# Asymptotic of the Green function for the diffraction by a perfectly conducting plane perturbed by a sub-wavelength rectangular cavity

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**Abstract**

This work is aimed at understanding the amplification and confinement of electromagnetic fields in open sub-wavelength metallic cavities. We present a theoretical study of the electromagnetic diffraction by a perfectly conducting planar interface which contains a sub-wavelength rectangular cavity. We derive a rigorous asymptotic of the Green function associated to the Helmholtz operator when the width of the cavity shrinks to zero. We show that the limiting Green function is that of a perfectly conducting plane with a dipole in place of the cavity. We give an explicit description of the effective dipole in terms of the wavelength and of the geometry of the cavity.

## 1 Introduction

Rough metallic surfaces containing subwavelength apertures are the object of intensive studies, due to their interesting optical properties. Experimental studies have shown enhancement of transmission through subwavelength holes [11, 10], and dips in the reflectivity of gratings containing subwavelength grooves [6]. The local amplification of the fields near the aperture can be strikingly high, sometimes by a factor of $10^6$. These amazing features of light localization and enhancement could find use in many applications in imaging, microscopy, spectroscopy or communication [23, 17]. For instance, commercial devices for detection of single hemoglobin proteins based on local enhancement of optical fields are already on the market.

A rigorous analytical treatment of these phenomena would be very helpful but proves quite challenging. Indeed, the particular light patterns observed in the far-field or in the near-field may be the results of very complex interactions of surface waves, cavity resonances, resonant tunnelling of plasmon waves, skin depth effects,... Even a qualitative description of the diffractive properties of surfaces with subwavelengths structures requires the resolution of the full Maxwell equations in non smooth geometries. As with most resonance phenomena,

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the localization or enhancement of light is very sensitive to the geometrical parameters and to frequency. It is likely that they would be hard to evidence in numerical calculations without a priori insight about the values of these parameters.

This motivates the study of simple geometries, where one could come as close as possible to obtaining an exact analytical description of the electromagnetic field. This is the approach taken in [6], where gratings with periodic rectangular grooves are studied. The authors seek an approximation of the fields as series expansion in the grooves and in the surface above the midplane of the grating. Assuming that the fields are constant in the grooves (only one mode is excited) allows them to compute a truncated scattering matrix and approximate reflectivity diagrams that are compared to experimental measurements. In [15], the transmission of light through a metallic slab that contains a periodic array of holes through is studied. In this work, the author computes an approximate transmission coefficient, corresponding to a limiting situation where the holes are considered to be infinitely long, and shows that the grating has the property of complete transmission at certain resonant frequencies. In [9], an asymptotic study of the solutions to the Helmholtz in a domain with a single thin slot is presented, using the method of matched asymptotic expansions (see also [13]).

In our work, we study cavities as those studied in [6], with the same idea of taking advantage of the simplicity of the geometry to obtain a nearly explicit description of the fields. More precisely, we consider a diffracting domain $\Omega \times \mathbb{R} \subset \mathbb{R}^3$ delimited by a perfectly conducting planar interface with one sub-wavelength rectangular cavity. Our goal is to study the resonant frequencies, i.e., those for which the Maxwell system (actually due to the symmetries, the Helmholtz equation) is not invertible.

The simple geometric set up allows us to represent the Green function in terms of two nearly explicit functions: One is the Green function of the half space above the plane interface, the other is the Green function of the rectangular cavity, both satisfying homogeneous Neumann boundary conditions. We derive an integral equation on the aperture of the cavity. The formalism of integral equations naturally lends itself to asymptotic analysis, as the width $w$ of the cavity tends to 0 while the wavelength is fixed. Using an operator version of the Rouché theorem [12], we can derive asymptotics of the resonant frequencies $k_n(w)$ of our system. As $w \to 0$, we show that $k_n(w)$ tends to the roots of a function $e(k)$ associated to the infinitely thin limiting cavity. These numbers are not the resonant frequencies $n\pi/h$ of the limiting cavity (of height $h$) with Dirichlet or Neumann boundary conditions at the endpoints. We describe quite explicitly the first and higher order terms in the expansion of $k_n(w)$, in terms of the geometric parameters. We show that the contribution of the shrinking cavity can be approximated by that of a radiative dipole placed on the interface.

The principal tools we use are integral equations, Fredholm theory and analytic spectral theory. The latter has been used previously to derive the asymptotics of perturbed eigenmodes in a planar waveguide [4, 3]. Our approach should be sufficiently versatile to allow treatment of more complicated cases. A forthcoming paper will address the case of two close subwavelength cavities, where we show how the interaction between cavities can enhance the fields more significantly than in the case addressed here. We think that one could also analyse situations where the modelling of the metallic coating is more realistic.

The outline of the paper is as follows. Section 2 describes the problem and presents the main results. In section 3, an integral representation of the solution to the diffraction problem is derived. We reduce the Helmholtz equation in the diffraction domain to a linear integral equation on the aperture of the cavity. Section 4 is the core of the paper. It is devoted to the asymptotic expansion of the integral equation and of the resonances, as the width of the cavity tends to 0. Section 5, contains the proof of the invertibility of the first order term in the rescaled integral operator. Finally, in the appendix, we recall some results of Ghoberg and Sigal [12] on the operator version of the Residue theorem.

# 2 Formulation of the problem and main results

## 2.1 Notations

Let $\omega$ be a smooth and connected domain in $\mathbb{R}^p$. Sobolev spaces are denoted by $H^s(\omega)$, and their norms and scalar product are denoted $\|.\|_s$ and $\langle .,.\rangle_s$. When $\omega \equiv \mathbb{R}^p$, $H^s$ coincides with the space of tempered distributions $u(X)$, the Fourier transform $\mathcal{F}(u)(\xi)$ of which belong to $L^2_{loc}$ and satisfy $\|(1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}(u)(\xi)\|_0 < \infty$.
Let $\mathcal{D}(\omega)$ denote the space of $C^\infty$ functions, that are compactly supported in $\omega$. We denote by $\widetilde{H}^{1/2}(\omega)$ and $\widetilde{H}^{-1/2}(\omega)$, the closures of $\mathcal{D}(\omega)$ in $H^{\frac{1}{2}}(\mathbb{R}^p)$ and $H^{-\frac{1}{2}}(\mathbb{R}^p)$ respectively.
If $\omega$ is a smooth domain, it is well-known that $\widetilde{H}^{\pm\frac{1}{2}}(\omega)$ coincides with the space of functions $u$ in $H^{\pm\frac{1}{2}}(\mathbb{R}^p)$ with support in $\overline{\omega}$ [19]. Moreover, if $\widetilde{u}$ denotes the extension by zero of a function $u$ defined on $\omega$, we have $\widetilde{H}^{\frac{1}{2}}(\omega) = \{u \in H^0(\omega) : \widetilde{u} \in H^{\frac{1}{2}}(\mathbb{R}^p)\}$.
We also recall [18] that $\widetilde{H}^{\pm\frac{1}{2}}(\omega) = (H^{\mp\frac{1}{2}}(\omega))'$ and that $(\widetilde{H}^{\pm\frac{1}{2}}(\omega))' = H^{\mp\frac{1}{2}}(\omega)$. We denote by $\langle .,.\rangle_{\frac{1}{2},-\frac{1}{2}}$ the duality product on $H^{\frac{1}{2}}(\omega) \times \widetilde{H}^{-\frac{1}{2}}(\omega)$.

We consider the geometry shown in Figure 1 to study the scattering of electromagnetic waves by an optical device, that contains a small subwavelength cavity. The scattering

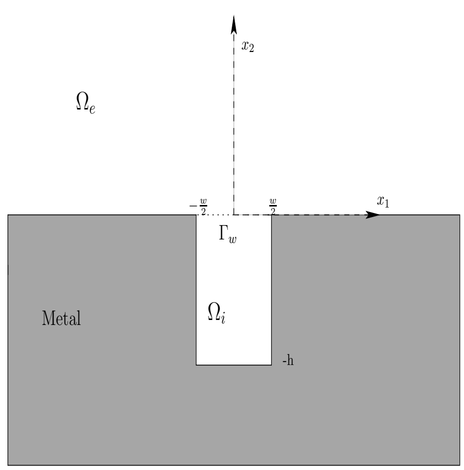


Figure 1: The diffracting domain $\Omega$

domain (the air above the device) is invariant in the $x_3$-direction and has the form $\Omega\times\mathbb{R}$. Its cross-section $\Omega$ consists of the union of the upper-half plane $\Omega_e = \mathbb{R}^2_+$ and of the open cavity

$\Omega_i = (-h, 0] \times (-\frac{w}{2}, -\frac{w}{2})$, of width $w$ and depth $h$. We denote by $\Gamma_w = (-w, w) \times \{x_2 = 0\}$ the aperture of the cavity, and by $X = (x_1, x_2)$ points in $\mathbb{R}^2$.

### 2.2 Problem Formulation

We assume that $\Omega$ is filled with a homogeneous dielectric material of magnetic permeability $\mu$, and electric permittivity $\varepsilon$. The time-dependent, linear Maxwell equations take the form

$$\begin{aligned} \nabla \times \mathbf{E} + \frac{\partial \mathbf{H}}{\partial t} &= 0 \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}_+, \\ \nabla \times \mathbf{H} - \frac{\partial \mathbf{E}}{\partial t} &= 0 \quad \text{in } \Omega \times \mathbb{R} \times \mathbb{R}_+, \end{aligned}$$

where $\mathbf{E} \in \mathbb{R}^3$ and $\mathbf{H} \in \mathbb{R}^3$ respectively denote the electric and magnetic fields. In this paper we only consider time-harmonic solutions, i.e., special solutions of the form

$$\mathbf{E}(x,t) = Re(E(x)e^{-i\omega t}) \quad \text{and} \quad \mathbf{H}(x,t) = Re(H(x)e^{-i\omega t}),$$

where $\omega$ denotes the time pulsation and the complex fields $E(x)$ and $H(x)$ satisfy

$$\nabla \times E - i\omega\mu H = 0, \tag{2.1}$$
$$\nabla \times H + i\omega\varepsilon E = 0. \tag{2.2}$$

We assume that the surface of the device is a perfect conductor, so that $E$ satisfies the following boundary condition in $\partial\Omega \times \mathbb{R}$

$$\mathbf{n} \times E = 0,$$

where $\mathbf{n} = (n, 0)$ if $n$ denotes the outward normal vector to $\partial\Omega$. In addition, we assume that the fields satisfy $\int_K (|E|^2 + |H|^2) dX < \infty$, for any compact subset $K \subset \Omega$. This finite energy condition ensures fulfilment of the edge condition at the corners $(\pm w, 0, 0)$ [5].
The diffraction of (time harmonic) incident fields $E^{inc}, H^{inc}$ by the surface $\partial\Omega$ gives rise to reflected and scattered fields. We write

$$\begin{aligned} E &= E^{inc} - \widetilde{E}^{inc}(x_1, -x_2, x_3) + E^s, \\ H &= H^{inc} + \widetilde{H}^{inc}(x_1, -x_2, x_3) + H^s, \end{aligned}$$

with $\widetilde{v} = (v_1, -v_2, v_3)$ if $v$ is a vector of the form $(v_1, v_2, v_3)$. Since the scattering domain is unbounded, we require that the scattered fields $(E^s, H^s)$ satisfy the Silver-Müller radiation condition. As in [6], we focus on the transverse electric polarization (TE), where the electric field is transverse to the invariant dimension. In this case, $H = (0, 0, H_3(x_1, x_2))$ and the $x_3-$component of the magnetic field verifies the Helmholtz equation

$$\begin{cases} \Delta H_3(X) + k^2 H_3(X) = 0, & X \in \Omega, \\ \partial_n H_3 = 0, & X \in \partial\Omega, \end{cases}$$

with $k = w\sqrt{\varepsilon\mu}$. The whole system is reduced to a Helmholtz equation, as the components of $E$ can be recovered from $H_3$ by (2.2).

It is known that the problem (2.2) has a unique solution whenever $Im(k) \geq 0$ [8]. The mapping $R(k) \; : H^{inc} \to H$ defines an operator-valued function which is holomorphic in $Im(k) \geq 0$. It has a meromorphic extension to the whole complex plane, except for a countable number of poles: These values of $k$ are the resonant frequencies. In other words, they are the values $k$ for which (2.2) has non-trivial solutions when $H^{inc} \equiv 0$. The space of such non-trivial solutions, called characteristic functions, has finite dimension. When the pole $k_j$ is simple, then the solution operator $R(k)$ can be factorized in the form

$$R(k) \quad = \quad \frac{R_{-1,j}}{k - k_j} + R_{0,j}(k),$$

where $R_{-1,j}$ is a finite rank operator, and where $R_{0,j}(k)$ is an operator-valued function which is holomorphic near $k_j$ [14]. The confinement of the electromagnetic fields around the cavity occurs at frequencies $k \in \mathbb{R}_+$ close to $Re(k_j)$, if the imaginary part $Im(k_j)$ is small enough. In this case $\frac{\|R_{-1,j}\|}{|Im(k_j)|}$ represents the factor of enhancement of the fields.
As for the time-dependent Maxwell equations, the behavior of the field $\mathbf{H}$, for large time is related to the resonant frequencies and their associated characteristic functions. Its energy decays exponentially with $t$. More precisely, in each compact subset $K \subset \Omega$, we have [16]

$$\int_K |\mathbf{H}(x,t) - \sum_{j=1}^{N} e^{-ick_j t} H_j(x)|^2 \Big) dx \quad \leq \quad C_N e^{cIm(k_{N+1})t}.$$

where $H_j$ are the characteristic functions associated to the resonance $k_j$, where $c = \frac{1}{\sqrt{\varepsilon\mu}}$ is the speed of light in $\Omega$, and where $C_N > 0$. The resonances $(k_j)_{j\geq 1}$ can be ordered such that

$$0 > Im(k_1) \geq \cdots \geq Im(k_j) \geq Im(k_{j+1}) \cdots \tag{2.3}$$

Thus $Im(k_j)$, also represents the lifetime of the confinement phenomena, which plays an important role in the applications.

We define the Green function associated to (2.2) by

$$\begin{aligned} \Delta\mathcal{G}(w,k;X,Y) + k^2\mathcal{G}(w,k;X,Y) &= \delta_Y(X), \qquad X \in \Omega, \\ \partial_n \mathcal{G} &= 0, \qquad X \in \partial\Omega, \end{aligned}$$

which also satisfies the following Sommerfeld Radiation condition far from the interface [8]

$$\lim_{|X|\to\infty} |X|^{1/2}(\partial_{|X|}\mathcal{G} - ik\mathcal{G}) \quad = \quad 0, \tag{2.4}$$

where $|X| = (x_1^2 + x_2^2)^{1/2}$.

**Remark 2.1.** *When $w$ and $k$ are fixed and no confusion arises, we simply write $\mathcal{G}(X,Y)$ for the Green function $\mathcal{G}(w,k;X,Y)$.*

By Green's formula (see [25]) it follows that

$$H_3^s(X) \;=\; -\int_{\partial\Omega} \mathcal{G}(w,k;X,Y)(\partial_n H_3^{inc}(Y) + \partial_n \widetilde{H}_3^{inc}(\widetilde{Y}))dY, \tag{2.5}$$

where $\widetilde{Y} = (y_1, -y_2)$ is the image of $Y = (y_1, y_2)$. Since the function $H_3^{inc}(X) + \widetilde{H}_3^{inc}(\widetilde{X})$ is a solution to the Helmholtz equation in the whole space it follows that

$$H_3(X) \;=\; H_3^{inc}(X) + \widetilde{H}_3^{inc}(\widetilde{X}) - \int_{\Gamma_w} \partial_{x_2}\mathcal{G}(w,k;X,Y)(H_3^{inc}(Y) + \widetilde{H}_3^{inc}(\widetilde{Y}))dY,$$

for any fixed $X \in \mathbb{R}^2_+$. The integral term in (2.5) represents all the effect of the cavity on the magnetic field. The diffractive properties of the device are thus completely encoded in $\partial_{x_2}\mathcal{G}(w,k;X,Y)$. Since we are interested in the response of a narrow cavity to the diffraction of incident plane waves, our purpose is to determine how $\partial_{x_2}\mathcal{G}(w,k;X,Y)$ behaves when $w$ becomes small.
The above equation shows that $\partial_{x_2}\mathcal{G}(w,k;X,Y)$ is also the principal kernel of $R(k)$. Therefore, the resonance frequencies as defined above, are exactly the poles of the Green function $\mathcal{G}(w,k;X,Y)$. One can easily prove that $\overline{\mathcal{G}(w,k;X,Y)} = \mathcal{G}(w,-\overline{k};X,Y)$. It follows that the poles of $\mathcal{G}(w,k;X,Y)$ are symmetric with respect to the imaginary axis ( $k_j$ and $-\overline{k}_j$ are simultaneously poles of $\mathcal{G}$).

