On the spectrum of Poincaré variational problem for two close-to-touching inclusions in 2D

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Abstract

We study the spectrum of the Poincaré variational problem for two close to touching inclusions in $I\!R^2$. We derive the asymptotics of its eigenvalues as the distance between the inclusions tends to zero.

1 Introduction

This work concerns the regularity of solutions of elliptic PDE's in composite media that contain touching or close to touching inhomogeneities, embedded in a matrix phase. In mechanics, regions where the presence of hard inclusions form narrow gaps are likely to concentrate stress, and therefore are prone to fracture. In optics, electromagnetic fields are likely to concentrate in narrow channels where the parameter contrast with surrounding regions is large, a fact that could be useful in applications such as microscopy, spectroscopy or bio-sensing.

Over the last decade, adressing a question originally raised by I. Babuška [8], a number of mathematical works have focused on estimating the size of gradients in composite media containing inhomogeneities with *smooth boundaries*, in terms of inter-inclusion distances and coefficient contrast. The case of 2 circular inclusions with finite conductivity, separated by a distance δ, was studied for the conduction equation in [11]. There, a $W^{1,\infty}$ bound, uniform in δ, is established using the maximum principle. YanYan

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Li and M. Vogelius [20] extended this result to the case of a scalar equation in piecewise Hölder media (i.e., in which the boundary of each inclusion has $C^{1,\alpha}$ regularity). The uniform boundedness of the gradient was then generalized to strongly elliptic systems, including the system of elasticity, by YanYan Li and L. Nirenberg [19].

In contrast, when the material coefficients are degenerate (perfectly conducting or insulating inclusions), the gradients may blow up as the inclusions come to touching (see e.g. [11]). In this case, the dependance of the bounds on the inter-inclusion distance was explicited in [9], who studied the case of two perfectly conducting $\mathcal{C}^{2,\alpha}$ inhomogeneities embedded in a domain $\Omega \subset I\!R^n$ of conductivity $\gamma = 1$. The gradient of the potential is shown there to satisfy

$$\left\{ \begin{array}{lllr} ||\nabla u||_{L^\infty} & \leq & \frac{C}{\sqrt{\delta}}||u||_{L^2(\partial\Omega)} & \text{for } n=2, \\ ||\nabla u||_{L^\infty} & \leq & \frac{C}{\delta|\ln\delta|}||u||_{L^2(\partial\Omega)} & \text{for } n=3, \\ ||\nabla u||_{L^\infty} & \leq & \frac{C}{\delta}||u||_{L^2(\partial\Omega)} & \text{for } n=4. \end{array} \right. \tag{1}$$

The case $n = 2$ was derived independently by Yun, using conformal mapping techniques [26].

How the bounds blow up when both the inclusions come to touching and their conductivities degenerate has been analyzed in particular geometries, where the potential u may have a series representation that lends itself to asymptotic analysis [5, 4, 2, 12, 21]. Optimal upper and lower bounds on the potential gradients are derived in [5, 4] for nearly touching pairs of circular inclusions. Spherical inclusions are studied in [2].

In [6], the case of 2 discs $D_1, D_2 \subset I\!R^2$ of constant conductivities, separated by a distance δ, is considered. The potential u satisfies the PDE

$$\left\{ \begin{array}{lll} \text{div}(\gamma(X)\nabla u(X)) & = 0 & \text{in } I\!R^2 \\ u(X) - H(X) & \to 0 & \text{as } |X| \to \infty, \end{array} \right. \tag{2}$$

where the conductivity γ is constant in each inclusion and in $I\!R^2 \setminus \overline{D_1 \cup D_2}$, and where H is a given harmonic function. The potential u_δ is shown to decompose as the sum $u_r + u_s$ of a regular and a singular part. Specified to the case when the discs have the same radius $r = 1$ and the same conductivity k and are touching at the point 0 tangentially to the direction e_1, the result states that for some constant C_1, C_2, C_3, independent of δ and k

$$\left\{ \begin{array}{lll} |\nabla u_s|^+(\pm\delta, 0) & \geq & C_1 \frac{|\nabla H(0)|\cdot e_1^\perp}{2k+\sqrt{\delta}} \\ ||\nabla u_s||_{\infty,\Omega} & \leq & C_2 \frac{|\nabla H(0)|\cdot e_1^\perp}{2k+\sqrt{\delta}} \\ ||\nabla u_r||_{\infty,\Omega} & \leq & C_3, \end{array} \right. \tag{3}$$

if $k > 1$, while for $0 < k < 1$ the estimates read

$$\left\{ \begin{array}{lll} |\nabla u_s|^+(\pm\delta, 0) & \geq & C_1 \frac{|\nabla H(0)|\cdot e_1}{2k^{-1}+\sqrt{\delta}} \\ ||\nabla u_s||_{\infty,\Omega} & \leq & C_2 \frac{|\nabla H(0)|\cdot e_1}{2k^{-1}+\sqrt{\delta}} \\ ||\nabla u_r||_{\infty,\Omega} & \leq & C_3. \end{array} \right. \quad (4)$$

Further development has recently led to obtaining an asymptotic expansion of the potential with an explicit characterization of the singular part [14].

