MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. 2001; **24**:1021–1042 (DOI: 10.1002/mma.254)
MOS subject classification: 35 Q 60; 78 A 40

Analysis of the radiation properties of a planar antenna on a photonic crystal substrate

Habib Ammari[*,†], Natacha Béreux[‡] and Eric Bonnetier[§]

Centre de Mathématiques Appliquées, Unité Mixte de Recherche CNRS, UMR 7641, Ecole Polytechnique, 91128 Palaiseau Cedex, France

Communicated by J.C. Nedelec

SUMMARY

This paper is concerned with the rigorous investigation of the radiation properties of a planar patch antenna on a photonic crystal substrate. Under the assumptions that the driving frequency of the antenna lies within the band gap of the photonic crystal substrate and that the crystal satisfies a symmetry condition, we prove that the power radiated into the substrate decays exponentially. To do this, we reduce the radiation problem to the study of the well-posedness of a weakly singular integral equation on the patch antenna, and to the study of the asymptotic behaviour of the corresponding Green's function. We also provide a mathematical justification of the use of a photonic crystal substrate as a perfect mirror at any incidence angle. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: photonic crystals; patch antennas; dielectric reflectors; perfect mirrors; integral equations; Green's functions

1. INTRODUCTION

A photonic crystal is a periodic dielectric structure that possesses the feature that electromagnetic waves of certain frequencies cannot propagate inside. The range of the prohibited frequencies is called the complete band gap. The propagation of electromagnetic waves in photonic crystals has been the subject of intensive study, since new fabrication techniques have been developed to construct photonic crystals with a band gap in the region of visible wavelengths. Photonic crystal technology promises many significant technological applications in devices where one wishes to guide electromagnetic waves with little loss of energy (fibre optics, cellular telephones, spectroscopy, etc). The reader can refer for the physical aspects

[*]Correspondence to: H. Ammari, Centre de Mathématiques Appliquées, Unité Mixte de Recherche CNRS, UMR 7641, Ecole Polytechnique, 91128 Palaiseau Cedex, France
[†]E-mail: ammari@cmapx.polytechnique.fr
[‡]E-mail: natacha@cmapx.polytechnique.fr
[§]E-mail: eopus@cmapx.polytechnique.fr

 Received 6 March 2001

of the photonic crystals to the available surveys and proceedings [1–4, 43] devoted to this topic, and especially to the book [5]. The bibliography [6] is also very useful.

It turns out that the Maxwell system provides a very accurate description of the photonic band gap phenomena. Although some significant mathematical results have already been obtained on this topic, see Reference [7] for a survey, most of the mathematical problems in the area of photonic crystals are still not explored or explored only tangentially. We hope that this publication will play some role in publicizing this exciting and important topic in the applied mathematical community. Photonic band gap structures provide a wealth of interesting open problems. For example, it would be very interesting to study the effects of the finiteness of the samples and whether their geometry gives rise to surface waves. The development of accurate numerical methods is very challenging, with obvious applications to problem design. These involve the design of the crystals themselves, to optimize the features of their band gaps, or the design of devices, such as planar antennas like those we are concerned with in this paper.

Recent progress in the manufacturing of photonic crystals has opened the door to applications in the microwave range, and in particular to planar patch antennas [8–22]. Photonic band-gap substrates have also been recognized as an important feature of dielectric mirrors, resonant cavities, high Q filters, novel filter design for planar antennas and frequency selective surfaces. The reader is referred to References [5, 1] for an overview of these potential applications. Classical planar antennas are mounted on a uniform dielectric substrate inside which most of the radiated power is trapped, thus an efficient use of such antennas is in applications where loss of power and formation of surface waves are undesirable, in particular in communication systems.

It has been suggested in Reference [23] that it may be possible to use photonic crystals as substrates for planar antennas. They have observed experimentally that highly efficient planar antennas can be made on photonic crystal regions made of periodic mixtures of semiconductors. When driven at a frequency that lies in the band gap of the photonic crystal, the antenna radiates predominantly into the air rather than into the substrate. The basic idea, behind this significant advance in planar antenna technology, is that no power should be radiated into the substrate at any incidence angle if the driving frequency of the antenna lies within the band gap, since no propagation is allowed into the substrate. The photonic band-gap substrates have also the possibility to eliminate losses due to waves propagating on the surface.

The subject of this paper is the rigorous investigation of the radiation properties of a planar antenna placed on a photonic crystal substrate. Our focus is on the mathematical concepts and methods that give a solid mathematical basis to the work [23], for the quantities describing the radiation properties of planar antenna on photonic crystal substrates. Throughout the paper, we assume that the photonic crystal is symmetric with regards to the Ox_1-axis. This assumption allows us to show that the band gap of the photonic crystal, considered as a medium over the whole of $\mathbf{R}^2$, lies in the band gap of the crystal considered as a medium over the half-plane $x_2 < 0$ only. We believe that this property of band gaps is not true in general, as physicists have observed, in similar devices, surface waves which could reflect the existence of exponentially decaying eigenfunctions inside the band gap of the photonic crystal [24–26].

Under this symmetry assumption, we prove that the radiated electromagnetic wave decays exponentially in the photonic crystal substrate and also along the crystal/air interface. This suggests that it would be advantageous to use crystals, that are periodic along the axes tangent and normal to the interface along which an antenna or a printed circuit is mounted, and that have a symmetric structure with respect to the interface: most of the energy would be radiated in the region where it is desirable.

From a mathematical point of view, we reduce the propagation problem to a weakly singular integral equation on the patch antenna. The well-posedness of this equation in an appropriate framework is established via the construction of a Green's function for the medium crystal/air. We give a representation formula for this Green's function, comparing it to a homogeneous Green's function in the air, and to a Green's function for a crystal that would fill the whole of $\mathbf{R}^2$. Checking the consistency of the representation formula reduces the problem to solving another integral equation, posed on the interface. We solve this equation in spaces of functions with exponential decay: this very feature guarantees the 'coercivity' of the associated integral operator. The behaviour at infinity of the Green's function is then easily established, which in turn gives the behaviour of the radiated wave. We note that the qualitative description of the Green's function that follows from our analysis, suggests that a numerical method based on boundary integral representation could prove very efficient for computing the radiation pattern of such antennas.

The organization of this paper is as follows: in Section 2 we review the essential features of photonic band gap materials, and explain what makes them attractive for many applications. In Section 3 we formulate a model problem and review some useful results. Section 4 is devoted to studying the asymptotic behaviours of the Green's function of the model problem. In Section 5, we derive a weakly singular integral equation on the patch and prove that it is well posed. In Section 6 we show that if the driving frequency of the patch antenna lies within the band gap of the photonic crystal then the unique weak solution to the integral equation decays exponentially in the photonic crystal substrate. In Section 7, we formulate the radiation problem for an array of patch antennas placed on a photonic crystal, and we prove the exponential decay of the solution in the photonic crystal substrate. The paper concludes with some remarks on the generalization of our approach. We believe that the mathematical techniques developed here may be applied to study the three-dimensional radiation problem for a single patch antenna, or for an array of patch antennas, placed on a photonic crystal substrate. The integral equation formulation should also permit comprehension of surface wave formation due to the finiteness of the photonic band-gap structures.

2. A QUICK SURVEY OF PHOTONIC CRYSTALS

We only present here some essential results about photonic band gap materials (for a very nice and complete survey, see Reference [7]). A photonic crystal is a periodic mixture of dielectrics. The propagation of electromagnetic waves in a photonic crystal is governed by the macroscopic Maxwell equations. Assuming that the crystal is non-magnetic (i.e. that its magnetic permeability is equal to 1), they take the form

$$\nabla \times E = -\partial_t H, \quad \nabla \cdot H = 0$$

$$\nabla \times H = -\varepsilon_{\mathrm{p}} \partial_t E, \quad \nabla \cdot (\varepsilon_{\mathrm{p}} E) = 0$$

Here, ε_{p} denotes the electric permittivity: it is assumed to be a periodic function and characterizes the material properties. As the coefficients are time-independent, Fourier transform in time reduces the study of the system to the case of monochromatic waves, where the fields have the form $E(x,t)=\mathrm{e}^{\mathrm{i}\omega t}E(x)$, $H(x,t)=\mathrm{e}^{\mathrm{i}\omega t}H(x)$. The functions $E(x), H(x)$ satisfy then

$$\nabla\times E=-\mathrm{i}\omega H,\quad \nabla\cdot H=0$$
$$\nabla\times H=\mathrm{i}\omega\varepsilon_{\mathrm{p}}E,\quad \nabla\cdot(\varepsilon_{\mathrm{p}}E)=0$$

We will consider, in particular, photonic crystals the electric permittivity of which is independent of one of the co-ordinates, say the x_3-co-ordinate. The Maxwell system can then be decoupled in two scalar equations, in $\mathbf{R}^2$: the transverse magnetic (TM) Helmholtz equation where the electric field is parallel to the x_3-direction, and the transverse electric (TE) Helmholtz equation where the magnetic field is parallel to the x_3-direction. In particular, we will focus on the TM equation at the frequency ω, which takes the form

$$(\Delta+\omega^2\varepsilon_{\mathrm{p}})u=0 \tag{1}$$

The main property of photonic crystal, which make them so attractive for many applications, is the possible existence of gaps in the spectrum of the Maxwell operator or of the TM or TE Helmholtz operators, considered as operators acting on $L^2(\mathbf{R}^2)$. The presence of gaps is essentially due to the periodic character of the material coefficients: it is well known that the spectrum of dielectrics with constant coefficients is a half-line. For photonic crystals, the spectrum may consist of a finite union of intervals.

