Asymptotics in the presence of inclusions of small volume for a conduction equation: A case with a non-smooth reference potential

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ABSTRACT. We derive an asymptotic formula for the Green function of a conduction equation in the presence of a small inhomogeneity, that perturbs a background conductivity, which is piecewise constant on an angular sector. We show how the elliptic corner singularity of the background potential affects the form of the first order term in the expansion.

1. Introduction

Asymptotics in the presence of small inclusions have been the subject of several studies in recent years. For a conduction equation in a bounded domain $\Omega \subset \mathbb{R}^n$, with given Neumann boundary data f, as in [**FMV**] one seeks an asymptotic expansion of the difference of u_0, the potential in a reference medium, and u_ε, the potential in that medium perturbed by p inclusions of diameter ε. One can show that the first correction term, in the expansion of $(u_\varepsilon - u_0)(z)$ for z far from the inclusions, is of order ε^n and has the form

$$(u_\varepsilon - u_0)(z) \quad = \quad \varepsilon^n \sum_{j=1}^{p} M_j \nabla u_0(z_j) \cdot \nabla N(z, z_j) \ + \ o(\varepsilon^n). \tag{1.1}$$

In this expression, M_j is a polarization tensor, that contains some information about the coefficient contrast and the geometry of each inclusion. The function $N(\cdot, z_j)$ is a Neumann function with a singularity at z_j, the center of the j-th inclusion [**FMV, AK**].

To derive this asymptotic expansion, one can make the ansatz that the perturbed potential u_ε near the j-th inclusion takes the form

$$u_\varepsilon(x) \quad = \quad u_0(x) + \varepsilon v_j(\frac{x - z_j}{\varepsilon}) + r_\varepsilon(x),$$

where the corrective term v_j satisfies the PDE

$$\left\{ \begin{array}{lcl} \operatorname{div}(a_1(y)\nabla[v_j(y) + \nabla u_0(z_j) \cdot y] & = & 0 \quad \text{in } \mathbb{R}^n \\ \lim_{|y|\to\infty} v_j(y) & = & 0, \end{array} \right. \tag{1.2}$$

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where $a_1(y) = a_\varepsilon(z_j + \varepsilon y)$ is the conductivity rescaled from a_ε defined in (1.4) below. The coefficients of M_j are computed as moments of the function v_j [**FMV, AK**].

Asymptotics of this type have also been derived for the Helmholtz equation, for the system of elasticity and for the Maxwell system (see the book [**AK**] and the references therein). Letting $z \to \partial\Omega$, the above expression provides an approximation of the Neumann to Dirichlet operator, which only depends on a finite number of parameters: the points z_j and the coefficients of the matrices M_j. The inverse problem of detecting the inclusions can thus be approximated by a finite dimensional problem, which explains the interest such asymptotics have stirred. Efficient numerical strategies for detection, based on such expansions have been proposed [**BHV, AIL**].

In all the cases studied so far, one of the main ingredients in the convergence proof of the expansion is the smoothness of the background potential u_0. We study here a case where u_0 is singular. We consider a conduction equation in a disk where the reference conductivity $a_0(x)$ takes a constant value a_1 in the sector $0 < \theta < 2\pi\alpha$, and a different value a_2 in the rest of the disk. We chose the case of a disk for simplicity of exposition: Our analysis generalizes to bounded domains in $\mathbb{R}^2$ that contain isolated polygonal subsets, where the conductivity takes distinct constant values. Indeed, isolating a circular region around any corner of such a polygonal set, brings one to the situation studied here.
More precisely, we let $B_R = B(0,R)$ and for $x = (r\cos(\theta), r\sin(\theta)) \in B_R$,

$$a_0(x) \quad = \quad \begin{cases} a_1 & \text{if } 0 < \theta < 2\pi\alpha \\ a_2 & \text{otherwise} \end{cases} \tag{1.3}$$

This reference medium is perturbed by a small inclusion ω_ε of size $\varepsilon < R$, centered at 0, i.e. centered on the corner of the sector:

$$\omega_\varepsilon \;=\; \varepsilon\omega, \quad \operatorname{diam}(\omega) \;=\; 1.$$

We define the perturbed conductivity by

$$a_\varepsilon(x) \quad = \quad \begin{cases} k & \text{if } x \in \omega_\varepsilon \\ a_0(x) & \text{otherwise.} \end{cases} \tag{1.4}$$

Given a function $f \in L^2(\partial B_R)$, that satisfies $\displaystyle\int_{\partial B_R} f = 0$, we denote u_0 the reference potential, solution to

$$\begin{cases} \operatorname{div}(a_0 \nabla u_0) &= \; 0 \quad \text{in } B_R \\ a_0 \partial_n u_0 &= \; f \quad \text{on } \partial B_R, \end{cases} \tag{1.5}$$

and u_ε the perturbed potential, solution to

$$\begin{cases} \operatorname{div}(a_\varepsilon \nabla u_\varepsilon) &= \; 0 \quad \text{in } B_R \\ a_\varepsilon \partial_n u_\varepsilon &= \; f \quad \text{on } \partial B_R, \end{cases} \tag{1.6}$$

normalized by the conditions

$$\int_{\partial B_R} a_0 u_0 \quad = \quad \int_{\partial B_R} a_0 u_\varepsilon \;=\; 0. \tag{1.7}$$

We show how the first term in the corresponding small volume expansion is affected by the presence of the elliptic corner singularity in the reference potential. More precisely, let $0 < \lambda_1 < 1$ denote the first non-zero eigenvalue of the periodic transmission problem (2.2) below. It is well known that λ_1 determines the exponent of the elliptic singularity due to the corner [**G**]. We prove the following

THEOREM 1.1. *Let $z \in \partial B_R$ such that $|R| >> \varepsilon$. Then*

$$u_\varepsilon(z) - u_0(z) \quad = \quad \varepsilon^{2\lambda_1} u_{1,1}(z) + o(\varepsilon^{2\lambda_1}).$$

The term $u_{1,1}$ is defined for $z \in \partial B_R$ by

$$u_{1,1}(z) \quad = \quad m_{1,1} \left(\int_0^{2\pi} f(\beta)\varphi_1(\beta) d\beta \right) \varphi_1(\theta_z),$$

where φ_1 is the eigenfunction associated to λ_1 and the constant $m_{1,1}$ is given in (4.4).

Thus, the main difference with the case of a homogeneous background is that the first term in the expansion scales like $\varepsilon^{2\lambda_1}$, and not like the volume of the inclusion. Theorem 1.1 indicates that it should be easier to detect the corners (and edges in 3D) of inhomogeneities buried in a reference medium. This could prove useful in a tomography experiment when one knows *a priori* that the objects to be localized are polyhedral.

In the context of impedance tomography by elastic perturbation [**ABCTF**] where one performs tomography measurements while focusing localized ultrasound waves into the medium, it has been observed that sharp corners could be reconstructed very accurately. It may be due to the fact that the leading term in the expansion of $u_\varepsilon - u_0$ is of order smaller than the volume, when the ultrasound spot hits the neighborhood of the corner. Also, the exponent λ_1 depends on the angle α of the sector and on the conductivity contrast. Since one may vary the size of the ultrasound spot, it might be possible to deduce the value of λ_1 from the measurements, hence, obtain information on the angle or on the conductivity contrast.

The paper is structured as follows: In section 2, we derive a series representation of the fundamental solution G_0 which satisfies

$$\operatorname{div}(a_0(x)\nabla G_0(x,y)) \quad = \quad \delta(x-y) \quad \text{in } \mathbb{R}^2. \tag{1.8}$$

In section 3, we give an asymptotic expansion of the difference between the perturbed Green function G_ε and the reference Green function G_0. Section 4 is devoted to the proof of theorem 1.1. Finally, in the appendix, we show how G_ε can be constructed. Throughout this paper, the polar coordinates of a point $x \in \mathbb{R}^2$ are denoted (r_x, θ_x).