### 2.3 Main results

The following function plays an important role in our analysis:

$$e(k) \;:=\; -(\frac{1}{hk} + \cot(hk))\frac{2}{k}. \tag{2.6}$$

The number $e(k)$ is the response at the point $(0,0)$ of the infinitely thin cavity $\{0\}\times(-h,0)$ (limit of the shrinking cavity $\Omega_i$, (see Fig. 2) excited by a dipole $\delta_{x_2=0}$ at the frequency $k$: Indeed, if $g$ solves

$$\begin{cases} g''(s) + k^2 g(s) & = \delta_0(s) \quad \text{in } (-h,0), \\ g'(-h) \;=\; g'(0) & = 0, \end{cases}$$

then we have

$$e(k) \;=\; g(0) \;=\; \frac{4}{h}\sum_{n=0}^{\infty} \frac{1}{k^2 - (\frac{n\pi}{h})^2}.$$

We denote its real, non-negative zeros by $k_1(0) < k_2(0) < \cdots < k_n(0) < \ldots$. A simple calculation shows that (see Fig 3)

$$\frac{(n-1)\pi}{h} < k_n(0) < \frac{n\pi}{h}, \quad n \in \mathbb{N}^*.$$

Given $r > 0$, we set $n_r$ to be the number of values $k_n(0)$ inside the interval $(0,r)$. Throughout the paper, $w_0 > 0$ is a fixed parameter that measures the size of a disk $D_{\frac{\pi}{w_0}}$ in the complex plane, in which we let the frequency $k$ vary. We now state the main results of this paper.

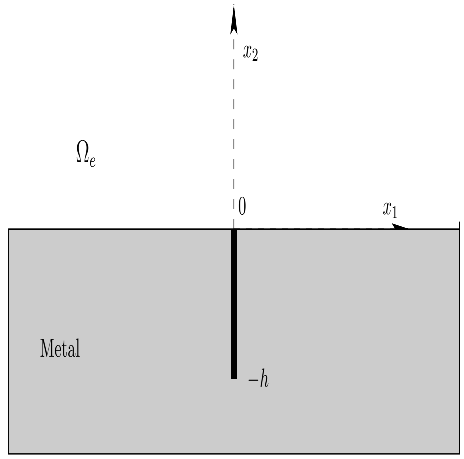


Figure 2: The limiting cavity

**Theorem 2.1.** *Let $0 < w < w_0$. The number of resonances of the open cavity $\Omega_i$ contained in $D_{\frac{\pi}{w_0}}$ is equal to $n_{w_0}$. Moreover, for $1 \le n \le n_{w_0}$, the resonance $k_n(w)$ has the asymptotic expansion, as $w \to 0$*

$$k_n(w) \quad = \quad k_n(0) + \eta_{n,1}(h) w \ln(w) + \eta_{n,2}(h) w + o(w).$$

*The imaginary part of $k_n(w)$ behaves like*

$$Im(k_n(w)) \quad = \quad -\eta_{n,3}(h) w + o(w).$$

*The constants $\eta_{n,j}(h), j = 1, 2, 3$ only depend on the height $h$ of the rectangular cavity, and are given explicitly in theorem (4.5). Further, the constant $\eta_{n,3}(h)$ is strictly positive.*

**Remark 2.2.** *The expression (4.13) shows that $\eta_{n,3}$ is an increasing function of $k_n(0)$. The ordering of the resonances in the statement of the theorem is consistent with (2.3), when $w$ is sufficiently small.*

Let $\mathcal{G}_e(X, Y)$ be the Green function of the Helmholtz equation with Neumann boundary condition in the half-space $\mathbb{R}^2_+$. Using the method of images, $\mathcal{G}_e$ can be derived explicitly from the Green function of the Helmholtz equation in the whole space:

$$\mathcal{G}_e(X, Y) = -\frac{i}{4} H_0^{(1)}(k|X - Y|) - \frac{i}{4} H_0^{(1)}(k|X - \tilde{Y}|),$$

where $H_0^{(1)}(z)$ is the Hankel function of the first kind of order zero and $\tilde{Y} = (y_1, -y_2)$ is the image of $Y = (y_1, y_2)$.

The next results give the asymptotic form of the Green function $\mathcal{G}$, when both the positions of the observer and the source are in $\Omega_e$ and are far from the aperture of the cavity.

**Theorem 2.2.** *Let $0 < w < w_0$. Let $Y, Z \in \Omega_e$. Then, for $k \in D_{\frac{\pi}{w_0}}$, the Green function $\mathcal{G}$ has the following behavior:*

**A**) *When $k$ is close to a resonance $k_n(w)$,*

$$\mathcal{G}(w,k;Z,Y) = \mathcal{G}_e(Y,Z) + \sum_{n=0}^{n_{w_0}} \big(k - k_n(w)\big)^{-1} g_n(w,Z) g_n(w,Y) + R(w,k;Z,Y), \tag{2.7}$$

*where $R$ is holomorphic in $k$ and smooth with respect to the space variables, and where the functions $g_n(w,\cdot)$ only depend on the depth $h$. The explicit expressions of the latter are given in* (4.20)*. In addition, we have*

$$g_n(w,Y)g_n(w,Z) = C_n H_0^{(1)}(k_n(0)|Y|) H_0^{(1)}(k_n(0)|Z|) w \ln(w) + o(w), \tag{2.8}$$

*with $C_n$ is a constant that only depends on $h$ and is given in* (4.22)*. Moreover, we have*

$$g_n(w,Y) \sim \frac{\beta_n(w)}{|Y|^{\frac{1}{2}}} e^{Im(k_n(w))|Y|}, \ \text{as } |Y| \to +\infty, \tag{2.9}$$

*where $\beta_n(w)$ is a constant that depends on $w$ and $h$.*

**B**) *Let $k$ be a fixed frequency that satisfies $e(k) \neq 0$. Then, we have*

$$\mathcal{G}(w,k;Z,Y) = \mathcal{G}_e(Y,Z) - \frac{1}{4e(k)} H_0^{(1)}(k|Z|) H_0^{(1)}(k|Y|) w + o(w), \tag{2.10}$$

This theorem shows that the subwavelength open cavity acts like a dipole $p(k;Y)\delta_0(Z)$ placed at the center of its aperture. The function $p(k;Y)$ represents the dipole moment, and only depends on $k$ and on the position $Y$ of the source. When $k$ is equal to one of the values $Re(k_n(w)), 1 \leq n \leq n_{w_0}$ the above expressions show that

$$p(k;Y) = \sum_{n=0}^{n_{w_0}} \frac{w \ln(w)}{k - k_n(w)} C_n H_0^{(1)}(k|Y|),$$

and for $k$ fixed far away from $Re(k_n(w))$ we have

$$p(k;Y) = \frac{-w}{4e(k)} H_0^{(1)}(k|Y|).$$

We further deduce from the asymptotic (2.9) that the enhancement is local and specific to the region close to the aperture $\Gamma_w$.

Finally, we consider the case when the source is away from the cavity, but the observer is located near its aperture:

**Theorem 2.3.** *Let $0 < w < w_0$. Assume that $Y \in \Omega_e$ and $\overline{Z} \in \Omega_e$ Then, for $k \in D_{\frac{\pi}{w_0}}$,*

**A**) *When $k$ is close to a resonance $k_n(w)$,*

$$\mathcal{G}(w,k;w\overline{Z},Y) = \mathcal{G}_e(Y,w\overline{Z}) + \sum_{n=0}^{n_{w_0}} \big(k - k_n(w)\big)^{-1} g_n(w,w\overline{Z}) g_n(w,Y) + R(w,k;w\overline{Z},Y), \tag{2.11}$$

*where $g_n(w,X)$ and $R(w,k;Z,Y)$ are defined as in theorem 2.2. In addition, we have*

$$g_n(w,Y)g_n(w,Z) = C_{n2}H_0^{(1)}(k_n(0)|Y|)w\ln^2(w) + o(w\ln^2(w)), \tag{2.12}$$

*with $C_{n2}$ is a constant that only depends on $h$ and is given in (4.23).*

**B**) *Let $k$ be a fixed frequency that satisfies $e(k) \neq 0$. Then, we have*

$$\mathcal{G}(w,k;Z,Y) = \mathcal{G}_e(Y,Z) + Q_1(Y)w\ln(w) + Q_2(Y,\overline{Z})w + o(w). \tag{2.13}$$

*where $Q_1$ and $Q_2$ defined respectively in (4.17) and (4.18) only depend on the parameters $k$ and $h$.*

Thus, when the observer $Z$ is close to the aperture, the cavity still behaves like a dipole when $k$ is close to one of the $Re(k_n(w)), 1 \leq n \leq n_{w_0}$. However, the corresponding moment is of order $\dfrac{w\ln^2(w)}{k-k_n(w)}$, and thus greater than that of theorem 2.2.

The proof of these results is based on a particular integral representation for $\mathcal{G}$.

# 3 Integral Representation

The Green function of the Helmholtz operator in the rectangle satisfies

$$\begin{cases} \Delta\mathcal{G}_i(X,Y) + k^2\mathcal{G}_i(X,Y) & = & \delta_Y(X), & X \in \Omega_i, \\ \partial_{n_i}\mathcal{G}_i & = & 0, & X \in \partial\Omega_i, \end{cases}$$

where $n_i$ is the outward normal on $\partial\Omega_i$.
Since the width $w$ of the cavity tends to zero, we may assume that $k^2$ is not an eigenvalue of the Laplacian with Neumann boundary condition in the cavity. Thus the Green function of the Helmholtz operator in $\Omega_i$ exists and can be expressed as ([7])

$$\mathcal{G}_i(X,Y) = \frac{4}{hw}\sum_{m=0}^{+\infty}\sum_{n=0}^{+\infty}\frac{\cos(\frac{m\pi}{w}(x_1+\frac{w}{2}))\cos(\frac{m\pi}{w}(y_1+\frac{w}{2}))}{k^2-(\frac{m\pi}{w})^2-(\frac{n\pi}{h})^2}\cos(\tfrac{n\pi}{h}(x_2+h))\cos(\tfrac{n\pi}{h}(y_2+h)). \tag{3.1}$$

It follows from the Green formula in $\Omega_e$ and $\Omega_i$, that for any fixed $Y$ in $\Omega_e \cup \Omega_i$ we have

$$\mathcal{G}(Z,Y)\chi_{\Omega_e}(Z) \;=\; \mathcal{G}_e(Y,Z)\chi_{\Omega_E}(Y) + \int_{\Gamma_w}\partial_{x_2}\mathcal{G}(x_1,0,Y)\mathcal{G}_e(x_1,0,Z)d\sigma(x_1), \tag{3.2}$$

and

$$\mathcal{G}(Z,Y)\chi_{\Omega_i}(Z) \;=\; \mathcal{G}_i(Y,Z)\chi_{\Omega_i}(Y) - \int_{\Gamma_w}\partial_{x_2}\mathcal{G}(x_1,0,Z)\mathcal{G}_i(x_1,0,Z)d\sigma(x_1), \tag{3.3}$$

where $\chi_{\Omega_i}(Z)$ and$\chi_{\Omega_e}(Z)$ are the characteristic functions of the domains $\Omega_i$ and $\Omega_e$, respectively. Taking the limit $Z \to \Gamma_w$ while $Y$ is fixed in $\Omega_e$, and adding (3.2) and (3.3), we obtain

$$\int_{\Gamma_w}(\mathcal{G}_e(x_1,0,z_1,0) + \mathcal{G}_i(x_1,0,z_1,0))\partial_{x_2}\mathcal{G}(x_1,0,Y)d\sigma(x_1) \;=\; -\mathcal{G}_e(Y,z_1,0). \tag{3.4}$$

It is convenient to rescale this equation: We let $\Gamma = (-\frac{1}{2}, \frac{1}{2})$, and we consider the integral operator

$$S(w,k)\phi(x) := \int_\Gamma (\mathcal{G}_e(wx,0,wz,0) + \mathcal{G}_i(wx,0,wz,0))\phi(z)dz, \tag{3.5}$$

to discover that $\partial_n \mathcal{G}_{\Gamma_w}$ satisfies the following integral equation

$$S(w,k)\partial_{x_2}\mathcal{G}(w,k;wx,0,Y) = -w^{-1}\mathcal{G}_e(Y,wx,0), \text{ on } \Gamma. \tag{3.6}$$

# 4 Asymptotics

In this section, we study the integral operator (3.5). We note that once $\partial_{x_2}\mathcal{G}(w,k;X,Y)$ and its asymptotic behavior are determined, $\mathcal{G}(w,k;Z,Y)$ itself is obtained from (3.2) and (3.3). In the rescaled equation(3.6), $Z$ represents the position of the observer. We derive two types of asymptotics: When the observer is asymptotically far from the cavity ($Z$ is fixed in $\Omega_e$) and when the observer is close to the cavity ($Z = w\overline{Z}$, with $\overline{Z}$ fixed in $\Omega_e$).

This section is divided in three. In the first part, we show that $S(w,k)$ is a bounded operator from $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$. We derive its asymptotics when $w$ tends to zero, while the other parameters (the frequency, the depth of the cavity) remain fixed. Based on Fredholm theory and pseudo-differential techniques, we then prove that $S(w,k)$ is invertible.
In the second part, we show that the resonances of the open cavity are exactly the poles of the function $S^{-1}(w,.)$, and we determine their asymptotics. We derive the asymptotics of $S^{-1}(w,k)$ as $w$ tends to zero, when the frequency $k$ is close or far from the resonances.
The third part is devoted to the proofs of theorems 2.2 and 2.3.

## 4.1 Asymptotic of $S(w,k)$

**Lemma 4.1.** *Let $0 < w < w_0$, and fix the parameters $h$ and $k$. The kernel of $S(w,k)$ has the following asymptotics:*

$$\mathcal{G}_e(wx,0,wz,0) + \mathcal{G}_i(wx,0,wz,0) \quad = \quad \theta_w + s_1 + s_2 w + s_3 w^2 \ln(w) + s_4 w^2,$$

*where*

$$\begin{cases} \theta_w(k) := \frac{e(k)}{w} + \frac{1}{\pi}\ln(w) + \frac{1}{\pi}\ln(k) + \delta, \\ s_1(x,z) := \frac{1}{\pi}\ln\left(2\left|(x-z)\sin(\frac{\pi}{2}(x-z))\sin(\frac{\pi}{2}(x+z+1))\right|\right), \\ s_2(x,z) := -\frac{2}{h}\left(\frac{1}{6} + \frac{1}{8}((x+z+1)^2 + (x-z)^2) - \frac{1}{4}(x+z+1+|x-z|)\right), \\ s_3(k,x,z) := -\frac{1}{4\pi}k^2(x-z)^2, \end{cases}$$

*and where $\delta = \frac{\gamma}{\pi} - \frac{i}{2}$ ($\gamma$ is the Euler constant). The function $s_4(w,k,.,.)$ is of class $C^{1,\nu}$ ( $0 \leq \nu < 1$). Moreover, its $C^{1,\nu}$-norm is uniformly bounded with respect to $(w,k) \in (0,w_0) \times D_{\frac{\pi}{\omega_0}}$.*

*Proof.* The asymptotic expansions of the Green functions $\mathcal{G}_e$ and $\mathcal{G}_i$ follow from their explicit expressions. In particular, the term $s_4$ in the expansion of the kernel is a sum $s_{4,e}+s_{4,i}$ of contributions from $\mathcal{G}_e$ and $\mathcal{G}_i$.
The asymptotic of the Hankel function near zero is (see for instance [1])

$$\frac{i}{2}H_0^{(1)}(z) \quad = \quad -\frac{1}{\pi}\ln(|z|)+\Gamma_1+\frac{1}{4\pi}z^2\ln(|z|)+\Gamma_2 z^2+o(z^2), \tag{4.1}$$

with $\Gamma_1=\frac{i}{2}+\frac{1}{\pi}(\ln(2)-\gamma)$ and $\Gamma_2=\frac{i}{4\pi}-\frac{i}{8}-\frac{1}{4\pi}(\ln(2)-\gamma)$.
Since $\mathcal{G}_e(wx,0,wz,0)=-\frac{i}{2}\,H_0^{(1)}(kw|x-z|)$, we see that

$$\begin{aligned}\mathcal{G}_e(wx,0,wz,0) \quad &= \quad \frac{1}{\pi}\ln(kw)+\delta+\frac{1}{2\pi}\ln(2)+\frac{1}{\pi}\ln(|x-z|)\\ &\quad -\frac{1}{4\pi}k^2(x-z)^2w^2\ln(w)+s_{4,e}(w,k,x,z)\end{aligned}$$

with

$$s_{4,e}(w,k,x,z) \quad = \quad -\left[\frac{1}{4\pi}k^2\ln(k|x-z|)+\Gamma_2k^2\right](x-z)^2w^2+o(w^2).$$

Using the series expansion of the Hankel function near zero, one can even derive all the terms in the above asymptotics and one can prove that the remainder $o(z^2)$ in (4.1), is smoother than the term

$$\left[\frac{1}{4\pi}k^2\ln(k|x-z|)-\Gamma_2k^2\right](x-z)^2,$$

as a function of the variables $(x,z)$. In addition, the $C^2$ norm of the remainder is uniformly bounded with respect to $(w,k)\in(0,r)\times D_{\frac{\pi}{r}}$, for any fixed $r>0$. It follows that the regularity of $s_{4,e}(w,k,x,z)$ is that of the term above, and it suffices to show that this term, or equivalently that the function $X\to X^2\ln|X|$, has the regularity announced in the lemma. This is the object of proposition 4.1 below.