The results in [6] are based on the representation of u_δ as a series derived through the method of images. In a recent work [10], we studied the same configuration of 2 discs from an integral equation point of view. More precisely, we considered (2) when D_1, D_2 are the discs of radius 1, respectively centered at the points $(0, 1 + \delta/2)$ and $(0, -1 - \delta/2)$ so that the discs meet tangentially to the x_1–axis as $\delta \to 0$. One can seek the solution u of (2) in the form

$$u(X) \; = \; H(X) + S[\varphi](X) \; := \; H(X) + \begin{pmatrix} S_1 & 0 \\ 0 & S_2 \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \quad (5)$$

where S_i denotes the single layer potential operator on D_i

$$S_i[g](X) \; = \; \frac{1}{2\pi} \int_{\partial D_i} \ln(|X - Y|) g(Y) \, d\sigma_Y.$$

Expressing the transmission conditions satisfied by u, one sees that the layer potential $\varphi = (\varphi_1, \varphi_2)$ satisfies

$$(\lambda I - K^*) \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \; = \; \begin{pmatrix} \partial_{\nu_1} H_{|\partial D_1} \\ \partial_{\nu_2} H_{|\partial D_2} \end{pmatrix}, \quad (6)$$

where ν_i denotes the outward normal to D_i, $\lambda = \frac{k+1}{2(k-1)}$, and where

$$K^* \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \; = \; \begin{pmatrix} K_1^* & \frac{\partial}{\partial \nu_1} S_2 \\ \frac{\partial}{\partial \nu_2} S_1 & K_2^* \end{pmatrix} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}. \quad (7)$$

The operators K_i^* in the above expression are the standard layer potential operators defined on a single inclusion: For $g \in L^2(\partial D_i)$,

$$K_i^*[g](X) \; = \; \int_{\partial D_i} \frac{(X - Y) \cdot \nu_i(X)}{|X - Y|^2} g(Y) \, d\sigma_Y, \qquad X \in \partial D_i. \quad (8)$$

One of the nice features of (6) is that it decouples the contrast k and the distance δ. If K^* had a spectral decomposition with eigenvectors $(\varphi_n)_{n\geq 1}$ associated to eigenvalues λ_n, then one could easily construct a solution of (6) in the form

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \sum_{n\geq 1} \frac{\left\langle \varphi_n, \begin{pmatrix} \partial_{\nu_1} H_{|\partial D_1} \\ \partial_{\nu_2} H_{|\partial D_2} \end{pmatrix} \right\rangle}{\lambda - \lambda_n} \varphi_n.$$

Injecting this expression in (5) shows that the blow-up of ∇u as $\delta \to 0$ and as $k \to 0$ or $k \to +\infty$ (i.e. $\lambda \to -1/2$ or $\lambda \to 1/2$ respectively) depends on the behavior of $\lambda - \lambda_n$ in the above expression. And indeed, in [10] we show that $\lambda - \lambda_1 = 2k - \sqrt{\delta} + O(\delta)$ when $k > 1$, that $\lambda - \lambda_1 = 2k^{-1} + \sqrt{\delta} + O(\delta)$ when $0 < k < 1$, and that one does recover the estimates (3) through the spectral analysis of K^*.

The operator K^* is however not self adjoint, which complicates matters with respect to its spectral decomposition. One can nevertheless symmetrize K^* following a technique originally due to J. Plemelj [24] (see also the work of Korn [16, 17]) and revisited in modern terms in [15] (see also [18]): The Plemelj symmetrization principle implies that the operator K^*, and its adjoint, the Poincaré Neumann operator K, satisfy

$$SK^* = KS.$$

Since S is non-positive and self-adjoint, one can define a new inner-product on the space $L^2(\partial D_1) \times L^2(\partial D_2)$ by setting

$$\begin{aligned} < \varphi, \psi >_S &= < -S[\varphi], \psi >_{L^2} \\ &:= -\int_{\partial D_1} S_1[\varphi_1]\psi_1 - \int_{\partial D_2} S_2[\varphi_2]\psi_2, \end{aligned} \tag{9}$$

for which K^* becomes a compact self-adjoint operator, which therefore has a spectral decomposition. See [15] and also [1], where an operator similar to K^* is studied in the context of cloaking. Moreover, the eigenvalues of K^* can be obtained via a min-max principle. This idea was originally developed by Poincaré, when he was trying to extend Neumann's proof of existence of solutions to the Dirichlet problem to the case of non-convex domains [25, 13, 15]. The Poincaré variational problem (in the terminology of [15]) consists in optimizing the ratio $J'(u)/J(u)$ where (in the context described above)

$$J(u) = \int_D |\nabla u|^2 \qquad J'(u) = \int_{D'} |\nabla u|^2$$

among all functions $u \in W^{1,2}(I\!R^2)$ whose restriction to $D = D_1 \cup D_2$ and to $D' = I\!R^2 \setminus \overline{D_1 \cup D_2}$ are harmonic.

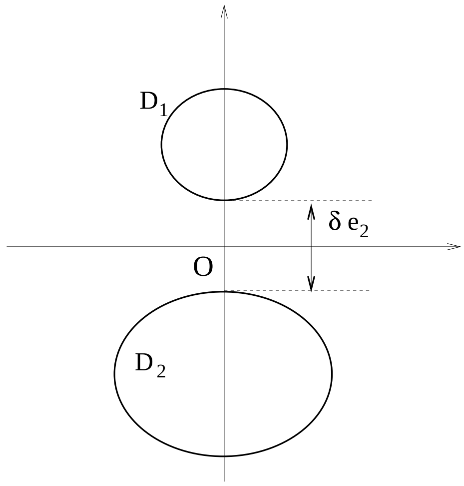

Figure 1: The inclusions D_1 and D_2 .

In this work, we extend our work on discs to more general inclusions $D_1, D_2 \subset I\!R^2$ and study the spectrum of the corresponding Poincaré variational problem. To fix ideas, we assume that D_1 and D_2 are translates of 2 reference touching inclusions

$$D_1 \;=\; D_1^0 + (0, \delta/2) \qquad D_2 \;=\; D_2^0 + (0, -\delta/2).$$

We assume that D_1^0 lies in the lower half–plane $x_2 < 0$, D_2^0 in the upper half–plane, and that they meet at the point 0 tangentially to the x_1–axis. We make the following additional assumptions on the geometry:

A1. The inclusions are strictly convex and only meet at the point 0.