2. The Green function for the reference medium

2.1. A Sturm-Liouville spectral problem. We seek the Green function G_0 solution to (1.8) via an expansion in polar coordinates

$$G_0(x,y) \quad = \quad \sum_{n\geq 0} w_n(r,r')\varphi_n(\theta)\varphi_n(\theta'), \tag{2.1}$$

with $x = (r, \theta), y = (r', \theta')$, and where φ_n are the eigenfunctions associated to the following Sturm-Liouville problem

$$\left\{ \begin{array}{l} (a_0(\theta)\varphi'(\theta))' + \lambda^2 a_0(\theta)\varphi(\theta) \;=\; 0 \quad \text{in } (0,2\pi) \\ \varphi \in H^1(0,2\pi), \; \varphi \text{ periodic} \\ \displaystyle\int_0^{2\pi} a_0(\theta)\varphi^2(\theta)\, d\theta \;=\; 1. \end{array} \right. \tag{2.2}$$

It is well known that there exists a system of eigenvectors (φ_n) that forms an orthonormal basis of $L^2((0,2\pi), a_0)$. The corresponding eigenvalues $\lambda_0 = 0 < \lambda_1 < \cdots < \lambda_n < \ldots$, are simple and the Rayleigh-Ritz min-max principle shows that

$$\frac{\min(a_1,a_2)n^2}{\max(a_1,a_2)\pi} \;\leq\; \lambda_n^2 \;\leq\; \frac{\max(a_1,a_2)n^2}{\min(a_1,a_2)\pi}. \tag{2.3}$$

In particular, the first eigenfunction is equal to the constant

$$\varphi_0 \;=\; \left(\int_0^{2\pi} a_0 \right)^{-1/2}. \tag{2.4}$$

Seeking φ_n as a linear combination of $\cos(\lambda_n\theta)$ and $\sin(\lambda_n\theta)$ in each sector, one sees that the eigenvalues are the roots of the determinant of a linear system, and solve

$$(a_1^2 + a_2^2)\sin(2\pi\alpha\lambda)\sin(2\pi(1-\alpha)\lambda) + 2a_1a_2(1 - \cos(2\pi\alpha\lambda)\cos(2\pi(1-\alpha)\lambda)) \;=\; 0.$$

It follows that the eigen-elements (λ_n, φ_n) are entire functions of the contrast $\kappa = \frac{2a_1a_2}{a_1^2+a_2^2}$. Further, one can check that if $0 < a_1, a_2 < \infty$, and if $a_1 \neq a_2$, then

$$0 < \lambda_1 < 1. \tag{2.5}$$

2.2. The Green function $G_0(x,y)$. We are looking for a solution to (1.8) which also satisfies

$$\left\{ \begin{array}{lcl} \lim_{r_x\to\infty} \sqrt{r_x}\,\partial_{r_x} G_0(x,y) & = & 0 \\ \lim_{r_x\to 0} G_0(x,y) & = & O(1). \end{array} \right. \tag{2.6}$$

Substituting the expression (2.1) into (1.8) we find that w_n must solve

$$\frac{\partial^2}{r^2} w_n(r,r') + \frac{1}{r}\partial_r w_n(r,r') - \frac{\lambda_n^2}{r^2} w_n(r,r') \;=\; \frac{\delta(r-r')}{r}. \tag{2.7}$$

The solutions to the homogeneous equation are linear combinations of

$$\beta_n(r) \;=\; r^{\lambda_n} \quad \text{and} \quad \gamma_n(r) \;=\; r^{-\lambda_n},$$

the Wronskian of which is

$$\beta_n(r)\gamma_n'(r) - \beta_n'(r)\gamma_n(r) \;=\; \frac{-2\lambda_n}{r}.$$

Moreover, the boundary conditions (2.6) imply that w_n satisfies $w_n(r) = O(1)$ as $r \to 0$, and that $w_n \to 0$ as $r \to \infty$. It follows that

$$w_n(r,r') \;=\; \left\{ \begin{array}{ll} -\dfrac{1}{2\lambda_n}\left(\dfrac{r'}{r}\right)^{\lambda_n} & \text{if } r' < r \\ -\dfrac{1}{2\lambda_n}\left(\dfrac{r}{r'}\right)^{\lambda_n} & \text{if } r < r'. \end{array} \right.$$

A similar computation can be carried out when $n = 0$: The solutions to the homogeneous equation (2.7) are linear combinations of $\beta_0(r) = 1$ and $\gamma_0(r) = \ln(r)$, the Wronskian of which equals $1/r$, so that

$$w_0(r, r') = \begin{cases} \ln(r) & \text{if } r' < r \\ \ln(r') & \text{if } r < r'. \end{cases}$$

We finally obtain the expression

$$G_0(x, y) = \begin{cases} \ln(r)\varphi_0^2 - \sum_{n\geq 1} \dfrac{1}{2\lambda_n} \left(\dfrac{r'}{r}\right)^{\lambda_n} \varphi_n(\theta)\varphi_n(\theta') & \text{if } r' < r \\ \ln(r')\varphi_0^2 - \sum_{n\geq 1} \dfrac{1}{2\lambda_n} \left(\dfrac{r}{r'}\right)^{\lambda_n} \varphi_n(\theta)\varphi_n(\theta') & \text{if } r < r'. \end{cases} \tag{2.8}$$

2.3. The radiation condition. Let $\psi \in H^1_{loc}(\mathbb{R}^2)$ and define for $n \geq 1$

$$\psi_n(r) = \int_0^{2\pi} a_0(\theta)\varphi_n(\theta)\psi(r, \theta)\, d\theta.$$

We also define

$$\begin{cases} \alpha_n(\psi, r) = -\dfrac{r^{-\lambda_n}}{2\lambda_n} \left(r\dfrac{\partial \psi_n}{\partial r}(r) - \lambda_n \psi_n(r) \right), \quad n \geq 1 \\ \alpha_0(\psi, r) = r\ln(r)\dfrac{\partial \psi_0}{\partial r} - \psi_0(r). \end{cases} \tag{2.9}$$

We say that ψ satisfy the radiation condition, if for any compact set $K \subset \mathbb{R}^n$,

$$\sup_{y\in K} \sum_{n=0}^{+\infty} \alpha_n(\psi, R)|y|^{\lambda_n} \longrightarrow 0, \quad \text{as } R \to \infty. \tag{2.10}$$

LEMMA 2.1. *Let $\psi \in H^1_{loc}(\mathbb{R}^2)$, which satisfies the radiation condition (2.10). Then*

$$\int_{\partial B_R} \frac{\partial \psi}{\partial r_x}(x)G_0(x, y) - \psi(x)\frac{\partial G_0}{\partial r_x}(x, y) = o(1), \quad \text{as } R \to +\infty,$$

uniformly with respect to y in any fixed compact subset of B_R.