Now, we focus our attention on the asymptotics of the other Green function in the kernel of $S(w,k)$: Recalling (3.1), its expression on $\Gamma_w$ is

$$\mathcal{G}_i(wx,0,wz,0) \quad = \quad \frac{4}{hw}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{k^2-(\frac{m\pi}{w})^2-(\frac{n\pi}{h})^2}.$$

We set

$$\begin{cases} R_m(w,k) \;=\; \sum_{n=0}^{\infty}\frac{1}{k^2-(\frac{m\pi}{w})^2-(\frac{n\pi}{h})^2},\\ r_m(w,k) \;=\; \left((\frac{mh}{w})^2-(\frac{kh}{\pi})^2\right)^{\frac{1}{2}}.\end{cases}$$

We remark that $r_m(w,k)$ is well-defined for $k\in D_{\pi/w_0}$ and $m\geq 1$. Using the fact that

$$\sum_{n=0}^{\infty}\frac{1}{n^2+r^2} \quad = \quad \frac{1}{2r^2}+\frac{\pi}{2r}\frac{\sinh(\pi r)}{\cosh(\pi r)} \qquad \text{for} \qquad r\neq 0,$$

one easily checks that for $m \geq 1$,

$$R_m(w,k) \quad = \quad -\frac{h^2}{\pi^2}\Big(\frac{1}{2r_m^2} + \frac{\pi}{2r_m}\frac{\sinh(\pi r_m)}{\cosh(\pi r_m)}\Big),$$

and a straightforward computation shows that

$$R_m(w,k) \quad = \quad -\frac{h}{2\pi}\frac{w}{m} - \frac{1}{2\pi^2}\frac{w^2}{m^2} - \frac{kh^2}{4\pi^3}\frac{w^3}{m^3} + O(\frac{w^4}{m^4}), \quad \text{as} \quad \frac{w}{m} \to 0.$$

Noting that

$$\mathcal{G}_i(wx,0,wz,0) \quad = \quad \frac{4}{hw}\sum_{m=0}^{\infty} R_m(w,k)\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2})),$$

we substitute the quantity $R_m(w,k)$ by its asymptotic in the expression of $\mathcal{G}_i$, and obtain

$$\mathcal{G}_i(wx,0,wz,0) \quad = \quad \frac{4}{h}\sum_{n=0}^{\infty}\frac{1}{k^2-(\frac{n\pi}{h})^2}\frac{1}{w} - \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m} \tag{4.2}$$

$$-\frac{2}{h\pi^2}\sum_{m=1}^{\infty}\frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m^2}w \tag{4.3}$$

$$-\frac{2k^2}{\pi^3}\sum_{m=1}^{\infty}\frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m^3}w^2 + O(w^3). \tag{4.4}$$

On the other hand, the following sums can be computed explicitly (see [7]):

$$\begin{cases} \frac{4}{h}\sum_{n=0}^{\infty}\frac{1}{k^2-(\frac{n\pi}{h})^2} & = \quad -(\frac{1}{hk}+\cot(hk))\frac{2}{k} \;\; =: \;\; e(k) \\ \sum_{m=1}^{\infty}\frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m} & = \quad -\ln(2) - \frac{1}{2}\ln(|\sin(\pi\frac{x+z+1}{2})\sin(\pi\frac{x-z}{2})|), \\ \sum_{m=1}^{\infty}\frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m^2} & = \quad \frac{\pi^2}{6} + \frac{\pi^2}{8}((x+z+1)^2+(x-z)^2) \\ & \quad\; -\frac{\pi^2}{4}(x+z+1+|x-z|). \end{cases}$$

Let $s_{4,i}(w,k,x,z) = -\frac{2k^2}{\pi^3}\sum_{m=1}^{\infty}\frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m^3}w^2 + O(w^3)$. It follows from the definition of $e(k)$, that

$$\begin{aligned} \mathcal{G}_i(wx,0,wz,0) \quad = \quad & \frac{e(k)}{w} + \frac{2}{\pi}\ln(2) + \frac{1}{\pi}\ln\Big(\Big|\sin(\frac{\pi}{2}(x-z))\sin(\frac{\pi}{2}(x+z+1))\Big|\Big) \\ & -\frac{2}{h}\Big(\frac{1}{6} + \frac{1}{8}((x+z+1)^2+(x-z)^2) \\ & -\frac{1}{4}(x+z+1+|x-z|)\Big) + s_{4,i}(w,k,x,z). \end{aligned}$$

Next, we study the regularity of the remainder $s_{4,i}(w,k,x,z)$ as a function of the parameters $(w,k,x,z)$. From the asympotics of $R_m(w,k)$, we deduce that the term $O(w^3)$ is a $C^2$ function of the variables $(x,z)$. We remark that for $0 < w < r$, the poles of the function $\mathcal{G}_i$ in the complex domain $D_{\frac{\pi}{r}}$ are exactly $\{\pm\frac{n\pi}{h} : n \in \mathbb{N}\}$. These poles are only present in the term $e(k)$ in the expansion (4.2). Therefore the function $s_{4,i}(w,k,x,z)$ has no poles in

$(w,k) \in (0,r) \times D_{\frac{\pi}{r}}$, and its $C^2$ norm is uniform therein. Quoting again [7], the following asymptotics hold when $r$ tends to zero

$$\sum_{m=1}^{\infty} \frac{\cos(mr)}{m^3} = \sum_{m=1}^{\infty} \frac{1}{m^3} + r^2 \ln(r) + O(r^2),$$

which implies that

$$\sum_{m=1}^{\infty} \frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m^3}$$
$$= \frac{1}{2}\sum_{m=1}^{\infty} \frac{\cos(m\pi(x+z+1))}{m^3} + \frac{1}{2}\sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{\pi^2}{2}(x-z)^2 \ln(|x-z|) + O(|x-z|^2),$$

when $(x-z)$ tends to zero, and

$$\sum_{m=1}^{\infty} \frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m^3} =$$
$$\frac{1}{2}\sum_{m=1}^{\infty} \frac{\cos(m\pi(x-z))}{m^3} + \frac{1}{2}\sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{\pi^2}{2}(x+z+1)^2 \ln(x+z+1) + O(|x+z+1|^2),$$

when $(x+z+1)$ tends to zero. Therefore, the term

$$\frac{2k^2}{\pi^3}\sum_{m=1}^{\infty} \frac{\cos(m\pi(x+\frac{1}{2}))\cos(m\pi(z+\frac{1}{2}))}{m^3},$$

as a function of $(x,z)$ has the same regularity as the function $X^2 \ln(|X|)$.

**Proposition 4.1.** *Fix $\alpha$ in $(0,1]$. Then, the function $|x|^\alpha \ln(|x|)$ belongs to $C^{0,\nu}([-\frac{1}{2},\frac{1}{2}])$ for all $\nu \in [0,\alpha)$.*

*Proof.* (proposition) Set $f(x) = |x|^\alpha \ln(|x|)$ with $\alpha \in (0,1]$. We first remark that for $\varepsilon > 0$, the function $g_\epsilon(x) = |x|^{1+\epsilon} \ln(|x|)$ is in $C^1([-1,1])$, and satisfies

$$|g_\epsilon(x) - g_\epsilon(y)| \leq C_\epsilon |x-y|,$$

for all $x,y$ in $[-1,1]$, where $C_\epsilon = \sup_{t\in[-1,1]} (|t|^\epsilon\, |1+\ln|t||$.
Since $\nu \in [0,\alpha)$ this result is valid for $\epsilon = \frac{\alpha-\nu}{\nu}$. Applying it to the function $f(x) = \nu^{-1} g_{\frac{\alpha-\nu}{\nu}}(|x|^\nu)$, we get

$$|f(x)-f(y)| \leq \frac{1}{\nu} C_{\frac{\alpha-\nu}{\nu}} ||x|^\nu - |y|^\nu| \leq \frac{1}{\nu} C_{\frac{\alpha-\nu}{\nu}} |x-y|^\nu,$$

for all $x,y$ in $[-\frac{1}{2},\frac{1}{2}]$, which proves the proposition. □
We deduce from the proposition (4.1), that $X^2 \ln(|X|)$ is of class $C^{1,\nu}$ for any $0 \leq \nu < 1$. This implies that the remainders $s_{4,e}(w,k,x,z)$ and $s_{4,i}(w,k,x,z)$ are $C^{1,\nu}$, with respect to

$(x,z)$, for any $0 \le \nu < 1$, and so does $s_4(w,k,x,z) = s_{4,e}(w,k,x,z) + s_{4,i}(w,k,x,z)$, which completes the proof of the lemma. $\square$

We define the following integral operators

$$\begin{cases} \widetilde{H}^{-\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{1}{2}}(\Gamma), \\ \Theta(w,k)\phi(x) := \theta_w(k)\langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}}, \\ S_j\phi(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} s_j(x,z)\phi(z)dz, \quad j=1,2, \\ S_3(k)\phi(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} s_3(k,x,z)\phi(z)dz, \\ S_4(w,k)\phi(x) := \int_{-\frac{1}{2}}^{\frac{1}{2}} s_4(w,k,x,z)\phi(z)dz. \end{cases}$$

The next theorem concerns the leading part of the operator $S(w,k)$. Its proof is exposed in section 6.

**Theorem 4.1.** *The linear operator $S_1$ is invertible from $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$.*
*The linear operators $S_j, j=2,3,4$ are compact from $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$.*

**Corollary 4.1.** *The linear operator $S(w,k)$ is bounded from $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$ and satisfies the following asymptotics:*

$$S(w,k) \quad = \quad \Theta_w(k) + S_1 + wS_2 + w^2\ln(w)S_3(k) + w^2S_4(w,k). \tag{4.5}$$

*Moreover, given $w_0 > 0$ small enough, there exists a constant $C(w_0)$ such that: $\|S_4(w,k)\| \le C(w_0)$ for all $(w,k)$ in $(0,w_0) \times D_{\frac{\pi}{w_0}}$.*

*Proof.* The result is a direct consequence of lemma (4.1) and theorem (4.1). The only poles of the kernel (as a function of the frequency $k$) in $D_{\frac{\pi}{w_0}}$ are $\{\pm\frac{n\pi}{h} : n \in \mathbb{N}, n < \frac{h}{w_0}\}$, and $\theta_w(k)$ is the only singular term in (4.5). In particular, the kernel of the operator $S_4(w,k)$ is holomorphic on $D_{\frac{\pi}{w_0}}$. $\square$

Since the function 1 belongs to $H^{\frac{1}{2}}(\Gamma)$, which is isomorphic to $(\widetilde{H}^{-\frac{1}{2}}(\Gamma)'$, the dual product $\langle 1,.\rangle_{\frac{1}{2},-\frac{1}{2}}$ in the above corollary is well defined. Moreover, by construction of the space $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$, we have $\langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}} = \lim_{n\to+\infty}\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi_n(x)dx$, for any sequence $(\phi_n)_n$ in $\mathcal{D}(\Gamma)$ such that $\phi_n \to \phi$ in $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$. By a slight abuse of notation, we may sometimes write $\int_{-\frac{1}{2}}^{\frac{1}{2}} \phi(x)dx$ instead of $\langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}}$.

We deduce from the corollary (4.1), that $S(w,k)$ is a compact perturbation of the operator $S_1$. Therefore $S(w,k)$ is a Fredholm operator with index zero. In the next subsection we derive an asymptotic of its inverse and of its characteristic values.

## 4.2 Asymptotics of $S^{-1}(w,k)$

**Theorem 4.2.** *i. The operator-valued function $S(w,k)$ is finitely meromorphic and of Fredholm type on $\mathbb{C}\setminus\mathbb{R}_-$.*

*ii. Its poles are the numbers $\pm\big((\frac{n\pi}{h})^2+(\frac{m\pi}{w})^2\big)^{\frac{1}{2}}, \quad n,m\in\mathbb{N}$.*

*iii. The operator $S(w,k)$ is invertible from $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$, for $Im(k)\geq 0$.*

*iv. The operator valued function $S^{-1}(w,k)$ is finitely meromorphic on $\mathbb{C}\setminus\mathbb{R}_-$ and its poles are exactly the resonances of the scattering domain $\Omega$.*

*Proof.* It is known that the Hankel function is holomorphic on $\mathbb{C}\setminus\mathbb{R}_-$ (see for instance [1]). On the other hand, we deduce from the explicit expression of Green function $\mathcal{G}_i$ that the kernel of the operator $S(w,k)$ is finitely meromorphic on $\mathbb{C}\setminus\mathbb{R}_-$. The only poles of $S(w,k)$ come from the function $\mathcal{G}_i$ and are given by $\{\pm\big((\frac{n\pi}{h})^2+(\frac{m\pi}{w})^2\big)^{\frac{1}{2}}: n,m\in\mathbb{N}\}$.
Again, it follows from the expression of the Green function $\mathcal{G}_i$ that these poles are simple with multiplicity one. Corollary (4.1) implies that $S(w,k)$ is a Fredholm operator with index zero. On the other hand the integral equations (3.2)-(3.4) show that the uniqueness of the Green function $\mathcal{G}(w,k)$ is equivalent to the uniqueness of the integral equation (3.6). When $Im(k)\geq 0$, the Helmholtz equation in $\Omega$ with Neumann boundary condition has a unique solution [5]. Thus, the Fredholm alternative implies that $S(w,k)$ is invertible for $Im(k)\geq 0$. Finally, we deduce from the generalized Steinberg theorem 6.3, that the operator valued function $S^{-1}(w,k)$ is finitely meromorphic on $\mathbb{C}\setminus\mathbb{R}_-$ and that its poles are the resonances of $\Omega$.

□

We note that the inverse of the operator $S(w,k)$ has a holomorphic continuation at the poles $\{(\frac{n\pi}{h},\frac{m\pi}{w}): n,m\in\mathbb{N}\}$. In fact those poles are transformed into characteristic values of $S^{-1}(w,k)$. We set

$$\mathcal{L}(w,k) \quad = \quad S_1+wS_2+w^2\ln(w)S_3(k)+w^2S_4(w,k). \tag{4.6}$$

It follows from theorem 4.1 and corollary 4.1 that the operator $\mathcal{L}(w,k)$ is Fredholm of index zero and it is invertible for $w$ small enough. Using the Neumann series, its inverse can be written as

$$\mathcal{L}^{-1}(w,k) \quad = \quad S_1^{-1}+\sum_{p=1}^{\infty}\Big(S_1^{-1}\big(-(S_2+w(\ln(w)S_3(k)+S_4(w,k)))S_1^{-1}\big)^p\Big)w^p. \tag{4.7}$$

We next derive an explicit expression of the inverse of the operator $S(w,k)$.

**Theorem 4.3.** *Fix $w_0>0$ small enough. For $w<w_0$ the following expression holds:*

$$S^{-1}(w,k) \quad = \quad \mathcal{L}^{-1}(w,k)-\frac{\mathcal{L}^{-1}(w,k)1}{\Theta(w,k)\mathcal{L}^{-1}(w,k)1+1}\Theta(w,k)\mathcal{L}^{-1}(w,k). \tag{4.8}$$

*The resonances of $\Omega$ in $D_{\frac{\pi}{w_0}}$, are exactly the zeros of the function*

$$f_w(k) \quad := \quad \Theta(w,k)\mathcal{L}^{-1}(w,k)1+1. \tag{4.9}$$

*Proof.* For the sake of brevity we may sometimes denote $\mathcal{L}$ instead of $\mathcal{L}(w,k)$. By lemma 4.1, we choose $\omega_0 > 0$ small enough so that the series (4.7) converges, uniformly for $k \in D_{\pi/w_0}$. Moreover, lemma 4.1, also implies that $\mathcal{L}(w,k)$ is holomorphic in the complex domain $D_{\frac{\pi}{w_0}}$. Consequently, $\mathcal{L}^{-1}(w,k)$ is well-defined and holomorphic on the same domain $D_{\frac{\pi}{w_0}}$.