The main tool to study the spectral properties of these materials is the Floquet–Bloch transform, which is to PDEs with periodic coefficients what the Fourier transform is to PDEs with constant coefficients. We present it in the context of the TM Helmholtz equation, when the medium is assumed to be periodic with period $Y=[0,1]^2$.

Let $k\in\mathbf{R}^2$. A regular function v is called k-quasi-periodic if and only if

$$\forall x\in\mathbf{R}^2,\ \forall j\in Z^2,\quad v(x+j)=\mathrm{e}^{\mathrm{i}k\cdot j}v(x) \tag{2}$$

Let $u(x)$ be a regular function defined on $\mathbf{R}^2$, that decays sufficiently fast at infinity. We define its Floquet transform by

$$\mathscr{F}u(x,k)=\sum_{j\in\mathbf{Z}^2}u(x-j)\mathrm{e}^{\mathrm{i}k\cdot j}$$

One easily checks that $\mathscr{F}u(.,k)$ is k-quasi-periodic with respect to the first variable. Moreover, it is periodic with respect to the variable k, called quasi-momentum. It is then sufficient to know the function $\mathscr{F}u$ for $(x,k)\in Y\times B$, where B is any period relative to k. Solid-state physicists usually choose $B=[-\pi,\pi]^2$, which is called the first Brillouin zone.

It turns out that the Floquet transform commutes with differential operators with periodic coefficients. In the particular case we are interested in, Equation (1) transforms into

$$\Delta\mathscr{F}u(x,k)+\omega^2\varepsilon_{\mathrm{p}}(x)\mathscr{F}u(x,k)=0$$

The Floquet transform allows us to represent a function of $L^2(\mathbf{R}^2)$ as a sum of quasi-periodic functions, a result analogous to the Plancherel Theorem for the Fourier transform. Indeed, the Floquet transform defines an isometry between $L^2(\mathbf{R}^2)$ and $L^2(B,L^2_{\#}(Y))$. Also, the following inversion formula holds:

$$(\mathscr{F}^{-1}v)(x) = \frac{1}{|B|}\int_B v(x,k)\,\mathrm{d}k$$

for any function v in $L^2(B, L^2_\#(Y))$ which is k-quasi-periodic. Together with its commutation properties on operators with periodic coefficients, this isometric character makes the Floquet transform very useful to study spectral problems. Indeed, the spectral problem for an operator L, acting on functions defined on the whole $\mathbf{R}^2$, becomes a family of spectral problems for operators $L(k)$ (formally the same operator, but with a domain that depends on the quasi-momenta k) acting on functions defined on a bounded set. For the TM Helmholtz operator, because of positivity, these spectral problems take the form: for any $k \in B$, find ω and $v: Y \to \mathbf{C}$, k-quasi-periodic, such that

$$\Delta v(x) + \omega^2 \varepsilon_{\mathrm{p}}(x) v(x) = 0$$

Each of the operators $L(k)$ is self-adjoint and has compact resolvent, hence has a discrete spectrum of countably many eigenvalues $\lambda_n(k)$. The spectrum of L can be reconstructed as the union of these discrete spectra. Since the eigenvalues $\lambda_n(k)$ (counted with their multiplicity) are continuous functions of k, the spectrum consists of the collection of intervals $[\min_k \lambda_n(k), \max_k \lambda_n(k)]$.

Normally the bands overlap, however it may happens that the spectrum has gaps. This feature strongly depends on the form of ε_{p}. When the crystal considered consists of a periodic mixture of two phases, the existence of gaps depends on the geometry of the arrangement. Rigorous proofs for existence of gaps have been obtained in Reference [27] (see also Reference [47]) for the TM and TE operators, when the contrast between the permittivities is large, for special geometries: periodic array of large bubbles of material with a low permittivity (e.g. air) separated by thin walls of a material with high permittivity. Some interesting numerical results are provided in Reference [45].

Another important property of the Floquet transform concerns the characterization of the decay of functions in $L^2(\mathbf{R}^n)$ ($n=2$ or 3) in term of the smoothness of their transform, such in the same spirit as the Paley–Wiener theorem [28]. Consider the Green's function of the TM operator in $\mathbf{R}^2$, defined by

$$(\Delta + \omega^2 \varepsilon_{\mathrm{p}})G_{\mathrm{p}} = \delta(x-y) \quad \text{in } \mathbf{R}^2 \tag{3}$$

where $\delta(x-y)$ denotes the Dirac delta function at 0. When the frequency ω lies in a band gap, it has been established in Reference [29], that the Floquet transform of G_{p} is analytic with respect to ω, in a complex neighbourhood of the real axis. In view of the Paley–Wiener-type theorems just mentioned, the analyticity of $\mathscr{F}G_{\mathrm{p}}$ is the key ingredient to the proof of the following result (see Reference [29] for a proof):

Lemma 2.1. There exists two positive constants C_1 and C_2 such that

$$|G_{\mathrm{p}}(\omega; x, y)| \leqslant C_1 \mathrm{e}^{-C_2|x-y|} \quad \text{for } |x-y| \to +\infty \tag{4}$$

The behaviour at infinity of G_{p} is the essential feature of PBG materials: it explains why localized defects in photonic crystals may act as perfect cavities, when the frequency lies in a band gap. Electromagnetic waves can be represented in terms of G_{p} and thus inherit the exponential decay property.

Many applications take advantage of this principle: defects are made in the periodic structure, to create channels where waves, that propagate at frequencies within a gap, will be (practically) localized. Coupled with a pair of waveguides, such a device will act as a very sharp filter, and could be used to enhance the efficiency of a laser, or in telecommunications to sort out signals. Wave guide bends can be created in this manner too, where one can sharply bend a channel of light with nearly no losses, on very small distances (of order of the wavelength).

3. NOTATION AND PRELIMINARY RESULTS

Let (e_1, e_2) be an orthonormal basis of $\mathbf{R}^2$. We first present the electromagnetic structure that we study: a planar patch antenna on a photonic crystal substrate.

The antenna consists of a perfectly conducting segment $\Gamma =]-h, h[\times \{0\}, h>0$, see Figure 1. The photonic crystal fills in the lower domain $\Omega_p = \mathbf{R} \times]-\infty, 0[$.

The domain $\Omega_e = \mathbf{R} \times]0, +\infty[$, above the photonic crystal, is occupied by a homogeneous medium (say air), with constant electromagnetic characteristics $(\varepsilon_e, \mu_e = 1)$. Here, ε_e is a fixed positive constant. The characteristics of the photonic crystal are $(\varepsilon_p, \mu_p = \mu_e = 1)$, where the dielectric function ε_p is real valued and periodic and belongs to the set $\{\varepsilon_p \in L^\infty, 0 < \varepsilon_1 \leqslant \varepsilon_p \leqslant \varepsilon_2 \text{ a.e.}\}$ where ε_1 and ε_2 are fixed.

We assume that the crystal is periodic with period $[0,1]^2$, i.e. that $\varepsilon_p(x+n) = \varepsilon_p(x)$ for almost all $x \in \mathbf{R}^2$ and all $n \in Z^2$. We note that the interface is thus assumed to be parallel to one of the directions of periodicity. Furthermore, we make a symmetry assumption that bear on both the structure of the crystal and the position of the interface: we suppose that

$$\varepsilon_p(x_1, -x_2) = \varepsilon_p(x_1, x_2)$$

for almost all $(x_1, x_2) \in \mathbf{R}^2$.

Throughout this paper, we set

$$\varepsilon = \begin{cases} \varepsilon_e & \text{in } \Omega_e \\ \varepsilon_p & \text{in } \Omega_p \end{cases}$$

Let Σ denote the interface air/photonic crystal substrate: $\Sigma = (\mathbf{R} \backslash [-h, h]) \times \{0\}$.

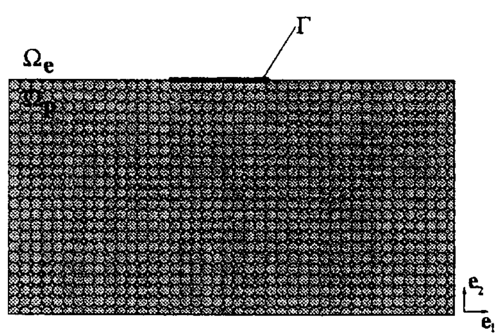

Figure 1. The electromagnetic structure.

