Characterization of the essential spectrum of the Neumann-Poincaré operator in 2D domains with corner via Weyl sequences

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Abstract

The Neumann-Poincaré (NP) operator naturally appears in the context of metamaterials as it may be used to represent the solutions of elliptic transmission problems via potentiel theory. In particular, its spectral properties are closely related to the well-posedness of these PDE's, in the typical case where one considers a bounded inclusion of homogeneous plasmonic metamaterial embedded in a homogeneous background dielectric medium. In a recent work [32], M. Perfekt and M. Putinar have shown that the NP operator of a 2D curvilinear polygon has an essential spectrum, which depends only on the angles of the corners. Their proof is based on quasi-conformal mappings and techniques from complex-analysis. In this work, we characterise the spectrum of the NP operator for a 2D domain with corners in terms of elliptic corner singularity functions, which gives insight on the behaviour of generalized eigenmodes.

1 Introduction

Plasmonic metamaterials are composite structures, in which some parts are made of media with negative indices. Their fascinating properties of subwavelength confinement and enhancement of electro-magnetic waves have drawn considerable interest from the physics and mathematics communities.

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The progress in the controlled production of composites with characteristic features of the order of optical wavelengths contributes to this activity, as it may enable many applications to nano-optical-mechanical systems, cancer therapy, neuro-science, energy and information storage and processing.

From the mathematical modelling point of view, these studies have also renewed interest in the Neumann-Poincaré operator, the integral operator derived from the normal derivative of the single layer potential. Indeed, it proves to be an interesting tool to construct, represent and derive properties of solutions to diffusion-like equations, in situations where the Lax-Milgram theory does not apply, which is typically the case of negative index materials.

The spectral properties of this operator have proved interesting in several contexts [1, 2, 13, 14, 15]. They are particularly relevant to metamaterials, as they are closely related to the existence of surface plasmons, i.e., solutions of the governing PDE (Maxwell, Helmholtz, acoustic equations) which are supported in the vicinity of the interfaces where the coefficients change signs.

To fix ideas, we consider a single inclusion D made of negative index material (typically metals, such as gold or silver at optical frequencies). It is embedded in a homogeneous dielectric background medium and we by denote $\mathcal{K}_D^*$ the associated NP operator (its precise definition is given in section 2). For particular frequencies, called plasmonic resonant frequencies, an incident wave may excite electrons on the surface of the inclusion into a resonant state, that generates highly oscillating and localised electromagnetic fields. For gold and silver, plasmonic resonances occur when the diameter of the particles is small compared to the wavelength. From the modelling point of view, one may rescale the governing Maxwell or Helmholtz equations, with respect to particule size, and take the limit of the resulting equations to obtain the quasi-static regime, where only the higher-order terms of the original PDE remain [29, 21, 3, 4]. Plasmonic resonances have been investigated via layer potential techniques in [1]–[6].

When D has a smooth boundary (say $\mathcal{C}^2$) the operator $\mathcal{K}_D^*$ is compact. Its spectrum is real, contained in the interval $(-1/2, 1/2]$, and consists in a countable number of eigenvalues that accumulates to 0. In the context of plasmonics, domains with corners present an obvious interest when one attempts to concentrate electro-magnetic fields, and several authors have considered geometries where the negative index materials are distributed in regions with corners [11, 9, 23, 24]. When D has corners, $\mathcal{K}_D^*$ is not compact [33]. In a recent work, M.-K. Perfekt and M. Putinar have shown, relying on the relationship between complex analysis and potential theory, that the NP operator associated to a planar domain with corners has essential spectrum, which they characterised to be

$$\sigma_{ess}(\mathcal{K}_D^*) \quad = \quad [\lambda_-, \lambda_+], \quad \lambda_+ \; = \; -\lambda_- \; = \; \frac{1}{2}(1 - \frac{\alpha}{\pi}),$$

where α is the most acute angle of D. See [32, 31].

The objective of our paper is to give an alternative derivation of the essential spectrum of $\mathcal{K}_D^*$ when D has corners, and to establish a close connection between the fact that $\mathcal{K}_D^*$ has essential spectrum and the theory of elliptic corner singularities initiated by Kondratiev in the 1970's and developed in many directions. See [27] and also [22, 17, 28] and the many references therein. We emphasise that throughout the paper we use the word 'elliptic' to refer to scalar transmission problems of the form $\operatorname{div}(a\nabla u) = f$ where the conductivity is real, positive and bounded away from 0, so that the associated bilinear form is coercive and the Lax-Milgram theory applies. This has been the usual framework of the work on corner singularities, until negative index materials became an active topic of research, about a decade ago [12, 19].

The Kondratiev theory shows that the solution u to an elliptic scalar equation in a domain O with corners splits as the sum $u = u_{reg} + u_{sing}$ of a regular part $u_{reg} \in H^2(O)$ and a singular part $u_{sing} \in H^1(O) \setminus H^2(O)$, locally around each corner. Up to a scaling factor, the expression of the latter part, which we call 'singularity function', only depends on the geometry of the corner, and on the nature of the boundary conditions. In the case of a transmission problem, it depends on the angle and on the contrast in material coefficients. Typically, u_{sing} is a non-trivial solution of a homogeneous problem for the associated operator in the infinite domain obtained by zooming around the vertex of the corner. For a transmission problem in 2D, it has the form

$$u_{sing} \quad = \quad C r^{\eta} \varphi(\theta), \tag{1}$$

where (r, θ) denote the polar coordinates with origin at the vertex of the corner under consideration. The exponent η is the root of a dispersion relation, and φ is a smooth function (or piecewise smooth in the case of a transmission problem).

This paper is organised in the following way. Section 2 of the paper describes the setting and notations and reviews useful facts about the NP operator.

Our analysis relies on the connection between the NP operator and the conductivity equation

$$\operatorname{div}(a(x)\nabla u(x)) \quad = \quad f. \tag{2}$$

To avoid technicalities related to the behaviour of solutions at infinity, we consider this equation in a bounded domain Ω and impose homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. We thus seek to construct solutions to the conductivity equation in the Sobolev space $H^1_0(\Omega)$, the subspace of the Sobolev space $H^1(\Omega)$ of functions with 0-trace on $\partial\Omega$. These solutions are related to a Neumann-Poincaré operator, defined with the Green function of (2) that vanishes on $\partial\Omega$. Considering the conductivity equation in the whole of $\mathbf{R}^2$ would have involved a different Neumann-Poincaré operator, defined with the Newtonian potential. As the difference between these integral operators is compact, their essential spectra are however identical.

In Section 3, we study how elliptic corner singularity functions depend on the conductivity contrast. In the very interesting papers [10, 11, 9], it is shown that functions of the form (1) only exist when the conductivity contrast λ lies outside a critical interval $[\lambda_-, \lambda_+]$. When $\lambda \in [\lambda_-, \lambda_+]$, the elliptic corner singularity functions still have the form u_{sing} but their expression involves a complex exponent η. In [9], the use of the Mellin transform converts the search of these singular functions to that of propagative mode in an infinite wave-guide. These functions are called plasmonic black-hole waves, reflecting the fact that they are not in the energy space $H^1(\Omega)$. In Section 4, we show that the critical interval is contained in the essential spectrum $\sigma_{ess}(\mathcal{K}^*_D)$, by generating singular Weyl sequences [8] using the singularity functions. In Section 5, the reverse inclusion is proved. In particular, we use a construction inspired by [30] to transform, around the vertex of the corner, the PDE with sign changing conditions into a system of PDE's defined in the inhomogeneity only, that satisfies complementing boundary conditions in the sense of Agmon, Douglis and Nirenberg, and for which we prove well-posedness. Finally in Section 6, we show that these results extend to smooth curvilinear polygons.

2 The Neumann-Poincaré operator and the Poincaré variational operator

Throughout the text, $\Omega \subset \mathbf{R}^2$ denotes a bounded open set with smooth boundary, that strictly contains a connected inclusion D. For simplicity, we assume that ∂D is smooth, except for one corner point, of angle $\alpha, 0 < \alpha < \Pi$, located at the origin. We also assume that ∂D has straight edges in the vicinity of the corner, in other words, we assume that for some $R_0 > 0$,

$$D \cap B_{R_0} = \{x = (r\cos(\theta), r\sin(\theta)), 0 \le r < R_0, |\theta| < \alpha/2\}, \qquad (3)$$

where, for any $\rho > 0$, B_ρ denotes the ball of radius ρ centred at 0. We make these simplifying assumptions so as to focus only on the core mechanisms that are responsible for the creation of essential spectrum, and so as to relate them to the generic behaviour of solutions to elliptic PDE's near corners. However, we show in Section 6 that our results extend to smooth curvilinear polygons.

The space $H_0^1(\Omega)$ is equipped with the following inner product and associated norm

$$< u, v >_{H_0^1} = \int_\Omega \nabla u \cdot \nabla v \, dx, \qquad ||u||_{H_0^1} = \left(\int_\Omega |\nabla u|^2 \, dx \right)^{1/2}.$$

Our work concerns the following diffusion equation: given a function $f \in L^2(\Omega)$, we seek u such that

$$\left\{ \begin{array}{rcll} -\text{div}(a(x)\nabla u(x)) & = & f & \text{in } \Omega, \\ u(x) & = & 0 & \text{on } \partial\Omega, \end{array} \right. \qquad (4)$$

where the conductivity a is piecewise constant

$$a(x) = \left\{ \begin{array}{ll} k \in \mathbf{C} & x \in D, \\ 1 & x \in \Omega \setminus \overline{D}. \end{array} \right. \qquad (5)$$

It is well known that when k is strictly positive, or when $k \in \mathbf{C}$ and $Im(k) \neq 0$, this problem has a unique solution in $H^1(\Omega)$, and that

$$||u||_{H_0^1} \le C(k) \, ||f||_{L^2},$$

for some constant $C(k) > 0$ that depends on k.