A2. Around the point 0, ∂D_1^0 and ∂D_2^0 are parametrized by 2 curves $(x, \psi_1(x))$ and $(x, -\psi_2(x))$ respectively. The graph of ψ_1 (resp. ψ_2) lies above (resp. below) the x–axis.

A3. The boundary ∂D_i^0 of each inclusion is globally $\mathcal{C}^{1,\alpha}$ for some $0 < \alpha$.

A4. The function $\psi_1(x) + \psi_2(x)$ is equivalent to $C|x|^m$ as $x \to 0$, where $m \geq 2$ is a fixed integer and C is a positive constant.

To simplify the notations, we set $D := D_1 \cup D_2$, $D' = I\!R^2 \setminus \overline{D}$, and $\psi = \psi_1 + \psi_2 \geq 0$.

Our main result relates the behavior of the non-degenerate eigenvalues of K^* as $\delta \to 0$ to the geometry of the contact when the inclusions touch.

Theorem 1. *For two close to touching inclusions with contact of order m, the eigenvalues of the Poincaré variational problem contained in $]-1/2, 1/2[$ split in two families $(\lambda_n^{\delta,\pm})_{n\geq 1}$, with*

$$\begin{cases} \lambda_n^{\delta,+} \sim 1/2 - c_n^+ \delta^{1-\frac{1}{m}} + o(\delta^{1-\frac{1}{m}}) \\ \lambda_n^{\delta,-} \sim -1/2 + c_n^- \delta^{1-\frac{1}{m}} + o(\delta^{1-\frac{1}{m}}) \end{cases} \tag{10}$$

where $(c_n^\pm)_{n\geq 1}$ are increasing sequences of positive numbers, that only depend on the shapes of the inclusions, and that satisfy $c_n^\pm \sim n$ as $n \to \infty$.

The paper is organized as follows: Section 2 introduces some elements of potential theory and spaces of functions. In section 3, we show that the solution to (2) can be expanded on a basis of eigenfunctions associated to the Poincaré variational problem. Our version is slightly different from that exposed in [15] (see also [7]) as we optimize

$$\frac{J(u)}{J(u)+J'(u)} \quad = \quad \frac{\int_D |\nabla u|^2}{\int_{\mathbb{R}^2} |\nabla u|^2} \tag{11}$$

on the subspace of single layer potentials $\mathfrak{H}_S$. Roughly speaking, these are H^1_{loc} functions which are harmonic in D and in D', and which tend to 0 at infinity, see the precise definition (16). In Section 4, we show that the ratio (11) defines a norm on $\mathfrak{H}_S$. We also show that if $\varepsilon > 0$ is sufficiently small, this norm is equivalent to equivalent to

$$\frac{\sum_j ||u||^2_{\dot{H}^{1/2}(\partial D_j)}}{\sum_j ||u||^2_{\dot{H}^{1/2}(\partial D_j)} + \int_{-\varepsilon}^{\varepsilon} \frac{\left|u_{|\partial D_1} - u_{|\partial D_2}\right|^2}{\psi(x_1)+\delta}\, dx_1}.$$

This result is inspired from the work of Mazy'a, who derived asymptotic expansions of the solutions to elliptic problems in domains with cusps [22]. Essentially, the weight in the integral constrains the behavior of u as $\delta \to 0$ so as to keep the electrostatic energy $\int_{\mathbb{R}^2} |\nabla u|^2$ uniformly bounded. Finally, we prove Theorem 1 in Section 5.

As explained in [15], H. Poincaré had foreseen that a min-max principle was lurking behind the spectral properties of the operators K and K^*. However, at that time, he did not have all the Hilbert-space theoretical tools to make his insights rigorous. He suggested in particular that K and K^* would only have a positive spectrum. Our study provides another example (with those described in [15]) where these operators have both positive and negative eigenvalues.

2 Preliminaries

To cope with the usual difficulties related with the unboundedness of the Newtonian potential in 2D, we introduce the weighted Sobolev space

$$\mathcal{W}^{1,-1}(I\!R^2) \quad := \quad \left\{u;\ \frac{u(X)}{(1+|X|^2)^{1/2}\log(2+|X|^2)} \in L^2(I\!R^2),\ \nabla u \in L^2(I\!R^2)\right\}.$$

This space contains the constant functions and may be used to invert the Laplace operator in the plane [23]:

Proposition 1. *Let g be in $H^{1/2}(\partial D)$. The PDE*

$$\begin{cases} \Delta w(X) &= 0 \quad \text{in } D', \\ w(X) &= g(X) \quad \text{on } \partial D, \end{cases} \tag{12}$$

has a unique solution $w \in \mathcal{W}^{1,-1}(D')$. In addition, w satisfies

$$\int_{D'} |\nabla w(X)|^2\, dX \leq \int_{D'} |\nabla v(X)|^2\, dX,$$

for all v in $\mathcal{W}^{1,-1}(D')$ such that $v(X) = g(X)$ on ∂D.