The propagation of electromagnetic waves is governed by the Maxwell's equations. It is common to reduce these equations to two sets of scalar equations. The first set of equations are associated with the terminology transverse magnetic (TM), the second set with the terminology transverse electric (TE). In this paper we focus on the TM case. Assuming now that the planar patch antenna is radiating into $\mathbf{R}^2\backslash\bar{\Gamma}$ at the frequency ω, the problem is modeled in the TM case by the Helmholtz equations

$$(\Delta + \omega^2\varepsilon_{\mathrm{p}})u = 0 \quad \text{in } \Omega_{\mathrm{p}} \tag{5}$$

$$(\Delta + \omega^2\varepsilon_{\mathrm{e}})u = 0 \quad \text{in } \Omega_{\mathrm{e}} \tag{6}$$

with the boundary condition

$$u = f \quad \text{on } \Gamma \tag{7}$$

the transmission relations across the interface Σ

$$[u]_\Sigma = [\partial_{x_2}u]_\Sigma = 0 \tag{8}$$

and the radiation condition

$$\lim_{r\to+\infty} \sqrt{r}(\partial_r u - \mathrm{i}\omega\sqrt{\varepsilon_{\mathrm{e}}}u) = 0 \quad \text{for } x_2 > 0 \tag{9}$$

Here, $r = \sqrt{x_1^2 + x_2^2}$, f is a given function in $H^{1/2}(\Gamma)$ and $[g]_\Sigma$ denotes the jump of the function g across Σ.

As for the local behaviour of G_{e} and G_{p}, i.e. when $|x-y|\to 0$, the following result of logarithmic singularity holds:

Lemma 3.1. The functions $G_{\mathrm{p}} - (1/2\pi)\log|x-y|$ and $G_{\mathrm{e}} - (1/2\pi)\log|x-y|$ are continuous for $|x-y|\to 0$.

Proof. We first recall that G_{p} is defined so that the function

$$w_y(x) = \left(G_{\mathrm{p}} - \frac{1}{2\pi}\log|x-y|\right)$$

lies in $H^1(\mathbf{R}^2)$ and satisfies the Helmholtz equation

$$\Delta w_y + \omega^2\varepsilon_{\mathrm{p}} w_y = -\frac{1}{2\pi}\omega^2\varepsilon_{\mathrm{p}}\log|x-y| \quad \text{in } \mathbf{R}^2 \tag{10}$$

Since ω is in a band gap of the Helmholtz operator $(\Delta + \omega^2\varepsilon_{\mathrm{p}})$ in $\mathbf{R}^2$, w_y can be defined as the convolution product

$$w_y(x) = -\frac{\omega^2}{2\pi}\int_{\mathbf{R}^2} G_{\mathrm{p}}(\omega; x, y')\varepsilon_{\mathrm{p}}(y')\log|y-y'|\,\mathrm{d}y'$$

The exponential decay of G_{p} guarantees that such $w_y \in H^1(\mathbf{R}^2)$. Further, since the right-hand side in (10) is locally square integrable, classical H^2-interior estimates [30] together with the continuous Sobolev embedding of H^2 into C^0 give that $G_{\mathrm{p}} - (1/2\pi)\log|x-y|$ is continuous for $|x-y| \to 0$. Classical arguments [31] and [48] show the result for $G_e - (1/2\pi)\log|x-y|$. □

As a consequence of Lemma 3.1, the potentials

$$\Phi(x) = \int_{\mathbf{R}} G_p(\omega; x_1, x_2, y_1, 0)\varphi(y_1)\,\mathrm{d}y_1$$

$$\Psi(x) = \int_{\mathbf{R}} \partial_{y_2} G_p(\omega; x_1, x_2, y_1, 0)\varphi(y_1)\,\mathrm{d}y_1$$

defined for $x_2 \neq 0$ and for φ sufficiently smooth, satisfy the well-known jump relations for the (logarithmic) surface potentials, on $x_2 = 0$ [31].

We recall now a result on the unique solvability of a weak singular integral equation on the open boundary Γ. Following Reference [32], we introduce the Sobolev spaces

$$H^{\pm 1/2}(\Gamma) = \{\tilde{\varphi}|_{\Gamma}, \tilde{\varphi} \in H^{\pm 1/2}(\mathbf{R})\}$$

$$\tilde{H}^{-1/2}(\Gamma) = (H^{1/2}(\Gamma))'$$

$$\tilde{H}^{1/2}(\Gamma) = (H^{-1/2}(\Gamma))'$$

Let S be the operator defined by

$$S: \varphi \in \tilde{H}^{-1/2}(\Gamma) \mapsto \int_{\Gamma} \log|x_1 - y_1|\varphi(y_1)\,\mathrm{d}y_1 \in H^{1/2}(\Gamma)$$

In Reference [32], the following lemma is proven.

Lemma 3.2. There exists a positive constant C such that

$$|(S(\varphi), \varphi)_{H^{1/2}(\Gamma), \tilde{H}^{-1/2}(\Gamma)}| \geqslant C\|\varphi\|^2_{\tilde{H}^{-1/2}(\Gamma)} \tag{11}$$

for all $\varphi \in \tilde{H}^{-1/2}(\Gamma)$.

4. THE GREEN'S FUNCTION

In this section we construct a Green's function for the propagation problem (5)–(9). To do this, we state some preliminary results.

For $\gamma > 0$, we introduce the weighted Sobolev spaces:

$$H^s_\gamma(\mathbf{R}) = \left\{ f \in H^s(\mathbf{R}) : \int_{\mathbf{R}} (1+\xi_1^2)^s |\mathscr{F}(\mathrm{e}^{\gamma x_1} f)(\xi_1)|^2\,\mathrm{d}\xi_1 < +\infty \right\}$$

$$H^s_\gamma(\mathbf{R}, H^s_\gamma(]0,+\infty[) = \left\{ f \in H^s(\mathbf{R}\times]0,+\infty[) : \int_{\mathbf{R}}\int_0^{+\infty} (1+\xi_1^2+\xi_2^2)^s |\mathscr{F}(\mathrm{e}^{\gamma(x_1+x_2)} f)(\xi_1,\xi_2)|^2\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 < +\infty \right\}$$

where $\mathscr{F}$ denotes the Fourier transform and $s > 0$. The embedding $H^{s'}_\gamma(\mathbf{R}) \hookrightarrow H^s_\gamma(\mathbf{R})$ is compact, for any $0 < s < s'$. Further, functions in $H^s_\gamma(\mathbf{R})$ are exponentially decreasing, thus their Fourier transform is analytic: this property is essential in the proof of the uniqueness Lemma 3.3 below, and is the reason for the choice of such spaces.

The following results hold.

Lemma 4.1. Let $\gamma > 0$. Let $f \in H^{1/2}_\gamma(\mathbf{R})$. There exists a unique solution v to the Helmholtz equation

$$(\Delta + \omega^2 \varepsilon_e) v = 0 \quad \text{in } x_2 > 0 \tag{12}$$

$$v|_{x_2=0} = f \tag{13}$$

$$\lim_{r \to +\infty} \sqrt{r}(\partial_r v - \mathrm{i}\omega\sqrt{\varepsilon_e} v) = 0 \tag{14}$$

Furthermore,

$$\partial_{x_2} v|_{x_2=0} = N(f)$$

where the pseudo-differential operator N is defined by

$$N(f) = \frac{\mathrm{i}}{2\pi} \int_{\mathbf{R}} \sqrt{\omega^2 \varepsilon_e - \xi_1^2}\, \mathscr{F}(f)(\xi_1) e^{\mathrm{i}\xi_1 x_1}\, d\xi_1$$

The proof of Lemma 4.1 easily follows from the Fourier representation of the solution w to (12)–(14).

Lemma 4.2. There exists $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$ the pseudo-differential operator L:

$$f \in H^{1/2}_\gamma(\mathbf{R}) \mapsto \frac{1}{2} f + \int_{\mathbf{R}} G_p N(f) - \int_{\mathbf{R}} \partial_{y_2} G_p f \in H^{1/2}_\gamma(\mathbf{R})$$

is of Fredholm type with index 0.

Proof. Since G_p decays exponentially, there exists $\gamma_0 > 0$ (small enough) such that for any $0 < \gamma < \gamma_0$ and for any $f \in H^{1/2}_\gamma(\mathbf{R})$, we have $L(f) \in H^{1/2}_\gamma(\mathbf{R})$. So, to prove the claim, it suffices to consider the operator L as an operator from $H^{1/2}_\gamma(\mathbf{R})$ into $H^{1/2}(\mathbf{R})$.