Let $P(x, y)$ denote the Poisson kernel associated to Ω, defined by

$$P(x, y) = G(x, y) + R_x(y), \quad x, y \in \Omega,$$

where $G(x, y)$ denotes the free space Green function

$$G(x, y) = \frac{1}{2\pi} \ln |x - y|,$$

and where $R_x(y)$ is the smooth solution to

$$\begin{cases} \Delta_y R_x(y) &= 0 \qquad y \in \Omega, \\ R_x(y) &= -G(x,y) \quad y \in \partial\Omega. \end{cases}$$

With the Poisson kernel, we define the single layer potential $\mathcal{S}_D\varphi \in L^2(\partial D)$ of a function $\varphi \in L^2(\partial D)$ by

$$\mathcal{S}_D\varphi(x) = \int_{\partial D} P(x,y)\varphi(y)\, ds(y), \quad x \in D \cup (\Omega \setminus \overline{D}).$$

where ds denotes the surface measure on ∂D. It is well known [20, 33] that $\mathcal{S}_D\varphi$ is harmonic in D and in $\Omega \setminus \overline{D}$, continous in $\overline{\Omega}$, and that its normal derivatives satisfy the Plemelj jump conditions

$$\frac{\partial \mathcal{S}_D\varphi}{\partial\nu}|^{\pm}(x) = (\pm\frac{1}{2}I + \mathcal{K}_D^*)\varphi(x), \quad x \in \partial D. \tag{6}$$

where $\mathcal{K}_D^*$ is the Neumann-Poincaré operator, defined by

$$\mathcal{K}_D^*\varphi(x) = \int_{\partial D} \frac{\partial P}{\partial \nu_y}(x,y)\varphi(y)\, ds(y).$$

It is shown in [16] that this definition makes sense for Lipschitz domains, and in that case, the operator $\mathcal{K}_D^*$ is continuous from $L^2(\partial D) \to L^2(\partial D)$, which extends as an operator $H^{-1/2}(\partial D) \to H^{-1/2}(\partial D)$.

The solution u to (4) can then be represented in the form

$$u(x) = \mathcal{S}_D\varphi(x) + H(x), \tag{7}$$

where the harmonic part is given by

$$H(x) = \int_{\Omega} P(x,y)f(y)\, ds(y).$$

The jump conditions (6), constrain the layer potential $\varphi \in H^{-1/2}(\partial D)$ to satisfy the integral equation

$$(\lambda I - \mathcal{K}_D^*)\varphi(x) = \partial_\nu H_{|\partial D}(x), \quad x \in \partial D,$$

where $\lambda = \frac{k+1}{2(k-1)}$ (when $k \neq 1$).

We also introduce the Poincaré variational operator $T_D : H_0^1(\Omega) \to H_0^1(\Omega)$, defined for $u \in H_0^1(\Omega)$ by

$$\forall\, v \in H_0^1(\Omega), \quad \int_{\Omega} \nabla T_D u \cdot \nabla v\, dx = \int_D \nabla u \cdot \nabla v\, dx. \tag{8}$$

Some of its properties are described in the following proposition (see [13] for a proof).

Proposition 1. *The operator T_D is bounded, self-adjoint, and satisfies $||T_D|| = 1$. Moreover,*

(i) Its spectrum $\sigma(T_D)$ is contained in the interval $[0,1]$.

(ii) Its kernel, the eigenspace associated to $\beta = 0$, is

$$Ker(T_D) = \{u \in H^1_0(\Omega), u = const \text{ on } D\}.$$

(iii) $1 \in \sigma(T_D)$ and the associated eigenspace is

$$Ker(I - T_D) = \{u \in H^1_0(\Omega), u = 0 \text{ in } \Omega \setminus \overline{D}\},$$

(and thus, can be identified with $H^1_0(D)$).

(iv) The space $H^1_0(\Omega)$ decomposes as

$$H^1_0(\Omega) = Ker(T_D) \oplus Ker(I - T_D) \oplus \mathcal{H},$$

where $\mathcal{H}$ is the closed subspace defined by

$$\mathcal{H} = \{u \in H^1_0(\Omega), \Delta u = 0 \text{ in } D \cup (\Omega \setminus \overline{D}), \int_{\partial D} \frac{\partial u^+}{\partial \nu}\, ds = 0\}.$$

We denote $\mathcal{H}_S = \mathcal{H} \oplus Ker(T_D)$ the space of single layer potentials. It is isomorphic to

$$H_0^{-1/2}(\partial D) = \{\varphi \in H^{1/2}(\partial D), < \varphi, 1 >_{H^{-1/2}, H^{1/2}} = 0\}.$$

This latter space is equipped with the inner product

$$< \varphi, \psi >_S = -\int_{\partial D} \varphi S \psi \, d\sigma \tag{9}$$

for which the operator $\mathcal{K}^*_D : H_0^{-1/2}(\partial D) \to H_0^{-1/2}(\partial D)$ is self-adjoint as a result of the Calderón identity [26]. We denote by $|| \cdot ||_S$ the associated norm. In particular, if $u, v \in \mathcal{H}_S$ are such that $u = S_D\varphi, v = S_D\psi$, then the jump conditions (6) and integration by parts show that

$$\int_\Omega \nabla u \cdot \nabla v = < \varphi, \psi >_S . \tag{10}$$

When the domain D has a $\mathcal{C}^2$ boundary, the Poincaré-Neumann operator $\mathcal{K}_D^* \ : H_0^{-1/2}(\partial D) \to H_0^{-1/2}(\partial D)$ is compact. Its spectrum $\sigma(\mathcal{K}_D^*)$ is contained in $[-1/2, 1/2]$, and consists of a sequence of real eigenvalues that accumulates to 0. In this case, $\sigma(\mathcal{K}_D^*)$ is directly related to $\sigma(T_D)$. Indeed, if $u \in H_0^1(\Omega)$ and $\beta \in \mathbf{R}, \beta \neq 1$, satisfy $T_D u = \beta u$, it follows from (8) that

$$\forall\, v \in H_0^1(\Omega), \quad \beta \int_{\Omega \setminus D} \nabla u \cdot \nabla v \, dx + (\beta - 1) \int_D \nabla u \cdot \nabla v \, dx \;\; = \;\; 0,$$

so that u is a non-zero solution to

$$\left\{ \begin{array}{rcll} \mathrm{div}(a(x)\nabla u(x)) & = & 0 & \text{in } \Omega, \\ u(x) & = & 0 & \text{on } \partial\Omega, \end{array} \right. \tag{11}$$

where the conductivity a equals β in $\Omega \setminus \overline{D}$ and $(\beta - 1)$ in D. Expressing u in the form $u = \mathcal{S}_D \varphi$ yields the integral equation

$$(\lambda I - \mathcal{K}_D^*)\varphi(x) \;\; = \;\; 0, \quad x \in \partial D,$$

where $\lambda = 1/2 - \beta$ is thus an eigenvalue of $\mathcal{K}_D^*$. It follows that

$$\sigma(T_D) \;\; = \;\; (1/2 - \sigma(\mathcal{K}_D^*)) \cup \{0, 1\}.$$

As recalled above, when D is a domain with corners, $\sigma(\mathcal{K}_D^*)$ contains an interval of essential spectrum [32]. We have

Proposition 2. *The essential spectra of T_D and $\mathcal{K}_D^*$ are related by $\sigma_{ess}(T_D) = 1/2 - \sigma_{ess}(\mathcal{K}_D^*)$.*

Proof: Let $\lambda \in \sigma_{ess}(\mathcal{K}_D^*)$. By definition, there exists a singular Weyl sequence, i.e., a sequence of functions $(\varphi_\varepsilon) \subset H_0^{-1/2}$ such that

$$\left\{ \begin{array}{rcll} (\lambda I - \mathcal{K}_D^*)\varphi_\varepsilon & \to & 0 & \text{strongly in } H_0^{-1/2}, \\ ||\varphi_\varepsilon||_S & = & 1, & \\ \varphi_\varepsilon & \rightharpoonup & 0 & \text{weakly in } H_0^{-1/2}. \end{array} \right.$$

Let $\beta = 1/2 - \lambda$ and $u_\varepsilon = S_D \varphi_\varepsilon \in \mathcal{H}_S$. Let $v \in \mathcal{H}_S$ so that $v = S_D \psi$ for some $\psi \in H_0^{-1/2}(\partial D)$. It follows from (10) that

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla v \;\; = \;\; < \varphi_\varepsilon, \psi > \to \; 0.$$

This equality also holds for $v \in Ker(I - T_D)$ since this subspace is orthogonal to $\mathcal{H}_S$, and thus

$$u_\varepsilon \rightharpoonup 0 \quad \text{weakly in } H^1_0(\Omega). \tag{12}$$