Next, for a fixed function $g(x) \in H^{\frac{1}{2}}(\Gamma)$, we derive the solution $\phi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to the equation $S(w,k)\phi(x) = g(x)$, in terms of the constant $\theta_w(k)$ and of the operator $\mathcal{L}^{-1}(w,k)$. It follows from the asymptotics (4.5) that the equation satisfied by $\phi$ can be rewritten in the form

$$S(w,k)\phi \quad = \quad \Theta(w,k)\phi + \mathcal{L}(w,k)\phi = \theta_w(k)\langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}} + \mathcal{L}(w,k)\phi = g. \qquad (4.10)$$

Applying the operator $\mathcal{L}^{-1}(w,k)$ to both sides of the equation, we discover

$$\theta_w(k)\langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}}\mathcal{L}^{-1}1 + \phi \quad = \quad \mathcal{L}^{-1}g.$$

Since the constant function 1 belongs to $H^{\frac{1}{2}}(\Gamma) = (\widetilde{H}^{-\frac{1}{2}}(\Gamma))'$, we deduce from the previous equality that

$$\theta_w(k)\langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}}\langle 1,\mathcal{L}^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}} + \langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}} \quad = \quad \langle 1,\mathcal{L}^{-1}g\rangle_{\frac{1}{2},-\frac{1}{2}}.$$

Consequently,

$$\langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}} = \frac{\langle 1,\mathcal{L}^{-1}g\rangle_{\frac{1}{2},-\frac{1}{2}}}{\theta_w(k)\langle 1,\mathcal{L}^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}} + 1}.$$

Thus, substituting the quantity $\langle 1,\phi\rangle_{\frac{1}{2},-\frac{1}{2}}$ into (4.10), we obtain

$$S^{-1}(w,k)g \;=\; \phi \quad = \quad \mathcal{L}^{-1}(w,k)g - \frac{\theta_w(k)\langle 1,\mathcal{L}^{-1}(w,k)g\rangle_{\frac{1}{2},-\frac{1}{2}}\mathcal{L}^{-1}(w,k)1}{\theta_w(k)\langle 1,\mathcal{L}^{-1}(w,k)1\rangle_{\frac{1}{2},-\frac{1}{2}} + 1},$$

which leads to the announced expression of $S^{-1}(w,k)$. We remark that $\mathcal{L}(w,k)$ is symmetric (the kernels of the integral operators are symmetric), and thus $\langle 1,\mathcal{L}^{-1}(w,k)g\rangle_{\frac{1}{2},-\frac{1}{2}} = \langle g,\mathcal{L}^{-1}(w,k)1\rangle_{\frac{1}{2},-\frac{1}{2}}$.

Since the operator-valued function $\mathcal{L}^{-1}(w,k)$ is holomorphic on the complex domain $D_{\frac{\pi}{w_0}}$, only the second term in (4.8) may have poles. In view of applying the generalized Rouché theorem, we compute

$$\begin{aligned} S^{-1}(w,k)\partial_k S(w,k) \;&=\; \mathcal{L}^{-1}(w,k)\partial_k\mathcal{L}(w,k) \\ &+ \frac{\mathcal{L}^{-1}(w,k)1}{\Theta(w,k)\mathcal{L}(w,k)1 + 1}\big(\partial_k\Theta(w,k) \;-\; \Theta_w(k)\mathcal{L}^{-1}(w,k)\partial_k\mathcal{L}(w,k)\big). \end{aligned}$$

The first term in the above right-hand side is holomorphic in $k$. Therefore, we only need to compute the trace of the second term to find the characteristic values of $S^{-1}(w,k)$. The latter is a sum of projections on a one-dimensional subspace.

It is easy to check [14] that if $\varphi_0 \in H^{-\frac{1}{2}}(\Gamma)$ and $\psi_0 \in \widetilde{H}^{\frac{1}{2}}(\Gamma)$, and if $P\ : \widetilde{H}^{\frac{1}{2}}(\Gamma) \longrightarrow H^{\frac{1}{2}}(\Gamma)$ is the projection operator defined by $P(\psi) = \langle \psi,\varphi_0\rangle_{\frac{1}{2},-\frac{1}{2}}\psi_0$, then

$$tr(P) \quad = \quad \langle \psi_0,\varphi_0\rangle_{\frac{1}{2},-\frac{1}{2}}.$$

As a consequence, we see that

$$\begin{aligned} & tr\Big(\frac{\mathcal{L}^{-1}(w,k)1}{\Theta(w,k)\mathcal{L}(w,k)1+1}\big(\partial_k\Theta(w,k)-\Theta_w(k)\mathcal{L}^{-1}(w,k)\partial_k\mathcal{L}(w,k)\big)\Big) \\ &= f_w^{-1}(k)\left(\partial_k\theta_w(k)\langle 1,\mathcal{L}^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}-\theta_w(k)\langle 1,\mathcal{L}^{-1}\partial_k\mathcal{L}\mathcal{L}^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}\right) \\ &= \partial_k f_w(k) f_w^{-1}(k). \end{aligned}$$

It thus follows from theorem (6.4) that the characteristic values of $S^{-1}(w,k)$ are exactly those of $f_w$. Moreover, considering the form of $S^{-1}(w,k)$, the poles (respectively the zeros) of $S^{-1}(w,k)$ are the zeros (respectively the poles) of $f_w$.

□

**Remark 4.1.** *For $w<w_0$, the poles of $f_w(k)$ in $D_{\frac{\pi}{w_0}}$ are in fact the poles of the function $\theta_w(k)=\frac{e(k)}{w}+\frac{1}{\pi}\ln(w)+\frac{1}{\pi}\ln(k)+\delta$. We note that $e(k)=-(\frac{1}{hk}+\cot(hk))\frac{2}{k}$ can be rewritten in the form:*

$$e(k)=\frac{4}{h}\sum_{n=0}^{\infty}\frac{1}{k^2-(\frac{n\pi}{h})^2}.$$

*Therefore, the poles of the function $f_w(k)$ in $D_{\frac{\pi}{w_0}}$ are precisely $\{\frac{n\pi}{h}, n\in\mathbb{Z}$ and $|n|<\frac{h}{w_0}\}$. This function originates from the asymptotics of the Green function $\mathcal{G}_i$, inside the small rectangular cavity $\Omega_i$, as its width shrinks. It represents the modes trapped in the narrow cavity, and it is not surprising that its poles are among the resonances of a flat vertical cavity of height $h$ (the asymptotic limit of the rectangular cavity). We deduce from theorem 4.2 that $\{\frac{n\pi}{h}, n\in\mathbb{Z}$ and $|n|<\frac{h}{w_0}\}$ are the characteristic values of the operator-valued function $S^{-1}(w,k)$ on $D_{\frac{\pi}{w_0}}$. Thus, for any $n\in\mathbb{Z}$ that satisfies $|n|<\frac{h}{w_0}$, there exists a source function $\zeta_n(X)$, such that $S^{-1}(w,k)\big(\mathcal{G}_e*\zeta_n\big)(X)$ vanishes on $\Gamma$ at $k=\frac{n\pi}{h}$. In view of the integral equations derived in section 3, we deduce from equations (3.2), (3.3) and (3.4), that the wave generated by the source function $\zeta_n(X)$ at the frequency $k=\frac{n\pi}{h}$ in $\Omega$ coincides with the one generated by the same source but in the half space $\mathbb{R}^2_+$ (without cavity). From a physical point of view, the small rectangular cavity becomes transparent for a source $\zeta_n(X)$ at the frequency $k=\frac{n\pi}{h}$.*

We prove later in this section that $\partial_k f_w(k)f_w^{-1}(k)$ tends to $\partial_k e(k)e^{-1}(k)$ when $w\to 0$, uniformly on every compact that does not contain any isolated zeros or poles of the function $e(k)$. Therefore, we expect that when $w$ is small, the resonances of the open cavity approach the zeros of the function $e(k)$. Based on this, we first provide a localization result for the resonances in the complex disc $D_{\frac{\pi}{w_0}}$. Then, we derive their asymptotics as $w\to 0$.
Recall that $(k_j(0))_{1\le j\le\infty}$ are the set of ordered zeros of the function $e(k)$. A simple computation shows that these zeros are simple, i.e., that their multiplicity is equal to one.

**Theorem 4.4.** *Let $k_n(0)$ be a fixed zero of the function $e(k)$, that verifies $k_n(0)<\frac{\pi}{w_0}$, and $r_n$ be a fixed positive real such that the set $\{k_j(0), j\pi : j\in\mathbb{N}^*\}\cap B(k_n(0),r_n)$is reduced to $k_n(0)$. Then, for $w<w_1<w_0$ where $w_1$ is small enough, there exists a unique resonance $k_n(w)$ in $D_{r_n}(k_n(0))$.*

*Proof.* From theorem 4.3, we know that the resonance of $\Omega$ are the zeros of the complex function $f_w(k) = \Theta_w(k)\mathcal{L}^{-1}(w,k)1+1$, while lemma 3.6 and the expression (4.7) show that the functions $\theta_w(k)$ and $\langle 1, \mathcal{L}^{-1}(w,k)1\rangle_{\frac{1}{2},-\frac{1}{2}}$ are respectively meromorphic and holomorphic on $D_{\frac{\pi}{w_0}}$ for $w < w_0$. Recalling (4.7), we see that

$$\begin{aligned}
\langle 1, \mathcal{L}^{-1}(w,k)1\rangle_{\frac{1}{2},-\frac{1}{2}} &= \langle 1, S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}} - \langle 1, S_1^{-1}S_2S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}w + o(w), &(4.11)\\
\langle 1, \partial_k\mathcal{L}(w,k)1\rangle_{\frac{1}{2},-\frac{1}{2}} &= o(w), &(4.12)
\end{aligned}$$

as $w \to 0$.
To simplify the exposition, we introduce the notation

$$\begin{cases} q_0 := \langle 1, S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}},\\ q_1 := \langle 1, S_1^{-1}S_2S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}.\end{cases}$$

The following lemma is proved in section 6:

**Lemma 4.2.** *The constant* $q_0 = \langle 1, S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}$ *is different from zero.*

Using again the explicit expression of $\theta_w(k)$, and using (4.11-4.12) yields

$$\partial_k f_w(k) f_w^{-1}(k) = \big(\partial_k e(k) + O(w)\big)\big(e(k) + O(w)\big)^{-1},$$

for $k$ far from the zeros and poles of the function $e(k)$. Therefore, the function $\frac{\partial_k f_w(k)}{f_w(k)}$ tends to $\frac{\partial_k e(k)}{e(k)}$ when $w$ goes to zero, uniformly on every compact far from the isolated zeros and poles of the function $e(k)$. Note that the zeros of the function $e(k)$ are simple and intertwined with its poles: $\frac{(n-1)\pi}{h} < k_n(0) < \frac{n\pi}{h}$ (see figure 3). Since $k_n(0)$ is a simple zero

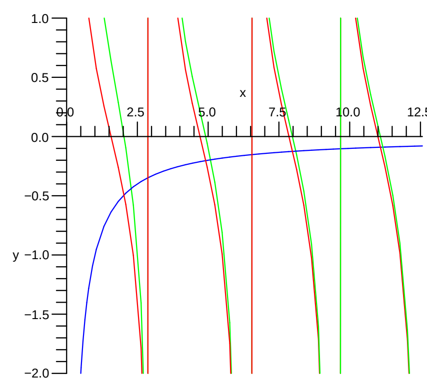


Figure 3: The zeros of the function $e(k)$ ( Green: $\cot(x) + \frac{1}{x}$, blue: $-\frac{1}{x}$ and red: $\cot(x)$ )

of the function $e(k)$, we have for $r_n > 0$ small enough

$$\frac{1}{2i\pi}\int_{|k-k_n(0)|=r_n} \frac{\partial_k e(k)}{e(k)} dk = 1.$$

Moreover, as $\lim_{w\to 0} \frac{\partial_k f_w(k)}{f_w(k)} = \frac{\partial_k e(k)}{e(k)}$ uniformly with respect to $k$ on the circle $|k-k_n(0)| = r_n$, we see that

$$\lim_{w\to 0} \frac{1}{2i\pi} \int_{|k-k_n(0)|=r_n} \frac{\partial_k f_w(k)}{f_w(k)} dk \quad = \quad 1,$$

and it follows from the Rouché theorem that $\frac{1}{2i\pi} \int_{|k-k_n(0)|=r_n} \frac{\partial_k f_w(k)}{f_w(k)} dk = 1$, for $w$ small enough. Thus, there exists a unique resonance $k_n(w)$ in the complex disc $|k-k_n(0)| < r_n$. □

We can now derive the asymptotic of the resonances as $w \to 0$

**Theorem 4.5.** *Let $k_n(0)$ be a fixed zero of the function $e(k)$, that verifies $k_n(0) < \frac{\pi}{w_0}$. Then, for $w < w_0$, the unique resonance $k_n(w)$ located near $k_n(0)$ has the following expansion*

$$\begin{aligned} k_n(w) \;&=\; k_n(0) + \frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))} \frac{k_n(0)}{2\pi h} w \ln(w) \\ &\qquad + \big(\delta + \frac{1}{\pi} \ln(k_n(0)) + \frac{1}{q_0}\big) \frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))} \frac{k_n(0)}{2h} w + o(w). \end{aligned}$$

*In particular, as $w$ goes to zero, its imaginary part behaves like*

$$Im(k_n(w)) \quad = \quad -\frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))} \frac{k_n(0)}{4h} w + o(w) \;=\; -\frac{w}{2\partial_k e(k_n(0))} + o(w). \quad (4.13)$$

*Proof.* Since $k_n(0)$ is a simple pole of the operator-valued function, The generalized Rouché theorem 6.4 yields

$$k_n(w) - k_n(0) \quad = \quad -\frac{1}{2i\pi} tr\Big[\int_{|k-k_n(0)|=r_n} (k-k_n(0)) S^{-1}(w,k) \frac{\partial S(w,k)}{\partial k} dk\Big],$$

which by (4.11) reduces to

$$k_n(w) - k_n(0) \quad = \quad -\frac{1}{2i\pi} \int_{|k-k_n(0)|=r_n} (k-k_n(0)) \frac{\partial_k f_w(k)}{f_w(k)} dk. \quad (4.14)$$

Using the asymptotics (4.11) – (4.12) and the explicit form of the function $\theta_w(k)$ we have

$$\begin{aligned} f_w(k) \quad &= \quad \frac{1}{w}\Big(e(k) q_0 + \frac{q_0}{\pi} w \ln(w) + ((\delta + \frac{1}{\pi}\ln(k)) q_0 + e(k) q_1 + 1) w + o(w)\Big). \\ \partial_k f_w(k) \quad &= \quad \frac{1}{w}\Big(\partial_k e(k) q_0 + (\partial_k e(k) q_1 + \frac{q_0}{\pi k}) w + o(w)\Big). \end{aligned}$$

We compute

$$\partial_k e(k) \quad = \quad \frac{2h}{k \sin^2(hk)} + \frac{2\cot(hk)}{k^2} + \frac{4}{hk^3}.$$

The equation $e(k_n(0)) = 0$ implies that

$$\partial_k e(k_n(0)) = \frac{1+\cos^2(hk_n(0))}{\sin^2(hk_n(0))}\frac{2h}{k_n(0)}, \tag{4.15}$$

which can also be written as

$$\partial_k e(k_n(0)) = \frac{2(h^2k_n(0)^2+2)}{hk_n(0)^3}.$$

Substituting the above asymptotics in the Cauchy integral (4.14) and using (4.15) gives the desired result.

□

In the rest of the section we study the behavior of the meromorphic function $S_w^{-1}(k)$ near to and away from its poles. To this end, we distinguish two main zones in the complex domain $D_{\frac{\pi}{w_0}}$: The resonance zone where the contribution of the singular part of the function $\frac{1}{k-k_n(w)}$ is important compared to the regular part; the non-resonance zone where the contribution of the singular part of the function $\frac{1}{k-k_n(w)}$ is negligible (it is a term of order $w$) with respect to the regular part. In each zone we derive asymptotics of the dominant terms as $w$ tends to zero.