Proof: Assume that $|y| < R$. We form

$$h(y) = \int_{\partial B_R} a_0(\theta_x) \left(\frac{\partial \psi}{\partial r_x}(x)G_0(x, y) - \psi(x)\frac{\partial G_0}{\partial r_x}(x, y) \right) Rd\theta_x.$$

Substituting the expression (2.8) for G_0 in this integral, we obtain

$$h(y) = \sum_{n\geq 0} h_n(r_y)\varphi_n(\theta_y),$$

where

$$\begin{aligned} h_n(r_y) &= -\frac{1}{2\lambda_n}\frac{r_y^{\lambda_n}}{R^{\lambda_n}} \left(R\frac{\partial \psi_n}{\partial r}(R) - \lambda_n\psi_n(R) \right) \\ &= \alpha_n(\psi, R)r_y^{\lambda_n} \quad \text{for } n \geq 1, \\ h_0(r_y) &= R\ln(R)\frac{\partial \psi_0}{\partial r}(R) - \psi_0(R) \\ &= \alpha_0(\psi, R), \end{aligned}$$

and the result follows from the assumption (2.10) on ψ. $\square$

An easy consequence is the following

Proposition 2.2. The Green function G_0 given by (2.8) is the unique solution to (1.8) that satisfies the radiation conditions (2.10).

Proof: Firstly, we note that G_0 satisfies (2.10). Secondly, assume that G_0^* is another solution to (1.8) which satisfies (2.10). Let $y \in \mathbb{R}^2$ and define $w(x) = G_0^*(x,y) - G_0(x,y)$, a solution to

$$\mathrm{div}(a_0 \nabla w) \;\; = \;\; 0 \quad \text{in } \mathbb{R}^2.$$

Elliptic regularity theory shows that $w \in H^1_{loc}(\mathbb{R}^2)$. Thus, one can multiply the above equation by $G_0(x,z)$ and integrate on a ball B_R, to obtain for R large enough

$$w(z) \;\; = \;\; \int_{\partial B_R} a_0 \left(\frac{\partial G_0}{\partial r_x}(x,z) w(x) \; - \; G_0(x,z)\frac{\partial w}{\partial r}(x) \right) d\sigma_x,$$

which tends to 0 as $R \to \infty$. It follows that $w \equiv 0$ and thus that $G_0^* \equiv G_0$. □

3. Asymptotic of the Green function in the perturbed medium

Let G_ε denote the Green function for a_ε, that satisfies

$$\left\{ \begin{array}{l} \mathrm{div}(a_\varepsilon(x)\nabla G_\varepsilon(x,y)) \; = \; \delta_y(x) \quad \text{in } \mathbb{R}^2 \\ \text{the radiation condition (2.10)} \end{array} \right. \tag{3.1}$$

We show in the Appendix how one can construct G_ε. Its uniqueness can be proved with the same argument as that of proposition 2.2. We first study how G_ε scales with ε:

Lemma 3.1. *Let $\varepsilon > 0$ and $Z, Y \in \mathbb{R}^2$ such that $Z \neq Y$. Then*

$$G_\varepsilon(\varepsilon Z, \varepsilon Y) \;\; = \;\; G_1(Z,Y) + \varphi_0^2 \ln(\varepsilon).$$

Proof: The function $\Phi(Z,Y) = G_\varepsilon(\varepsilon Z, \varepsilon Y) - G_1(Z,Y) - \varphi_0^2 \ln(\varepsilon)$ satisfies

$$\mathrm{div}_Z(a_1(Z)\nabla_Z \Phi(Z,Y)) \; = \; \varepsilon^2 \mathrm{div}_x(a_\varepsilon(\varepsilon Z)\nabla_x G_\varepsilon(\varepsilon Z, \varepsilon Y)) - \delta(Z-Y) \; = \; 0. \tag{3.2}$$

Let $y, z \in \mathbb{R}^2, y \neq z$. It follows from (1.8) and (3.1) that

$$\mathrm{div}(a_\varepsilon(x)\nabla[G_\varepsilon(x,y) - G_0(x,y)]) \;\; = \;\; \mathrm{div}([a_0(x) - a_\varepsilon(x)]\nabla G_0(x,y)).$$

Multiplying this equation by $G_\varepsilon(x,z)$ and integrating over $\mathbb{R}^2$ yields

$$\begin{aligned} G_\varepsilon(z,y) - G_0(z,y) \;\; &= \;\; \int_{\mathbb{R}^2} [a_\varepsilon - a_0]\nabla G_0(x,z)\nabla G_\varepsilon(x,y)\, dx \\ &= \;\; \int_{\varepsilon\omega} [k - a_0(\frac{x}{|x|})]\nabla G_0(x,z)\nabla G_\varepsilon(x,y)\, dx. \end{aligned} \tag{3.3}$$

In particular,

$$\begin{aligned} G_\varepsilon(\varepsilon Z, \varepsilon Y) - G_0(\varepsilon Z, \varepsilon Y) \;\; &= \;\; \int_{\varepsilon\omega} [k - a_0(\frac{x}{|x|})]\nabla_x G_0(x, \varepsilon Z)\nabla_x G_\varepsilon(x, \varepsilon Y)\, dx \\ &= \;\; \int_{\omega} [k - a_0(\frac{X}{|X|})](\nabla_x G_0)(\varepsilon X, \varepsilon Z)\nabla_x G_\varepsilon(X, \varepsilon Y)\varepsilon^2\, dX, \end{aligned}$$

Differentiating the expression (2.8) of G_0 shows that $(\nabla_x G_0)(\varepsilon X, \varepsilon Z) = \varepsilon^{-2}\nabla_X G_0(X,Z)$, and so

$$G_\varepsilon(\varepsilon Z, \varepsilon Y) - G_0(\varepsilon Z, \varepsilon Y) \;\; = \;\; \int_{\omega} [k - a_0(\frac{X}{|X|})]\nabla_X G_0(X,Z)\nabla_x G_\varepsilon(X, \varepsilon Y)\, dX.$$

As $\nabla_X G_0(X,Z)$ satisfies the radiation condition (2.10), so does $G_\varepsilon(\varepsilon Z,\varepsilon Y) - G_0(\varepsilon Z,\varepsilon Y)$. On the other hand, again using the form (2.8) of G_0, we see that

$$G_1(Z,Y) - G_0(\varepsilon Z,\varepsilon Y) + \ln(\varepsilon)\varphi_0^2 \quad = \quad G_1(Z,Y) - G_0(Z,Y),$$

which also satisfies the radiation condition. It follows that $\Phi(Z,Y) = G_\varepsilon(\varepsilon Z,\varepsilon Y) - G_0(\varepsilon Z,\varepsilon Y) - (G_1(Z,Y) - G_0(\varepsilon Z,\varepsilon Y) + \ln(\varepsilon)\varphi_0^2)$ satisfies the radiation condition. Recalling (3.2), uniqueness implies that $\Phi \equiv 0$, and the lemma is proved. □

We now derive a representation formula for G_ε.

LEMMA 3.2. *Let $y \in \mathbb{R}^2 \setminus \varepsilon\overline{\omega}$ and $z \in \varepsilon\omega$. Then we have*

$$\begin{aligned} G_\varepsilon(z,y) - G_0(z,y) \;&=\; \varepsilon^2 \int_\omega (k - a_0(X))\nabla_x G_0(\varepsilon X,y)\nabla_x G_0(\varepsilon X,z)\,dX \\ +\varepsilon^2 \int_\omega (k - a_0(X))&\nabla_x G_0(\varepsilon X,y)\nabla_X\Big(\int_\omega (k - a_0(S))\nabla_S G_1(S,X)\nabla G_0(\varepsilon S,y)dS\Big)dX. \end{aligned}$$

Proof: Let $y, z \in \mathbb{R}^2, y \neq z$. We start from (3.3)

$$G_\varepsilon(z,y) - G_0(z,y) \quad = \quad \int_{\varepsilon\omega} [k - a_0(\frac{x}{|x|})]\nabla G_0(x,y)\nabla G_\varepsilon(x,z)\,dx.$$

Changing variables to $X = x/\varepsilon$ in the integral, we find that

$$\begin{aligned} &G_\varepsilon(z,y) - G_0(z,y) \\ &\quad = \; \varepsilon^2 \int_\omega [k - a_0(\frac{X}{|X|})](\nabla_x G_0)(\varepsilon X,y)(\nabla_x G_\varepsilon)(\varepsilon X,z)\,dX. \end{aligned} \tag{3.4}$$