We denote by $\mathcal{W}_0^{1,-1}(I\!R^2)$ the subset of functions $u \in \mathcal{W}^{1,-1}(I\!R^2)$ such that $u(X) = o(1)$ as $|X| \to \infty$. It follows from Hardy's inequality that $\mathcal{W}_0^{1,-1}(I\!R^2)$ endowed with the scalar product

$$(u,v)_{\mathcal{W}} \quad := \quad \int_{I\!R^2} \nabla u(X)\nabla v(X) dX, \tag{13}$$

is a Hilbert space (see for instance [23]). Given a harmonic function H, the Lax-Milgram lemma shows that there exists a unique solution to

$$\begin{cases} \operatorname{div}\left([1+(k-1)1_D(x)]\nabla u(x)\right) &= 0 \quad \text{in } I\!R^2 \\ u(x) - H(x) &\in \mathcal{W}_0^{1,-1}(I\!R^2), \end{cases} \tag{14}$$

Let $G(X,Y) = \frac{1}{2\pi}\ln(|X-Y|)$ denote the fundamental solution to the Laplace operator in dimension 2. We denote the single and double layer potentials on ∂D of functions $f \in H^{-\frac{1}{2}}(\partial D)$ and $g \in H^{\frac{1}{2}}(\partial D)$ as $S[f]$ and $D[g]$ respectively, where

$$\begin{aligned} S[f](X) &= \int_{\partial D} G(X,Y)f(Y)d\sigma_Y \qquad X \in I\!R^2 \\ D[g](X) &= \int_{\partial D} \frac{\partial G(X,Y)}{\partial \nu(Y)} g(Y)d\sigma_Y \qquad X \in D'. \end{aligned}$$

These operators are well defined, even if $D = D_1 \cup D_2$ is not connected, since $\delta > 0$ and since each boundary ∂D_i is smooth. They satisfy the jump conditions across ∂D [23]:

$$\begin{aligned} S[f]^+(X) &= S[f]^-(X) \qquad X \in \partial D, \\ D[g]^{\pm}(X) &= \left(\mp\frac{1}{2} + K\right)[g](X) \qquad X \in \partial D, \\ \frac{\partial}{\partial \nu} S[f]^{\pm}(X) &= \left(\pm\frac{1}{2} + K^*\right)[f](X) \qquad X \in \partial D, \end{aligned}$$

where $K^* : H^{-\frac{1}{2}}(\partial D) \to H^{-\frac{1}{2}}(\partial D)$ is a compact operator defined by

$$K^*[f](X) := \frac{1}{2\pi} \int_{\partial D} \frac{(X-Y)\cdot \nu(X)}{|X-Y|^2} g(Y)d\sigma_Y,$$

and $K : H^{\frac{1}{2}}(\partial D) \to H^{\frac{1}{2}}(\partial D)$ is the L^2 adjoint of K^*, defined by

$$K[g](X) = \frac{1}{2\pi} \int_{\partial D} \frac{(Y-X)\cdot \nu(Y)}{|X-Y|^2} g(Y)\, d\sigma_Y.$$

In dimension two the single layer potential is not always invertible. The following result can be found in [3] for connected domains.

Proposition 2. *The operator* $\mathbb{S} : H^{-\frac{1}{2}}(\partial D) \times I\!R \longrightarrow H^{\frac{1}{2}}(\partial D) \times I\!R$ *defined by*

$$\mathbb{S}[f,a] = \left(S[f]|_{\partial D} + a, \sum_{j=1}^{2} \int_{\partial D_j^\delta} f(Y)d\sigma_Y \right),$$

is invertible.

Proof. The operator S is Fredholm of index zero [23] and thus $\mathbb{S}$ has the same property. To show that it has a bounded inverse, we only need to prove that it is injective.
Let $(f,a) \in H^{-\frac{1}{2}}(\partial D) \times I\!R$ such that $\mathbb{S}[f,a] = (0,0)$, i.e.,

$$S[f]|_{\partial D} = -a, \quad \text{and} \quad \sum_{j=1}^{2} \int_{\partial D_j^\delta} f(Y) d\sigma_Y = 0.$$

The asymptotic properties of the Newtonian potential and the second equation above imply that

$$S[f](X) = O(\frac{1}{|X|}) \qquad \text{as } |X| \to +\infty,$$

and it follows that

$$\begin{aligned} \int_{I\!R^2} |\nabla S[f](X)|^2 \, dX &= -\int_{\partial D} S[f]|_D(X) f(X) d\sigma_X, \\ &= a \sum_{j=1}^{2} \int_{\partial D_j} f(Y) d\sigma_Y = 0. \end{aligned}$$

It follows that $S[f]$ is constant, $S[f](X) = -a = 0$ for all $X \in I\!R^2$. Finally, the jump of the normal derivative of the single layer potential gives $f = 0$ which finishes the proof. ∎

Let $\mathfrak{H}$ be the space of harmonic functions on $D \cup D'$, which vanish at infinity and with finite energy semi-norm

$$\|h\|_{\mathfrak{H}}^2 = \int_{D \cup D'} |\nabla h|^2 \, dX. \tag{15}$$

An element $h \in \mathfrak{H} \subset \left(C_0^\infty(I\!R^2)\right)'$ can be viewed as a distribution defined on $I\!R^2$, which satisfies

$$\Delta h = \mu \in \left(C_0^\infty(I\!R^2)\right)',$$

with $\text{supp}(\mu) \subset \partial D$. We note that a function $h \in \mathfrak{H}$ lies in $\mathcal{W}_0^{1,-1}(I\!R^2)$ if and only if $h^+ = h^-$ on ∂D. We define

$$\begin{aligned} \mathfrak{H}_S &:= \mathfrak{H} \cap \mathcal{W}_0^{1,-1}(I\!R^2) \\ &= \left\{ h \in \mathfrak{H};\ h^+ = h^- \text{ on } \partial D \right\}. \end{aligned} \tag{16}$$

Proposition 3. *The space $\mathfrak{H}_S$ is the subspace of single layer potentials in $\mathcal{W}_0^{1,-1}(I\!R^2)$, i.e.*

$$\mathfrak{H}_S \;=\; \{S[\varphi]; \varphi \in H^{-1/2}(\partial D), \int_{\partial D_1} \varphi|_{D_1} \;+\; \int_{\partial D_2} \varphi|_{D_2} \;=\; 0\}.$$