We first rewrite the first part of the above operator as follows:

$$\frac{1}{2} f + \int_{\mathbf{R}} G_p N(f) = \frac{1}{2} f + \frac{1}{2\pi} \int_{\mathbf{R}} \log|x_1 - y_1| N(f)(y_1)\, dy_1 + \int_{\mathbf{R}} R N(f)$$

The kernel R is continuous by Lemma 2.2, and the last term is compact. Thus, it suffices to show that the operator

$$\begin{aligned} &f \in H^{1/2}_\gamma(\mathbf{R}) \\ &\quad \mapsto \frac{1}{2} f + \frac{\mathrm{i}}{4\pi^2} \int_{\mathbf{R}} \log|x_1 - y_1| \int_{\mathbf{R}} \sqrt{\omega^2 \varepsilon_e - \xi_1^2}\, \mathscr{F}(f)(\xi_1) e^{\mathrm{i}\xi_1 y_1}\, d\xi_1\, dy_1 \in H^{1/2}(\mathbf{R}) \end{aligned} \tag{15}$$

is of Fredholm type with index 0. To do this, we determine its symbol, when treated as a pseudo-differential operator.

Recalling that

$$\Delta\left(\frac{1}{2\pi}\log|x-y|\right)=\delta(x-y)\quad \text{for } x=(x_1,x_2),\ \ y=(y_1,y_2)\in\mathbf{R}^2$$

we have $|\xi|^2\mathscr{F}((1/2\pi)\log|x-y|)=-1$, so that

$$\frac{1}{2\pi}\log|x_1-y_1|=-\frac{1}{2\pi}\int_{\mathbf{R}^2}\frac{\mathrm{e}^{\mathrm{i}\xi_1(x_1-y_1)}}{|\xi|^2}\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2$$

For $x=(x_1,0)$ we have:

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbf{R}}\log|x_1-y_1|N(f)(y_1)\,\mathrm{d}y_1&=-\frac{1}{2\pi}\int_{\mathbf{R}}\int_{\mathbf{R}^2}\frac{\mathrm{e}^{\mathrm{i}\xi_1(x_1-y_1)}}{|\xi|^2}\,\mathrm{d}\xi\,N(f)(y_1)\,\mathrm{d}y_1\\ &=-\frac{1}{2\pi\mathrm{i}}\int_{\mathbf{R}}\left[\int_{\mathbf{R}}\frac{\mathrm{e}^{\mathrm{i}\xi_1(x_1-y_1)}}{2|\xi_1|}\left(\int_{\mathbf{R}}\frac{1}{\xi_2-\mathrm{i}|\xi_1|}-\frac{1}{\xi_2+\mathrm{i}|\xi_1|}\right)\mathrm{d}\xi_2\,\mathrm{d}\xi_1\right]\\ &\quad\times N(f)(y_1)\,\mathrm{d}y_1\end{aligned}$$

which gives by the Residue Theorem

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbf{R}}\log|x_1-y_1|N(f)(y_1)\,\mathrm{d}y_1&=-\int_{\mathbf{R}^2}\frac{\mathrm{e}^{\mathrm{i}\xi_1(x_1-y_1)}}{2|\xi_1|}N(f)(y_1)\,\mathrm{d}y_1\,\mathrm{d}\xi_1\\ &=-\int_{\mathbf{R}}\frac{\mathrm{e}^{\mathrm{i}\xi_1x_1}}{2|\xi_1|}\int_{\mathbf{R}}\mathrm{e}^{-\mathrm{i}\xi_1y_1}N(f)(y_1)\,\mathrm{d}y_1\,\mathrm{d}\xi_1\\ &=-\mathrm{i}\int_{\mathbf{R}}\frac{\mathrm{e}^{\mathrm{i}\xi_1x_1}}{2|\xi_1|}\sqrt{\omega^2\varepsilon_{\mathrm{e}}-\xi_1^2}\mathscr{F}(f)(\xi_1)\,\mathrm{d}\xi_1\end{aligned}$$

To obtain the last equality, we have used the explicit form of the operator N. Thus, the symbol of the pseudo-differential operator defined by (15) is

$$\frac{1}{2}\left(1-\mathrm{i}\frac{\sqrt{\omega^2\varepsilon_{\mathrm{e}}-|\xi_1|^2}}{|\xi_1|}\right)=1+O\left(\frac{1}{|\xi_1|^2}\right),\quad |\xi_1|\to+\infty$$

which yields the desired claim [33].

We now turn to the remaining term

$$\int_{\mathbf{R}}\partial_{y_2}G_{\mathrm{p}}f$$

We can prove from Lemma 2.2 that the kernel $\partial_{y_2}(G_{\mathrm{p}}-(1/2\pi)\log|x-y|)$ has a logarithmic singularity when $|x-y|\to 0$. Since the operator

$$f\in H_{\gamma}^{1/2}(\mathbf{R})\mapsto\int_{\mathbf{R}}\log|x_1-y_1|f(y_1)\,\mathrm{d}y_1\in H^{1/2}(\mathbf{R})$$

is compact and

$$\int_{\mathbf{R}} \partial_{y_2}(\log|x-y|)|_{x_2=y_2=0} f(y_1)\,\mathrm{d}y_1 = 0$$

for any $f \in H_\gamma^{1/2}(\mathbf{R})$, we obtain that the operator

$$f \in H_\gamma^{1/2}(\mathbf{R}) \mapsto \int_{\mathbf{R}} \partial_{y_2} G_{\mathrm{p}} f \in H_\gamma^{1/2}(\mathbf{R})$$

is also compact. The proof is now complete. □

Therefore, the classical Fredholm alternative holds. Existence follows from uniqueness of solutions.

Lemma 4.3. There exists $\gamma_0 > 0$ such that for any $0 < \gamma < \gamma_0$ and for any g in $H_\gamma^{1/2}(\mathbf{R})$, there exists a unique solution $f \in H_\gamma^{1/2}(\mathbf{R})$ to the integral equation

$$\frac{1}{2} f - \int_{\mathbf{R}} \partial_{y_2} G_{\mathrm{p}} f + \int_{\mathbf{R}} G_{\mathrm{p}} N(f) = g \quad \text{on } \mathbf{R}$$

Proof. From Lemma 4.2, it follows that the classical Fredholm alternative holds. Existence follows then from the uniqueness of solutions. The proof is divided into two steps.

Let us first show that the trivial solution is the unique solution in $H_\gamma^1(\mathbf{R}, H_\gamma^1(]0,+\infty[))$ to the Dirichlet problem in the half-space:

$$(\Delta + \omega^2 \varepsilon_{\mathrm{p}})v = 0 \text{ in } \mathbf{R} \times]0,+\infty[, \text{ and } v = 0 \text{ on } x_2 = 0 \tag{16}$$

Defining

$$\tilde{v}(x_1,x_2) = \begin{cases} v(x_1,x_2) & \text{for } x_2 > 0 \\ -v(x_1,-x_2) & \text{for } x_2 \leqslant 0 \end{cases}$$

the extension $\tilde{v} \in H^1(\mathbf{R}^2)$ and it satisfies

$$(\Delta + \omega^2 \varepsilon_{\mathrm{p}})\tilde{v} = \omega^2(\varepsilon_{\mathrm{p}}(x_1,x_2) - \varepsilon_{\mathrm{p}}(x_1,-x_2))\tilde{v} = 0$$

It is precisely here, that we use our assumption on the symmetry of the crystal. Now, since ω lies in a band gap of the photonic crystal, it follows that $\tilde{v} = 0$ in $\mathbf{R}^2$.

Next, let $f \in H_\gamma^{1/2}(\mathbf{R})$ be a solution to the homogeneous integral equation

$$\frac{1}{2} f - \int_{\mathbf{R}} \partial_{y_2} G_{\mathrm{p}} f + \int_{\mathbf{R}} G_{\mathrm{p}} N(f) = 0 \quad \text{on } \mathbf{R}$$

and consider the function v defined by

$$v = \int_{\mathbf{R}} \partial_{y_2} G_{\mathrm{p}} f - \int_{\mathbf{R}} G_{\mathrm{p}} N(f) \quad \text{in } \mathbf{R}^2 \backslash (\mathbf{R} \times \{0\})$$