Next, we assume that $z = \varepsilon Z \in \omega_\varepsilon$ and that $y \in \mathbb{R}^2 \setminus \overline{\omega_\varepsilon}$. The above relation and lemma 3.1 show that

$$\begin{aligned} G_\varepsilon(\varepsilon Z,y) - G_0(\varepsilon Z,y) \;&=\; \varepsilon^2 \int_\omega [k - a_0(X)](\nabla_x G_0)(\varepsilon X,y)(\nabla_x G_\varepsilon)(\varepsilon X,\varepsilon Z)\,dX \\ &=\; \varepsilon \int_\omega [k - a_0(X)](\nabla_x G_0)(\varepsilon X,y)\nabla_X G_1(X,Z)\,dX. \end{aligned}$$

Given the assumptions on y and z, the functions $Z \to G_\varepsilon(\varepsilon Z,y)$ and $Z \to G_0(\varepsilon Z,y)$ are in $H^1(\omega)$. We can thus differentiate the previous equality with respect to Z to obtain

$$(\nabla_x G_\varepsilon)(\varepsilon Z,y) \quad = \quad (\nabla_x G_0)(\varepsilon Z,y) + \nabla_Z \int_\omega [k - a_0(X)]\nabla_X G_1(X,Z)(\nabla_x G_0)(\varepsilon X,y)\,dX.$$

Inserting the above expression into (3.4) yields the desired result. □

LEMMA 3.3. *Let $\varepsilon_0 > 0$, let $y \in \mathbb{R}^2 \setminus \varepsilon_0\overline{\omega}$, and let $X \in \omega$. Then, for $0 < \varepsilon < \varepsilon_0$,*

$$\nabla G_0(\varepsilon X,y) \quad = \quad \sum_{n\geq 1} \varepsilon^{\lambda_n - 1} g_n(X,y), \tag{3.5}$$

where the vector-valued functions g_n are defined by

$$g_n(X,y) \quad = \quad -\frac{r_X^{\lambda_n - 1}}{2\lambda_n r_y^{\lambda_n}}\varphi_n(\theta_y)\begin{pmatrix} \lambda_n\varphi_n(\theta_X)\cos(\theta_X) - \varphi_n'(\theta_X)\sin(\theta_X) \\ \lambda_n\varphi_n(\theta_X)\sin(\theta_X) + \varphi_n'(\theta_X)\cos(\theta_X) \end{pmatrix} \tag{3.6}$$

Proof: For $r_x = |x| < |y| = r_y$, $G_0(x, y)$ is equal to

$$G_0(x,y) \quad = \quad \ln(r_y) \; - \; \sum_{n\geq 1} \frac{1}{2\lambda_n}\left(\frac{r_x}{r_y}\right)^{\lambda_n} \varphi_n(\theta_x)\varphi_n(\theta_y),$$

which converges exponentially. Its gradient $\nabla_x G_0$ can be computed by differentiating each term in the series. In polar coordinates, one obtains

$$\begin{aligned}
\frac{\partial G_0}{\partial r_x}(x,y) &= -\sum_{n\geq 1} \frac{1}{2\lambda_n} \frac{\lambda_n r_x^{\lambda_n - 1}}{r_y^{\lambda_n}} \varphi_n(\theta_x)\varphi_n(\theta_y) \\
\frac{\partial G_0}{\partial \theta_x}(x,y) &= -\sum_{n\geq 1} \frac{1}{2\lambda_n}\left(\frac{r_x}{r_y}\right)^{\lambda_n} \varphi_n'(\theta_x)\varphi_n(\theta_y),
\end{aligned}$$

so that

$$\nabla_x G_0(x,y) \quad = \quad -\sum_{n\geq 1} \frac{r_x^{\lambda_n - 1}}{2\lambda_n r_y^{\lambda_n}} \varphi_n(\theta_y) \begin{pmatrix} \lambda_n \varphi_n(\theta_x)\cos(\theta_x) - \varphi_n'(\theta_x)\sin(\theta_x) \\ \lambda_n \varphi_n(\theta_x)\sin(\theta_x) + \varphi_n'(\theta_x)\cos(\theta_x) \end{pmatrix}.$$

Inserting $x = \varepsilon X$ in these expressions, which is legitimate since by assumption $|\varepsilon X| < |y|$, yields the result. $\square$

Finally, we derive the full pointwise asymptotic expansion of the Green function $G_\varepsilon(x, y)$.

THEOREM 3.4. *Let $0 < \varepsilon < \varepsilon_0$, $y, z \in \mathbb{R}^2 \setminus \varepsilon_0\overline{\omega}$ such that $y \neq z$. Then*

$$\begin{aligned}
G_\varepsilon(z,y) \quad &= \quad G_0(z,y) \; + \; \varepsilon^{2\lambda_1} G_{(1,1)}(z,y) \; + \; \varepsilon^{\lambda_1+\lambda_2} G_{(1,2)}(z,y) \\
&\quad + \sum_{\substack{4\leq m+n \\ n\leq m}}^{+\infty} \varepsilon^{\lambda_n+\lambda_m} G_{(m,n)}(z,y). \qquad (3.7)
\end{aligned}$$

The functions $G_{(m,n)}$ are symmetric, and defined by

$$\begin{aligned}
G_{(m,n)}(z,y) \quad &= \quad (2-\delta_{mn})\,\varphi_n(\theta_y)\varphi_n(\theta_z) \\
&\quad \left(\int_0^{2\pi}\!\!\int_0^{2\pi} a_0(\theta)a_0(\beta)(G_1 - G_0)((r_y,\theta),(r_z,\beta))\varphi_n(\theta)\varphi_n(\beta)d\theta d\beta\right), \qquad (3.8)
\end{aligned}$$

where $\delta_{mn} = 1$ if $m = n$ and 0 otherwise.

Proof: The result is a direct consequence of lemma 3.2 and 3.3: It suffices to insert the expansion (3.5) into the expression of $G_\varepsilon(z, y) - G_0(z, y)$ given in theorem 3.2. One obtain the expansion (3.7), with

$$\begin{aligned}
G_{(m,n)}(z,y) \;=\; (2-\delta_{mn}) \Bigg(&\int_\omega (k - a_0(X)) g_m(X,y) g_n(X,z)\, dX \\
&+ \int_\omega (k - a_0(X)) g_m(X,y) \nabla_X \int_\omega (k - a_0(S)) \nabla_S G_1(X,S) g_n(S,z)\, dS\, dX \Bigg).
\end{aligned}$$

The expression of the functions $G_{(m,n)}$ can be simplified. Indeed, due to the radial symmetry of the conductivity a_0 the vector-valued functions $g_n(X, y)$ are $y-$orthogonal and can be rewritten as

$$g_n(X,y) \quad = \quad \left(\int_0^{2\pi} a_0(\theta)\nabla G_0(X,(r_y,\theta))\varphi_n(\theta)d\theta\right)\varphi_n(\theta_y)$$

Therefore the functions $G_{(m,n)}(z,y)$ can be expressed as

$$\int_0^{2\pi}\int_0^{2\pi} a_0(\theta)a_0(\beta)\widetilde{G}((r_y,\theta),(r_z,\beta))\varphi_n(\theta)\varphi_n(\beta)d\theta d\beta\varphi_n(\theta_y)\varphi_n(\theta_z),$$

where

$$\begin{aligned}\widetilde{G}(z,y) &= \int_\omega (k-a_0(X))\nabla G_0(X,y)\nabla G_0(X,z)\,dX\\ &+\int_\omega (k-a_0(S))\nabla G_0(S,z)\nabla_S\big(\int_\omega (k-a_0(X))\nabla_X G_1(X,S)\nabla G_0(X,y)dX\big)dS.\end{aligned}$$

Using twice the equality (3.3) with $\varepsilon = 1$,

$$G_1(z,y) - G_0(z,y) = \int_\omega [k-a_0(\frac{x}{|x|})]\nabla G_0(x,y)\nabla G_1(x,z)\,dx, \tag{3.9}$$

we obtain the given expression of $G_{(m,n)}$. □

4. Asymptotics of the potential

In this section, we derive an asymptotic expansion of $u_\varepsilon - u_0$ in $H^{1/2}(\partial B_R)$. To this end, we first derive an asymptotic expansion of the difference $G_\varepsilon - G_0$ in $H^1(\mathcal{O}\times\mathcal{O})$, where $\mathcal{O}$ is any open set contained in $B_R\setminus\overline{\omega_\varepsilon}$. This result is then applied to expand the integral operators with which the potential u_ε can be represented.