Proof. The properties of the Newtonian potential at infinty show that the set on the right is contained in $\mathfrak{H}_S$. Conversely, if $u \in \mathfrak{H}_S$, then we infer from Proposition 2 that for some $\varphi \in H^{-1/2}(\partial D)$ and $a \in I\!R$,

$$u|_{\partial D} \;=\; S[\varphi] + a \quad \text{and} \quad \int_{\partial D_1} \varphi|_{D_1} \;+\; \int_{\partial D_2} \varphi|_{D_2} \;=\; 0.$$

By proposition 1 and by the uniqueness to solutions of the Dirichlet problem in a bounded domain

$$u(X) \;=\; S[\varphi](X) + a, \qquad \forall\, X \in I\!R^2.$$

The requirement that $u \in \mathcal{W}_0^{-1,1}(I\!R^2)$ shows that $a = 0$, hence the result. ■

The orthogonal space to $\mathfrak{H}_S$ with respect to the scalar product

$$< f, g >_{\mathfrak{H}} \;=\; \int_D \nabla f \cdot \nabla g + \int_{D'} \nabla f \cdot \nabla g$$

is the subspace of *double layer potentials.* It is shown in [15] that

$$\mathfrak{H}_D \;:=\; \left\{ h \in \mathfrak{H};\; \frac{\partial h}{\partial \nu}^{+} = \frac{\partial h}{\partial \nu}^{-} \text{ on } \partial D \right\}.$$

3 The Poincaré variational problem

In this paragraph, we introduce an operator whose spectrum is obtained by optimizing (11).
For $u \in \mathcal{W}_0^{1,-1}(I\!R^2)$, we infer from the Riesz Theorem that there exists a unique $T_\delta u(X) \in \mathcal{W}_0^{1,-1}(I\!R^2)$ such that for all $v \in \mathcal{W}_0^{1,-1}(I\!R^2)$

$$\int_{I\!R^2} \nabla T_\delta u(X) \nabla v(X) dX \;=\; \int_D \nabla u(X) \nabla v(X) dX. \tag{17}$$

The operator $T_\delta : \mathcal{W}_0^{1,-1}(I\!R^2) \to \mathcal{W}_0^{1,-1}(I\!R^2)$ is easily seen to be self-adjoint and bounded with with norm $\|T_\delta\| \leq 1$. The spectral problem for T_δ writes as:
Find $(w, \beta) \in \mathcal{W}_0^{1,-1}(I\!R^2) \times (I\!R \setminus \{0,1\})$, $w \neq 0$, such that $\forall\, v \in \mathcal{W}_0^{1,-1}(I\!R^2)$,

$$\beta \int_{I\!R^2} \nabla w(X) \nabla v(X) dX \;=\; \int_D \nabla w(X) \nabla v(X) dX. \tag{18}$$

Integrating by parts, one immediately obtains that any eigenfunction w is harmonic in D and in D', and for $X \in \partial D$ satisfies the transmission conditions

$$w|^+_{\partial D}(X) \;=\; w|^-_{\partial D}(X) \qquad \frac{\partial w}{\partial \nu}|^+_{\partial D}(X) \;=\; \left(1 - \frac{1}{\beta}\right) \frac{\partial w}{\partial \nu}|^-_{\partial D}(X),$$

where $w(X)|^{\pm} = \lim_{t \to 0} w(X + t\nu(X))$ for $X \in \partial D$. In other words, w is a solution to (14) for $k := 1 - 1/\beta$ and $H \equiv 0$. Further, we note that if w is a non-trivial solution of the homogeneous equation (14) for $k = 1/\beta$, then the harmonic conjugate $\tilde{w}$ of w (see for instance the construction in [5], section 3) satisfies

$$\begin{cases} \operatorname{div}\Big([1 + (\tilde{k} - 1)1_D(x)]\nabla \tilde{w}(x)\Big) & = \;\; 0 \qquad \text{in } I\!R^2 \\ \tilde{w}(x) & \in \;\; \mathcal{W}_0^{1,-1}(I\!R^2). \end{cases}$$

with $\tilde{k} = k^{-1}$. In other words, $\tilde{w}$ is an eigenvalue of T_δ associated to the eigenvalue $\tilde{\beta} = \frac{1}{1-\tilde{k}} = 1 - \beta$. We thus obtain:

Proposition 4. *The non-trivial solutions to the homogeneous equation* (14) *are the eigenfunctions of the spectral problem (18) associated to the non-degenerate eigenvalues (i.e., $\beta \neq 0$ and $\beta \neq 1$). Moreover, the eigenvalues are symmetric with respect to the eigenvalue $\beta = 1/2$.*

Lemma 1. *The following assertions hold.*

- *The eigenspace of T_δ associated to the eigenvalue $\beta = 1$ is*

$$\operatorname{Ker}(I - T_\delta) \;=\; \{v|_{D'} \equiv 0,\; v|_D \in H_0^1(D)\}.$$

This eigenspace does not contains any element of $\mathfrak{H}_S$ except $v = 0$.

- *The eigenspace of T_δ associated to the eigenvalue $\beta = 0$ is*

$$\mathrm{Ker}\,(T_\delta) \;=\; \{v|_{D'} \in \mathcal{W}_0^{1,-1}(D'),\ v|_D \equiv 0\} \,\cup\, I\!R\, w_0,$$

where w_0 is defined by

$$\begin{cases} \Delta w_0(X) &= 0 \quad \text{in } D', \\ w_0(X) &= C_j \quad \text{on } \partial D_j \;\; j = 1, 2, \\ \int_{\partial D_j} \frac{\partial w_0}{\partial \nu} &= (-1)^j \quad j = 1, 2. \end{cases} \tag{19}$$

where $C_1, C_2 \in I\!R$ are chosen so that $w_0 \in \mathcal{W}_0^{1,-1}(I\!R^2)$. Moreover, the only elements of $\mathfrak{H}_S$ in this eigenspace are the multiples of w_0.