It is easy to show, since G_{p} satisfies the logarithmic jump conditions, that v satisfies

$$(\Delta + \omega^2 \varepsilon_{\mathrm{p}})v = 0 \quad \text{in } \mathbf{R} \times]0,+\infty[, \quad v|_{x_2 \to 0^+} = 0$$

$$(\Delta + \omega^2 \varepsilon_{\mathrm{p}})v = 0 \quad \text{in } \mathbf{R} \times]-\infty,0[, \quad v|_{x_2 \to 0^-} = f$$

and that

$$[\partial_{x_2} v] = N(f)$$

The uniqueness of a solution to the Dirichlet problem (16) in the half-space shows that $\partial_{x_2} v|_{x_2 \to 0^+} = 0$. Let v_α (resp. f_α) be the Floquet transform of v (resp. of f):

$$v_\alpha(x_1, x_2) = \sum_{j \in Z} v(x_1 + j, x_2) \mathrm{e}^{-\mathrm{i}\alpha j}$$

$$f_\alpha(x_1, x_2) = \sum_{j \in Z} f(x_1 + j, x_2) \mathrm{e}^{-\mathrm{i}\alpha j}$$

for $\alpha \in [0, 2\pi]$. Each function v_α satisfies

$$((\partial_{x_1} + \mathrm{i}\alpha)^2 + \partial_{x_2}^2) v_\alpha + \omega^2 \varepsilon_\mathrm{p} v_\alpha = 0 \quad \text{in } x_2 < 0$$

$$v_\alpha = f_\alpha \quad \text{on } x_2 = 0$$

$$\partial_{x_2} v_\alpha = N_\alpha(f_\alpha) \quad \text{on } x_2 = 0$$

where the operator N_α is defined by

$$N_\alpha(g) = \sum_{j \in Z} \mathrm{i} \sqrt{\omega^2 \varepsilon_\mathrm{e} - (\alpha + 2\pi j)^2}\, g_j \mathrm{e}^{\mathrm{i}(\alpha + 2\pi j) x_1}$$

for $g = \sum_{j \in Z} g_j \, \mathrm{e}^{\mathrm{i}(\alpha + 2\pi j) x_1}$. Multiplying by $\overline{v_\alpha}$ and integrating by parts over $]0,1[\times]0,-\infty[$ yields

$$\int_{]0,1[\times]0,-\infty[} |(\partial_{x_1} + \mathrm{i}\alpha) v_\alpha|^2 + |\partial_{x_2} v_\alpha|^2 - \omega^2 \int_{]0,1[\times]0,-\infty[} \varepsilon_\mathrm{p} |v_\alpha|^2 = \int_0^1 N_\alpha(f_\alpha) \overline{f_\alpha}$$

which gives

$$\mathrm{Im} \int_0^1 N_\alpha(f_\alpha) \overline{f_\alpha} = 0$$

for any $\alpha \in [0, 2\pi]$. The Parseval equality for the Floquet transform yields

$$\int_\mathbf{R} N(f) \bar{f} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^1 N_\alpha(f_\alpha) \overline{f_\alpha} \, \mathrm{d}\alpha$$

Thus

$$\mathrm{Im} \int_\mathbf{R} N(f) \bar{f} = 0$$

which gives $\mathscr{F}(f) = 0$ for $|\xi_1| < \omega\sqrt{\varepsilon_\mathrm{e}}$. However, since f is exponentially decaying, $\mathscr{F}(f)$ is analytic [34], and thus $\mathscr{F}(f)$ vanishes identically. We conclude that $f = 0$ on $\mathbf{R}$, which completes the proof. □

Assume that $y \in \Omega_e$. Since G_p has exponential decrease, there exists $\gamma > 0$ such that $G_p(x_1, 0) \in H_\gamma^{1/2}(\mathbf{R})$. We seek a Green's function G to the propagation problem (5)–(9), solution to

$$(\Delta + \omega^2 \varepsilon) G = \delta(x - y) \quad \text{in } \mathbf{R}^2 \tag{17}$$

$$[G]_{\mathbf{R} \times \{0\}} = [\partial_{x_2} G]_{\mathbf{R} \times \{0\}} = 0 \tag{18}$$

$$\lim_{r \to +\infty} \sqrt{r} (\partial_r G - \mathrm{i}\omega \sqrt{\varepsilon_e} G) = 0 \tag{19}$$

Consider $\tilde{G}_e$ the unique solution of

$$(\Delta + \omega^2 \varepsilon_e) \tilde{G}_e = \delta(x - y) \quad \text{in } \Omega_e$$

$$\tilde{G}_e = G_p \quad \text{on } \mathbf{R} \times \{0\}$$

$$\lim_{r \to +\infty} \sqrt{r} (\partial_r \tilde{G}_e - \mathrm{i}\omega \sqrt{\varepsilon_e} \tilde{G}_e) = 0$$

Then, if G is a solution to (17)–(19), $G - \tilde{G}_e$ satisfies

$$(\Delta + \omega^2 \varepsilon_e)(G - \tilde{G}_e) = 0 \quad \text{in } \Omega_e \tag{20}$$

$$G - \tilde{G}_e = (G - \tilde{G}_e)|_{x_2 = 0} \quad \text{on } \mathbf{R} \times \{0\} \tag{21}$$

$$\lim_{r \to +\infty} \sqrt{r} (\partial_r (G - \tilde{G}_e) - \mathrm{i}\omega \sqrt{\varepsilon_e} (G - \tilde{G}_e)) = 0 \tag{22}$$

Thus, from the definition of the operator N, it follows that

$$\partial_{x_2} (G - \tilde{G}_e)|_{x_2 = 0} = N(f) \tag{23}$$

where

$$f = (G - \tilde{G}_e)|_{x_2 = 0}$$

Solving (20)–(22) by Fourier transform, we have

$$G = \tilde{G}_e + \frac{1}{2\pi} \int_{\mathbf{R}} \mathscr{F}(f)(\xi_1) \mathrm{e}^{\mathrm{i}\sqrt{\omega^2 \varepsilon_e - \xi_1^2} x_2} \mathrm{e}^{\mathrm{i}\xi_1 x_1} \, \mathrm{d}\xi_1 \quad \text{in } \Omega_e \tag{24}$$

Furthermore, $G - G_p$ is a solution to

$$(\Delta + \omega^2 \varepsilon_p)(G - G_p) = 0 \quad \text{in } \Omega_p$$

$$G - G_p = (G - \tilde{G}_e)|_{x_2 = 0} \quad \text{on } \mathbf{R} \times \{0\}$$

It follows that $(G - G_p)$ has the following integral representation in terms of G_p:

$$G - G_p = -\int_{\mathbf{R} \times \{0\}} G_p \partial_{y_2} (G - G_p)|_{y_2 = 0} + \int_{\mathbf{R} \times \{0\}} \partial_{y_2} G_p (G - G_p)|_{y_2 = 0} \quad \text{in } \Omega_p$$

Since $[\partial_{x_2} G]_{\mathbf{R}\times\{0\}}$ must be equal to 0, identity (23) together with the jump relations for the potential $G_{\rm p}$, constrain G to satisfy

$$\frac{1}{2}f - \int_{\mathbf{R}} \partial_{y_2} G_{\rm p}(x_1, 0, y_1, 0) f(y_1)\,\mathrm{d}y_1 + \int_{\mathbf{R}} G_{\rm p}(x_1, 0, y_1, 0) N(f)(y_1)\,\mathrm{d}y_1 = g(x_1) \quad \text{on } \mathbf{R} \tag{25}$$

where

$$g = -\int_{\mathbf{R}\times\{0\}} G_{\rm p}\partial_{y_2}(\tilde{G}_{\rm e} - G_{\rm p})|_{y_2=0}$$

lies in $H^{1/2}_\gamma(\mathbf{R})$, for γ small enough. Thus, solving (17)–(19) reduces to the resolution of the integral equation (25).

Lemma 4.3 shows that it has a unique solution $f \in H^{1/2}_\gamma(\mathbf{R})$. Knowing f, we construct G in $\Omega_{\rm e}$ by (24) and G in $\Omega_{\rm p}$ by the representation formula:

$$G = G_{\rm p} - \int_{\mathbf{R}\times\{0\}} G_{\rm p} N(f) + \int_{\mathbf{R}\times\{0\}} \partial_{y_2} G_{\rm p} f + g \quad \text{in } \Omega_{\rm p} \tag{26}$$

If $y \in \Omega_{\rm p}$ then G can be constructed by exactly the same procedure.

Representations (24)–(26) yield the following theorem, that describes the behaviour of the Green's function G.

Theorem 4.1. There exists two positive constants C_1, C_2, independent of x and y, such that

$$|G(\omega; x_1, x_2, y_1, y_2)| \leqslant C_1 \mathrm{e}^{-C_2|x_2|} \quad \text{for } x_2 \to -\infty \tag{27}$$

$$|(G - \tilde{G}_{\rm e})(\omega; x_1, x_2, y_1, y_2)| \leqslant C_1 \mathrm{e}^{-C_2 x_2} \quad \text{for } x_2 \to +\infty \tag{28}$$

$$|G(\omega; x_1, x_2 = 0, y_1, y_2)| \leqslant C_1 \mathrm{e}^{-C_2|x_1|} \quad \text{for } |x_1| \to +\infty \tag{29}$$

Moreover, the function $G - (1/2\pi)\log|x-y|$ is continuous for $|x-y| \to 0$.