**Theorem 4.6.** *Le $w_0 > 0$ be a fixed small constant.*
**A**) *For $w < w_0$, the operator*

$$S^{-1}(w,k) - \sum_{n=0}^{n_{w_0}} \frac{\ell_n(w)}{k-k_n(w)}, \tag{4.16}$$

*is holomorphic in $D_{\frac{\pi}{w_0}}$, where the operators $(\ell_n)_n$ are defined by*

$$\ell_n(w) := \frac{(\mathcal{L}^{-1}(w,k_n(w))1)\Theta_w(k_n(w))\mathcal{L}^{-1}(w,k_n(w))}{\partial_k\Theta_w(k_n(w))\mathcal{L}^{-1}(w,k_n(w))1 + \Theta_w(k_n(w))\partial_k\mathcal{L}^{-1}(w,k_n(w))1}.$$

*These operators have finite range and are bounded from $H^{\frac{1}{2}}(\Gamma)$ to $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$. In addition*

$$\ell_n(w) = l(w)\langle ., S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}} S_1^{-1}1 + o(w),$$

*where*

$$l(w) = \frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))}\frac{k_n(0)}{\pi^2 q_0 h} w\ln(w) + \big(2\delta + \frac{2}{\pi}\ln(k_n(0)) + \frac{1}{q_0}\big)\frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))}\frac{k_n(0)}{2\pi q_0 h} w.$$

**B**) *Let $k$ be a fixed frequency such that $e(k) \neq 0$. Then*

$$S^{-1}(w,k) = S_1^{-1} - \frac{\langle ., S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}}{\langle 1, S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}} S_1^{-1}1 + J(k)w + o(w).$$

*where $J(k)$ is given by*

$$J(k) \;:=\; -S_1^{-1}S_2S_1^{-1} + \Big(\frac{1}{q_0}S_1^{-1}S_2S_1^{-1}1 + \frac{e(k)q_1+1}{e(k)q_0^2}S_1^{-1}.1\Big)\langle ., S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}$$
$$+\frac{S_1^{-1}1}{q_0}\langle ., S_1^{-1}S_2S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}.$$

*Proof.* Part (A): Since the poles $k_n(w)$ are simple, the decomposition (4.16) of the operator-valued function $S^{-1}(w,k)$ on $D_{\frac{\pi}{w_0}}$ near its poles is a direct consequence of theorems 4.2, 4.3 and 4.4. Next, we derive asymptotics of the operator-valued function $\ell_n(w)$ as $w$ tends to zero. Considering the explicit expression of $\theta_w(k)$ and the asymptotics of the resonances derived in theorem 4.5 one can derive the following expansions

$$\theta_w(k_n(w)) \;=\; \frac{1}{w}\Big(\frac{2}{\pi}w\ln(w) + \big(2\delta + \frac{2}{\pi}\ln(k_n(0)) + \frac{1}{q_0}\big)w + o(w)\Big),$$
$$\partial_k\theta_w(k_n(w)) \;=\; \frac{1}{w}\Big(\partial_k e(k_n(0)) + \frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))}\frac{k_n(0)}{2\pi h}\partial_k^2 e(k_n(0))w\ln(w)$$
$$+\big((\delta+\frac{1}{\pi}\ln(k_n(0)) + \frac{1}{q_0})\frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))}\frac{k_n(0)}{2h}\partial_k^2 e(k_n(0)) + \frac{1}{\pi k_n(0)}\big)w + o(w)\Big).$$

We remind that the asymptotic of the operator-valued functions $\mathcal{L}^{-1}(w,k)$ and $\mathcal{L}(w,k)$ as $w$ tends to zero, are uniform with respect to $k$ on $D_{\frac{\pi}{w_0}}$. A straightforward computation based on the equalities (4.6), (4.7), (4.11) and the last asymptotics gives the result.
Part (B): The frequency $k$ satisfies $e(k) \neq 0$. Hence one can divide by $e(k)$ to directly derive the desired result, in view of the asymptotics (4.6), (4.7), (4.11) and the explicit expression of $\theta_w(k)$.

□

**Remark 4.2.** *We note that when $k$ is fixed away from the resonances (i.e. $e(k) \neq 0$) the function $S^{-1}(w,k)1$ has the following asymptotic: $\frac{w}{e(k)q_0}S_1^{-1}1 + o(w)$.*

## 4.3 Proof of theorems 2.2 and 2.3

### 4.3.1 Far from the resonance zone

Let $k$ be a fixed frequency satisfying the assumptions of part (B), of the two theorems: $k < \frac{\pi}{w_0}$ and $e(k) \neq 0$. We fix $Y$ in $\Omega_e = \mathbb{R}^2_+$. Classical results on the Hankel function [1] show that the series

$$\mathcal{G}_e(Y, wx, 0) = -\frac{i}{2}H_0^{(1)}(k|Y|) + \sum_{n=1}^{\infty} h_n(Y)(wx)^n,$$

is convergent uniformly for $x \in [-\frac{1}{2}, \frac{1}{2}]$. The invertibility of $S(w,k)$ and 3.6 yield

$$\partial_{x_2}\mathcal{G}(w,k;wx,0,Y) \;=\; \frac{1}{w}S^{-1}(w,k)\mathcal{G}_e(Y,w.,0),$$

and part (B) of theorem 4.6 implies

$$\begin{aligned}
\partial_{x_2}\mathcal{G}(wx,0,Y) &= -\frac{i}{2}H_0^{(1)}(k|Y|)\frac{1}{e(k)q_0}S_1^{-1}1 \\
&\quad + h_1(Y)S_1^{-1}x - \frac{1}{q_0}\langle x, S_1^{-1}1\rangle_{\frac{1}{2},-\frac{1}{2}}S_1^{-1}1 + o(1) \\
&=: \Psi(x,Y) + o(1).
\end{aligned}$$

We now discuss according to the position $Z$ of the observer: Assume firstly that $Z$ is fixed in $\Omega_e = \mathbb{R}^2_+$. Substituting the asymptotic of $\partial_{x_2}\mathcal{G}(w,k;wx,0,Y)$ in (3.2), we obtain

$$\mathcal{G}(w,k;Z,Y) = \mathcal{G}_e(Y,Z) - \frac{1}{4e(k)}H_0^{(1)}(k|Y|)H_0^{(1)}(k|Z|)w + o(w).$$

which proves (2.10)
Secondly, if $Z = w\overline{Z}$ with $\overline{Z}$ fixed in $\Omega_e$, we see from (4.1) that

$$\mathcal{G}_e(wx,0,w\overline{Z}) = \frac{1}{\pi}\ln(w) + \Phi(x,\overline{Z}) + o(w),$$

where

$$\Phi(x,\overline{Z}) = \frac{1}{\pi}\ln(k) - \Gamma_1 + \frac{1}{2\pi}\ln\left(|(x,0)-\overline{Z}||(x,0)-\overline{Z}|\right).$$

Substituting the asymptotics of $\partial_{x_2}\mathcal{G}(wx,0,Y)$ in (3.2), we find that

$$\mathcal{G}(w,k;Z,Y) = \mathcal{G}_e(Y,Z) + \langle 1, \Psi(.,Y)\rangle_{\frac{1}{2},-\frac{1}{2}}w\ln(w) + \langle \Phi(.,\overline{Z}), \Psi(.,Y)\rangle_{\frac{1}{2},-\frac{1}{2}}w + o(w),$$

so that letting

$$\begin{aligned}
Q_1(Y) &:= \langle 1, \Psi(.,Y)\rangle_{\frac{1}{2},-\frac{1}{2}}, && (4.17) \\
Q_2(Y,Z) &:= \langle \Phi(.,Z), \Psi(.,Y)\rangle_{\frac{1}{2},-\frac{1}{2}} && (4.18)
\end{aligned}$$

we obtain the desired result.

### 4.3.2 Within the resonance zone

Throughout this section we assume that $k$ lies in a small neighborhood of one of the values $k_n(0), n = 1,\dots,n_{w_0}$, but does not coincide with any of the resonances $k_n(w), n = 1,\dots,n_{w_0}$. Since $S(w,k)$ is invertible (see theorem 4.2) around its poles, the equation (3.6) implies

$$\partial_{x_2}\mathcal{G}(w,k;wx,0,Y) = \frac{1}{w}S^{-1}(w,k)\mathcal{G}_e(Y,w.,0).$$

The asymptotics of $S^{-1}(w,k)$ given in part (A) of theorem 4.6 yield

$$\partial_{x_2}\mathcal{G}(w,k;wx,0,Y) = \frac{1}{w}\Big(\sum_{n=0}^{n_0}\frac{\xi_n\widetilde{g}_n(w,Y)\mathcal{L}^{-1}(w,k_n(w))1}{k-k_n(w)} + H(k,w)\Big),$$

where $H(k,w)$ denotes a holomorphic function of $k$, where

$$\widetilde{g}_n(w,Y) \quad := \quad \langle -\frac{i}{2}H_0^{(1)}(k_n(w)|Y-(w.,0)|), \mathcal{L}^{-1}(w,k_n(w))1\rangle_{\frac{1}{2},-\frac{1}{2}}, \tag{4.19}$$

and where

$$\xi_n(w) \quad := \quad \frac{\theta_w(k_n(w))}{\partial_k\Theta_w(k_n(w))\mathcal{L}^{-1}(w,k_n(w))1+\Theta_w(k_n(w))\partial_k\mathcal{L}^{-1}(w,k_n(w))1}.$$

Define

$$g_n(w,Y) \quad = \quad \sqrt{\xi_n(w)}\widetilde{g}_n(w,Y). \tag{4.20}$$

Substituting the asymptotic of $partial_{x_2}\mathcal{G}(w,k;wx,0,Y)$ in the equation (3.2), we find that

$$\mathcal{G}(w,k;Z,Y) \quad = \quad \mathcal{G}_e(Y,Z)+\sum_{n=0}^{n_{w_0}}\frac{g_n(w,Y)g_n(w,Z)}{k-k_n(w)}+H(k,w),$$

where $H$ denotes a holomorphic function of $k$. Following the steps of the proof of theorem 4.6, we see that

$$\begin{aligned} \xi_n(w) \quad &= \quad \xi_{n,1}w\ln(w)+\xi_{n,2}w+o(w), \\ \widetilde{g}_n(w,Y) \quad &= \quad -\frac{i}{2}H_0^{(1)}(k_n(0)|Y|)q_0+g_{n,1}(Y)w\ln(w)+g_{n,2}(Y)w+o(w), \end{aligned} \tag{4.21}$$

where

$$\begin{aligned} \xi_{n,1} \quad &= \quad \frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))}\frac{k_n(0)}{\pi^2q_0h}, \\ \xi_{n,2} \quad &= \quad \big(2\delta+\frac{2}{\pi}\ln(k_n(0))+\frac{1}{q_0}\big)\frac{\sin^2(hk_n(0))}{1+\cos^2(hk_n(0))}, \end{aligned}$$

and where $g_{n,1}(Y)$ and $g_{n,2}(Y)$ are functions of $h$ only.

To prove (2.8), we assume that $Z$ is fixed in $\mathbb{R}^2_+$, $Z\neq Y$. Then, (4.19) and (4.21) show that

$$g_n(w,Y)g_n(w,Z) \quad = \quad -\xi_{n1}\frac{q_0^2}{4}H_0^{(1)}(k_n(0)|Y|)H_0^{(1)}(k_n(0)|Z|)w\ln(w)+o(w). \tag{4.22}$$

To prove (2.11), we assume that $Z=w\overline{Z}$, with $\overline{Z}$ fixed in $\mathbb{R}^2_+$. Then the expression of $\widetilde{g}_n$ can be expanded as

$$\widetilde{g}_n(w,Z) \quad = \quad \frac{q_0}{\pi}\ln(w)+\overline{g}_{n1}(\overline{Z})+\overline{g}_{n2}(\overline{Z})w\ln(w)+\overline{g}_{n2}(\overline{Z})w+o(w),$$

where the functions $\overline{g}_{n1}(\overline{Z})$ and $\overline{g}_{n2}(\overline{Z})$ only depend on $h$. We can then write

$$g_n(w,Y)g_n(w,Z) \quad = \quad -\xi_{n1}\frac{iq_0^2}{2\pi}H_0^{(1)}(k_n(0)|Y|)w\ln^2(w)+o(w\ln^2(w)). \tag{4.23}$$

### 4.3.3 Proof of (2.9)

As $|y| \to \infty$, the Hankel function $H_0^{(1)}(y)$ behaves like [1]

$$H_0^{(1)}(y) \;=\; \sqrt{\frac{2}{\pi y}} e^{-i(y-\frac{\pi}{4})}\big(1+O(\frac{1}{y})\big).$$

Therefore, we see that

$$H_0^{(1)}(k_n(w)|Y-(w.,0)| \sim \sqrt{\frac{2}{\pi k_n(w)|Y|}} e^{-i(k_n(w)|Y|-\frac{\pi}{4})} \quad \text{as} \quad |Y| \to +\infty.$$

Substituting the above asymptotic in (4.19) yields the desired expansion (2.9).

# 5 Proofs of theorem 4.1 and lemma 4.2

## 5.1 Proof of theorem (4.1).

Theorem (4.1) contains two independent parts that are treated separately.

### 5.1.1 Invertibility of $S_1$

The goal of this section is to show that the operator $S_1$ is invertible from $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$, i.e., that, for a fixed function in $H^{\frac{1}{2}}(\Gamma)$, there exists a unique solution $\phi$ in $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to the integral equation: $S_1\phi = f$. Some notation is due first.

- $\widehat{\Omega} = \widehat{\Omega}_E \cup \widehat{\Omega}_I$, where $\widehat{\Omega}_E = \mathbb{R}^2_+$ and $\widehat{\Omega}_I = (-1,1) \times \mathbb{R}_-$ ; $\widehat{\Omega}_\varepsilon = (-1,1) \times (0,-\varepsilon)$, see Fig 4.
- $\widehat{\Gamma} = (-1,1) \times \{0\}$ and $\widehat{\Gamma}_\varepsilon = (-1,1) \times \{-\varepsilon\}$.
- $u^{\pm}(X) = \lim_{t\to\pm 0} u(X+(t,0))$ for $X \in \widehat{\Gamma}$, and $u^{\mp}(X) = \lim_{t\to\pm 0} u(X+(t,0))$ for $X \in \widehat{\Gamma}_\varepsilon$.
- $[u]_{\widehat{\Gamma}}$ denotes the difference $u^+(X) - u^-(X)$ for $X \in \widehat{\Gamma}$.

On $(-1,1) \times (-1,1)$, we consider the kernel

$$\hat{s}_1(x,z) \;=\; \frac{1}{\pi}\ln(|x-z|) + \frac{1}{\pi}\ln\big(\sin(\frac{\pi}{4}(x-z))\sin(\frac{\pi}{4}(x+z+2))|\big),$$

and the associated integral operator

$$\widehat{S}_1\phi(x) \;=\; \int_{-1}^{1} \hat{s}_1(x,z)\phi(z)dz, \text{ for } x \in (-1,1) \text{and } \phi \in \widetilde{H}^{-\frac{1}{2}}(\Gamma).$$

By a simple change of variables one can easily prove that the invertibility of $S_1$, from $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{1}{2}}(\Gamma)$, is equivalent to that of $\widehat{S}_1$ from $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ to $H^{\frac{1}{2}}(\widehat{\Gamma})$. The latter is closely

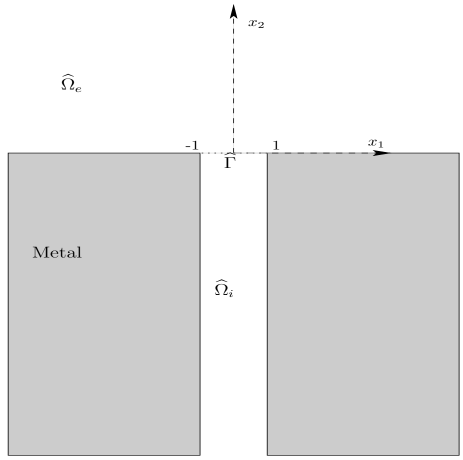


Figure 4: The domain $\widehat{\Omega}$

related to the well-posedness of the following transmission problem

$$
(P_1): \begin{cases}
\Delta u(X) = 0, \quad \text{in} \quad \widehat{\Omega}, \\
\partial_n u(X) = 0, \quad \text{on} \quad \partial\widehat{\Omega}, \\
\int_{\widehat{\Gamma}} u^-(X) ds_X = 0, \ [u]_{\widehat{\Gamma}} = f(X), \quad [\partial_{x_2} u]_{\widehat{\Gamma}} = 0, \\
u(X) - x_2 \int_{\widehat{\Gamma}} \partial_{x_2} u(X) ds_X = o(1), \quad \text{as} \quad x_2 \to -\infty, \\
|\nabla(u(X) - x_2 \int_{\widehat{\Gamma}} \partial_{x_2} u(X) ds_X)| = o(1) \quad \text{as} \quad x_2 \to -\infty, \\
u(X) - \frac{1}{\pi} \ln(|X|) \int_{\widehat{\Gamma}} \partial_{x_2} u(X) ds_X = O(\frac{1}{|X|}), \quad \text{as} \quad |X| \to +\infty, \quad x_2 > 0, \\
\nabla u(X). \frac{X}{|X|} - \frac{1}{\pi|X|} \int_{\widehat{\Gamma}} \partial_{x_2} u(X) ds_X = O(\frac{1}{|X|^2}), \quad \text{as} \quad |X| \to +\infty, \quad x_2 > 0.
\end{cases} \tag{5.1}
$$

