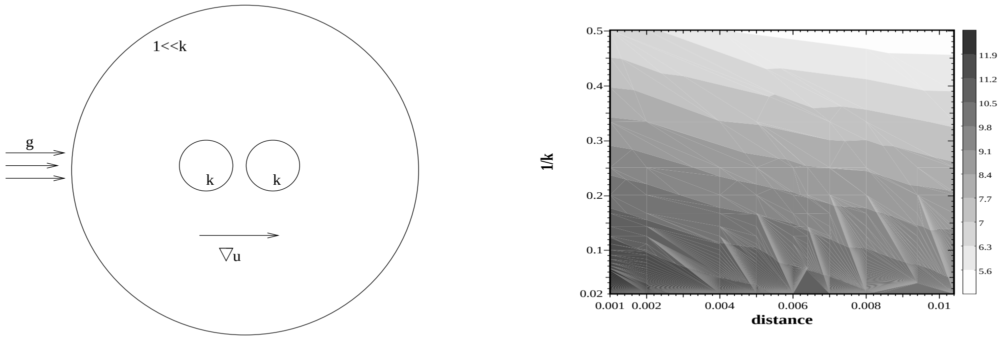


FIG. 5.1. *level lines of* e_2/e_1, $\varepsilon = .1$, $2 \le k \le 50$, $.001 \le \varepsilon\delta \le .011$

$$e_2(z) = u_\varepsilon(z) - u(z) + 2\int_{\partial\Omega} (u_\varepsilon(x) - u(x))\, \frac{\partial G}{\partial \nu_x}(x, z)\, d\sigma_x$$
$$-2\pi\varepsilon^2 \frac{(1-k)}{1+k} \sum_{j=1}^{2} \nabla_x G(z_j, z) \cdot \nabla_x u(z_j), \quad z \in \partial\Omega,$$

between the true solution u_ε and the asymptotic expansion (1.4). When the distance between the two fibers and the contrast vary, we compare the remainder terms $e_1(z)$ and $e_2(z)$, to find out when the asymptotic formula (3.9) is more accurate than (??), *i.e.* when can one consider the two inclusions as a single inhomogeneity rather than two well–separated objects. In all our computations, we use the background voltage potential $u(x) = x_1$ corresponding to the boundary current $g = \nu_1$. We also choose the background conductivity $\gamma(x)$ equal to one. The domain Ω is the unit ball and the inclusions are disks $z_j + \varepsilon B(0, 1)$.

To generate the data on $\partial\Omega$, we solve the direct Neumann problem (1.1) using P^1 finite elements. The boundary of each inclusions is meshed with 80 uniformly spaced points, while the outer boundary is discretized with 300 points.

Figure 5.1 shows the level lines of the ratio e_2/e_1 for two inclusions of radius $\varepsilon = 0.1$, centered along the x_1-axis, as we vary their conductivity k and the separating distance $\varepsilon\delta$ between them, $2 \leq k \leq 50$ and $.001 \leq \varepsilon\delta \leq .011$. As expected, the remainder e_1 is smaller than e_2 and our asymptotic expansion is more accurate in this case. In fact, for this configuration, which was analyzed by Keller in [15] and by L. Borcea and G. Papanicolaou in [7], the current flow is channeled horizontally through the fibers (there is a strong flow channeling through the gap between the fibers along a path which is an horizontal branch connecting the two fibers). Therefore, there is a strong interaction between the two inclusions which increases when $\varepsilon\delta$ goes to zero and k to infinity.

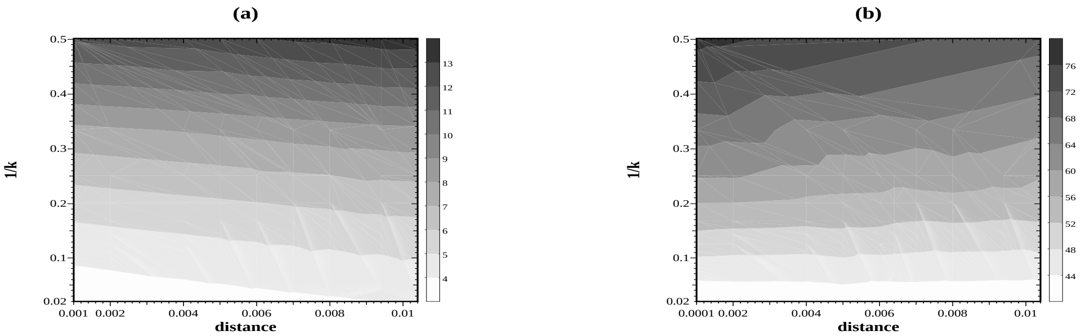


FIG. 5.2. *(a) level lines of $u_\varepsilon - u/e_2$, (b) level lines of $u_\varepsilon - u/e_1$*
$\varepsilon = .1, \quad 2 \leq k \leq 50, \quad 0.001 \leq \varepsilon\delta \leq 0.011$

Fig. 5.2(b) shows that expansion (3.9) is a good approximation of the potential $u(x)$: the error $u_\varepsilon - u$ is at least 40 times e_1. For the same configuration, as was already noted in [12], Fig. 5.2(a) shows that for $k >> 1$, $u_\varepsilon - u$ and e_2 are of the same order and the expansion (1.4) cannot be used to locate the inclusions with sufficient accuracy.