Note that the proof of (28) relies on the fact that $(G - \tilde{G}_{\rm e})|_{y_2=0}$ is exponentially decreasing and therefore its Fourier transform is analytic in ξ_1. Further, from References [32, 35], we obtain without any difficulty that problem (5)–(9) is equivalent to solving the integral equation

$$\int_\Gamma Gg = f \quad \text{on } \Gamma$$

where the unknown $g = [\partial_{x_2} u]_\Gamma \in \tilde{H}^{-1/2}(\Gamma)$.

Estimates (27)–(29), show, in particular, that the energy radiated in the substrate decays exponentially. Further, it also follows from Theorem 4.1 that the Green's function G behaves in $\Omega_{\rm e}$ like that associated to the homogeneous acoustic half-space with a Dirichlet boundary condition on $x_2 = 0$.

5. THE INTEGRAL EQUATION

In this section, we first establish the uniqueness of a solution to the integral equation: find $g \in \tilde{H}^{-1/2}(\Gamma)$ such that

$$\int_\Gamma G(\omega; x_1, x_2 = 0, y_1, y_2 = 0) g(y_1)\,\mathrm{d}y_1 = f(x_1) \quad \text{for all } x = (x_1, 0) \in \Gamma \tag{30}$$

Lemma 5.1. Let f be in $H^{1/2}(\Gamma)$. There exists at most one solution g in $\tilde{H}^{-1/2}(\Gamma)$ to the integral equation (30).

Proof. Let $g \in \tilde{H}^{-1/2}(\Gamma)$. Denote

$$u(x_1, x_2) = \int_\Gamma G(\omega; x_1, x_2, y_1, y_2 = 0) g(y_1)\,\mathrm{d}y_1 \quad \text{for } x = (x_1, x_2) \in \mathbf{R}^2 \backslash \bar{\Gamma} \tag{31}$$

Then, u satisfies

$$(\Delta + \omega\varepsilon) u = g \delta_\Gamma \quad \text{in } \mathbf{R}^2 \tag{32}$$

where δ_Γ is the delta function on Γ. Further, estimate (28) in Theorem 4.1 implies that u satisfies the radiation condition.

Assume that

$$\int_\Gamma G(\omega; x_1, x_2 = 0, y_1, y_2 = 0) g(y_1)\,\mathrm{d}y_1 = 0 \quad \text{for all } x = (x_1, 0) \in \Gamma \tag{33}$$

Multiplying (32) by $\bar{u}$ and integrating by parts over a ball B_R, for R large enough, gives

$$\int_{B_R \backslash \bar{\Gamma}} |\nabla u|^2 - \omega^2 \int_{B_R \backslash \bar{\Gamma}} \varepsilon |u|^2 - \int_{\partial B_R} \partial_r u\, \bar{u} = 0 \tag{34}$$

Taking the imaginary part of (34) implies

$$\operatorname{Im} \int_{\partial B_R} \partial_r u\, \bar{u} = 0 \tag{35}$$

Since

$$\int_{\partial B_R} |\partial_r u|^2 + \omega^2 \varepsilon_\mathrm{e} \int_{\partial B_R} |u|^2 + 2\omega\sqrt{\varepsilon_\mathrm{e}} \operatorname{Im} \int_{\partial B_R} \partial_r u\, \bar{u} = \int_{\partial B_R} |\partial_r u - \mathrm{i}\omega\sqrt{\varepsilon_\mathrm{e}} u|^2$$

which goes to 0 as $R \to +\infty$, according to the radiation condition, it follows that

$$\int_{\partial B_R} |u|^2 \to 0, \quad R \to +\infty$$

Hence

$$u = o\left(\frac{1}{\sqrt{R}}\right) \quad \text{for } R \to +\infty \tag{36}$$

Recalling the definition of the operator T and combining estimate (36) together with the asymptotic behaviour (29) of G in Theorem 4.1 we obtain from (31) that $\int_{-R}^{R} N(u)\bar{u}$ goes to 0 as $R \to +\infty$. Consequently, (36) together with the fact that $u|_{x_2=0}$ which is given by (31) is exponentially decaying when $|x_1| \to +\infty$ yields

$$u(x_1, x_2 = 0) = 0$$

for any x_1. By the unique solvability of the Dirichlet problem in both the homogeneous half-space and the photonic band-gap substrate u is identically zero in $\mathbf{R}^2$ and $[\partial_{x_2} u]$ must vanish identically on $x_2 = 0$, and thus, $g = 0$ on Γ. This completes the proof of uniqueness. □

The existence of a solution to the integral equation (30) follows now from the last statement in Theorem 4.1:

Theorem 5.1. Let $f \in H^{1/2}(\Gamma)$. There exists a unique solution g in $\tilde{H}^{-1/2}(\Gamma)$ to the integral equation (30).

6. THE RADIATION PROPERTIES

The following theorem is an immediate consequence of Lemma 2.1 and Theorem 4.1; it shows the exponential decay of the solution to the problem of propagation in the photonic crystal substrate.

Theorem 6.1. If ω is in a band gap of the periodic operator $(\Delta + \omega^2 \varepsilon_{\mathrm{p}})$ in $\mathbf{R}^2$ then the following estimate holds

$$|u(x)| \leqslant C_1 \mathrm{e}^{-C_2|x_2|} \quad \text{as } x_2 \to -\infty$$

where the positive constants C_1 and C_2 are independent of x.

A relevant question is then to estimate the decay of u at the interface photonic crystal/air as $|x_1| \to +\infty$. The following result which shows that, under the symmetry hypothesis on ε_{p}, a photonic crystal substrate permits reduction of the radiated energy even at the air/substrate interface, is also an easy consequence of Theorem 4.1.

Theorem 6.2. If ω is in a band gap of the periodic operator $(\Delta + \omega^2 \varepsilon_{\mathrm{p}})$ in $\mathbf{R}^2$ then there exists two positive constants C_1 and C_2 which are independent of the variable x_1, such that

$$|u(x_1, x_2 = 0)| \leqslant C_1 \mathrm{e}^{-C_2|x_1|} \quad \text{as } |x_1| \to +\infty$$

Theorems 4.1 and 6.2 provide a mathematical justification of the use of a photonic crystal substrate as a perfect mirror at any incidence angle. Mirrors, probably the most prevalent of optical devices, are used for imaging and solar energy collection and in laser cavities. One can distinguish between two types of mirrors, the age-old metallic and the more recent dielectric. The fabrication of dielectric reflectors may be the most significant advance in mirror technology. It promises to have significant applications in many fields, including fibre optics, cellular telephones, energy conservation, spectroscopy and even medicine. Unlike the dielectric reflectors, the metallic mirrors cannot be used in applications like communications and high-powered lasers, where minimizing energy loss is important. At infra-red and optical frequencies, a few percent of the incident power is typically lost because of absorption. For applications in which energy loss is important scientists depend on a more sophisticated device. As we have seen by dielectric structures one can make mirrors that are nearly perfect reflectors. The ability to reflect light with extremely low loss is associated with the existence of a band gap, which can exist only in a system with a dielectric function that is periodic. Another useful property of dielectric mirrors is that they can be designed to reflect only a small range of frequencies and let the rest pass unmolested. For example, periodic dielectric mirrors can be designed to reflect infra-red light but transmit visible light. Because of their very high reflectivity over a limited angular and spectral range they are also very useful for planar cavities [36]. Periodic dielectric mirrors could also be useful in improving thermophotovoltaic cells, devices that trap waste heat and convert it to energy. Since these mirrors could be made now to reflect radio waves, they could be used to boost the performance of cellular telephones [37, 38].

7. RADIATION PROPERTIES OF AN ARRAY OF PATCH ANTENNAS ON A PHOTONIC CRYSTAL SUBSTRATE

In this section, we consider an array of patch antennas on a photonic crystal substrate, as shown in Figure 2. These one-dimensional periodic structures are widely used devices, known for their polarization diplexing properties. Their main application concerns polarization twisters in antenna design. Power loss or coupling of different antennas may seriously modify the scattering characteristics of the device. The use of a periodic dielectric with a complete band gap as a substrate could lead to improved, more efficient devices.

Let $\Gamma_j = \{x = (x_1 + jh', 0) : (x_1, 0) \in \Gamma\}$, where $j \in Z$, $h' > 2h$. Let α be in $[0, 2\pi[$.

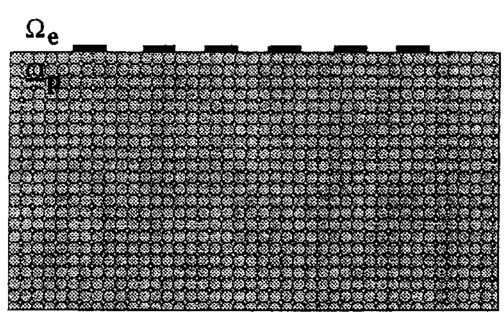

Figure 2. Array of patch antennas.