Instead of working on the unbounded domain $\widehat{\Omega}$, we transform $P_1$ into a problem set on $\widehat{\Omega}_\varepsilon = (-1, 1) \times (0, -\varepsilon)$, with two integral transmission boundary conditions on $\widehat{\Gamma}$ and $\widehat{\Gamma}_\varepsilon = (-1, 1) \times \{-\varepsilon\}$. To this end, we consider the Green function of the Laplace equation in $\widehat{\Omega}_E$

$$
\begin{cases}
\Delta \widehat{G}_E(X, Y) = -\delta_Y(X), \quad \text{in} \quad \widehat{\Omega}_E, \\
\partial_n \widehat{G}_E(X, Y) = 0, \quad \text{on} \quad \partial\widehat{\Omega}_E, \\
\widehat{G}_E(X, Y) + \frac{1}{\pi} \ln(|X|) = O(\frac{1}{|X|}), \qquad \text{as} \quad |X| \to +\infty, \\
\nabla_X \widehat{G}_E(X, Y). \frac{X}{|X|} + \frac{1}{\pi|X|} = O(\frac{1}{|X|^2}), \quad \text{as} \quad |X| \to +\infty.
\end{cases}
$$

The method of images shows that $\widehat{G}_E(X,Y) = -\frac{1}{2\pi}\ln(|X-Y|) - \frac{1}{2\pi}\ln(|X-\widetilde{Y}|)$, where $\widetilde{Y} = (y_1, -y_2)$. In the half slab $\widehat{\Omega}_I$, the Green function is defined by

$$\begin{cases} \Delta \widehat{G}_I(X,Y) = -\delta_Y(X), \quad \text{in} \quad \widehat{\Omega}_I, \\ \partial_n \widehat{G}_I(X,Y) = 0, \quad \text{on} \quad \partial\widehat{\Omega}_I, \\ \int_{\widehat{\Gamma}} \widehat{G}_I(X,Y) ds_X = 0, \\ \widehat{G}_I(X,Y) = o(1), \ |\nabla_X \widehat{G}_I(X,Y)| = o(1) \quad \text{as} \quad x_2 \to -\infty, \end{cases}$$

and can be represented as the series

$$\widehat{G}_I(X,Y) \;=\; \sum_{m=1}^{+\infty} \frac{1}{m\pi}\left(e^{-\frac{m\pi}{2}|x_2+y_2|} + e^{-\frac{m\pi}{2}|x_2-y_2|}\right)\cos\left(\frac{m\pi}{2}(x_1+1)\right)\cos\left(\frac{m\pi}{2}(y_1+1)\right).$$

Given the structure of the above Green's functions, classical results in potential theory yield

**Lemma 5.1.** *The single layer potentials with kernels $\widehat{G}_E$ and $\widehat{G}_I$ satisfy the following jump conditions*

We define two integral operators on $\widehat{\Gamma}$ and $\widehat{\Gamma}_\varepsilon$

$$\Lambda : \ \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}) \longrightarrow H^{\frac{1}{2}}(\widehat{\Gamma}),$$

$$\Lambda\phi(X) := -\int_{\widehat{\Gamma}} \widehat{G}_E(Z,X)\phi(Z) ds_Z,$$

and

$$\Lambda_\varepsilon : \ \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}_\varepsilon) \longrightarrow H^{\frac{1}{2}}(\widehat{\Gamma}_\varepsilon),$$

$$\Lambda_\varepsilon\phi(X) := \int_{\widehat{\Gamma}_\varepsilon} \widehat{G}_I(Z+(0,\varepsilon), X+(0,\varepsilon))\phi(Z) ds_Z.$$

We emphasize that $\Lambda_\varepsilon$ does not depend on $\varepsilon$, and that

$$\Lambda_\varepsilon 1(X) \;=\; \int_{\widehat{\Gamma}} \widehat{G}_I(X,Y) ds_X \;=\; 0.$$

Our goal is to reduce the study of $P_1$ to that of

$$(P_2) : \begin{cases} \Delta u(X) = 0, \quad \text{in} \quad \widehat{\Omega}_\varepsilon, \\ \int_{\widehat{\Gamma}} u(X) ds_X = 0, \\ \partial_n u(X) = 0 \quad \text{on} \quad x_1 = \pm 1, \\ \Lambda_\varepsilon(\partial_{x_2} u)(X) + \varepsilon \int_{\widehat{\Gamma}} \partial_{x_2} u(X) ds_X = u(X) \quad \text{on} \quad \widehat{\Gamma}_\varepsilon, \\ \Lambda(\partial_{x_2} u)(X) = u(X) + f(X) \quad \text{on} \quad \widehat{\Gamma}. \end{cases}$$

**Remark 5.1.** *The functions $\partial_{x_2} u(X)|_{\widehat{\Gamma}_\varepsilon}$ and $\partial_{x_2} u(X)|_{\widehat{\Gamma}}$ naturally lie in the spaces $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}_\varepsilon)$ and $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ respectively. This follows from the trace theorem and the Green formula. The functions $\Lambda_\varepsilon(\partial_{x_2} u)(X)$ and $\Lambda(\partial_{x_2} u)(X)$ are thus well-defined on $\widehat{\Gamma}_\varepsilon$ and $\widehat{\Gamma}$ respectively.*

### 5.1.2 Equivalence between $P_1$ and the inversion of $\widehat{S}_1$

**Lemma 5.2.** *The following propositions are equivalent:*

*(H1)* $\widehat{S}_1$ *is invertible from* $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ *to* $H^{\frac{1}{2}}(\widehat{\Gamma})$,

*(H2) for any* $f \in H^{\frac{1}{2}}(\widehat{\Gamma})$*, there exists a unique solution to* $P_1$.

*Proof.* We first prove that (H1) implies (H2). Let $f$ be a function in $H^{\frac{1}{2}}(\widehat{\Gamma})$, and let $\phi_f(X)$ in $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ be the unique solution to $\widehat{S}_1\phi_f(X) = -f(X)$. We define $u_f$ on $\widehat{\Omega}$ by

$$u_f(X) \quad = \quad \begin{cases} -\int_{\widehat{\Gamma}} \widehat{G}_E(X,Z)\phi_f(Z)ds_Z, & \text{for } X \in \widehat{\Omega}_E, \\ x_2 \int_{\widehat{\Gamma}} \phi_f(X)ds_X + \int_{\widehat{\Gamma}} \widehat{G}_I(X,Z)\phi_f(Z)ds_Z, & \text{for } X \in \widehat{\Omega}_I. \end{cases} \tag{5.2}$$

We note that the kernel $\hat{s}_1$ of $\widehat{S}_1$

$$\hat{s}_1(x,z) \quad = \quad \frac{1}{\pi}\ln(|x-z|) + \frac{1}{\pi}\ln\left(|\sin(\frac{\pi}{4}(x-z))\sin(\frac{\pi}{4}(x+z+2))|\right),$$

coincides with the function $-(\widehat{G}_E(x,0,z,0)+\widehat{G}_I(x,0,z,0))$. Due to the logarithmic singularity of the kernels $-\widehat{G}_E$ and $-\widehat{G}_I$, and due to their behavior at infinity (see for instance [21]) the function $u_f(X)$ is a solution to $P_1$. Next, we prove that it is the only solution.

Let $v_f$ be a solution to $P_1$, with $[v_f] = f$ on $\widehat{\Gamma}$. Applying the Green formula in $\widehat{\Omega}$, we obtain:

$$v_f(X) \quad = \quad -\int_{\widehat{\Gamma}} \widehat{G}_E(X,Z)\partial_{x_2}v_f(Z)ds_Z, \quad \forall\, X \in \widehat{\Omega}_E. \tag{5.3}$$

and

$$v_f(X) \quad = \quad x_2\int_{\widehat{\Gamma}} \partial_{x_2}v_f(X)ds_X + \int_{\widehat{\Gamma}} \widehat{G}_I(X,Z)\partial_{x_2}v_f(Z)ds_Z, \quad \forall\, X \in \widehat{\Omega}_I. \tag{5.4}$$

We justify later (see (5.7) and (5.8)) why we can use the Green formula in $\widehat{\Omega}_E$ and in $\widehat{\Omega}_I$, based on the radiation conditions satisfied by both the Green functions and the solutions to $P_1$.

Taking the trace of $v_f(X)$ on both sides of the boundary $\widehat{\Gamma}$, we obtain for $X = (x,0) \in \widehat{\Gamma}$

$$\begin{aligned} f(X) \quad &= \quad -\int_{-\frac{1}{2}}^{\frac{1}{2}} (\widehat{G}_E(x,0,z,0) + \widehat{G}_I(x,0,z,0))\partial_{x_2}v_f(z)dz \\ &= \quad \widehat{S}_1(\partial_{x_2}v_f)(X). \end{aligned}$$

We deduce from $(H1)$ that $\partial_{x_2}v_f(z) = \phi_f(z)$. In view of the integral representations (5.3) and (5.4), we conclude that $u_f \equiv v_f$ in $\widehat{\Omega}$, which proves uniqueness for $P_1$.

We now assume that (H2) holds. Using again the integral equations (5.3) and (5.4), we see that the equation

$$\widehat{S}_1\phi(X) \quad = \quad -f(X) \tag{5.5}$$

has at least the solution $\phi(z) = \partial_{x_2}u_f(z)$. Let $\phi_0$ be a function in the kernel of $\widehat{S}_1$ and, with the datum $\phi_0$, construct $u_0$ by (5.2). This function is a solution to $P_1$ with $f = 0$, and hence it follows from (H2) that $u_0 \equiv 0$. We conclude from the usual logarithmic jump relations that $\phi_0 = \partial_n u_0 \equiv 0$, that solutions to (5.5) are unique, and that $\widehat{S}_1$ is invertible.

□

#### 5.1.3 $P_1$ and $P_2$ are equivalent

**Lemma 5.3.** *$P_1$ has a unique solution iff $P_2$ has a unique solution.*

*Proof.* Let $u_f$ be a solution to $P_1$. Applying the Green formula in $(-1,1)\times(-\varepsilon,-\infty)$ (see its justification (5.8)) we find that

$$u_f(X) \;=\; x_2\int_{\widehat{\Gamma}} \partial_{x_2}u_f(Z)ds_Z + \int_{\widehat{\Gamma}_\varepsilon} \widehat{G}_I(Z+(0,\varepsilon),X+(0,\varepsilon))\partial_{x_2}u_f(Z)ds_Z, \quad (5.6)$$

for all $X\in(-1,1)\times(-\varepsilon,-\infty)$. Taking the trace of the integral equations (5.3) and (5.4) on $\widehat{\Gamma}$ and $\widehat{\Gamma}_\varepsilon$ respectively, we find that $\Lambda_\varepsilon(\partial_{x_2}u_f)(X)+\varepsilon\int_{\widehat{\Gamma}}\partial_{x_2}u_f(X)ds_X = u_f^+(X)$ on $\widehat{\Gamma}_\varepsilon$ and $\Lambda(\partial_{x_2}u_f)(X)=u_f^-(X)+f(X)$ on $\widehat{\Gamma}$. Thus $u_f$ is also a solution to $P_2$.

Now, let $u(X)$ be a solution of $P_2$. A direct consequence of the Green formula, and particularly equations (5.2) and (5.6), is that $u(X)$ can be extended to $\widehat{\Omega}$ as a solution to $P_1$. We claim that this extension is unique: Assume that $u_1(X)$ and $u_2(X)$ are two solutions to $P_1$, that coincide on $\widehat{\Omega}_\varepsilon$. Let $v(X):=u_1(X)-u_2(X)$, which satisfies

$$\begin{cases} \Delta v(X) = 0, & \text{in } \widehat{\Omega}\setminus\overline{\widehat{\Omega}_\varepsilon}, \\ \partial_n v(X) = 0, & \text{on } \partial\widehat{\Omega}_E\cup\{x_1=\pm1\}\times(-\varepsilon,-\infty), \\ v(X) = 0, & \text{on } \widehat{\Gamma}\cup\widehat{\Gamma}_\varepsilon, \\ v(X) = o(1), & |\nabla v(X)| = o(1), \qquad \text{as } x_2\to-\infty, \\ v(X) = O(\frac{1}{|X|}), & \nabla v(X).\frac{X}{|X|} = O(\frac{1}{|X|^2}), \qquad \text{as } |X|\to+\infty. \end{cases}$$

Let $B_R^+$ be the upper half disc of radius $R$ and center $(0,0)$ in $\widehat{\Omega}_E$, and let $S_R^+=\partial B_R^+\cap\widehat{\Omega}_E$. We multiply $\Delta v$ by the complex conjugate of $v(X)$ and integrate by parts over $B_R^+$ to obtain

$$\int_{B_R^+}|\nabla v|^2(X)dX = \int_{S_R^+}\nabla v(X).\frac{X}{|X|}\overline{v}(X)ds_X = O(\frac{1}{R^2}), \quad \text{as} \quad R\to+\infty. \quad (5.7)$$

Therefore $v(X)$ is constant on $\widehat{\Omega}_E$. Since $v_{/\widehat{\Gamma}}$, we deduce that $v(X)=0$ in $\widehat{\Omega}_E$.

Let $A$ be a positive constant larger that $\varepsilon$. We again multiply $\Delta v(X)$ by the complex conjugate of the solution $v(X)$, and integrate by parts over $(-1,1)\times(-\varepsilon,-A)$ to obtain

$$\int_{(-1,1)\times(-\varepsilon,-A)}|\nabla v|^2(X)dX = -\int_{\widehat{\Gamma}_A}\partial_{x_2}v(x,-A)\overline{v}(x,-1)dx = o(1), \text{ as } A\to+\infty. \quad (5.8)$$

Consequently $v(X)$ is constant on $(-1,1)\times(-\varepsilon,-\infty)$. Since $v(X)$ vanishes on $\widehat{\Gamma}_\varepsilon$, we deduce that $v(X)\equiv 0$ on $(-1,1)\times(-\varepsilon,-\infty)$, which proves uniqueness.

Finally, we note that our use of the Green formula above can be justified by integrating on truncated domains as in (5.7) and (5.8), given the the radiation conditions satisfied by solutions to $P_1$ and the far-field behavior of $\widehat{G}_E$ and $\widehat{G}_I$.