THEOREM 4.1. *Let $\mathcal{O}$ be a bounded open subset of $\mathbb{R}^2\setminus\overline{\omega}$. Then, there exists $\varepsilon_0 > 0$, such that for $0<\varepsilon<\varepsilon_0$, the expansion (3.7) converges in $H^1(\mathcal{O}\times\mathcal{O})$.*

The proof of this result uses the following lemmas:

LEMMA 4.2. *Let $\mathcal{O}$ be a bounded open set contained in $\mathbb{R}^2\setminus\overline{\omega}$. The functions $\nabla_x G_0(x,y)$ and $\nabla_y(\nabla_x G_0)(x,y)$ are in $\big(L^2(\omega\times\mathcal{O})\big)^2$ and $\big(L^2(\omega\times\mathcal{O})\big)^4$ respectively.*

PROOF. Fix $(x,y)\in\omega\times\mathcal{O}$. ¿From the explicit expression of G_0, we deduce that

$$\nabla_x G_0(x,y) = -\sum_{n\ge 1}\frac{r_x^{\lambda_n-1}}{2\lambda_n r_y^{\lambda_n}}\varphi_n(\theta_y)\begin{pmatrix}\lambda_n\varphi_n(\theta_x)\cos(\theta_x)-\varphi_n'(\theta_x)\sin(\theta_x)\\ \lambda_n\varphi_n(\theta_x)\sin(\theta_x)+\varphi_n'(\theta_x)\cos(\theta_x)\end{pmatrix}.$$

Therefore

$$\nabla_y(\nabla_x G_0)(x,y) = -\sum_{n\ge 1}\frac{r_x^{\lambda_n-1}}{2\lambda_n r_y^{\lambda_n+1}}\begin{pmatrix}\psi_{n,1}(\theta_x)\psi_{n,1}(\theta_y) & \psi_{n,1}(\theta_x)\psi_{n,2}(\theta_y)\\ \psi_{n,2}(\theta_x)\psi_{n,1}(\theta_y) & \psi_{n,2}(\theta_x)\psi_{n,2}(\theta_y)\end{pmatrix},$$

where

$$\begin{aligned}\psi_{n,1}(\theta) &= \lambda_n\varphi_n(\theta)\cos(\theta)-\varphi_n'(\theta)\sin(\theta),\\ \psi_{n,2}(\theta) &= \lambda_n\varphi_n(\theta)\sin(\theta)+\varphi_n'(\theta)\cos(\theta).\end{aligned}$$

We first remark that there exists a constant $c>0$ such that $\frac{r_x}{r_y}<c<1$ uniformly for (x,y) in $\overline{\omega}\times\overline{\mathcal{O}}$. It is easy to see that for fixed integers n, i and j the functions

$$\left(\frac{r_x}{r_y}\right)^{\lambda_n}\psi_{n,j}(\theta_y)\psi_{n,i}(\theta_x),$$

$$\left(\frac{r_x}{r_y}\right)^{\lambda_n}\varphi_n(\theta_y)\psi_{n,i}(\theta_x),$$

are in $L^2(\omega\times\mathcal{O})$. In addition,

$$\| \left(\frac{r_x}{r_y}\right)^{\lambda_n-1} \psi_{n,j}(\theta_y)\psi_{n,i}(\theta_x)\|_{L^2(\omega\times\mathcal{O})} \leq \frac{C_1}{\min(a_1,a_2)}\lambda_n c^{\lambda_n-1}, \tag{4.1}$$

$$\| \left(\frac{r_x}{r_y}\right)^{\lambda_n} \varphi_n(\theta_y)\psi_{n,i}(\theta_x)\|_{L^2(\omega\times\mathcal{O})} \leq \frac{C_2}{\min(a_1,a_2)}\lambda_n c^{\lambda_n-1}, \tag{4.2}$$

where the constants C_1 and C_2 only depend on $\mathcal{O}$ and ω. Since $(\lambda_n)_{n\geq 1}$ is a non-decreasing positive sequence and satisfies (2.3), the series in the expressions of the functions $\nabla_x G_0(x,y)$ and $\nabla_y(\nabla_x G_0)(x,y)$ are exponentially convergent in $\big(L^2(\omega\times\mathcal{O})\big)^2$ and $\big(L^2(\omega\times\mathcal{O})\big)^4$ respectively. □

Lemma 4.3. *Let $\mathcal{O}$ be a bounded open set of $\mathbb{R}^2\setminus\overline{\omega}$. The functions $\nabla_x G_1(x,y)$ and $\nabla_y(\nabla_x G_1)(x,y)$ are in $\big(L^2(\omega\times\mathcal{O})\big)^2$ and $\big(L^2(\omega\times\mathcal{O})\big)^4$ respectively.*

Proof. In the appendix, it is proved that for R large enough so that $\omega,\mathcal{O}\subset B_R$, and for $x,y\in B_R$,

$$\begin{aligned} G_1(x,y) &= g_1(x,y)+v_1(x,y) \\ &\quad + R\ln(R)\varphi_0\big(\int_{\partial B_R}[a_0\Lambda_0(g_1(.,y)) - a_0\partial_r g_1(.,y)]d\sigma\big). \end{aligned}$$

Therefore

$$\nabla_x G_1(x,y) \;=\; \nabla_x g_1(x,y)+\nabla_x v_1(x,y).$$

Denote $v_{1,0}(x,y) = v_1(x,y)$, $g_{1,0}(x,y) = g_1(x,y)$ and

$$\nabla_y v_1(x,y) = \begin{pmatrix} v_{1,1}(x,y)\\ v_{1,2}(x,y)\end{pmatrix}, \nabla_y g_1(x,y) = \begin{pmatrix} g_{1,1}(x,y)\\ g_{1,2}(x,y)\end{pmatrix}.$$

Since $\omega\cap\mathcal{O}=\emptyset$, $\nabla_x g_1(x,y), \nabla_y(\nabla_x g_1)(x,y)$ are in $\big(L^2(\omega\times\mathcal{O})\big)^2$ and $\big(L^2(\omega\times\mathcal{O})\big)^4$ respectively by elliptic regularity [**GT**].
The functions $v_{1,j}(x,y)$, $j=0,1,2$ satisfy (see the appendix)

$$\left\{\begin{array}{rcll} \mathrm{div}(a_1\nabla_x v_{1,j}(x,y)) &=& 0 & \text{in } B_R\\ a_0\partial_r v_{1,j} - a_0\Lambda_0(v_{1,j}) &=& h_{1,j} & \text{on } \partial B_R. \end{array}\right.$$

where

$$\begin{aligned} h_{1,j} &= a_0\Lambda_0(g_{1,j}(.,y)) - a_0\partial_r g_{1,j}(.,y) \\ &\quad -a_0\varphi_0\int_{\partial B_R}[a_0\Lambda_0(g_1(.,y)) - a_0\partial_r g_1(.,y)]d\sigma\,\delta_{0,j}. \end{aligned}$$