When the fibers are centered along the x_2-axis and $k < 1$, the configuration is the harmonic conjugate of the previous one. The electric current is concentrated in the channel between the fibers. However, in this case, it flows horizontally in the gap, avoiding the fibers. The interaction between the inclusions is weak and formula (1.4) gives results of the same order as the expansion e_1, as is shown in Figure 5.3. If $k > 1$, the interaction between the inclusions is also weak, and e_1 and e_2 are of the same order (Figure 5.4).

Finally, in Figure 5.5, we plot the level lines of e_1/e_2 when $2 \leq k \leq 50$ and $0.01 \leq \varepsilon\delta \leq 0.1$, for fibers of radius 0.05. We remark that the ratio e_1/e_2 increases as the radius of the fibers decreases and our formula is increasingly more accurate than the expansion (1.4).

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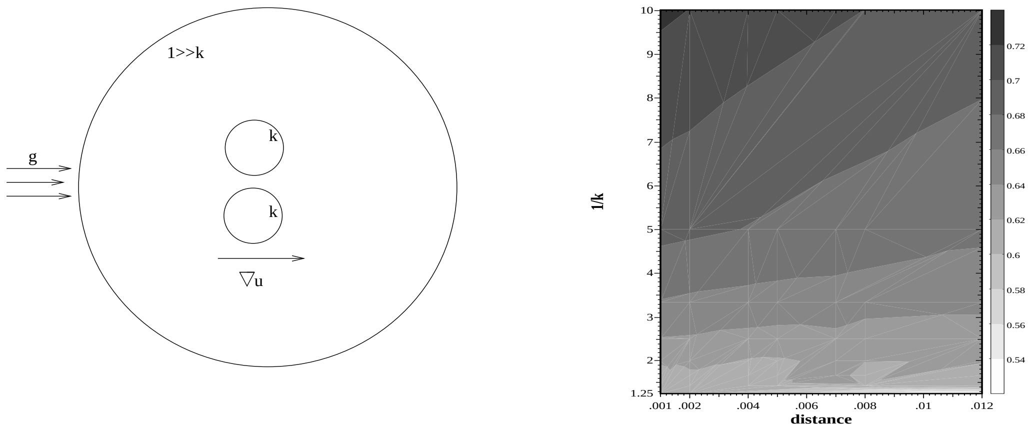

FIG. 5.3. *level lines of* e_2/e_1*,* $\varepsilon = .1$*,* $.1 \leq k \leq .8$*,* $.001 \leq \varepsilon\delta \leq .012$

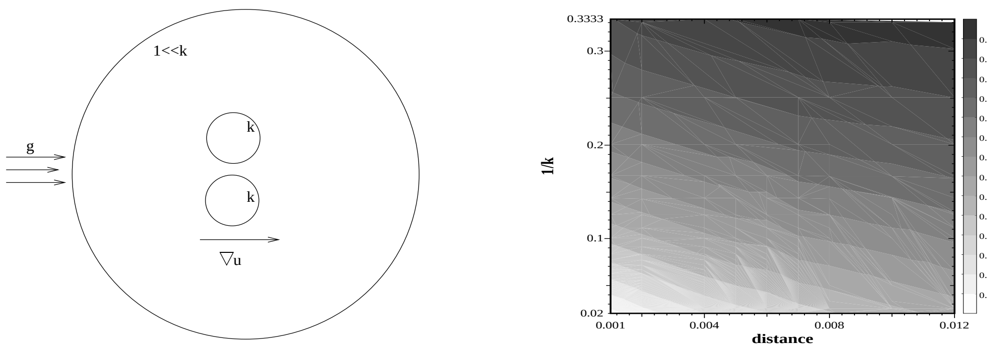

FIG. 5.4. *level lines of* e_2/e_1*,* $\varepsilon = .1$*,* $3 \leq k \leq 50$*,* $.001 \leq \varepsilon\delta \leq .012$

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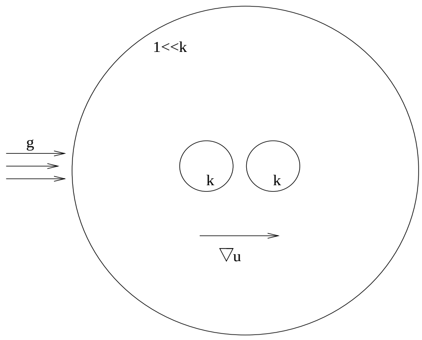

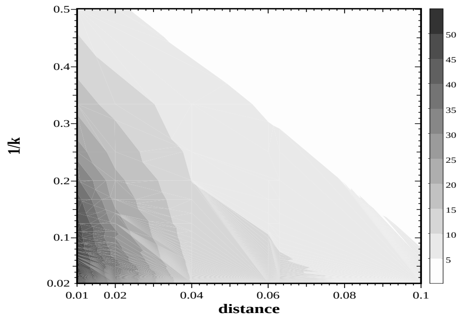

FIG. 5.5. *level lines of* e_2/e_1, $\varepsilon = .05$, $2 \leq k \leq 50$, $.01 \leq \varepsilon\delta \leq .1$

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