$\square$

### 5.1.4 Some properties of $\Lambda_\varepsilon$ and $\Lambda$.

Here, we establish some useful properties of the integral operators $\Lambda_\varepsilon$ and $\Lambda$. We consider the dual product $\langle .,. \rangle_{\frac{1}{2},-\frac{1}{2}}$ as a sesquilinear form on the complex space $H^{\frac{1}{2}}(\widehat{\Gamma}) \times \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$. Define

$$\widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma}) := \{\phi(X) \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}) : \int_{\widehat{\Gamma}} \phi(X) ds_X = 0\},$$

and

$$H_0^{\frac{1}{2}}(\widehat{\Gamma}) := \{\psi(X) \in H^{\frac{1}{2}}(\widehat{\Gamma}) : \int_{\widehat{\Gamma}} \psi(X) ds_X = 0\}.$$

**Lemma 5.4.** *The operator $\Lambda_\varepsilon$ is invertible from $\widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma}_\varepsilon)$ to $H_0^{\frac{1}{2}}(\widehat{\Gamma}_\varepsilon)$. In addition, the following inequality holds*

$$Re\big(\langle \Lambda_\varepsilon \phi, \phi \rangle_{\frac{1}{2},-\frac{1}{2}}\big) \geq 0 \qquad \text{for all } \phi \in \widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma}_\varepsilon). \tag{5.9}$$

*Proof.* From the expression of the Green function in $(-1,1) \times (-\infty, 0)$, it follows that the kernel of the operator $\Lambda_\varepsilon$ is given by

$$\widehat{G}_I(z,0,x,0) \;=\; \sum_{m=1}^{+\infty} \frac{2}{m\pi} \cos(\frac{m\pi}{2}(x_1+1)) \cos(\frac{m\pi}{2}(z_1+1)).$$

We claim that $\Lambda_\varepsilon$ is invertible on $\widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma}_\varepsilon)$, and that for a function

$$\psi(X) \;=\; \sum_{m=1}^{\infty} \psi_m \cos(\frac{m\pi}{2}(x_1+1)) \in H^{\frac{1}{2}}(\widehat{\Gamma}_\varepsilon),$$

we have

$$\Lambda_\varepsilon^{-1}\psi(X) \;=\; \sum_{m=1}^{\infty} \frac{m\pi}{2} \psi_m \cos(\frac{m\pi}{2}(x_1+1)). \tag{5.10}$$

Using the relation $\sum_{m=1}^{+\infty} \frac{\cos(2m\theta)}{m} = -\ln|2\sin(\theta)|$, we can express $\widehat{G}_I$ as

$$\widehat{G}_I(z,0,x,0) = -\frac{1}{\pi} \ln \big(4 \sin(\frac{\pi}{2}(x-z)) \sin(\frac{\pi}{2}(x+z+1))|.$$

We may consider $\Lambda_\varepsilon$ as an operator acting on the space $H^{-\frac{1}{2}}(\mathbb{R})$. To this end, we split its kernel into singular and smooth parts. We denote $\Lambda_\varepsilon^i, i = 1,2,3,4$ the integral operators respectively associated to the kernels: $-\frac{1}{\pi}\ln\big(|x-z|\big)$, $-\frac{1}{\pi}\ln\big(|x+z+1|\big)$, $-\frac{1}{\pi}\ln\big(|x+z-1|\big)$ and $-\frac{1}{\pi}\ln\big(\frac{4\sin(\frac{\pi}{2}(x-z))\sin(\frac{\pi}{2}(x+z+1))|}{|(x-z)||(x+z)^2-1|}\big)$.
Let $\tau$ be the isomorphism defined on $\widetilde{H}^{-\frac{1}{2}}(\Gamma_\varepsilon)$ by $\tau\phi(X) = \phi(-X)$. The symbols $\sigma_i(\xi), i = 1,2,3$ of $\Lambda_\varepsilon^i \tau, i = 2,3$, can be easily computed and one finds that

$$\sigma_1(\xi) = \frac{1}{|\xi|} \quad \sigma_2(\xi) = -\frac{e^{-i\xi}}{|\xi|} \quad \sigma_3(\xi) = -\frac{e^{i\xi}}{|\xi|}.$$

Classical pseudo-differential operator theory (see for instance [24]) shows that the operators $\Lambda_\varepsilon^i \tau, i = 1, 2, 3$, belong to the class $S^1_{1,0}$. They are thus bounded from $\widetilde{H}^{-\frac{1}{2}}(\Gamma_\varepsilon)$ to $H^{\frac{1}{2}}(\Gamma_\varepsilon)$. Since $\Lambda_\varepsilon^4$, has an analytic kernel, it follows that $\Lambda_\varepsilon = \sum_{i=1}^4 \Lambda_\varepsilon^i$ is also bounded from $\tilde{H}^{-\frac{1}{2}}(\Gamma_\varepsilon)$ to $H^{\frac{1}{2}}(\Gamma_\varepsilon)$.

In the rest of the paragraph, we prove the coercivity of $\Lambda_\varepsilon$. Let $\widetilde{\phi}(X) \in \mathcal{D}(\widehat{\Gamma})$, such that $\langle 1, \widetilde{\phi}\rangle_{\frac{1}{2},-\frac{1}{2}} = 0$. We can write $\widetilde{\phi}$ as the Fourier series

$$\widetilde{\phi}(X) \quad = \quad \sum_{m=1}^{\infty} \widetilde{\phi}_m \cos(\frac{m\pi}{2}(x_1+1)),$$

for $X \in \widehat{\Gamma}_\varepsilon$. We may also consider the smooth, compactly supported $\widetilde{\phi}$ as a function defined on the whole of $\mathbb{R}$ and derive its Fourier transform

$$\mathcal{F}(\widetilde{\phi})(\xi) \quad = \quad \sum_{m=1}^{\infty} \widetilde{\phi}_m \alpha_m(\xi),$$

where

$$2\pi\alpha_m(\xi) \quad := \quad e^{i\frac{(m+1)\pi}{2}} \frac{\sin(\frac{m\pi}{2}-\xi)}{\frac{m\pi}{2}-\xi} + e^{-i\frac{(m+1)\pi}{2}} \frac{\sin(\frac{m\pi}{2}+\xi)}{\frac{m\pi}{2}+\xi}.$$

Consequently, the norm of the function $\widetilde{\phi}(X)$ in the space $H^{-\frac{1}{2}}(\mathbb{R})$ can be rewritten in terms of its Fourier coefficients as

$$\|\widetilde{\phi}\|^2_{-\frac{1}{2}} \quad = \quad \int_{\mathbb{R}} (1+|\xi|^2)^{-\frac{1}{2}} |\sum_{m=1}^{\infty} \widetilde{\phi}_m \alpha_m(\xi)|^2 d\xi \leq \sum_{m=1}^{\infty} |\widetilde{\phi}_m|^2 \int_{\mathbb{R}} \frac{|\alpha_m(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi.$$

A straightforward computation shows that $\int_{\mathbb{R}} \frac{|\alpha_m(\xi)|^2}{(1+|\xi|^2)^{\frac{1}{2}}} d\xi \leq \frac{\frac{2^5}{\pi^2}}{m}$, and therefore we can estimate

$$\|\widetilde{\phi}\|^2_{-\frac{1}{2}} \leq \frac{2^5}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m} |\widetilde{\phi}_m|^2. \tag{5.11}$$

On the other hand, the explicit expressions of $\widehat{G}_I$ and $\Lambda_\varepsilon$ yield

$$\Lambda_\varepsilon \widetilde{\phi}(X) = \sum_{m=1}^{\infty} \frac{2}{m\pi} \widetilde{\phi}_m \cos(\frac{m\pi}{2}(x_1+1)).$$

Hence

$$\langle \Lambda_\varepsilon \widetilde{\phi}, \widetilde{\phi} \rangle_{\frac{1}{2},-\frac{1}{2}} \quad = \quad \sum_{m=1}^{\infty} \frac{1}{m} |\widetilde{\phi}_m|^2,$$

and we deduce from (5.11) that

$$\frac{\pi^2}{2^5} \|\widetilde{\phi}\|^2_{-\frac{1}{2}} \leq \langle \Lambda_\varepsilon \widetilde{\phi}, \widetilde{\phi} \rangle_{\frac{1}{2},-\frac{1}{2}}.$$

Invoking the density of $\mathcal{D}(\widehat{\Gamma})$ in $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}_\varepsilon)$, we conclude that $\Lambda_\varepsilon$ is coercive and so invertible on $\widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma})$. Finally, since $\Lambda_\varepsilon 1 = 0$, $\langle \Lambda_\varepsilon \widetilde{\phi}, \widetilde{\phi}\rangle_{\frac{1}{2},-\frac{1}{2}} \geq 0$ for any $\widetilde{\phi} \in \mathcal{D}(\widehat{\Gamma})$ and by density for any $\widetilde{\phi} \in \widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma})$.

□

**Lemma 5.5.** *The operator $\Lambda$ has a bounded inverse from $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ to $H^{\frac{1}{2}}(\widehat{\Gamma})$. In addition, the following inequality holds:*

$$Re\big(\langle \Lambda\phi, \phi\rangle_{\frac{1}{2},-\frac{1}{2}}\big) \leq 0, \qquad \textit{for all } \phi \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}).$$

*Proof.* We first consider the operator $\Lambda$ as a pseudo-differential operator acting on $H^{-\frac{1}{2}}(\mathbb{R})$, with kernel $\frac{1}{\pi}\ln(|x-z|)$. Its symbol is easily calculated to be $\frac{1}{|\xi|}$. It follows that $\Lambda$ belongs to the class $S^{-1}_{1,0}$ of pseudo-differential operators, which implies that it is bounded from $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ to $H^{\frac{1}{2}}(\widehat{\Gamma})$ ([24], page 2). In addition, a Gårding type inequality holds in $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$: There exists a compact operator $C : \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}) \to H^{\frac{1}{2}}(\widehat{\Gamma})$ and a constant $c > 0$ such that:

$$\Big|\langle \Lambda\phi + C\phi, \phi\rangle_{\frac{1}{2},-\frac{1}{2}}\Big| \;\; \geq \;\; c\|\phi\|^2_{-\frac{1}{2}}, \qquad \forall\, \phi \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}).$$

Consequently, $\Lambda$ is a Fredholm operator of index zero from $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ to $H^{\frac{1}{2}}(\widehat{\Gamma})$. Hence, proving its invertibility amounts to proving that it is injective. To this end, we first prove that

$$\phi \in \widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma}) \text{ and } \langle \Lambda\phi, \phi\rangle_{\frac{1}{2},-\frac{1}{2}} = 0 \implies \phi \equiv 0. \tag{5.12}$$

Let $\phi$ be a function in $\widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma})$ such that $\langle \Lambda\phi, \phi\rangle_{\frac{1}{2},-\frac{1}{2}} = 0$. Since $\int_{\widehat{\Gamma}} \phi(X) ds_X = 0$ the function $u_\phi(X) := \int_{\widehat{\Gamma}} \widehat{G}_E(X,Z)\phi(Z) ds_Z$ is a solution to

$$\begin{cases} \Delta u_\phi(X) &=& 0, \quad \text{in } \widehat{\Omega}_E, \\ \partial_n u_\phi(X) &=& 0, \quad \text{on } \partial\widehat{\Omega}_E \setminus \widehat{\Gamma}, \\ u_\phi(X) &=& 0, \quad \text{on } \widehat{\Gamma}, \\ u_\phi(X) &=& O(\frac{1}{|X|}), \quad \nabla u_\phi(X).\frac{X}{|X|} \;=\; O(\frac{1}{|X|^2}), \quad \text{as } |X| \to +\infty. \end{cases}$$

Multiplying $\Delta u_\phi(X)$ by the complex conjugate $u_\psi\bar{(}X)$ and integrating by parts over $B_R^+$, one finds that:

$$\int_{B_R^+} |\nabla u_\phi|^2(X) dX \;\;=\;\; \int_{S_R^+} \nabla u_\phi(X).\frac{X}{|X|}\overline{u_\phi}(X) ds_X \;=\; O(\frac{1}{R^2}), \quad \text{as } R \to +\infty.$$

Therefore $u_\phi(X)$ is constant on $\widehat{\Omega}_E$. Since $u_\phi(X)$ decreases like $\frac{1}{|X|}$ for large $|X|$, we deduce that $u_\phi(X) \equiv 0$ on $\widehat{\Omega}_E$. By taking its normal derivative on $\widehat{\Gamma}$, we conclude that $\phi(X) \equiv 0$, which proves (5.12), and consequently, that $\Lambda$ is injective on $\widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma})$.

Next, we prove that $\Lambda$ is injective on the whole space $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$. To this end, we introduce the capacity function:

$$\phi_e(x) \;\; := \;\; \frac{1/\pi}{\sqrt{1-x^2}}.$$

that satisfies

$$\begin{cases} \Lambda\phi_e = a \;:=\; -\pi\ln(2) \;<\; 0, \\ \phi_e \in \widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}),\; \langle\phi_e, 1\rangle = 1, \end{cases}$$

where $a = \frac{\pi}{4}\ln(2)$. Now, let $\psi$ be a function in $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ that belongs to the kernel of $\Lambda$. Setting $\psi_0 := \psi - (\int_{\widehat{\Gamma}}\psi ds_X)\phi_e$, we remark that:

$$\Lambda\psi_0 \;=\; -a\int_{\widehat{\Gamma}}\psi ds_X, \quad \text{and} \quad \langle 1, \psi_0\rangle_{\frac{1}{2},-\frac{1}{2}} = 0.$$

Hence $\langle\Lambda\psi_0, \psi_0\rangle_{\frac{1}{2},-\frac{1}{2}} = 0$, and from (5.12) we find that $\psi_0 = 0$ on $\widehat{\Gamma}$. Since $a \neq 0$, it follows that $\int_{\widehat{\Gamma}}\psi ds_X = 0$, and that $\psi = \psi_0 + (\int_{\widehat{\Gamma}}\psi ds_X)\phi_e \equiv 0$. Thus, $\Lambda$ is invertible.

Finally, the Fourier-Plancherel theorem implies that for all $\psi$ in $\widetilde{H}_0^{-\frac{1}{2}}(\widehat{\Gamma})$ we have

$$Re\langle\Lambda\psi, \psi\rangle_{\frac{1}{2},-\frac{1}{2}} \;\;=\;\; - \lim_{\substack{\widetilde{\psi}\in\mathcal{D}(\widetilde{\Gamma}) \\ \widetilde{\psi}\to\psi}} \int_{-\infty}^{+\infty}\frac{|\mathcal{F}(\widetilde{\psi})(\xi)|^2}{|\xi|}d\xi \;\leq\; 0.$$

We then see that

$$\begin{aligned} \langle\Lambda\psi, \psi\rangle_{\frac{1}{2},-\frac{1}{2}} \;&=\; \langle\Lambda\left(\psi_0 + \langle 1, \psi\rangle\phi_e\right), \psi_0 + \langle 1, \psi\rangle\phi_e\rangle_{\frac{1}{2},-\frac{1}{2}} \\ &=\; \langle\Lambda\psi_0, \psi_0\rangle_{\frac{1}{2},-\frac{1}{2}} \;+\; a\langle 1, \psi\rangle^2_{\frac{1}{2},-\frac{1}{2}} \;\leq\; 0. \end{aligned} \tag{5.13}$$

□

### 5.1.5 Well posdness of $P_2$.

We prove existence and uniqueness of solution to $P_2$ via the Lax-Milgram Theorem. We consider the space

$$V \;\;=\;\; \{v \in H^1(\widehat{\Omega}_\varepsilon) : \int_{\widehat{\Gamma}} v(X)ds_X = 0\}.$$

To derive its variational formulation, we multiply a solution $u$ to $P_2$ by the complex conjugate of a test function $v \in V$, and we integrate by parts over $\widehat{\Omega}_\varepsilon$, to get

$$\begin{aligned} &\int_{\widehat{\Omega}_\varepsilon}\nabla u(X)\nabla\overline{v}(X)dX \\ &\quad= \int_{\widehat{\Gamma}}\partial_{x_2}u(X)\overline{v}(X)ds_X - \int_{\widehat{\Gamma}_\varepsilon}\partial_{x_2}u(X)\overline{v}(X)ds_X. \end{aligned} \tag{5.14}$$

The integral $\int_{\widehat{\Gamma}} \partial_{x_2} u(X)\overline{v}(X) ds_X$ can be understood as the duality product $\overline{\langle v, \partial_{x_2} u\rangle}_{\frac{1}{2},-\frac{1}{2}}$. By construction we have $v|_{\widehat{\Gamma}} \in H^{\frac{1}{2}}(\widehat{\Gamma})$, so that $(\Lambda^{-1}v)(X)$ is well-defined in $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$. Since the Green function $\widehat{G}_E$ is symmetric, we also have $\overline{\langle v, \partial_{x_2} u\rangle}_{\frac{1}{2},-\frac{1}{2}} = \langle \Lambda\partial_{x_2} u, \Lambda^{-1} v\rangle_{\frac{1}{2},-\frac{1}{2}}$. Assuming that $u$ is a solution to $P_2$, we see that

$$\int_{\widehat{\Gamma}} \partial_{x_2} u(X)\overline{v}(X) ds_X \;=\; \int_{\widehat{\Gamma}} u(X)(\Lambda^{-1}\overline{v})(X) ds_X + \int_{\widehat{\Gamma}} u(X)(\Lambda^{-1} f)(X) ds_X.$$

Concerning the second term in (5.14), we note that $u$ solution to $P_2$ satisfies

$$\Lambda_\varepsilon \partial_{x_2} u(X) \;=\; u(X) - \varepsilon\langle 1, \partial_{x_2} u(X)\rangle.$$

Integrating over $\widehat{\Gamma}_\varepsilon$ and using the fact that functions in the range of $\Lambda_\varepsilon$ have 0-average, yields

$$2\varepsilon\langle 1, \partial_{x_2} u(X)\rangle \;=\; \int_{\widehat{\Gamma}_\varepsilon} u(X) dx. \tag{5.15}$$

We then write, setting $m(v) = 1/2 \int_{\widehat{\Gamma}_\varepsilon} v ds_X$

$$\int_{\widehat{\Gamma}_\varepsilon} \partial_{x_2} u(X)\overline{v}(X) ds_X \;=\; \int_{\widehat{\Gamma}_\varepsilon} \partial_{x_2} u(X) \left(\overline{v} - m(\overline{v})\right) ds_X \;+\; m(\overline{v})\langle 1, \partial_{x_2} u(X)\rangle.$$