Using the Lax Milgram theorem as in the appendix shows that there exists a constant $C>0$ such that

$$\|\nabla_x v_{1,j}(.,y)\|_{\left(L^2(B_R)\right)^2} \leq C\|h_{1,j}(.,y)\|_{H^{-\frac12}(\partial B_R)}.$$

On the other hand, we deduce from the properties of the operator Λ_0 that

$$\|h_{1,j}(.,y)\|_{H^{-\frac12}(\partial B_R)} \;\leq\; C\big(\|g_{1,j}(.,y)\|_{H^{\frac12}(\partial B_R)} + \|\partial_r g_{1,j}(.,y)\|_{H^{-\frac12}(\partial B_R)}\big).$$

Since $\mathcal{O} \subset B_R$, we infer by elliptic regularity [**GT**] that $g_{1,j}(x,y)$ and $\partial_r g_{1,j}(x,y)$ are in $L^2(\mathcal{O}, H^{\frac{1}{2}}(\partial B_R))$ and $L^2(\mathcal{O}, H^{-\frac{1}{2}}(\partial B_R))$ respectively, and the previous estimates imply that $\|\nabla_x v_{1,j}(x,y)\|_{\left(L^2(\omega\times\mathcal{O})\right)^2}$ is bounded by

$$C\left(\|g_{1,j}(x,y)\|_{L^2(\mathcal{O},H^{\frac{1}{2}}(\partial B_R))} + \|\partial_r g_1(x,y)\|_{L^2(\mathcal{O},H^{-\frac{1}{2}}(\partial B_R))}\right),$$

which achieves the proof.

□

Proof of theorem 4.1

The Lebesgue dominated convergence theroem and the regularity results of lemmas 4.2 and 4.3 show that the right hand side of (3.9) belongs to $H^1(\mathcal{O}\times\mathcal{O})$, and thus $G_1(z,y) - G_0(z,y) \in H^1(\mathcal{O}\times\mathcal{O})$.

Consequently, $G_{(m,n)} \in H^1(\mathcal{O}\times\mathcal{O})$, and additionnally, there exists a constant $C > 0$ such that

$$\|G_{(m,n)}\|_{H^1(\mathcal{O}\times\mathcal{O})} \leq C, \qquad \text{for } 2 \leq m+n.$$

Therefore, the Neumann series (3.7) converges in $H^1(\mathcal{O}\times\mathcal{O})$, which proves the theorem. □

Next, we derive the expansion of the potential u_ε, solution to (1.6). To this end, we introduce the following integral operators, defined for $\varepsilon \geq 0$

$$S_\varepsilon : H^{-\frac{1}{2}}(\partial B_R, a_0) \to H^{\frac{1}{2}}(\partial B_R, a_0),$$
$$S_\varepsilon\varphi(x) := \int_{\partial B_R} a_0 G_\varepsilon(x,y)\varphi(y)d\sigma_y.$$

$$D_\varepsilon : H^{\frac{1}{2}}(\partial B_R, a_0) \to H^{\frac{1}{2}}(\partial B_R, a_0),$$
$$D_\varepsilon\varphi(x) := \int_{\partial B_R} a_0 \partial_r G_\varepsilon(x,y)\varphi(y)d\sigma_y.$$

We start by giving some properties of the single and double layers S_0 and D_0.

THEOREM 4.4. *(i) For $R \neq 1$, S_0 is invertible from $H^{-\frac{1}{2}}(\partial B_R, a_0)$ to $H^{\frac{1}{2}}(\partial B_R, a_0)$. (ii) D_0 is a bounded operator from $H^{\frac{1}{2}}(\partial B_R, a_0)$ to $H^{\frac{1}{2}}(\partial B_R, a_0)$. In addition, $D_0 = \frac{1}{2}I$ on $H_0^{\frac{1}{2}}(\partial B_R, a_0)$.*

PROOF. (i) We first remark that

$$\begin{aligned} G_0(x,y)|_{\partial B_R\times\partial B_R} &= \sum_{n=0}^{\infty} w_n(R,R)\varphi_n(\theta_x)\varphi_n(\theta_y), \\ &= \ln(R)\varphi_0(\theta_x)\varphi_0(\theta_y) - \sum_{n=1}^{\infty}\frac{1}{2\lambda_n}\varphi_n(\theta_x)\varphi_n(\theta_y). \end{aligned}$$

Let $\phi \in H^{-\frac{1}{2}}(\partial B_R, a_0)$. Using the explicit expression of the kernel of S_0, we find that:

$$S_0\phi(\theta_x) = R\ln(R)\phi_0\varphi_0(\theta_x) - \sum_{n=1}^{\infty}\frac{R}{2\lambda_n}\phi_n\varphi_n(\theta_x),$$

and thus,

$$\|S_0\phi\|^2_{\frac{1}{2}} = R^2\ln^2(R)|\phi_0|^2 + R^2\sum_{n=1}^{\infty}\frac{(1+n^2)^{\frac{1}{2}}}{4\lambda_n^2}|\phi_n|^2.$$

The inequalities (2.3) imply the existence of a constant $C_0 > 0$, such that

$$\frac{1}{C_0}(1+n^2)^{-\frac{1}{2}} \leq \frac{(1+n^2)^{\frac{1}{2}}}{4\lambda_n^2} \leq C_0(1+n^2)^{-\frac{1}{2}}, \text{ for } n \geq 1$$

and thus, for some constant $C_i > 0,\ i = 1, 2$.

$$C_1\sum_{n=0}^{\infty}(1+n^2)^{-\frac{1}{2}}|\phi_n|^2 \leq \|S_0\phi\|^2_{\frac{1}{2}} \leq C_2\sum_{n=0}^{\infty}(1+n^2)^{-\frac{1}{2}}|\phi_n|^2,$$

Consequently, $S_0 : H^{-\frac{1}{2}}(\partial B_R, a_0) \longrightarrow H^{\frac{1}{2}}(\partial B_R, a_0)$ is bounded and coercive.

(ii) Again from the explicit expression of the Green function G_0 we obtain

$$\partial_{r_y}G_0(x,y)|_{\partial B_R\times\partial B_R} = \frac{1}{2R}\sum_{n=1}^{\infty}\varphi_n(\theta_x)\varphi_n(\theta_y),$$

so that

$$D_0\phi(\theta_x) = \frac{1}{2}\sum_{n=1}^{\infty}\phi_n\varphi_n(\theta_x) = \frac{1}{2}\phi(\theta_x) - \frac{1}{2}\phi_0\varphi_0(\theta_x).$$

We deduce that D_0 is bounded from $H^{\frac{1}{2}}(\partial B_R, a_0)$ to $H^{\frac{1}{2}}(\partial B_R, a_0)$. It is easy to see that $D_0 = \frac{1}{2}I$ on $H_0^{\frac{1}{2}}(\partial B_R, a_0)$. $\square$

THEOREM 4.5. *(i) For $R \neq 1$, S_ε is symmetric and invertible from $H^{-\frac{1}{2}}(\partial B_R, a_0)$ to $H^{\frac{1}{2}}(\partial B_R, a_0)$. In addition it has the following asymptotic*

$$S_\varepsilon = S_0 + \varepsilon^{2\lambda_1}S_{(1,1)} + \varepsilon^{\lambda_1+\lambda_2}S_{(1,2)} + o(\varepsilon^{\lambda_1+\lambda_2}),$$

The operators $S_{(1,i)}$, $i = 1, 2$ and $o(\varepsilon^{\lambda_1+\lambda_2})$ are bounded from $H^{-\frac{1}{2}}(\partial B_R, a_0)$ to $H^{\frac{1}{2}}(\partial B_R, a_0)$.