The radiation pattern induced by the array of patch antennas is modelled by the Helmholtz equations

$$(\Delta + \omega^2 \varepsilon_{\rm p})u = 0 \quad \text{in } \Omega_{\rm p} \tag{37}$$

$$(\Delta + \omega^2 \varepsilon_{\rm e})u = 0 \quad \text{in } \Omega_{\rm e} \tag{38}$$

with the boundary condition

$$u = {\rm e}^{{\rm i}\alpha j} f(x_1 - jh') \quad \text{on } \Gamma_j \text{ for } j \in Z \tag{39}$$

and the transmission relations across the interface $\Sigma = \mathbf{R} \times \{0\} \setminus \bigcup_{j\in Z} \bar{\Gamma}_j$

$$[u]_\Sigma = [\partial_{x_2} u]_\Sigma = 0 \tag{40}$$

To formulate the radiation condition on u, let us first introduce the following definition:

Definition 7.1. A regular function v is α-quasi-periodic in the x_1 variable if and only if

$$\forall (x_1, x_2) \in \mathbf{R}^2, \ \forall j \in Z, \quad v(x_1 + jh', x_2) = {\rm e}^{{\rm i}\alpha j} v(x_1, x_2) \tag{41}$$

Since the boundary condition on $\bigcup_{j\in Z} \Gamma_j$ is α-quasi-periodic, we may seek a solution u that satisfies (41).

Problem (37)–(41) is completed by the α-quasi-periodic radiation condition [39]: u consists of a sum of a finite number of outgoing plane waves plus an infinite number of exponentially vanishing plane waves, as $x_2 \to +\infty$.

Let $G_{\rm p}^\alpha$ be the α-quasi-periodic Green's function of the periodic Helmholtz equation in $\mathbf{R}^2$:

$$(\Delta + \omega^2 \varepsilon_{\rm p}) G_{\rm p}^\alpha = \delta^\alpha(x - y) = \sum_{j \in Z} {\rm e}^{-{\rm i}\alpha j} \delta(x_1 + jh' - y_1, x_2 - y_2)$$

Let

$$G_{\rm e}^\alpha(\omega; x_1, x_2, y_1, y_2) = \frac{\rm i}{4} \sum_{j\in Z} {\rm e}^{-{\rm i}\alpha j} H_0^{(1)}\left(\omega \sqrt{\varepsilon_{\rm e}} \sqrt{|x_1 + jh' - y_1|^2 + |x_2 - y_2|^2}\right)$$

From Reference [39], it follows that $G_{\rm e}^\alpha$ is the α-quasi-periodic Green's function of the homogeneous Helmholtz equation in $\mathbf{R}^2$:

$$(\Delta + \omega^2 \varepsilon_{\rm e}) G_{\rm e}^\alpha = \delta^\alpha(x - y)$$

According to Reference [40], the Green's functions $G_{\rm p}^\alpha$ and $G_{\rm e}^\alpha$ have the same logarithmic singularity when $|x - y| \to 0$: the functions $G_{\rm p}^\alpha - (1/2\pi)\log|x - y|$ and $G_{\rm e}^\alpha - (1/2\pi)\log|x - y|$ are continuous. Further, it can be shown that $G_{\rm p}^\alpha$ decays exponentially for $|x_2| \to +\infty$, if ω is in a band gap of the periodic Helmholtz operator in $\mathbf{R}^2$. This implies that the α-quasi-periodic Green's function of the problem

$$(\Delta + \omega^2 \varepsilon) G^\alpha = \delta^\alpha(x - y)$$

decays exponentially when $x_2 \to -\infty$. We may also arrive at the same conclusion by proving as in Reference [39] that G^α is, in fact, given by the following summation formula:

$$G^\alpha(\omega; x_1, x_2, y_1, y_2) = \sum_{j \in Z} e^{-i\alpha j} G(\omega; x_1 + jh' - y_1, x_2 - y_2) \tag{42}$$

where G is the Green's function which has been constructed in Section 3.

The radiation problem from the array of patch antennas described below can be reduced to the resolution of the following integral equation in $\tilde{H}^{-1/2}(\Gamma)$:

$$f = \int_\Gamma G^\alpha g \quad \text{on } \Gamma \tag{43}$$

The approach we developed to prove Theorems 4.1 and 5.1, can be easily adapted to the case of an array of patch antennas: it can be shown, following the same method step by step or making use of the summation formula (42), that the integral equation (43) has a unique solution in $\tilde{H}^{-1/2}(\Gamma)$. A statement analogue to that of Theorem 5.1 can be proved: more precisely, let u be defined by

$$u = \int_\Gamma G^\alpha g$$

for $-h'/2 + h < x_1 < h'/2$ and $x_2 \in \mathbf{R}$. Then the following estimate holds

$$|u| \leqslant C_1 e^{-C_2 |x_2|} \quad \text{for } x_2 \to -\infty$$

where the positive constants C_1 and C_2 are independent of $x = (x_1, x_2)$.

8. CONCLUDING REMARKS AND EXTENSIONS

We have investigated the radiating properties of a single-patch antenna, or of an array of phased patch antennas, placed on a substrate which is a photonic crystal.

For each situation, we have shown that a unique weak solution exists, when the driving frequency of the electromagnetic device lies within the band gap of the photonic crystal medium. Further, we have proven that the solution decays exponentially in the substrate, i.e. that the main part of the energy is radiated in the air.

Additionally, we found in the case of a simple patch antenna, that the solution also decays exponentially at the interface air/photonic crystal substrate.

Our analysis is based on the study of the asymptotic behaviour of the Green's function associated with the device: we have shown that its singular behaviour reduces to that of the fundamental solution for the Laplace operator, and that at infinity it behaves either like the Green's function of the homogeneous medium in $\mathbf{R}^2$ or like the Green's function of the photonic crystal in $\mathbf{R}^2$. Our analysis is based on two assumptions: the frequency ω lies in a band gap of the photonic crystal (in $\mathbf{R}^2$) and the crystal can be extended by reflection to a periodic structure in $\mathbf{R}^2$, i.e. that the periodic function ε_p satisfies the symmetry condition $\varepsilon_p(x_1, -x_2) = \varepsilon_p(x_1, x_2)$ for almost all $(x_1, x_2) \in \mathbf{R}^2$. This last assumption implies that ω also lies in a band gap for the operator $\Delta + \omega^2\varepsilon_{\mathrm{p}}$ in the half-plane with a Dirichlet boundary condition. We believe that this statement is not true in general. Indeed, assume that

$$(\Delta + \omega^2\varepsilon_{\mathrm{p}})v = 0 \quad \text{in } \mathbf{R} \times]0, +\infty[, v \in H^1_\gamma(\mathbf{R}, H^1_\gamma(]0, +\infty[)) \text{ and } v = 0 \text{ on } x_2 = 0$$

as in the proof of Lemma 3.3, and consider

$$\tilde{v}(x_1, x_2) = \begin{cases} v(x_1, x_2) & \text{for } x_2 > 0 \\ -v(x_1, -x_2) & \text{for } x_2 \leqslant 0 \end{cases}$$

The extension $\tilde{v} \in H^1(\mathbf{R}^2)$ and it satisfies

$$(\Delta + \omega^2\varepsilon_{\mathrm{p}})\tilde{v} = \omega^2(\varepsilon_{\mathrm{p}}(x_1, x_2) - \varepsilon_{\mathrm{p}}(x_1, -x_2))\tilde{v} \quad \text{in } \mathbf{R}^2$$

The term in the right-hand side is reminiscent of the effect of a defect in the medium. In References [29, 41, 42, 46], the implications of a compactly supported defect in a photonic crystal have been analysed. They have shown that the presence of a defect can create eigenvalues in the band gap, associated with exponentially decaying eigenfunctions. A similar phenomena might occur here. Moreover, experiments seem to consolidate our conjecture: surface waves have been observed in this type of devices in References [24–26, 44]. It would be very interesting to find out under which geometric conditions the band gap for the whole plane is embedded in the band gap of the half-plane.

Our analysis provides a mathematical justification of the use of a photonic crystal substrate as a perfect mirror at any incidence angle, with exponentially decaying energy in the crystal. According to Reference [5], this feature of photonic crystals has promising and significant technological applications in many fields including fibre optics, cellular telephones and energy conservation. It should also be noted that it also follows from our analysis that the use of photonic band-gap substrates which can be extended by reflection to periodic structures in all $\mathbf{R}^2$ as substrates for planar antennas and printed circuits eliminates the formation of surface waves on the interface $x_2 = 0$.

ACKNOWLEDGEMENTS

The authors are very grateful to H. Benisty (PMC, Ecole Polytechnique) for very insightful discussions. They are also very grateful to the referees for their comments which enable them to make many improvements in their presentation.