Additionally, invoking (10) again, we see that

$$\int_\Omega |\nabla u_\varepsilon|^2 \ = \ < \varphi_\varepsilon, \varphi_\varepsilon >_S = \ 1. \tag{13}$$

Finally, we compute, for $v = S_D\psi \in \mathcal{H}_S$,

$$\begin{aligned} \int_\Omega \nabla\left((\beta I - T_D)u_\varepsilon\right) \cdot \nabla v \ &= \ \int_\Omega \beta \nabla u_\varepsilon \cdot \nabla v \ - \ \int_D \nabla u_\varepsilon \cdot \nabla v \\ &= \ \int_{\Omega \setminus D} \beta \nabla u_\varepsilon \cdot \nabla v \ + \ \int_D (\beta - 1) \nabla u_\varepsilon \cdot \nabla v \\ &= \ -\beta \int_{\partial D} \partial_\nu u_\varepsilon|^+ v \ + \ (\beta - 1) \int_{\partial D} \partial_\nu u_\varepsilon|^- v. \end{aligned}$$

Inserting (6) in place of the normal derivatives of u_ε we see that

$$\begin{aligned} \left| \int_\Omega \nabla\left((\beta I - T_D)u_\varepsilon\right) \cdot \nabla v \right| \ &= \ |< (\lambda I - \mathcal{K}^*_D)\varphi_\varepsilon, \psi >_S| \\ &\leq \ ||(\lambda I - \mathcal{K}^*_D)\varphi_\varepsilon||_S \, ||\psi||_S. \end{aligned}$$

It follows that

$$||(\beta I - T_D)u_\varepsilon||_{H_1} \ \leq \ ||(\lambda I - \mathcal{K}^*_D)\varphi_\varepsilon||_S \ \to 0. \tag{14}$$

we conclude from (12–14) that u_ε is a singular Weyl sequence associated to β, so that $\beta \in \sigma_{ess}(T_D)$. The same argument proves the reverse inclusion $\sigma_{ess}(T_D) \subset (1/2 - \sigma_{ess}(\mathcal{K}^*_D))$. ∎

3 Corner singularity functions

Elliptic corner singularities have been the subject of much research since the pioneering works of Kondratiev [27], Grisvard [22] (see also [17, 28]). Essentially, the theory focuses on the regularity of solutions to elliptic PDEs near a corner of the domain, or in the case of a transmission problem such as (4), near a corner of the interface between several phases. The following is a typical statement:

Theorem 1. *Let $k > 0$. The solution $u \in H^1_0(\Omega)$ to (4) decomposes as*

$$u = u_{sing} + u_{reg},$$

where $u_{reg} \in H^2(\Omega)$ and where u_{sing} has the form

$$u_{sing}(x) = r^\eta \varphi(\theta)\zeta(x), \quad x \in \Omega. \tag{15}$$

Here $x = (r\cos(\theta), r\sin(\theta))$ in polar coordinates, ζ is a smooth cut-off function, such that, for some $s > 0$

$$\zeta(x) = \begin{cases} 1 & |x| < s, \\ 0 & |x| > 2s. \end{cases}$$

Moreover, for some constant $C = C(\alpha, k)$, the following estimate holds

$$||u_{sing}||_{H^1(\Omega)} + ||u_{reg}||_{H^2(\Omega)} \leq C \left(||u||_{H^1(\Omega)} + ||f||_{L^2(\Omega)} \right). \tag{16}$$

Most of these results have been derived in the case of strongly elliptic coefficients, i.e., when $k > 0$. In the context of plasmonic metamaterials, it is natural to try to extend them to complex values of k. To our best knowledge, the first steps in this direction have been obtained in [10, 11] and concern the existence of singularity functions of the form (15).

3.1 Regular corner singularity functions

In this paragraph, we investigate whether one can define singular functions such as (15) when k may also take negative values. More precisely, we seek $H^1_{loc}(\mathbf{R}^2)$ solutions to

$$\operatorname{div}(a(x)\nabla u(x)) = 0 \quad \text{in } \mathbf{R}^2, \tag{17}$$

of the form

$$u(x) = r^\eta \varphi(\theta), \eta \in \mathbf{R}, \tag{18}$$

when the conductivity $a(x)$ is defined in the whole of $\mathbf{R}^2$ by

$$a(x) = \begin{cases} k & |\theta| < \alpha/2, \\ 1 & \text{otherwise.} \end{cases} \tag{19}$$

Since we are only interested in singular solutions which belong to $H^1(\Omega) \setminus H^2(\Omega)$, we may restrict η to lie in $(0, 1)$. As u is harmonic in each sector $|\theta| < \alpha/2$ and $\alpha/2 < \theta < 2\pi - \alpha/2$, it follows that φ has the form

$$\varphi = \begin{cases} a_1 \cos(\eta(\theta + \alpha/2)) + b_1 \sin(\eta(\theta + \alpha/2)) & \text{if } -\alpha/2 < \theta < \alpha/2, \\ a_2 \cos(\eta(\theta + \alpha/2)) + b_2 \sin(\eta(\theta + \alpha/2)) & \text{if } \alpha/2 < \theta < 2\pi - \alpha/2 \end{cases} \tag{20}$$

for some $a_i, b_i, i = 1, 2$. Expressing the continuity of u and of $a(x)\partial_\nu u$ across the interfaces, shows that a non-trivial solution exists if and only if the following dispersion relation is satisfied

$$\det \begin{pmatrix} 1 & 0 & -\cos(2\pi\eta) & -\sin(2\pi\eta) \\ \cos(\alpha\eta) & \sin(\alpha\eta) & -\cos(\alpha\eta) & -\sin(\alpha\eta) \\ 0 & k & \sin(2\pi\eta) & -\cos(2\pi\eta) \\ -k\sin(\alpha\eta) & k\cos(\alpha\eta) & -\sin(\alpha\eta) & -\cos(\alpha\eta) \end{pmatrix} = 0,$$

which, after elementary manipulations, can be rewritten in the form

$$\frac{2k}{k^2+1} = \frac{\sin(\alpha\eta)\sin((2\pi-\alpha)\eta)}{1-\cos(\alpha\eta)\cos((2\pi-\alpha)\eta)} =: F(\eta,\alpha). \tag{21}$$

A Taylor expansion around $\eta = 0$ shows that $F(\eta, \alpha)$ can be extended by continuity to a function defined on the whole of $[0, 1]$ by setting

$$F(0,\alpha) = \frac{-2\alpha(2\pi-\alpha)}{\alpha^2+(2\pi-\alpha)^2}.$$

By solving

$$\frac{2k}{k^2+1} = F(0,\alpha),$$

we obtain two solutions

$$k_+ = \frac{-(2\pi-\alpha)}{\alpha}, \qquad k_- = \frac{-\alpha}{2\pi-\alpha}. \tag{22}$$

Additionally, it is easy to check that $|F(\eta, \alpha)| \leq 1$ and

$$\partial_\eta F = \frac{\cos((2\pi-\alpha)\eta)-\cos(\alpha\eta))\,[a\sin((2\pi-\alpha)\eta)-(2\pi-\alpha)\sin(\alpha\eta)]}{[1-\cos(\alpha\eta)\cos((2\pi-\alpha)\eta)]^2}.$$

We note that $\partial_\eta F(0, \alpha) = \partial_\eta F(1, \alpha) = 0$ and show below that $F(\cdot, \alpha)$ is strictly increasing.

Lemma 1. *For any $0 < \alpha < \pi$ and $0 \leq \eta \leq 1$, the following inequalities hold*

$$\begin{aligned} \cos((2\pi-\alpha)\eta)-\cos(\alpha\eta) &< 0, \\ a\sin((2\pi-\alpha)\eta)-(2\pi-\alpha)\sin(\alpha\eta) &< 0. \end{aligned} \tag{23}$$

Proof: To prove the first inequality, we first note that $\alpha < (2\pi - \alpha)$ so that $\alpha\eta < (2\pi - \alpha)\eta$. If $(2\pi - \alpha)\eta \leq \pi$, then (23) follows from the monotonicity of the cosine function on $[0, \pi]$. If $(2\pi - \alpha)\eta > \pi$, then

$$\cos((2\pi - \alpha)\eta) \;=\; \cos(\pi - \beta), \quad \text{with } (2\pi - \alpha)\eta =: \pi + \beta.$$

Noticing that

$$\alpha\eta \;\leq\; \alpha\eta + 2\pi(1 - \eta) \;=\; \pi - \beta \;<\; \pi,$$

we infer that $\cos(\pi - \beta) < \cos(\alpha\eta)$, which yields the result.

The second inequality follows from the fact that

$$\begin{aligned} &\partial_\eta \left[a \sin((2\pi - \alpha)\eta) - (2\pi - \alpha)\sin(\alpha\eta)\right] \\ &\quad = \;\alpha(2\pi - \alpha)\left[\cos((2\pi - \alpha)\eta) - \cos(\alpha\eta)\right], \end{aligned}$$

which according to (23) is negative. ■

As a consequence of (21), we obtain

Proposition 3. *Singular solutions in $H^1_{loc}(\mathbf{R}^2)$ of the form (18) exists for the equation (17) only when $k \in (-\infty, k_+) \cup (k_-, +\infty)$, see Figure 3.1. In terms of the contrast $\lambda = \frac{k+1}{2(k-1)}$ this condition is equivalent to*

$$\lambda \;\notin\; [\lambda_-, \lambda_+] \;:=\; [-\frac{1}{2}(1 - \frac{\alpha}{\pi}), \frac{1}{2}(1 - \frac{\alpha}{\pi})].$$

In other words, singular solutions of the form (18) only exist when $\lambda = \frac{k+1}{2(k-1)}$ is not in $\sigma_{ess}(\mathcal{K}^*_D)$.