(ii) $I - D_\varepsilon$ is invertible form $H_0^{\frac{1}{2}}(\partial B_R, a_0)$ to $H_0^{\frac{1}{2}}(\partial B_R, a_0)$. In addition it has the following asymptotics

$$(I - D_\varepsilon)^{-1} = (I - D_0)^{-1} + \varepsilon^{2\lambda_1}D_{(1,1)} + \varepsilon^{\lambda_1+\lambda_2}D_{(1,2)} + o(\varepsilon^{\lambda_1+\lambda_2}),$$

The operators $D_{(1,i)}$, $i = 1, 2$ and $o(\varepsilon^{\lambda_1+\lambda_2})$ are bounded from $H_0^{\frac{1}{2}}(\partial B_R, a_0)$ to $H_0^{\frac{1}{2}}(\partial B_R, a_0)$.

PROOF. We deduce from theorem 4.1 and from the expression of their kernels, that S_ε and D_ε converge strongly respectively to S_0 and D_0, as $\varepsilon \to 0$, and in addition,

$$\begin{aligned} D_\varepsilon &= D_0 + \varepsilon^{2\lambda_1}D'_{(1,1)} + \varepsilon^{\lambda_1+\lambda_2}D'_{(1,2)} + +o(\varepsilon^{\lambda_1+\lambda_2}), \\ S_\varepsilon &= S_0 + \varepsilon^{2\lambda_1}S_{(1,1)} + \varepsilon^{\lambda_1+\lambda_2}S_{(1,2)} + +o(\varepsilon^{\lambda_1+\lambda_2}). \end{aligned}$$

One can derive explicitly the expressions of the operators $S_{(1,i)}$, $D'_{(1,i)}$, $i = 1, 2$:

$$S_{(1,i)} : H^{-\frac{1}{2}}(\partial B_R, a_0) \to H^{\frac{1}{2}}(\partial B_R, a_0),$$
$$S_{(1,i)}\varphi(x) := \int_{\partial B_R} a_0 G_{(1,i)}(x,y)\varphi(y)d\sigma_y.$$

$$D_{(1,i)} : H_0^{\frac{1}{2}}(\partial B_R, a_0) \to H_0^{\frac{1}{2}}(\partial B_R, a_0),$$
$$D_{(1,i)}\varphi(x) := \int_{\partial B_R} a_0 \partial_r G_{(1,i)}(x,y)\varphi(y)d\sigma_y.$$

Therefore S_ε and $I - D_\varepsilon$ are invertible for ε small enough. Moreover, the asymptotic expansion of $I - D_\varepsilon$ can be derived from the Neumann series

$$\begin{aligned}(I - D_\varepsilon)^{-1} &= (I - D_0)^{-1}\big(I - (D_\varepsilon - D_0)(I - D_0)^{-1}\big)^{-1} \\ &= (I - D_0)^{-1} + \sum_{n=1}^{\infty}(I - D_0)^{-1}\big((D_\varepsilon - D_0)(I - D_0)^{-1}\big)^n.\end{aligned}$$

Taking $D_{(1,i)} = (I - D_0)^{-1}D'_{(1,i)}(I - D_0)^{-1}$ when $i = 1, 2$ we obtain the form given in the theorem. □

THEOREM 4.6. *Let u_ε be the solution of the problem* (1.6). *There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$,*

$$u_\varepsilon(x) = u_0(x) + \varepsilon^{2\lambda_1}u_{(1,1)}(x) + \varepsilon^{\lambda_1+\lambda_2}u_{(1,2)}(x) + o(\varepsilon^{\lambda_1+\lambda_2}),$$

holds in $H^{\frac{1}{2}}(\partial B_R, a_0)$, where $u_0(x)$ is the solution to the problem (1.5).

PROOF. We set $\tilde{f}(x) = f(x)/a_0(x)$, $x \in \partial B_R$. Using the Green formula, we see that the trace of u_ε, $\varepsilon \geq 0$, on the boundary ∂B_R, is a solution to the following integral equation

$$(I - D_\varepsilon)u_\varepsilon(x) = -S_\varepsilon\tilde{f}(x), \quad x \in \partial B_R. \tag{4.3}$$

Since $I + D_\varepsilon$ is invertible on $H_0^{\frac{1}{2}}(\partial B_R, a_0)$ we have

$$u_\varepsilon(x) = -(I - D_\varepsilon)^{-1}S_\varepsilon\tilde{f}(x), \quad x \in \partial B_R.$$

We deduce from theorem 4.5 that the following expansion for u_ε holds in $H_0^{\frac{1}{2}}(\partial B_R, a_0)$

$$u_\varepsilon(x) = u_0(x) + u_{1,1}\varepsilon^{2\lambda_1} + u_{1,2}\varepsilon^{\lambda_1+\lambda_2} + o(\varepsilon^{\lambda_1+\lambda_2}),$$

where

$$\begin{aligned}u_0(x) &= -(I - D_0)^{-1}S_0\tilde{f}(x), \\ u_{1,1}(x) &= -\big((I - D_0)^{-1}S_{(1,1)} + D_{(1,1)}S_0\big)\tilde{f}(x), \\ u_{1,2}(x) &= -\big((I - D_0)^{-1}S_{(1,2)} + D_{(1,2)}S_0\big)\tilde{f}(x),\end{aligned}$$

and $\|o(\varepsilon^{\lambda_1+\lambda_2})\|_{\frac{1}{2}} \leq C\varepsilon^{\lambda_1+\lambda_2}$.

Since the expression of the Green function G_0 is known explicitly, we can derive an analytic representation of the first term in the expansion:

$$u_{1,1}(\theta) = m_{1,1}\left(\int_0^{2\pi} f(\beta)\varphi_1(\beta)d\beta\right)\varphi_1(\theta),$$

where

$$m_{1,1} = -2R\int_0^{2\pi}\int_0^{2\pi} a_0(\gamma)a_0(\beta)[G_1 - G_0 - \frac{R}{\lambda_1}\partial_r(G_1 - G_0)](R,\gamma,R,\beta)\varphi_1(\gamma)\varphi_1(\beta)d\gamma d\beta.$$

Using the equality (3.9) shows that

$$\int_0^{2\pi} a_0(\gamma)\partial_r(G_1 - G_0)(R,\gamma,R,\beta)\varphi_1(\gamma) \;=\; -\frac{\lambda_1}{R}\int_0^{2\pi} a_0(\gamma)(G_1 - G_0)(R,\gamma,R,\beta)\varphi_1(\gamma).$$

Finally, we obtain

$$m_{1,1} \;=\; -4R\int_0^{2\pi}\int_0^{2\pi} a_0(\gamma)a_0(\beta)(G_1 - G_0)(R,\gamma,R,\beta)\varphi_1(\gamma)\varphi_1(\beta)d\gamma d\beta. \tag{4.4}$$

□

REMARK 4.7. The structure of $u_{1,1}$ can be related to the structure of the first order term in the expansion (1.1), when the background medium is smooth. However, in our context, the polarization properties of the inhomogeneity has a more complex form, which can be related to how the singular function $r^{\lambda_1}\varphi_1(\theta)$ interacts with the inhomogeneity.