Proof. Let $w \in \mathrm{Ker}\,(I - T_\delta)$, so that $(T_\delta w, v)_{\mathcal{W}} = (w, v)_{\mathcal{W}}$ for all $v \in \mathcal{W}_0^{1,-1}(I\!R^2)$, and thus

$$\int_{D'} \nabla w(X) \nabla v(X) dX \;=\; 0 \qquad \forall\, v \in \mathcal{W}_0^{1,-1}(I\!R^2).$$

Given the behavior of w at infinity, it follows that $w \equiv 0$ in D', and in particular $w|^+_{\partial D} = 0$. Since w has no jump across ∂D, it follows that $w|_D \in H_0^1(D)$. Conversely, the definition of T_δ shows that if $w|_D \in H_0^1(D)$ and $w|_{D'} \equiv 0$, then $T_\delta(w) = w$. Next, if $v \in \mathrm{Ker}\,(I - T_\delta) \cap \mathfrak{H}_S$, then $v|_{D'} \equiv 0, v|_D \in H_0^1(D)$ and $\Delta v = 0$ in D. It follows that $v \equiv 0$.

Concerning the second claim of the lemma, a straightforward computation shows that w_0 is an eigenfunction of T_δ associated to $\beta = 0$. Suppose next that $w \in \mathrm{Ker}(T_\delta)$. It follows from (17) that

$$\int_{D} \nabla w(X) \nabla v(X) dX \;=\; 0 \qquad \forall\, v \in \mathcal{W}_0^{1,-1}(I\!R^2).$$

Taking $v = w$ shows that $\nabla w(X) = 0$ in D. Thus, w is constant in D. If $w = 0$ on both inclusions, then $w|_{D'} \in \mathcal{W}_0^{1,-1}(D')$. Otherwise, Proposition 1 shows that w must be a linear combination of the two functions $w_{10}, w_{20} \in \mathcal{W}^{1,-1}(I\!R^2)$ solutions to

$$\begin{cases} \Delta w_{i0}(X) &= 0 \quad \text{in } D', \\ w_{i0}(X) &= \delta_{ij} \quad \text{on } \partial D_j \;\; j = 1, 2, \end{cases}$$

where δ_{ij} is the Kronecker delta. The existence of w_{10} and w_{20} is again guaranteed by Proposition 1. We also note that they are orthogonal for the

scalar product (13). Writing $w = c_1 w_{10} + c_2 w_{20}$ a straightforward computation shows that the condition that w vanishes when $|X| \to +\infty$ implies $w \in I\!Rw_0$.
Finally, we note that if $v \in \mathrm{Ker}(T_\delta) \cap \mathfrak{H}_S$ is not a multiple of w_0, then $v \equiv 0$ on ∂D, and since v has to be harmonic on D', Proposition 1 implies that $v \equiv 0$. ■

Remark 1. *i) Since the constant function $v \equiv 1$ is the only harmonic fonction in $\mathcal{W}^{1,-1}(I\!R^2)$ that takes the value 1 on ∂D, $w_{10} + w_{20} \equiv 1$ in $I\!R^2$.*

ii) When D_1 and D_2 are discs, the function w_0 can be calculated explicitely (see for instance [10]). For instance, in the case of discs of radius 1, one has

$$w_0(X) = \frac{1}{2\pi} \log\left(\frac{|X-a|}{|X+a|}\right),$$

where $a = \sqrt{\delta(2+\delta)}$.

In the rest of the section we study the spectrum of the restriction of T_δ to the space $\mathfrak{H}_S$. Since the Laplace operator is linear and since the trace operator is continuous with respect to the H^1 norm, $\mathfrak{H}_S$ is a closed subspace of $\mathcal{W}_0^{1,-1}(I\!R^2)$. Lemma 1 implies that $T_\delta \mathfrak{H}_S \subset \mathfrak{H}_S$. *We henceforth use the notation T_δ for the restriction of T_δ to $\mathfrak{H}_S$.*

For $u \in \mathfrak{H}_S$, there exists a unique function $R_\delta u \in \mathfrak{H}_S$, such that for all $v \in \mathfrak{H}_S$

$$2\int_{I\!R^2} \nabla R_\delta u(X) \nabla v(X) dX = \int_D \nabla u(X) \nabla v(X) dX - \int_{D'} \nabla u(X) \nabla v(X) dX, \qquad (20)$$

The operator $R_\delta : \mathfrak{H}_S \to \mathfrak{H}_S$ is a bounded operator with norm $\|R_\delta\| \leq 1$.

Theorem 2. *The operator $R_\delta : \mathfrak{H}_S \to \mathfrak{H}_S$ is compact and self-adjoint.*

Proof. From (17) and (20) we infer that $T_\delta = \frac{1}{2}I + R_\delta$. It follows that R_δ is a bounded self-adjoint operator. Next, given $u, v, \in \mathfrak{H}_S$, we see from (20) that since u and v are harmonic in $D \cup D'$ and continuous across ∂D

$$2\int_{I\!R^2} \nabla R_\delta u(X) \nabla v(X) = \int_{\partial D} \left(\frac{\partial u}{\partial \nu}^+ + \frac{\partial u}{\partial \nu}^-\right) v(X) dX,$$

Applying Proposition 3, we rewrite $u = S[\varphi]$ with $\varphi \in H^{-1/2}(\partial D)$ and

$$\int_{\partial D_1} \varphi|_{D_1} + \int_{\partial D_2} \varphi|_{D_2} \; = \; 0.$$