3.2 Singular corner singularity functions

We now construct local singular solutions when $k \in [k_+, k_-]$. By this we mean functions which satisfy the PDE (17), but which may only be in $H^1_{loc}(\mathbf{R}^2 \setminus \{0\})$. To this end, we seek $u(x) = r^\eta \varphi(\theta)$, with φ in the form (20), but assume now that $\eta \in \mathbf{C}$. The same algebra leads to the same dispersion relation (21). In particular if we restrict η to be a pure imaginary number, $\eta = i\xi$, this relation takes the form

$$\frac{2k}{k^2 + 1} \;=\; \frac{\sinh(\alpha\xi)\sinh((2\pi - \alpha)\xi)}{1 - \cosh(\alpha\xi)\cosh((2\pi - \alpha)\xi)} \;=:\; \tilde{F}(\xi, \alpha).$$

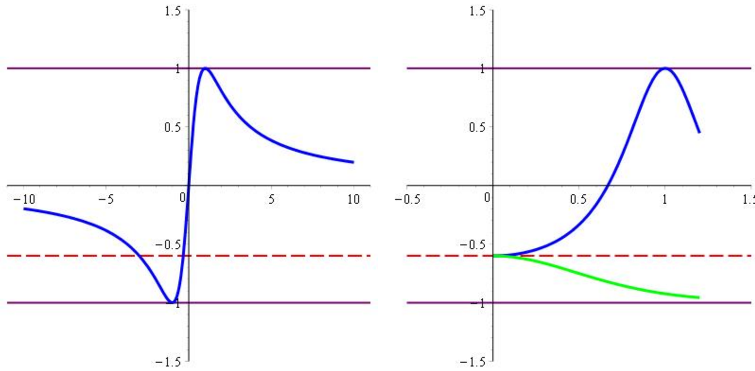

Figure 3.1: Left: Plot of the function $k \to 2k/(k^2+1)$. Right: Plot of $\eta \to F(\eta,\alpha)$ (blue), and of $\xi \to \tilde{F}(\xi,\alpha)$ (green), for $\alpha = \pi/2$. The dotted line indicates the value of $2k/(k^2+1)$ below which the dispersion relation has no solution $\eta \in \mathbf{R}$.

It is easy to check that the function $\xi \to \tilde{F}(\xi,\alpha)$ can be extended by continuity at $\xi = 0$ by setting

$$\tilde{F}(0,\alpha) = \frac{2\alpha(2\pi-\alpha)}{\alpha^2+(2\pi-\alpha)^2} = \frac{2k_\pm}{k_\pm^2+1}$$

(and we note that $\tilde{F}(0,\alpha) = F(0,\alpha)$). In addition, we compute

$$\partial_\xi \tilde{F} = \frac{(\cosh((2\pi-\alpha)\xi)-\cosh(\alpha\xi))\,[(2\pi-\alpha)\sinh(\alpha\xi)-\alpha\sinh((2\pi-\alpha)\xi)]}{[1-\cosh(\alpha\xi)\cosh((2\pi-\alpha)\xi)]^2}.$$

Just as in Proposition 1, one can show that for any $\xi > 0$ and $0 < \alpha < \pi$, the product of the two factors in the above numerator is negative, so that $\tilde{F}(\cdot,\alpha)$ is strictly decreasing on $\mathbf{R}^+$ and its range is equal to $[F(0,\alpha),-1)$, see Figure 3.1. We also note that $\lim_{\eta\to\infty}\tilde{F}(\eta,\alpha) = -1$, so that the value -1 (which corresponds to $k = -1$) is never attained. Summarizing, we have shown that

Proposition 4. *For any value of* $\lambda \in (\lambda_-,\lambda_+), \lambda \neq 0$*, there exists* $\xi > 0$ *and a function* $u(x) = r^{i\xi}\varphi(\theta)$*, which is a local solution to* $div(a(x)\nabla u(x)) = 0$*, where* a *is defined by (19), with* $\lambda = \frac{k+1}{2(k-1)}$.

4 Construction of singular Weyl sequences

In this section we prove

Theorem 2. *The set* $[\lambda_-, \lambda_+]$ *is contained in* $\sigma_{ess}(T_D)$*.*

Proof:
Since $\sigma_{ess}(\mathcal{K}_D^*)$ is a closed set, it is sufficient to show that $(\lambda_-, \lambda_+) \setminus \{0\} \subset \sigma_{ess}(T_D)$. We proceed as follows: We consider $\lambda \in (\lambda_-, \lambda_+), \lambda \neq 0$, and show that $\beta = 1/2 - \lambda \in \sigma_{ess}(T_D)$ by constructing a singular Weyl sequence, i.e., a sequence of functions $u_\varepsilon \in H_0^1(\Omega)$, such that

$$\left\{ \begin{array}{ccll} ||u_\varepsilon||_{H_0^1} & = & 1, & \\ (\beta I - T_D)u_\varepsilon & \to & 0 & \text{strongly in } H_0^1(\Omega), \\ u_\varepsilon & \rightharpoonup & 0 & \text{weakly in } H_0^1(\Omega). \end{array} \right. \tag{24}$$

According to Proposition 4, there exists $\xi > 0$ and coefficients $a_1, b_2, a_2, b_2 \in \mathbf{C}$, not all equal to 0, such that the function

$$u(x) = Re(r^{i\xi})\varphi(\theta) = \left\{ \begin{array}{l} Re(r^{i\xi})\,[a_1 \cos(i\xi(\theta+\alpha/2)) + b_1 \sin(i\xi(\theta+\alpha/2))] \\ \qquad \text{if } -\alpha/2 < \theta < \alpha/2, \\ \\ Re(r^{i\xi})\,[a_2 \cos(i\xi(\theta+\alpha/2)) + b_2 \sin(i\xi(\theta+\alpha/2))] \\ \qquad \text{otherwise,} \end{array} \right. \tag{25}$$

is harmonic in $(D \cap B_{R_0}) \setminus \{0\}$ and in $\big((\Omega \setminus \overline{D}) \cap B_{R_0}\big) \setminus \{0\}$, and satisfies the transmission conditions at the interfaces $\theta = \pm\alpha/2$.

Let $r_0 < R_0/2$ and let $\chi_1, \chi_2 : \mathbf{R}^+ \to [0, 1]$ denote two smooth cut-off functions, such that for some constant $C > 0$

$$\left\{ \begin{array}{llll llll} \chi_1(s) & = & 0 & |s| \leq 1, & \chi_2(s) & = & 0 & |s| \geq 2r_0, \\ \chi_1(s) & = & 1 & |s| \geq 2, & \chi_2(s) & = & 1 & |s| \leq r_0, \\ |\chi_1'(s)| & \leq & C, & & |\chi_2'(s)| & \leq & C. & \end{array} \right.$$

We set $\chi_1^\varepsilon(r) = \chi_1(r/\varepsilon)$, and define

$$u_\varepsilon(x) \quad = \quad s_\varepsilon \chi_1^\varepsilon(r)\chi_2(r)u(x), \quad x \in \Omega. \tag{26}$$

The function u is not in H^1 as its gradient blows up like r^{-1} near the corner, consequently

$$m_\varepsilon \;:=\; \int_\varepsilon^{r_0}\int_0^{2\pi} |\nabla u(x)|^2\, r dr d\theta \;\to\; \infty \quad \text{as } \varepsilon \to 0.$$

We choose s_ε in (26) so that $||u_\varepsilon||_{H_0^1} = 1$, in other words

$$\begin{aligned} s_\varepsilon^{-2} &= \int_\varepsilon^{2\varepsilon}\int_0^{2\pi} |u\nabla\chi_1^\varepsilon + \chi_1^\varepsilon\nabla u|^2 \; + \; m_\varepsilon \; + \; \int_{r_0}^{2r_0}\int_0^{2\pi} |u\nabla\chi_2 + \chi_2\nabla u|^2 \\ &=: \; J_1 + m_\varepsilon + J_2. \end{aligned}$$

The term J_2 is independent of ε and is $O(1)$, and in particular

$$J_2 \;\; = \;\; o(m_\varepsilon) \quad \text{as } \varepsilon \to 0.$$

The other term can be estimated as follows

$$\begin{aligned} J_1 &= \int_\varepsilon^{2\varepsilon}\int_0^{2\pi} \left| \frac{r^{i\xi} + r^{-i\xi}}{2}\varphi(\theta)\chi_1'(r/\varepsilon)/\varepsilon + i\xi\frac{r^{i\xi-1} - r^{-i\xi-1}}{2}\varphi(\theta)\chi_1(r/\varepsilon) \right|^2 \\ &\qquad\qquad + \left| \frac{r^{i\xi-1} - r^{-i\xi-1}}{2}\varphi'(\theta)\chi_1(r/\varepsilon) \right|^2 \; r dr d\theta \qquad\qquad (27) \\ &\leq C\int_0^{2\pi} \left(|\varphi(\theta)|^2 + |\varphi'(\theta)|^2\right) d\theta \int_\varepsilon^{2\varepsilon} \left(||\chi_1'||_\infty^2/\varepsilon^2 + r^{-2}||\chi_1||_\infty\right) r dr \\ &\leq C\int_0^{2\pi} \left(|\varphi(\theta)|^2 + |\varphi'(\theta)|^2\right) d\theta \, \left(3/2||\chi_1'||_\infty^2 + (\ln(2\varepsilon) - \ln(\varepsilon))||\chi_1||_\infty\right). \end{aligned}$$

Since φ is independent of ε, we see that

$$J_1 \;\; = \;\; O(1) \; = \; o(m_\varepsilon), \quad \text{as } \varepsilon \to 0,$$

and so $s_\varepsilon \sim m_\varepsilon^{-1/2} \to 0$.