5. Appendix

In this section, we construct the Green function for the perturbed problem, solution to

$$\begin{cases} \mathrm{div}(a_\varepsilon(x)\nabla G_\varepsilon(x,y)) \;=\; \delta_y(x) \quad \text{in } \mathbb{R}^2 \\ \text{the radiation condition (2.10),} \end{cases} \tag{5.1}$$

Let $y \in \mathbb{R}^2$ and $R > \sup(\varepsilon\,\mathrm{diam}(\omega)), |y|)$. Let $H^s(B_R, a_0)$ denote the space of functions

$$\phi(x) \quad = \quad \sum_{n\geq 0}\phi_n\varphi_n(\theta_x),$$

defined on ∂B_R, such that

$$\sum_{n\geq 0}(1+n^2)^s|\phi_n|^2 \quad < \quad \infty.$$

One easily checks that solutions to

$$\mathrm{div}(a_0\nabla u_0(x)) \quad = \quad 0 \quad \text{in } \mathbb{R}^2\setminus\overline{B_R}, \tag{5.2}$$

that satisfy the radiation condition (2.10), can be written in the form

$$u(x) \quad = \quad u_0\ln(r_x) \;+\; \sum_{n\geq 1}u_n r_x^{-\lambda_n}\varphi_n(\theta_x).$$

In particular, when $x \in \partial B_R$

$$\begin{aligned} u(x) \quad &= \quad u_0\ln(R) \;+\; \sum_{n\geq 1}u_n R^{-\lambda_n}\varphi_n(\theta_x) \\ \partial_r u(x) \quad &= \quad \frac{u_0}{R} \;-\; \sum_{n\geq 1}\lambda_n u_n R^{-\lambda_n-1}\varphi_n(\theta_x). \end{aligned}$$

It follows that the Dirichlet to Neumann operator for (5.2) can be defined on $L^2(\partial B_R)$ by

$$\Lambda_0(\phi)(x) \quad = \quad \frac{\phi_0}{R\ln(R)} - \sum_{n\geq 1} \frac{\lambda_n}{R}\phi_n\varphi_n(\theta_x),$$

where, for $n \geq 0$,

$$\phi_n \quad = \quad \int_0^{2\pi} a_0(y)\phi(y)\varphi_n(\theta_y)\, d\sigma_y. \tag{5.3}$$

Recalling (2.3), one easily sees that Λ_0 is a continuous operator from $H^s(\partial B_R, a_0)$ into $H^{s+1}(\partial B_R, a_0)$, for any $s \geq 0$.

Let g_ε denote the solution to

$$\begin{cases} \operatorname{div}(a_\varepsilon(x)g_\varepsilon(x,y)) & = & \delta_y(x) & \text{in } B_R \\ g_\varepsilon(x,y) & = & 0 & \text{on } \partial B_R. \end{cases}$$

The existence of g_ε, solution to an elliptic equation with L^∞ coefficients, with a right-hand side which is a measure, follows from the De Giorgi-Nash theorem [**LSW**]. We also define the space

$$H \quad = \quad \{v \in H^1(B_R), \quad \int_{\partial B_R} a_0(y)v(y)\, d\sigma_y \ = \ 0\}.$$

PROPOSITION 5.1. let $\varepsilon > 0$, $y \in \mathbb{R}^2$ and $R > 0$, such that $R > \max(\varepsilon \operatorname{diam}(\omega), |y|)$. Then, the equation

$$\begin{cases} \operatorname{div}(a_\varepsilon \nabla v_\varepsilon) & = & 0 & \text{in } B_R \\ a_0\partial_r v_\varepsilon - a_0\Lambda_0(v_\varepsilon) & = & h_\varepsilon & \text{on } \partial B_R, \end{cases} \tag{5.4}$$

where

$$\begin{aligned} h_\varepsilon \quad = \quad & a_0\Lambda_0(g_\varepsilon(.,y)) - a_0\partial_r g_\varepsilon(.,y) \\ & -a_0\varphi_0 \int_{\partial B_R} a_0[\Lambda_0(g_\varepsilon(.,y)) - \partial_r g_\varepsilon(.,y)]\, d\sigma_y, \end{aligned} \tag{5.5}$$

has a unique solution $v_\varepsilon = v_\varepsilon(.,y)$ in H.

Proof: Let $a\ : H \times H \to \mathbb{R}$ denote the bilinear form

$$a(u,v) \quad = \quad \int_{B_R} a_0 \nabla u \cdot \nabla v \ - \int_{\partial B_R} a_0\Lambda_0(u)\, v\, d\sigma,$$

and we set, for $v \in H$,

$$l(v) \quad = \quad \int_{B_R} h_\varepsilon v\, d\sigma.$$

For $u, v, \in H$, we can estimate

$$|a(u,v)| \quad \leq \quad ||a_0||_\infty ||\nabla u||_{0,B_R}||\nabla v||_{0,B_R} \ + \ ||\Lambda_0||\, ||u||_{1/2,B_R}||v||_{1/2,B_R},$$

which shows that a is continuous. Further, we note that for $u \in H$, the expression of $\Lambda_0(u)$ takes the form

$$\Lambda_0(u) \quad = \quad -\sum_{n\geq 1}\frac{\lambda_n}{R}u_n\varphi_n(\theta_x)$$

with the notation (5.3), so that

$$\int_{\partial B_R} a_0 \Lambda_0(u)\, u\, d\sigma \quad \leq \quad 0.$$

Hence, for $u \in H$,

$$a(u,u) \quad \geq \quad \min a_\varepsilon \int_{B_R} |\nabla u|^2,$$

and an easy adaptation of the proof of the Poincaré-Friedrichs inequality shows that a is coercive on H.

Finally, since $|y| < R$, the Green's function g_ε is in $H^1(B_R \setminus \overline{B_{R'}})$, for some $|y| < R' < R$, the form l is easily seen to be continous on H. The Lax Milgram Lemma shows that there exists a unique solution in H to the variationnal problem

$$\forall\, v \in H, \quad a(u,v) \quad = \quad l(v),$$

which proves the proposition. □

Next, we note that the function

$$w_\varepsilon(x,y) \quad = \quad v_\varepsilon(x,y) + R\ln(R)\left(\int_{B_R} a_0\left[\Lambda_0(g_\varepsilon) - \partial_r g_\varepsilon\right] d\sigma\right)\varphi_0,$$

is a solution to

$$\begin{cases} \operatorname{div}(a_\varepsilon \nabla w_\varepsilon) &= \quad 0 \quad \text{in } B_R \\ a_0 \partial_r w_\varepsilon &= \quad a_0 \Lambda_0(g_\varepsilon) - \alpha_0 \partial_r g_\varepsilon \quad \text{on } \partial B_R. \end{cases}$$

Defining $G_\varepsilon(x,y) = g_\varepsilon(x,y) + w_\varepsilon(x,y)$, we obtain a function that satisfies

$$\begin{cases} \operatorname{div}(a_\varepsilon \nabla G_\varepsilon) &= \quad \delta_y(x) \quad \text{in } B_R \\ a_0 \partial_r G_\varepsilon &= \quad a_0 \Lambda_0(G_\varepsilon) \quad \text{on } \partial B_R, \end{cases}$$

which can be extended to the whole of $\mathbb{R}^2$ by setting for $|x| > R$

$$G_\varepsilon(x,y) \quad = \quad \int_{\partial B_R} a_0 \left[\Lambda_0(G_\varepsilon(z,x)) G_0(z,y) - G_\varepsilon(x,y)\partial_n G_0(z,y)\right] d\sigma_z.$$

The extended function clearly satisfies

$$\operatorname{div}(a_\varepsilon \nabla G_\varepsilon(x,y)) \quad = \quad \delta_y(x) \quad \text{in } \mathbb{R}^2.$$

and due to the form (2.8) of G_0, satisfies the radiation condition (2.10). Finally, uniqueness of G_ε can be proven as in proposition 2.2.

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