The jump conditions yield that for $X \in \partial D$,

$$\frac{\partial u}{\partial \nu}|^+ + \frac{\partial u}{\partial \nu}^- \; = \; 2K^*[\varphi] \; = \; 2K^* \left[i_{H^{-1/2}} o \mathbb{S}^{-1} \left[(u|_{\partial D}, 0) \right] \right] (X),$$

where $i_{H^{-1/2}} : H^{-\frac{1}{2}}(\partial D) \times I\!R \longrightarrow H^{-\frac{1}{2}}(\partial D)$ is defined by $i_{H^{-1/2}}(f, a) = f$. Recalling the Calderon identity [15, 23] $SK^* = KS$, we find

$$\int_{I\!R^2} \nabla R_\delta u(X) \nabla v(X) dX \; = \; \int_{\partial D} i_{H^{-1/2}} o \mathbb{S}^{-1} \left[(K \left[u|_{\partial D}, 0 \right]) \right] (X) v(X),$$

and since K is a compact operator, we deduce that R_δ is also compact. ■

Since $T_\delta \; = \; \frac{1}{2} I + R_\delta$, an immediate consequence of the above Theorem is that T_δ is a Fredholm operator of index zero and that its spectrum is real, discrete, contained in $(0, 1)$ and symmetric with respect to $\frac{1}{2}$, with only $\frac{1}{2}$ as an accumulation point. We denote by $\left(\beta_n^{\delta, \pm} \right)_{n \geq 1}$ the nontrivial eigenvalues of T_δ, ordered as follows:

$$0 < \beta_1^{\delta,+} \leq \beta_2^{\delta,+} \leq \cdots \leq \frac{1}{2},$$

the eigenvalues in $(0, 1/2]$ and, similarly,

$$1 > \beta_1^{\delta,-} \geq \beta_2^{\delta,-} \geq \cdots \geq \frac{1}{2},$$

the eigenvalues in $[1/2, 1)$. The eigenvalue $1/2$ is the unique accumulation point of the spectrum. In view of the symmetry discussed in Proposition 4, we actually have $\beta_n^{\delta,-} = 1 - \beta_n^{\delta,+}$. The notation $\pm$ is consistent with Proposition 4 and with the notation used in the introduction:

$$\begin{cases} k_n^+ \; = \; 1 - 1/\beta_n^+ \in \,] - \infty, -1[& \lambda_n^+ \; = \; \frac{k_n^+ + 1}{2(k_n^+ - 1)} \in \,]0, 1/2[\\ k_n^- \; = \; 1 - 1/\beta_n^- \in \,]1, 0[& \lambda_n^- \; = \; \frac{k_n^- + 1}{2(k_n^- - 1)} \in \,] - 1/2, 0[. \end{cases} \tag{21}$$

In addition, $\left(\beta_n^{\delta, \pm} \right)_{n \geq 1}$ satisfy the following min-max principle:

Lemma 2. *Let* $\left(\beta_n^{\delta,\pm}\right)_{n\geq 1}$ *be the eigenvalues of* T_δ *repeated according to their multiplicity, and let* $\left(w_n^{\delta,\pm}\right)_{n\geq 1}$ *be the corresponding eigenfunctions. Then*

$$\begin{aligned}
\beta_n^{\delta,+} &= \min_{u\in\mathfrak{H}_S,\,\perp w_0, w_1^{\delta,+},\cdots,w_n^{\delta,+}} \frac{\int_D |\nabla u(X)|^2\,dX}{\int_{\mathbb{R}^2} |\nabla u(X)|^2\,dX} \\
&= \max_{\substack{F_n \subset \mathfrak{H}_S \\ dim(F_n) = n+1}} \min_{u\in F_n} \frac{\int_D |\nabla u(X)|^2\,dX}{\int_{\mathbb{R}^2} |\nabla u(X)|^2\,dX}
\end{aligned}$$

and similarly

$$\begin{aligned}
\beta_n^{\delta,-} &= \max_{u\in\mathfrak{H}_S,\, w_1^{\delta,-},\cdots,w_n^{\delta,-}} \frac{\int_D |\nabla u(X)|^2\,dX}{\int_{\mathbb{R}^2} |\nabla u(X)|^2\,dX}, \\
&= \min_{\substack{F_n \subset \mathfrak{H}_S \\ dim(F_n) = n}} \max_{u\in F_n} \frac{\int_D |\nabla u(X)|^2\,dX}{\int_{\mathbb{R}^2} |\nabla u(X)|^2\,dX}.
\end{aligned}$$

Proof. It is a direct consequence of the Theorem 2 and of the min-max principle for the compact self-adjoint operator R_δ. ■

It follows from Theorem 2 and Lemma 1 that any function $u \in \mathfrak{H}_S$ can be decomposed as

$$u = u_0 w_0 + \sum_{n\geq 1} \frac{(u, w_n^{\delta,\pm})_{\mathcal{W}}}{||w_n^{\delta,\pm}||_{\mathcal{W}}^2} w_n^{\delta,\pm}, \tag{22}$$

with $u_0 \in \mathbb{R}$, and where the series is convergent in $\mathcal{W}_0^{1,-1}(\mathbb{R}^2)$.

Proposition 5. *The solution* u *to* (14) *has the following decomposition in* $\mathfrak{H}_S$*:*

$$u(X) = H(X) + \sum_{n\geq 1} u_n^{\pm} w_n^{\delta,\pm}(X), \qquad X \in \mathbb{R}^2, \tag{23}$$

where

$$u_n^{\pm} = \frac{\beta_n^{\delta,\pm}}{\frac{1}{1-k} - \beta_n^{\delta,\pm}} \frac{\int_D \nabla H(X) \nabla w_n^{\delta,\pm}(X) dX}{\int_D \left|\nabla w_n^{\delta,\pm}(X)\right|^2 dX}. \tag{24}$$

The series is convergent with respect to the norm $\|\cdot\|_{\mathcal{W}}$.