We next show that $||(\beta I - T_D)u_\varepsilon||_{H^1} \to 0$. Indeed, let $v \in H_0^1(\Omega)$ and consider

$$\begin{aligned} J &= \int_\Omega \nabla(\beta I - T_D)u_\varepsilon \cdot \nabla v \\ &= \int_{\Omega\setminus D} \beta\nabla u_\varepsilon \cdot \nabla v \; + \; \int_D (\beta - 1)\nabla u_\varepsilon \cdot \nabla v, \end{aligned}$$

in view of the definition of T_D. Inserting the expression (26) of u_ε, we see that

$$\begin{aligned} J \;\; = \;\; & s_\varepsilon \int_{\Omega\setminus D} \beta\nabla u \cdot \nabla(\chi_1^\varepsilon\chi_2 v) \; + \; s_\varepsilon \int_D (\beta - 1)\nabla u \cdot \nabla(\chi_1^\varepsilon\chi_2 v) \\ & + s_\varepsilon \int_{\Omega\setminus D} \beta u\nabla(\chi_1^\varepsilon\chi_2) \cdot \nabla v \; + \; s_\varepsilon \int_D (\beta - 1)u\nabla(\chi_1^\varepsilon\chi_2) \cdot \nabla v \\ & - s_\varepsilon \int_{\Omega\setminus D} \beta\nabla u \cdot v\nabla(\chi_1^\varepsilon\chi_2) \; - \; s_\varepsilon \int_D (\beta - 1)\nabla u \cdot v\nabla(\chi_1^\varepsilon\chi_2). \end{aligned}$$

Since u is a local solution to (17), the sum of the first two integrals vanishes, and we are left with

$$\begin{aligned} J &= \left(s_\varepsilon \int_\Omega au\nabla(\chi_1^\varepsilon \chi_2) \cdot \nabla v \ + \ s_\varepsilon \int_{\Omega\cap(B_{2r_0}\setminus B_{r_0})} au\nabla(\chi_2)\cdot\nabla v \right) \\ &\quad + s_\varepsilon \int_{\Omega\cap(B_{2\varepsilon}\setminus B_\varepsilon)} av\nabla(\chi_1^\varepsilon)\cdot\nabla u \ =: \ s_\varepsilon(J_3+J_4), \end{aligned} \tag{28}$$

where $a=\beta$ in $\Omega\setminus\overline{D}$ and $a=\beta-1$ in D. The Cauchy-Schwarz inequality allows us to estimate the first two terms on the right-hand side by

$$\begin{aligned} |J_3| &\leq C\,||v||_{H^1} \left\{ \int_0^{2\varepsilon}\int_0^{2\pi} \left(|u|^2|\chi_1'|^2/\varepsilon^2 + |\nabla u|^2|\chi_1|^2\right) rdrd\theta \right. \\ &\quad \left. + \int_{r_0}^{2r_0}\int_0^{2\pi} \left(|u|^2|\chi_2'|^2 + |\nabla u|^2|\chi_2|^2\right) rdrd\theta \right\}. \end{aligned} \tag{29}$$

and the same arguments as those used to control the term J_1 in (27) show that the two integrals above are $O(1)$. As for the last term in (28), we write

$$\begin{aligned} J_4 &:= \int_{B_{2\varepsilon}\setminus B_\varepsilon} a\nabla u\cdot v\nabla\chi_\varepsilon \\ &= \int_{B_{2\varepsilon}\setminus B_\varepsilon} a\nabla u\cdot\overline{v}\nabla\chi_\varepsilon \ + \int_{B_{2\varepsilon}\setminus B_\varepsilon} a\nabla u\cdot(v-\overline{v})\nabla\chi_\varepsilon, \end{aligned}$$

where $\overline{v}=|B_{2\varepsilon}|^{-1}\int_{B_{2\varepsilon}} v(x)\,dx$. We note that the first integral in the above right-hand side reduces to

$$\overline{v}\int_0^{2\pi} a(\theta)\varphi(\theta)\,d\theta \int_\varepsilon^{2\varepsilon} i\xi\left(\frac{r^{i\xi-1}-r^{-i\xi-1}}{2}\right)\frac{\chi_1'(r/\varepsilon)}{\varepsilon}\,rdr \ = \ 0.$$

Indeed, since φ is a solution to $(a(\theta)\varphi'(\theta))' - \xi^2 a(\theta)\varphi(\theta)=0$, with periodic boundary conditions, it satisfies

$$\int_0^{2\pi} a(\theta)\varphi(\theta)\,d\theta \ = \ 0.$$

It follows that

$$|J_4| \ \leq \ \left(\int_{B_{2\varepsilon}\setminus B_\varepsilon} a^2|\nabla u\cdot\nabla\chi_\varepsilon|^2\,dx\right)^{1/2}\left(\int_{B_{2\varepsilon}} |v-\overline{v}|^2\right)^{1/2}.$$

Using the following Poincaré inequality

$$\int_{B_{2\varepsilon}} |v - \overline{v}|^2 \quad \leq \quad 4|B_{2\varepsilon}| \int_{B_{2\varepsilon}} |\nabla v|^2,$$

we obtain

$$\begin{aligned} |J_4| &\leq C\varepsilon ||v||_{H^1(\Omega)} \left(\int_0^{2\pi} a(\theta)^2 |\varphi(\theta)|^2 \, d\theta \right)^{1/2} \\ &\quad \left(\int_\varepsilon^{2\varepsilon} |i\xi \frac{r^{i\xi - 1} - r^{-i\xi-1}}{2}|^2 \frac{[\chi'(r/\varepsilon)]^2}{\varepsilon^2} \, rdr \right)^{1/2} \\ &\leq C \left(\int_\varepsilon^{2\varepsilon} r^{-1} dr \right)^{1/2} ||v||_{H^1(\Omega)} \\ &\leq C\sqrt{ln(2)} ||v||_{H^1(\Omega)} \; = \; O(1) ||v||_{H^1(\Omega)}. \end{aligned}$$

Altogether, (28, 29) and the above estimate show that

$$\forall \, v \in H^1_0(\Omega), \qquad \left| \int_\Omega \nabla(\beta I - T_D) u_\varepsilon \cdot \nabla v \right| \quad \leq \quad O(s_\varepsilon) ||v||_{H^1},$$

which proves the claim since $s_\varepsilon \to 0$.

Finally, we show that $u_\varepsilon \to 0$ weakly in $H^1(\Omega)$. In fact, since this sequence is uniformly bounded in H^1, it suffices to show that $u_\varepsilon \to 0$ strongly in L^2, which follows from (26), from the boundedness of χ_1 and χ_2 and from the fact that $s_\varepsilon \to 0$. ■

5 Characterisation of the essential spectrum

In this section, we consider $\lambda \notin [\lambda_-, \lambda_+]$, $\beta = 1/2 - \lambda$, and $k = 1 - 1/\beta$. The latter satisfies

$$k < k_+ = \frac{-(2\pi - \alpha)}{\alpha} < 0 \quad \text{or} \quad k_- = \frac{-\alpha}{2\pi - \alpha} < k < 0. \tag{30}$$

We show that $\beta \notin \sigma_{ess}(T_D)$, so that according to Proposition 2, $\lambda \notin \sigma_{ess}(\mathcal{K}^*_D)$.

We proceed by contradiction: If $\beta \in \sigma_{ess}(T_D)$, then there exists a singular Weyl sequence u_ε, that satisfies the conditions (24). In the next three sections, we show

Proposition 5. *The sequence* u_ε *converges to* 0 strongly *in* $H^1(\Omega)$.

This contradicts the fact that $||u_\varepsilon||_{H^1} = 1$. Consequently, in view of Theorem 2, this proves

Theorem 3. *The essential spectrum of* $\mathcal{K}_D^*$ *is exactly*

$$\sigma_{ess}(\mathcal{K}_D^*) \quad = \quad [\lambda^-, \lambda^+].$$

5.1 Controlling the energy of u_ε away from the corner

Let $z_\varepsilon = \beta u_\varepsilon - T_D u_\varepsilon \in H_0^1(\Omega)$. Let $\rho < R_0$ and let χ_ρ denote a smooth, radial cut-off function, such that

$$\chi_\rho(x) \quad = \quad \begin{cases} 1 & \text{if } |x| \leq \rho/2, \\ 0 & \text{if } |x| \geq \rho. \end{cases} \tag{31}$$

Let $v_\varepsilon = (1 - \chi_\rho)u_\varepsilon$. We show that

Proposition 6. *The sequence* v_ε *converges strongly to* 0 *in* H^1.