REFERENCES

1. Bowden CM, Dowling JP, Everitt HO (eds). Special issue on development and applications of materials exhibiting photonic band gaps. *Journal of the Optical Society of America* B 1993; **10**(2):280–413.
2. Hui PM, Johnson NF. Photonic band-gap materials. In *Solid State Physics*, Ehrenreich H, Spaepen F (eds), vol. 49. Academic Press: New York, 1995, 151–203.
3. John S. The localization of waves in disordered media. In *Scattering and Localization of Classical Waves in Random Media.* Ping Sheng (ed). World Scientific: Singapore, 1990; 1–66.
4. Kurizki G, Haus JW (eds). Photonic band structures (special issue). *Journal of Modern Optics* 1994; **41**(2).
5. Joannopoulos JD, Meade RD, Win JN. *Photonic Crystals*, *Molding the Flow of Light.* Princeton University Press: Princeton, NJ, 1995.
6. Photonic and Acoustic Band-Gap Bibliography. http://www.ee.ucla.edu/labs/photon/biblio.html

7. Kuchment P. Mathematics of photonic crystals. *Mathematical Modelling in Optical Science*, *SIAM Frontiers in Applied Mathematics*, to appear.
8. Coccioli R, Yang FR, Ma KP, Itoh T. Aperture-coupled patch antenna on UC-PBG substrate. *IEEE Transactions on Microwave Theory and Techniques* 1999; **47**:2123–2130.
9. Colburn JS, Rahmat-Samii Y. Printed antenna pattern improvement through substrate perforation of high dielectric constant material: an FDTD evaluation. *Microwave and Optical Technology Letters* 1998; **18**:27–32.
10. Falcone F, Lopetegi T, Sorolla M. 1-D and 2-D photonic bandgap microstrip structures. *Microwave and Optical Technology Letters* 1999; **22**:411–412.
11. Falcone F, Lopetegi T, Irisarri M, Laso MAG, Erro MJ, Sorolla M. Compact photonic bandgap microstrip structures. *Microwave and Optical Technology Letters* 1999; **23**:233–236.
12. Gonzalo R, de Maagt P, Sorolla M. Enhanced patch-antenna performance by suppressing surface waves using photonic-bandgap substrates. *IEEE Transactions on Microwave Theory* 1999; **47**:2131–2138.
13. Gonzalo R, Martinez B, de Maagt P, Sorolla M. Improved patch-antenna performance by using photonic-bandgap substrates. *Microwave and Optical Technology Letters* 2000; **24**:213–215.
14. Horii Y, Tsutsumi M. Harmonic control by photonic bandgap on microstrip patch antenna. *IEEE Microwave Guided Waves* 1999; **9**:13–15.
15. Sigalas MM, Biswas R, Ho KM. Theoretical study of dipole antennas on photonic band-gap materials. *Microwave and Optical Technology Letters* 1996; **13**:205.
16. Sigalas MM, Biswas R, Li Q, Leung W, Jacobs-Woodbury R, Lough B, Nielsen S, Tuttle G, Ho KM. Dipole antennas on photonic band-gap crystals: experiment and simulation. *Microwave and Optical Technology Letters* 1997; **15**:153.
17. Sigalas MM, Biswas R, Ho KM, Leung W, Tuttle G, Crouch DD. The effect of photonic crystals on dipole antennas. *Electromagnetics* 1999; **19**:291–303.
18. Smith DR, Kessler MP, Maloney JG. Dipole antennas used with all-dielectric, woodpile photonic-bandgap reflectors: Gain, field patterns, and input impedance. *Microwave and Optical Technology Letters* 1999; **21**: 191–196.
19. Thevenot M, Denis MS, Reineix A, Jecko B. Design of a new photonic cover to increase antenna directivety. *Microwave and Optical Technology Letters* 1999; **22**:136–139.
20. Thevenot M, Cheype C, Reineix A, Jecko B. Directive photonic-bandgap antennas. *IEEE Transactions on Microwave Theory and Techniques* 1999; **47**:2115–2122.
21. Yang HYD. Theory of microstrip lines on artificial periodic substrates. *IEEE Transactions on Microwave Theory and Techniques* 1999; **47**:629–635.
22. Yang HYD. Theory of antenna radiation from photonic band-gap materials. *Electromagnetics* 1999; **19**: 255–276.
23. Brown ER, Parker CD, Yablonovitch E. Radiation properties of a planar antenna on a photonic-crystal substrate. *Journal of the Optical Society of America* B 1993; **10**:404–407.
24. Cheng SD, Biswas R, Ozbay E, McCalmont S, Tuttle G, Ho KM. Optimized dipole antennas on photonic band gap crystals. *Applied Physics Letters* 1995; **67**:3399–3401.
25. Meade RD, Brommer KD, Rappe AM, Joannopoulos JD. Electromagnetic Bloch waves at the surface of a photonic crystal. *Physical Review B* 1991; **44**:10 961–10 964.
26. Robertson WM, Arjavallingam G, Meade RD, Brommer KD, Rappe AM, Joannopoulos JD. Observation of surface photons on periodic dielectric array. *Optics Letters* 1993; **18**:528–530.
27. Figotin A, Kuchment P. Band-gap structure of spectra of periodic dielectric and acoustic media. I. Scalar model. *SIAM Journal of Applied Mathematics* 1996; **56**:68–88.
28. Kuchment P. *Floquet Theory for Partial Differential Equations.* Birkhäuser-Verlag: Basel, 1993.
29. Figotin A, Klein A. Localized classical waves created by defects. *Journal of Statistical Physics* 1997; **86**: 165–177.
30. Gilbarg D, Trudinger NS. *Elliptic Partial Differential Equations of Second Order* (2nd edn). Springer: Berlin, 1983.
31. Colton D, Kress R. *Inverse Acoustic and Electromagnetic Scattering Theory* (2nd edn). Applied Mathematical Sciences, Vol. 93. Springer: Berlin, 1998.
32. Wendland WL, Stephan EP. A hypersingular boundary integral method for two-dimensional screen crack problems. *Archive for Rational Mechanics and Analysis.* 1990; **112**:363–390.
33. Taylor ME. *Partial Differential Equations II. Qualitative Studies of Linear Equations*, Applied Mathematical Sciences, vol. 116. Springer: Berlin, 1996.
34. Reed R, Simon B. *Methods of Modern Mathematical Physics II*: *Fourier Analysis*, *Self-adjointness*. Academic Press: New York, 1975.
35. Stephan EP. Boundary integral equation for screen problems in R^3. *Integral Equations Operator Theory* 1987; **10**:236–257.
36. Baets R, Bienstman P, Bockstaele R. Basics of dipole emission from a planar cavity. In *Confined Photon Systems. Fundamentals and Applications*, Benisty H, Gérard JM, Houdré R, Rarity J, Weibuch C (eds), Lecture Notes in Physics, Springer: Berlin, 1999; 38–79.

37. Fink Y, Winn JN, Fan S, Chen C, Michel J, Joannopoulos DJ, Thomas EL. A dielectric omnidirectional reflector. *Science* 1998; **282**:1679–1682.
38. Winn JN, Fink Y, Fan S, Joannopoulos JD. Omnidirectional reflexion from a one-dimensional photonic crystal. *Optics Letters* 1998; **23**:1573–1575.
39. Bruno O, Reitich F. Solution of a boundary value problem for Helmholtz equation via variation of the boundary into the complex domain. *Proceedings of the Royal Society of Edinburgh Section A* 1992; **122**:317–340.
40. Ammari H, Béreux N, Nédélec J-C. Resonances of Maxwell's equations in a periodic structure. *Japan Journal of Industrial Applications and Mathematics* 2000; **17**:149–198.
41. Figotin A, Klein A. Midgap defect modes in dielectric and acoustic media. *SIAM Journal of Applied Mathematics* 1998; **58**:1748–1773.
42. Figotin A, Klein A. Localization of light in lossless inhomogeneous dielectrics. *Journal of the Optical Society of America* A 1998; **15**:1423–1435.
43. Benisty H, Gérard JM, Houdré R, Rarity J, Weibuch C (eds). *Confined Photon Systems. Fundamentals and Applications*, Lecture Notes in Physics. Springer: Berlin, 1999.
44. Brown ER, Parker CD, McMahon OB. Effect of surface composition on the radiation pattern from photonic-crystal planar-dipole antenna. *Applied Physics Letters* 1994; **64**:3345.
45. Cox SJ, Dobson D. Maximizing band gaps in two-dimensional photonic crystals. *SIAM Journal of Applied Mathematics* 1999; **59**:2108–2120.
46. Figotin A, Klein A. Localization of classical waves I: acoustic waves. *Communications in Mathematical Physics* 1996; **180**:439–482.
47. Figotin A, Kuchment P. Band-gap structure of spectra of periodic dielectric and acoustic media. II. 2D photonic crystals. *SIAM Journal of Applied Mathematics* 1996; **56**:1561–1620.
48. Lebedev NN. *Special Functions and Their Applications*. Prentice-Hall: Engelwood Cliffs, NJ, 1965.