We infer from the invertibility of $\Lambda_\varepsilon$ $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma}_\varepsilon) \to H_0^{\frac{1}{2}}(\widehat{\Gamma}_\varepsilon)$, from its symmetry and from (5.15), that

$$\begin{aligned}\int_{\widehat{\Gamma}_\varepsilon} \partial_{x_2} u(X)\overline{v}(X) ds_X &= \int_{\widehat{\Gamma}_\varepsilon} \Lambda_\varepsilon \partial_{x_2} u \, \Lambda_\varepsilon^{-1} \left(\overline{v} - m(\overline{v})\right) ds(X) \;+\; m(\overline{v})\langle 1, \partial_{x_2} u(X)\rangle \\ &= \int_{\widehat{\Gamma}_\varepsilon} \left(u - m(u)\right) \Lambda_\varepsilon^{-1} \left(\overline{v} - m(\overline{v})\right) ds(X) \;+\; \varepsilon^{-1} m(u) m(\overline{v}).\end{aligned}$$

Thus, we introduce

$$\begin{aligned} F(v) &:= \int_{\widehat{\Gamma}} (\Lambda^{-1} f)(X)\overline{v}(X) ds_X, \\ a(u,v) &:= \int_{\Omega_\varepsilon} \nabla u(X)\nabla\overline{v}(X) dX - \int_{\widehat{\Gamma}} u(X)(\Lambda^{-1}\overline{v})(X) ds_X \\ &\quad + \int_{\widehat{\Gamma}_\varepsilon} \left(u - m(u)\right) \Lambda_\varepsilon^{-1} \left(\overline{v} - m(\overline{v})\right) ds(X) \;+\; \varepsilon^{-1} m(u) m(\overline{v}). \end{aligned}$$

**Theorem 5.1.** *1. The linear form $F(v)$ is bounded from $V$ to $\mathbb{C}$. The bilinear form $a(u,v)$ is bounded and coercive on $V \times V$: There exists constants $C_0 > 0$ and $C_1 > 0$, such that*

$$\begin{aligned} |a(u,v)| \le C_0 \|u\|_1 \|v\|_1, &\qquad \textit{for all } u, v \in V, \\ Re\big(a(u,u)\big) \ge C_1 \|u\|_1^2, &\qquad \textit{for all } u \in V. \end{aligned}$$

*2. For any $f \in H^{\frac{1}{2}}(\hat{\Gamma})$, there exists a unique solution to $P_2$.*

*Proof.* It follows from lemma 5.5 that $\Lambda^{-1}$ is a bounded operator from $H^{\frac{1}{2}}(\widehat{\Gamma})$ to $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$. Since the trace operator on $\widehat{\Gamma}$ is a continuous operator from $V$ to $H^{\frac{1}{2}}(\widehat{\Gamma})$, $F(v)$ is a continuous linear form from $V$ to $\mathbb{C}$.
Invoking the trace theorem, the boundedness of $\Lambda$ and $\Lambda_\varepsilon^{-1}$, the bilinear form $a$, is easily seen to be bounded on $V \times V$. Next, for a fixed $u \in V$, we have:

$$\begin{aligned} a(u,u) \quad &:= \quad \int_{\Omega_\varepsilon} |\nabla u(X)|^2 dX - \int_{\widehat{\Gamma}} u(X)(\Lambda^{-1}\overline{u})(X) ds_X \\ &\quad + \int_{\widehat{\Gamma}_\varepsilon} (u - m(u))\, \Lambda_\varepsilon^{-1} \left(\overline{u} - m(\overline{u})\right)\, ds(X) \; + \; \varepsilon^{-1} |m(u)|^2. \end{aligned}$$

It follows from lemma 5.5 that the second term in the right-hand side above is positive. So is the third term by lemma 5.9. The coercivity of $a$ is thus a direct consequence of the Poincaré-Friedrichs inequality.

Finally, the Lax-Milgram applies and shows the existence and uniqueness of a weak solution to $P_2$. □

To conclude this part, we infer from theorem 5.1, lemmas 5.2 and 5.3 that $\widehat{S}_1$ is invertible from $\widetilde{H}^{-\frac{1}{2}}(\widehat{\Gamma})$ to $H^{\frac{1}{2}}(\widehat{\Gamma})$.

### 5.1.6 Compactness of the operators $S_j, j = 2,3,4$

In this paragraph, we prove the compactness of the operators $S_j \; : \widetilde{H}^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), j = 2,3,4$, introduced in lemma 4.1. From theorem 4.1, we see that the kernels of the operators $S_3$ and $S_4$ are of class $C^{1,\nu}$ with $0 \leq \nu < 1$, and thus these operators map the space $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ into $H^1(\Gamma)$. The compactness of the embedding $H^1(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ yield the desired result for $S_3$ and $S_4$.
Recalling the terms in the kernel of $S_2$,

$$s_2(x,z) \quad = \quad -\frac{2}{h}\Big(\frac{1}{6} + \frac{1}{8}((x+z+1)^2 + (x-z)^2) - \frac{1}{4}(x+z+1+|x-z|)\Big),$$

we note that all of them but one are $C^\infty$. Arguing as above shows that the contributions of the smooth term to $S_2$ are compact. The only singular term is $\frac{1}{2h}|x-z|$. The associated operator has a symbol equal to $\frac{-1}{2h\xi^2}$, and so belongs to the class $S^{-2}_{1,0}$ of pseudo-differential operators: In particular it is bounded from $\widetilde{H}^{-\frac{1}{2}}(\Gamma)$ to $H^{\frac{3}{2}}(\Gamma)$ ([24], page 2). The compactness of the injection $H^{\frac{3}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ yields that of $S_2$.

□

## 5.2 The constant $q_0 = \langle 1, S_1^{-1} 1 \rangle_{\frac{1}{2},-\frac{1}{2}}$ is different from 0

It is sufficient to prove that $\langle 1, \widehat{S}_1^{-1} 1 \rangle_{\frac{1}{2},-\frac{1}{2}} \neq 0$, where $\widehat{S}_1$ is the operator introduced in the previous section. Assume that $\langle 1, \widehat{S}_1^{-1} 1 \rangle_{\frac{1}{2},-\frac{1}{2}} = 0$. Then, the solution $u_1$ to $P_1$, with datum

$f \equiv 1$ satisfies

$$\begin{cases} \Delta u_1(X) & = & 0, & \text{in } \widehat{\Omega}, \\ \partial_n u_1(X) & = & 0, & \text{on } \partial\widehat{\Omega}, \\ \int_{\widehat{\Gamma}} u_1^-(X) ds_X & = & 0, & [u_1]_{\widehat{\Gamma}} \ = \ 1, \quad [\partial_{x_2} u_1]_{\widehat{\Gamma}} = 0, \\ u_1(X) & = & o(1), & \text{as } x_2 \to -\infty, \\ |\nabla u_1(X)| & = & o(1), & \text{as } x_2 \to -\infty, \\ u_1(X) & = & O(\frac{1}{|X|}), & \text{as } |X| \to +\infty, \quad x_2 > 0, \\ \nabla u_1(X).\frac{X}{|X|} & = & O(\frac{1}{|X|^2}), & \text{as } |X| \to +\infty, \quad x_2 > 0, \end{cases}$$

given that (see the previous section)

$$\partial_{x_2} u_1 = \widehat{S}_1^{-1} 1 \quad \text{and} \quad \int_{\widehat{\Gamma}} \partial_{x_2} u_1 ds_X = \langle 1, \widehat{S}_1^{-1} 1 \rangle_{\frac{1}{2}, -\frac{1}{2}} = 0.$$

Multiplying $\Delta u_1$ by $\overline{u_1}$, and integrating by parts over $B_R^+ \cup (-1,1) \times (-R, 0)$,we obtain

$$\begin{aligned} &\int_{B_R^+ \cup (-1,1) \times (-R,0)} |\nabla u_1|^2(X) dX \\ &= \int_{S_R^+} \nabla u_1(X).\frac{X}{|X|} \overline{u_1}(X) ds_X - \int_{\widehat{\Gamma}_R} \partial_{x_2} u_1(x, -R) \overline{u_1}(x, -1) dx \\ &= o(1), \text{ as } R \to +\infty, \end{aligned}$$

so that $u_1(X)$ is constant in $\widehat{\Omega}_E$ and in $\widehat{\Omega}_I$. Since $u_1(X) \to 0$ as $|X| \to \infty$ or as $x_2 \to -\infty$, it vanishes everywhere. Thus, $\widehat{S}_1^{-1} 1 = \partial_{x_2} u_1 = 0$, which contradicts the invertibility of $\widehat{S}_1$. We conclude that $\langle 1, \widehat{S}_1^{-1} 1 \rangle_{\frac{1}{2}, -\frac{1}{2}} \neq 0$, which proves the claim.

# 6 Appendix

In this section, we recall for convenience the main results of the theory developed by Ghoberg and Sigal in [12] about the operator version of the Residue theorem.

## 6.1 Definitions

Let $\mathcal{H}$ and $\mathcal{H}'$ be two Banach spaces and let $\mathcal{L}(\mathcal{H}, \mathcal{H}')$ be the algebra of all bounded functions acting from $\mathcal{H}$ into $\mathcal{H}'$.
Let $k_0 \in \mathbb{C}$ and let $D_\varepsilon(k_0)$ be the disk of center $k_0$ and radius $\varepsilon > 0$. We denote by $S(k)$ an operator-valued function, acting from $D_\varepsilon(k_0)$ into $\mathcal{L}(\mathcal{H}, \mathcal{H}')$. The number $k_0$ is called a *characteristic* value of $S(k)$ if

(i) $S(k)$ is holomorphic in some neighborhood of $k_0$, except possibly at this point itself.

(ii) There exists a vector valued function $\phi$: $D_\varepsilon(k_0) \to \mathcal{H}$, holomorphic at $k_0$, such that $\phi(k_0) \neq 0$, $S(k)\phi(k)$ is holomorphic at $k_0$ and vanishes at this point. $\phi(k)$ is called a *root* function of $S(k)$ associated to $k_0$, and the vector $\phi_0 = \phi(k_0)$ is called an eigenvector. The closure of the linear set of eigenvectors corresponding to $k_0$ is denoted by $KerS(k_0)$.

Suppose that $k_0$ is a characteristic value of the function $S(k)$ and $\phi$ is a root function. Then there exists a number $m(\phi) \geq 1$ and a holomorphic vector valued function $\psi(k)$ : $D_\varepsilon(k_0) \to \mathcal{H}$ such that

$$S(k)\phi(k) = (k-k_0)^{m(\phi)}\psi(k),$$
$$\psi(k_0) \neq 0.$$

The number $m(\phi)$ is called the *multiplicity* of the root function $\phi(k)$. Let $\phi_0$ be an eigenvector corresponding to $k_0$ and let
$\mathcal{R}(\phi_0) = \{m(\phi); \phi(k),$ is a root function such $\phi(k_0) = \phi_0\}$.
Then by *rank* of $\phi_0$ we mean $rank(\phi_0) = \max \mathcal{R}(\phi_0)$. Suppose that $n = dimKerS(k_0) < +\infty$ and that the ranks of all vectors in $KerS(k_0)$ are finite. A system of eigenvectors $\phi_0^j$, $j = 1, \dots, n$, is called a *canonical system of eigenvectors* of $S(k)$ associated to $k_0$ if their ranks possess the following property: $rank(\phi_0^j)$ is the maximum of the ranks of all eigenvectors in some direct complement in $dimKerS(k_0)$ of the linear span of the vectors $\phi_0^1, \dots, \phi_0^{j-1}$. Let $r_j = rank(\phi_0^j)$. Then $(r_j)_j$ determines the function $S(k)$ uniquely. We call

$$N(S(k_0)) = \sum_{j=1}^{n} r_j,$$

the *null multiplicity of the characteristic* value $k_0$ of $S(k)$.
If $k_0$ is not a characteristic value of $S(k)$ we put $N(S(k_0)) = 0$.
Suppose that $S^{-1}(k)$ exists and is holomorphic in some neighborhood of $k_0$, except possibly at this point itself. Then the number

$$M(S(k_0)) = N(S(k_0)) - N(S^{-1}(k_0)),$$

is called the *multiplicity* of the characteristic value $k_0$ of $S(k)$. Suppose that $k_1$ is a pole of the operator valued function. The Laurent expansion of $S(k)$ in $k_1$ is given by

$$S(k) = \sum_{j \geq -s} (k-k_1)^j S_j$$

If in the last expression the operators $S_{-j}$, $j = 1, \dots, s$, are finite dimensional, Then $S(k)$ is called *finitely meromorphic* at $k_1$.
The operator-valued function $S(k)$ is said to be of Fredholm type at the point $k_1$ if the operator $A_0$ in the last expansion is a Fredholm operator.
If $S(k)$ is holomorphic at the point $k_0$ and the operator $S(k)$ is invertible, then $k_0$ is called a *regular point* of $S(k)$.

## 6.2 The Generalized Rouché Theorem

A value $k_0$ is called a *normal point* of $S(k)$, if there exists a constant $0 < \varepsilon_0 \leq \varepsilon$ such that $S(k)$ is finitely meromorphic and of Fredholm type at $k_0$, and if all the points of $D_{\varepsilon_0}(k_0) \setminus \{k_0\}$ are regular for $S(k)$.

**Lemma 6.1.** *Every normal point $k_0$ of $S(k)$ is a normal point of $S^{-1}(k)$.*

Let $\partial D_{\varepsilon_0}$ is the contour bounding the domain $D_{\varepsilon_0}(k_0)$. An operator-valued function $S(k)$ which is finitely meromorphic and of Fredholm type in $D_{\varepsilon_0}(k_0)$ and continuous at $\partial D_{\varepsilon_0}$ is called *normal with respect to* $\partial D_{\varepsilon_0}$, if the operator $S(k)$ is invertible in $\overline{D_{\varepsilon_0}(k_0)}$, except at a finite number of interior points which are normal points of $S(k)$. If $S(k)$ is normal with respect to the contour $\partial D_{\varepsilon_0}$, and if $k_i$, $i = 1, \ldots, \sigma$ are all its characteristic values **and poles** in $D_{\varepsilon_0}(k_0)$, we set

$$\mathcal{M}(S(k); \partial D_{\varepsilon_0}) = \sum_{i=1}^{\sigma} M(S(k_i)).$$

**Theorem 6.1.** *Suppose that the operator-valued $S(k)$ is normal with respect to $\partial D_{\varepsilon_0}$. Then, the operator $\int_{\partial D_{\varepsilon_0}} S^{-1}(k) \frac{d}{dk} S(k) dk$ has finite range (and thus belongs to the trace class) and*

$$\mathcal{M}(S(k); \partial D_{\varepsilon_0}) = \frac{1}{2i\pi} tr \int_{\partial D_{\varepsilon_0}} S^{-1}(k) \frac{d}{dk} S(k) dk.$$

Rouché 's theorem generalizes to operators as follows:

**Theorem 6.2.** *Let $S(k)$ be an operator-valued function which is normal with respect to $\partial D_{\varepsilon_0}$. If an operator-valued function $B(k)$ which is finitely meromorphic in $D_{\varepsilon_0}(k_0)$ and continuous at $\partial D_{\varepsilon_0}$ satisfies the condition*

$$|S^{-1}(k)B(k)|_{\mathcal{L}(\mathcal{H},\mathcal{H})} < 1, \quad k \in \partial D_{\varepsilon_0},$$

*then $S(k) + B(k)$ is also normal with respect to $\partial D_{\varepsilon_0}$, and*

$$\mathcal{M}(S(k); \partial D_{\varepsilon_0}) = \mathcal{M}(S(k) + B(k); \partial D_{\varepsilon_0}).$$

We also state the operator version of Steinberg's theorem.

**Theorem 6.3.** *Suppose that $S(k)$ is an operator-valued function which is finitely meromorphic and of Fredholm type in the domain $D_{\varepsilon_0}(k_0)$. If the operator $S(k)$ is invertible at one point of $D_{\varepsilon_0}$, then $S(k)$ has a bounded inverse for all $k \in D_{\varepsilon_0}$, except possibly for a finite number of isolated points.*

Proof for both the above theorems can be found in [12], as well as the following result:

**Theorem 6.4.** *Suppose that $S(k)$ is an operator-valued function which is normal with respect to $\partial D_{\varepsilon_0}$. Let $f(k)$ be a scalar function which is analytic in $D_{\varepsilon_0}(k_0)$ and continuous in $\overline{D_{\varepsilon_0}(k_0)}$. Then,*

$$\frac{1}{2i\pi}\ tr \int_{\partial D_{\varepsilon_0}} f(k) S^{-1}(k) \frac{d}{dk} S(k) dk = \sum_{j=1}^{\sigma} M(S(k_j)) f(k_j),$$

*where $k_j$, $j = 1, \ldots, \sigma$ are all the points in $D_{\varepsilon_0}(k_0)$ which are either poles or characteristic values of $S(k)$.*

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