Proof: Assume that it is not the case. Then there exists $\delta > 0$ and a subsequence (still labeled with ε) such that

$$||v_\varepsilon||_{H_0^1} \quad \geq \quad \delta. \tag{32}$$

We note that for any $v \in H_0^1(\Omega)$,

$$\begin{aligned} \int_\Omega \nabla z_\varepsilon \cdot \nabla v \quad &= \quad \int_\Omega \nabla(\beta u_\varepsilon - T_D u_\varepsilon) \cdot \nabla v \\ &= \quad \beta \int_\Omega \nabla u_\varepsilon \cdot \nabla v - \int_D \nabla u_\varepsilon \cdot \nabla v \\ &= \quad \int_\Omega a \nabla u_\varepsilon \cdot \nabla v, \end{aligned} \tag{33}$$

where $a(x) = \beta$ for $x \in \Omega \setminus \overline{D}$, and $a(x) = \beta - 1$ for $x \in D$. Given $v \in H_0^1(\Omega)$, we compute

$$\begin{aligned} \int_\Omega a \nabla v_\varepsilon \cdot \nabla v \quad &= \quad \int_\Omega a \nabla \left[(1 - \chi_\rho)u_\varepsilon\right] \cdot \nabla v \\ &= \quad \int_\Omega a \left[(1 - \chi_\rho)\nabla u_\varepsilon - u_\varepsilon \nabla \chi_\rho\right] \cdot \nabla v \\ &= \quad \int_\Omega a \nabla u_\varepsilon \cdot \left[\nabla \left((1 - \chi_\rho)v\right) + v \nabla \chi_\rho\right] \; - \; a u_\varepsilon \nabla \chi_\rho \cdot \nabla v \\ &= \quad \int_\Omega \nabla z_\varepsilon \cdot \nabla \left((1 - \chi_\rho)v\right) \; - \; u_\varepsilon \nabla \cdot (a v \nabla \chi_\rho) \; - \; a u_\varepsilon \nabla \chi_\rho \cdot \nabla v. \end{aligned}$$

Invoking the Cauchy-Schwarz and the Poincaré inequality, it follows that

$$\begin{aligned}\left|\int_\Omega \nabla\left((\beta I - T_D)v_\varepsilon\right)\cdot\nabla v\right| &= \left|\int_\Omega a\nabla v_\varepsilon\cdot\nabla v\right| \\ &\le C\left(||u_\varepsilon||_{L^2} + ||z_\varepsilon||_{H^1_0}\right)||v||_{H^1_0}.\end{aligned}$$

As $u_\varepsilon \to 0$ strongly in $L^2(\Omega)$ since it converges weakly to 0 in H^1, we conclude that

$$(\beta I - T_D)v_\varepsilon \quad\to\quad 0 \quad \text{strongly in } H^1_0(\Omega). \tag{34}$$

We note that since v_ε has support in $\Omega\setminus B_{\rho/2}$,

$$T_D v_\varepsilon \quad = \quad T_{\tilde{D}} v_\varepsilon,$$

where $\tilde{D}$ denotes any *smooth* connected inclusion, such that $(D\setminus B_{\rho/2}) \equiv (\tilde{D}\setminus B_{\rho/2})$, and thus (34) also reads

$$(\beta I - T_{\tilde{D}})v_\varepsilon \quad\to\quad 0 \quad \text{strongly in } H^1_0(\Omega).$$

It is easily seen that $v_\varepsilon \rightharpoonup 0$ weakly in $H^1_0(\Omega)$, and, upon rescaling in view of (32), we conclude from the above estimate that $v_\varepsilon/||v_\varepsilon||_{H^1_0}$ is a singular Weyl sequence for $T_{\tilde{D}}$. But $\tilde{D}$ is smooth, so that the associated Neumann-Poincaré operator is compact and does not have essential spectrum, which contradicts this fact, and proves the Proposition. ■

5.2 Controlling the energy of u_ε near the corner

We now focus on $w_\varepsilon := \chi_\rho u_\varepsilon$, which has compact support in B_ρ. In view of (33), it is easy to check that w_ε satisfies

$$\partial^2_{rr}w_\varepsilon + 1/r\partial_r w_\varepsilon + 1/r^2\partial^2_{\theta\theta}w_\varepsilon \quad = \quad \tilde{f}_\varepsilon,$$

in $D\cap B_\rho$ and in $(\Omega\setminus\overline{D})\cap B_\rho$. The right-hand side is defined as

$$\tilde{f}_\varepsilon \quad = \quad \frac{1}{a}\chi_\rho\Delta z_\varepsilon + \nabla\chi_\rho\cdot\nabla u_\varepsilon + \nabla u_\varepsilon\cdot\nabla\chi_\rho + u_\varepsilon\Delta\chi_\rho,$$

where $a(x) = \beta$ for $x\in\Omega\setminus\overline{D}$, and $a(x) = \beta - 1$ for $x \in D$. We note that $\tilde{f}_\varepsilon$ converges strongly to 0 in $H^{-1}(\Omega)$. Moreover, since the function χ_ρ is radial, w_ε satisfies the following transmission conditions on the edges of the corner

$$\begin{cases} w_\varepsilon(r, \frac{\alpha}{2}|_-) &= w_\varepsilon(r, \frac{\alpha}{2}|_+) \\ w_\varepsilon(r, -\frac{\alpha}{2}|_-) &= w_\varepsilon(r, -\frac{\alpha}{2}|_+), \\ (\beta-1)\partial_\theta w_\varepsilon(r, \frac{\alpha}{2}|_-) &= \beta\partial_\theta w_\varepsilon(r, \frac{\alpha}{2}|_+), \\ (\beta-1)\partial_\theta w_\varepsilon(r, -\frac{\alpha}{2}|_-) &= \beta\partial_\theta w_\varepsilon(r, -\frac{\alpha}{2}|_+), \end{cases}$$

where the notations $|_-$, $|_+$ indicate taking the limit from left and right sides respectively.
We set

$$A = \frac{\alpha}{2\pi - \alpha} \in (0,1), \tag{35}$$

and consider the change of variables $(r,\theta) \in (0,\rho) \times (-\alpha/2, \alpha/2) \to (r, \pi - \theta/A)$, which maps $D \cap B_\rho$ into $(\Omega \setminus \overline{D}) \cap B_\rho$. We define

$$\begin{cases} v_\varepsilon(r,\theta) &= w_\varepsilon(r, \pi - \theta/A) \\ \tilde{g}_\varepsilon(r,\theta) &= \tilde{f}_\varepsilon(r, \pi - \theta/A), \end{cases} \qquad \text{for } (r,\theta) \in D \cap B_\rho.$$

It is easy to check that when $(f,g) = (\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)$, the functions $(w,v) = (w_\varepsilon|_{D\cap B_\rho}, v_\varepsilon)$ satisfy the following system

$$\begin{cases} \partial^2_{rr} w + 1/r\partial_r w + 1/r^2 \partial^2_{\theta\theta} w &= f, \\ \partial^2_{rr} v + 1/r\partial_r v + A^2/r^2 \partial^2_{\theta\theta} v &= g, \end{cases} \tag{36}$$

with the boundary conditions

$$\begin{cases} v(\rho,\theta) &= w(\rho,\theta) = 0, \\ v(r, \pm\alpha/2) &= w(r, \pm\alpha/2), \\ \partial_\theta v(r, \pm\alpha/2) &= \frac{-k}{A}\partial_\theta w(r, \pm\alpha/2). \end{cases} \tag{37}$$

In other words, w_ε and v_ε both satisfy an elliptic equation and take nearly the same Cauchy data on the edges of the corner.

To study the above system, we introduce the (closed) subspace $V_1 \subset H^1(D \cap B_\rho) \times H^1(D \cap B_\rho)$ of functions (w,v) that satisfy

$$\begin{cases} v(\rho,\theta) &= w(\rho,\theta) = 0 & |\theta| < \alpha/2 \\ v(r, \pm\alpha/2) &= w(r, \pm\alpha/2) & 0 < r < \rho. \end{cases}$$

Theorem 4. *The system (36–37) has a unique solution $(w,v) \in V_1$. Moreover, there exists a constant $C > 0$, such that*

$$||\nabla w||_{L^2(D\cap B_\rho)} + ||\nabla v||_{L^2(D\cap B_\rho)} \leq C \left(||f||_{H^{-1}(D\cap B_\rho)} + ||g||_{H^{-1}(D\cap B_\rho)}\right).$$

Proof: On V_1 we consider the norm

$$||(w,v)|| \;:=\; \left(\int_{D\cap B_\rho} |\nabla w|^2 + |\nabla v|^2 \right)^{1/2}. \qquad (38)$$

We multiply the equations (36) by two functions $\phi, \psi \in H^1(D \cap B_\rho)$ that vanish on $D \cap \partial B_\rho$, and integrate to obtain

$$\begin{aligned}
&\int_{D\cap B_\rho} f\phi \;+\; g\psi \\
&= \int_0^\rho \int_{-\alpha/2}^{\alpha/2} \begin{pmatrix} \partial_r w \\ \frac{1}{r}\partial_\theta w \end{pmatrix} \cdot \begin{pmatrix} \partial_r \phi \\ \frac{1}{r}\partial_\theta \phi \end{pmatrix} + \begin{pmatrix} \partial_r v \\ \frac{A^2}{r}\partial_\theta v \end{pmatrix} \cdot \begin{pmatrix} \partial_r \psi \\ \frac{1}{r}\partial_\theta \psi \end{pmatrix} r dr d\theta \\
&\quad - \int_{\theta=\pm\alpha/2} \left(\frac{1}{r}\partial_\theta w \phi \;+\; \frac{A^2}{r}\partial_\theta v \psi \right).
\end{aligned}$$

We note that that the last integral can be rewritten as

$$\int_{\theta=\pm\alpha/2} \frac{1}{r}\partial_\theta w(\phi - Ak\psi) \;+\; \frac{A^2}{r}\left[\partial_\theta v + \frac{k}{A}\partial_\theta w\right]\psi.$$

To satisfy the natural boundary conditions in (37), we are thus led to introduce the subspace $V_2 \subset H^1(D \cap B_\rho) \times H^1(D \cap B_\rho)$ of functions (ϕ, ψ) that satisfy

$$\begin{cases} \phi(\rho,\theta) \;=\; \psi(\rho,\theta) \;=\; 0, & |\theta| < \alpha/2 \\[2ex] \phi(r,\pm\frac{\alpha}{2}) - Ak\psi(r,\pm\frac{\alpha}{2}) \;=\; 0, & 0 < r < \rho, \end{cases}$$

which we also equip with the norm (38). We also introduce the following bilinear form B on $V_1 \times V_2$ by

$$\begin{aligned}
B\left(\begin{pmatrix} w \\ v \end{pmatrix}, \begin{pmatrix} \phi \\ \psi \end{pmatrix} \right) \;=\; & \int_0^\rho \int_{-\alpha/2}^{\alpha/2} \begin{pmatrix} \partial_r w \\ \frac{1}{r}\partial_\theta w \end{pmatrix} \cdot \begin{pmatrix} \partial_r \phi \\ \frac{1}{r}\partial_\theta \phi \end{pmatrix} \\
& + \begin{pmatrix} \partial_r v \\ \frac{A^2}{r}\partial_\theta v \end{pmatrix} \cdot \begin{pmatrix} \partial_r \psi \\ \frac{1}{r}\partial_\theta \psi \end{pmatrix} r dr d\theta.
\end{aligned}$$

Thus, solving (36–37) amounts to solving the variational problem : find $W = (w,v) \in V_1$ such that

$$\forall\, \Phi = (\phi,\psi) \in V_2, \quad B(W,\Phi) \;=\; \int_{D\cap B_\rho} f\phi + g\psi.$$

It is easily checked that the above right-hand side defines a continuous linear form on V_2, and that

$$\forall\,(W,\Phi)\in V_1\times V_2,\quad |B(W,\Phi)| \;\le\; ||W||\,||\Phi||.$$

Therefore, the theorem will be proved upon showing that B satisfies the inf-sup condition (for instance the version in [7]), i.e., that there exists $\delta>0$ such that

$$\inf_{W\in V_1,||W||=1}\left(\sup_{\Phi\in V_2,||\Phi||=1} B(W,\Phi)\right) \;\ge\; \delta. \tag{39}$$

Let $W=(w,v)\in V^1$ and $p,q,d\in\mathbf{R}$. We set

$$\phi \;=\; (Akp+d)w+(Akq-d)v, \qquad \psi \;=\; pw+qv,$$

so that both ϕ and ψ vanish on the curve $r=\rho, -\alpha/2<\theta<\alpha/2$, and $\phi-Ak\psi=d(w-v)=0$ on the edges $\theta=\pm\alpha/2, 0<r<\rho$. It follows that $(\phi,\psi)\in V_2$ is an admissible test function. The integrand in the expression of $B(W,\phi)$ takes the form

$$\begin{aligned} e \;:=\;& \partial_r w\partial_r\left[(Akp+d)w+(Akq-d)v\right] \;+\; \partial_r v\partial_r(pw+qv)\\ &+r^{-2}\partial_\theta w\partial_\theta\left[(Akp+d)w+(Akq-d)v\right] \;+\; \frac{A^2}{r^2}\partial_\theta v\partial_\theta(pw+qv)\\ =\;& (Akp+d)\xi_1^2+(Akq-d+p)\xi_1\xi_3+q\xi_3^2\\ &+(Akp+d)\xi_2^2+(Akq-d+A^2p)\xi_2\xi_4+A^2q\xi_4^2, \end{aligned}$$

where $\xi_1=\partial_r w, \xi_2=r^{-1}\partial_\theta w, \xi_3=\partial_r v, \xi_4=r^{-1}\partial_\theta v$. Fixing $q=1$, it follows that e defines a positive definite quadratic form (pointwise) provided that the polynomials

$$\begin{aligned} P_1(\xi) &= (Akp+d)+(Ak-d+p)\xi+\xi^2,\\ P_2(\xi) &= (Akp+d)+(Ak-d+A^2p)\xi+A^2\xi^2, \end{aligned}$$

are strictly positive, in other words, provided that

$$\left\{\begin{array}{lcl} (Ak-d+p)^2-4(Akp+d) &<& 0,\\ (Ak-d+A^2p)^2-4A^2(Akp+d) &<& 0. \end{array}\right. \tag{40}$$

We regard these expressions as polynomials in p, the roots of which are respectively

$$\begin{aligned} f_\pm(d) &= (d+Ak)\pm 2\sqrt{d(1+Ak)},\\ g_\pm(d) &= \frac{1}{A^2}\left[(d+Ak)\pm 2A\sqrt{d(1+k/A)}\right]. \end{aligned}$$

We remark that the roots are real if and only if

$$\begin{cases} k \ < \ -1/A & \text{if } d < 0, \\ -A < k < 0 & \text{if } d > 0, \end{cases}$$

i.e., recalling (35, 30), if and only if $\lambda \notin [\lambda_-, \lambda_+]$, which is our hypothesis.

It only remains to show that we can indeed find parameters p, d for which (40) is satisfied, i.e. that we can find d such that

$$(f_-(d), f_+(d)) \cap (g_-(d), g_+(d)) \quad \neq \quad \emptyset \tag{41}$$

(and then pick p in the intersection).

To this end, assume first that $-A < k < 0$, so that $d_+ := -Ak > 0$. We note that

$$\frac{f_+(d_+) + f_-(d_+)}{2} \quad = \quad 0,$$

and that

$$\begin{aligned} g_+(d_+) \quad &= \quad \frac{2}{A^2}\sqrt{Ak(k/A-1)} \ > 0, \\ g_-(d_+) \quad &= \quad \frac{-2}{A^2}\sqrt{Ak(k/A-1)} \ < 0, \end{aligned}$$

which yields (41).

If $k < -1/A$, one can see that $d_- = A^2 + kA < 0$ and that

$$\frac{f_+(d_-) + f_-(d_-)}{2} \quad = \quad 2Ak + A^2 \ < \ -1 \ = \ g_+(d_-).$$

On the other hand, since $0 < A < 1$ and $k < -1/A$, we have

$$\frac{-2}{A}\sqrt{d(1+k/A)} \quad < \quad -2\sqrt{d(1+Ak)},$$

so that for any $d < 0$, $g_-(d) < f_-(d)$, and in particular $g_-(d_-) < \frac{f_+(d_-)+f_-(d_-)}{2}$. It follows that (41) also holds in this case. ∎

5.3 Proof of Proposition 5

We come back to the singular Weyl sequence u_ε, which we split as $u_\varepsilon = (1-\chi_\rho)u_\varepsilon + \chi_\rho u_\varepsilon$. Proposition 6 shows that $(1-\chi_\rho)u_\varepsilon$ converges strongly to 0. On the other hand, Theorem 4 applied to $\chi_\rho u_\varepsilon$ shows that

$$\begin{aligned} ||\nabla(\chi_\rho u_\varepsilon)||_{L^2(B_\rho)} &\le C\left(||\tilde{f}_\varepsilon||_{H^{-1}(D\cap B_\rho)} + ||\tilde{g}_\varepsilon||_{H^{-1}(D\cap B_\rho)}\right) \\ &\le C\left(||z_\varepsilon||_{H^1(\Omega)} + ||u_\varepsilon||_{L^2(\Omega)}\right) \to 0. \end{aligned}$$

It thus follows that u_ε converges strongly to 0 in $H^1(\Omega)$, which contradicts the assumption that $||u_\varepsilon||_{H^1(\Omega)} = 1$, so that $\beta \notin \sigma_{ess}(T_D)$. ■

6 The case of smooth curvilinear polygons

This section has been added to answer the second referee's question about extension of the results to curvilinear polygons

So far, we assumed that for $\rho > 0$ small enough, the set $D \cup B_\rho$ was a perfect cone. The techniques developed in [31, 32] allow extension of the characterisation (1) of $\sigma_{ess}(K_D^*)$ to $\mathcal{C}^2$ curvilinear polygons (see the precise definition in [31], chap.4). In particular, Lemmas 4.3 and 4.4. in [31] imply that the Neumann-Poincaré operator associated to a $\mathcal{C}^2$ curvilinear polygon is unitarily equivalent to that of a domain with the same number of corners and which has straight edges in the neighbourhood of its vertices.

In this section, we show how our analysis, based on the Poincaré variational operator, can be extended to curvilinear polygons. Let $E \subset \Omega$, and $F \subset \Omega$ be $\mathcal{C}^{1,1}$ curvilinear polygons. For simplicity, we assume that both E and F have a single corner, at $x = 0$, with the same angle $0 < \alpha < \pi$.
Let $\Phi\ : \Omega \to \Omega$ be a $\mathcal{C}^{1,1}$-diffeomorphism such that for some $0 < \rho_1 \le \rho_0$,

$$\begin{cases} \Phi(E\cap B_{\rho_1}) \ \subset (F \cap B_{\rho_0}), \\ \Phi(\partial E\cap B_{\rho_1}) \subset (\partial F \cap B_{\rho_0}), \\ \Phi(0) = 0, \text{ and } \quad D\Phi(0) = I. \end{cases} \tag{42}$$

Denoting T_E and T_F the Poincaré variational operators associated to E and F respectively, we show the following

Proposition 7. *Let $0 < \beta < 1$. Assume that $(U_n)_{n\ge1}$ is a singular Weyl sequence associated to β for T_F, such that Suppt$(U_n) \subset B_{\rho_0/n}$. Then $(u_n)_{n\ge1} = (U_n \circ \Phi)_{n\ge1}$ is a singular Weyl sequence associated to β for T_E.*

Proof: Let $v \in H^1_0(\Omega)$ and let $V(x) = v(\Phi(x))$ for $x \in \Omega$. Then $V \in H^1_0(\Omega)$ and there is a constant C that only depends on $||\Phi||_{\mathcal{C}^{1,1}}$ such that

$$||V||_{H^1_0(\Omega)} \leq C\,||v||_{H^1_0(\Omega)}. \tag{43}$$

Let $0 < \beta < 1$ and assume that $(U_n)_{n\geq 1}$ is a singular Weyl sequence for T_F associated to β, i.e. that $(U_n)_{n\geq 1}$ satisfies (24) with T_F instead of T_D. Let us also assume that for $n \geq 1$, $\mathrm{Suppt}(U_n) \subset B_{\rho_0/n}$. Let $u_n(x) = U_n(\Phi(x))$ for $x \in \Omega$ and notice that

$$\mathrm{Suppt}(u_n) \subset \Phi^{-1}(B_{\rho_0/n}).$$

Since Φ is a diffeomorphism and since $\Phi(0) = 0$, it follows that the support of u_n shrinks to $\{0\}$, and we may assume that for n large enough, $\Phi^{-1}(B_{\rho_0/n}) \subset B_{\rho_1}$. We estimate

$$\begin{aligned} I &= \int_\Omega \nabla(\beta I - T_E)u_n \cdot \nabla v \\ &= \int_{\Phi^{-1}(B_{\rho_0/n})} \nabla(\beta I - T_E)u_n \cdot \nabla v \\ &= \beta \int_{\Phi^{-1}(B_{\rho_0/n})} \nabla u_n \cdot \nabla v - \int_{\Phi^{-1}(B_{\rho_0/n})\cap E} \nabla u_n \cdot \nabla v \\ &= \beta \int_{B_{\rho_0/n}} D\Phi^T \nabla U_n \cdot D\Phi^T \nabla v\, J_\Phi - \int_{B_{\rho_0/n}\cap F} D\Phi^T \nabla U_n \cdot D\Phi^T \nabla v\, J_\Phi, \end{aligned}$$

where J_Φ denotes the Jacobian of Φ, and where we have used (42). Further, we note that

$$||D\Phi(x) - I||_{L^\infty(B_{\rho_0/n})} = ||D\Phi(x) - D\Phi(0)||_{L^\infty(B_{\rho_0/n})} \leq \frac{||\Phi||_{\mathcal{C}^{1,1}(B_{\rho_0/n})}}{n},$$

and a similar estimate holds for $||J_\Phi - 1||_{L^\infty(B_{\rho_0/n})}$. Recalling the normalisation $||U_n||_{H^1_0(\Omega)} = 1$ and (43), it follows that

$$\begin{aligned} |I| &\leq \left| \beta \int_{B_{\rho_0/n}} \nabla U_n \cdot \nabla V - \int_{B_{\rho_0/n}\cap F} \nabla U_n \cdot \nabla V \right| \\ &\quad + C\,\frac{||\Phi||_{\mathcal{C}^{1,1}(B_{\rho_0/n})}}{n} ||\nabla U_n||_{L^2(\Omega)}\, ||\nabla V||_{L^2(\Omega)} \\ &\leq \left| \beta \int_\Omega \nabla U_n \cdot \nabla V - \int_F \nabla U_n \cdot \nabla V \right| + C\,\frac{||\Phi||_{\mathcal{C}^{1,1}(B_{\rho_0/n})}}{n} ||\nabla V||_{L^2(\Omega)} \\ &\leq C\left(||(\beta I - T_F)U_n||_{H^1_0(\Omega)} + \frac{1}{n} \right) ||v||_{H^1_0(\Omega)}, \end{aligned}$$

where C is independent of n. It follows that $||(\beta I - T_E)u_n||_{H_0^1(\Omega)} \to 0$ as $n \to \infty$.

A similar argument shows that

$$\begin{aligned}\int_\Omega |\nabla u_n|^2 &= \int_{\Phi^{-1}(B_{\rho_0/n})} |\nabla u_n|^2 \\ &= \int_{B_{\rho_0/n}} |D\Phi^T \nabla U_n|^2 J_\Phi \\ &= \int_\Omega |\nabla U_n|^2 \;+\; O(\frac{1}{n})||\nabla U_n||^2_{L^2(\Omega)} \;=\; 1 + O(\frac{1}{n}).\end{aligned}$$

Finally, since $u_n(x) = U_n(\Phi(x))$ for $x \in \Omega$, we also see that $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. We conclude that $\frac{u_n}{||u_n||_{H_0^1(\Omega)}}$ is a singular Weyl sequence for T_E, associated to β. ■

We now consider an inclusion $D \subset \Omega$ with a corner of aperture $0 < \alpha < \pi$ at $x = 0$, that satisfies the hypothesis of Section 2 : we assume that D has straight edges in a neighbourhood of 0. Let $\tilde{D}$ be a $\mathcal{C}^{1,1}$ curvilinear polygon, also with a single corner of aperture α at $x = 0$. Let Φ be a $C^{1,1}$ change of variable that satisfies (42) with $(E, F) = (D, \tilde{D})$. Proposition 7 together with theorems 2 and 3 imply the following

Proposition 8.

$$\sigma_{ess}(T_{\tilde{D}}) \;=\; \sigma_{ess}(T_D) \;=\; [\lambda_-, \lambda_+].$$

Proof: Let $0 < \beta < 1$ such that $\lambda = 1/2 - \beta \in [\lambda_-, \lambda_+]$. Let ξ, u, χ_1 be defined as in the proof of theorem 2, and, for $0 < \rho < \rho_0$, let χ_ρ be as in (31). In the spirit of (26), we set

$$u_{\varepsilon,\rho}(x) \;=\; s_{\varepsilon,\rho}\chi_1^\varepsilon(r)\chi_\rho(r)u(x), \quad x \in \Omega,$$

where s_ε is chosen so that $||u_{\varepsilon,\rho}||_{H_0^1(\Omega)} = 1$. The proof of theorem 2 shows that for fixed ρ, $u_{\varepsilon,\rho}$ is a singular Weyl sequence for T_D associated to β and in particular that

$$\left\{\begin{array}{ccll} ||u_{\varepsilon,\rho}||_{H_0^1} & = & 1, & \\ (\beta I - T_D)u_{\varepsilon,\rho} & \to & 0 & \text{strongly in } H_0^1(\Omega), \\ u_{\varepsilon,\rho} & \to & 0 & \text{a.e. in } \Omega. \end{array}\right.$$

Letting $\rho = \rho_0/n$ and using a diagonal extraction process yields a singular Weyl sequence (U_n), associated to β for T_D such that $\text{Suppt}(U_n) \subset B_{\rho_0/n}$.

Proposition 7, with the choice $(E, F) = (D', D)$ and Φ^{-1} in place of Φ, shows that $(u_n)_{n\geq 1} = (U_n \circ \Phi^{-1})_{n\geq 1}$ is a singular Weyl sequence for $T_{\tilde{D}}$ associated to β, and thus $\sigma_{ess}(T_D) \subset \sigma_{ess}(T_{\tilde{D}})$.

To prove the reverse inclusion, we proceed by contradiction and assume that there exists $\beta \in \sigma_{ess}(T_{\tilde{D}}) \setminus \sigma_{ess}(T_D)$. Let $\tilde{u}_\varepsilon$ denote a singular Weyl sequence associated to β for $T_{\tilde{D}}$ and let χ_ρ as in (31). Arguing as in Proposition 6 shows that the functions $\tilde{w}_{\varepsilon,\rho} = (\frac{\chi_\rho \tilde{u}_\varepsilon}{||\tilde{u}_\varepsilon||_{H^1_0(\Omega)}})$ form a singular Weyl sequence for $T_{\tilde{D}}$ associated to β. Choosing $\rho = 1/n$, a diagonalization process allows us to construct from the $\tilde{w}_{\varepsilon,\rho}$'s a singular Weyl sequence for $T_{\tilde{D}}$, say $(U_n)_{n\geq 1}$, such that for each $n \geq 1$, $\text{Suppt}(U_n) \subset B_{\rho_0/n}$. Proposition 7 implies then that $(u_n)_{n\geq 1} = (U_n \circ \Phi)_{n\geq 1}$ is a singular Weyl sequence for T_D associated to β, which contradicts the hypothesis $\beta \notin \sigma_{ess}(T_D)$. ■

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