Proof. We remark that $u - H$ lies in $\mathfrak{H}_S$ so that it decomposes as (22) on the eigenfunctions w_0 and $w_n^{\delta,\pm}$. We first note that

$$\begin{aligned}
\int_{\mathbb{R}^2} \nabla w_0 \cdot \nabla(u-H) &= \int_{D'} \nabla w_0 \nabla(u-H) \\
&= \int_{\partial D} \partial_n (u-H)|^+ w_0 \\
&= \sum_{i=1,2} k\, C_i \int_{\partial D_i} \partial_n(u-H)|^- \;=\; 0.
\end{aligned}$$

Next, since $T_\delta w_n^{\delta,\pm} = \beta_n^{\delta,\pm}$, we have

$$\begin{aligned}
\int_D \nabla w_n^{\delta,\pm} \cdot \nabla(u-H) &= \beta_n^{\delta,\pm} \int_{\mathbb{R}^2} \nabla w_n^{\delta,\pm} \cdot \nabla(u-H) \\
\int_{D'} \nabla w_n^{\delta,\pm} \cdot \nabla(u-H) &= (1-\beta_n^{\delta,\pm}) \int_{\mathbb{R}^2} \nabla w_n^{\delta,\pm} \cdot \nabla(u-H).
\end{aligned}$$

It follows that

$$\begin{aligned}
\beta_n^{\delta,\pm} u_n ||w_n^{\delta\pm}||^2_{\mathcal{W}} &= \beta_n^{\delta,\pm} \int_{\mathbb{R}^2} \nabla w_n^{\delta,\pm} \cdot \nabla(u-H) \\
&= \int_D \nabla w_n^{\delta,\pm} \cdot \nabla(u-H) \\
&= \int_{\partial D} w_n^{\delta,\pm} \frac{\partial u}{\partial \nu}|^- \;-\; \int_D \nabla w_n^{\delta,\pm} \cdot \nabla H \\
&= \frac{1}{k} \int_{\partial D} w_n^{\delta,\pm} \frac{\partial}{\partial \nu}(u-H)|^+ \;+\; (\frac{1}{k}-1) \int_D \nabla w_n^{\delta,\pm} \cdot \nabla H \\
&= \frac{-1}{k} \int_{D'} \nabla w_n^{\delta,\pm} \cdot \nabla(u-H) \;+\; (\frac{1}{k}-1) \int_D \nabla w_n^{\delta,\pm} \cdot \nabla H \\
&= \frac{-(1-\beta_n^{\delta\pm})}{k}\, u_n\, ||w_n^{\delta\pm}||^2_{\mathcal{W}} \;+\; (\frac{1}{k}-1) \int_D \nabla w_n^{\delta,\pm} \cdot \nabla H.
\end{aligned}$$

Noticing that

$$\int_D |\nabla w_n^{\delta,\pm}|^2 \;=\; \beta_n^{\delta,\pm} \int_{\mathbb{R}^2} |\nabla w_n^{\delta,\pm}|^2,$$

yields (24). ∎

4 Behavior of the eigenvalues as $\delta \to 0$

In this section we prove Theorem 1, which gives the first order asymptotics of the eigenvalues as $\delta \to 0$. Reformulated in terms of $\beta_n^{\delta,\pm} = 1/2 - \lambda_n^{\delta,\pm}$, see (21), the Theorem states that for $n \geq 1$,

$$\left\{ \begin{array}{lcl} \beta_n^{\delta,+} & = & c_n \delta^{1-\frac{1}{m}} + o(\delta^{1-\frac{1}{m}}) \\ \beta_n^{\delta,-} & = & 1 - c_n \delta^{1-\frac{1}{m}} + o(\delta^{1-\frac{1}{m}}), \end{array} \right. \tag{25}$$

where $c_n \geq 0$ is a real increasing sequence that only depends on the shape of the inclusions D_1^0, D_2^0 (and therefore is independent of δ) and satisfies

$$c_n \sim n \qquad \text{as} \quad n \to \infty. \tag{26}$$

4.1 An estimate of the exterior energy

Let $\varepsilon \in (0, \frac{1}{2})$ be small enough such that

$$c_1 |x|^m \leq \psi(x) \leq c_2 |x|^m \qquad \text{for } x \in (-\varepsilon, \varepsilon),$$

for some $c_1, c_2 \in I\!R$. We consider the spaces $\dot{H}^{1/2}(\partial D) := H^{1/2}(\partial D)/I\!R$ and $\dot{H}^{1/2}(\partial D_j) = H^{1/2}(\partial D_j)/I\!R$, $j = 1, 2$, with the norms

$$\begin{array}{lcl} \|w\|_{\dot{H}^{1/2}(\partial D)} & = & \inf_{c \in I\!R} \|w + c\|_{H^{1/2}(\partial D)}, \\ \|w\|_{\dot{H}^{1/2}(\partial D_j)} & = & \inf_{c \in I\!R} \|w + c\|_{H^{1/2}(\partial D_j)}, \ j = 1, 2. \end{array} \tag{27}$$

For $w \in \dot{H}^{1/2}(\partial D)$, we introduce the norm

$$[w]_{1/2}^2 = \sum_{j=1}^{2} \|w\|_{\dot{H}^{1/2}(\partial D_j)}^2 + \int_{-\varepsilon}^{\varepsilon} \frac{|w^+(x) - w^-(x)|^2}{\psi(x) + \delta} dx, \tag{28}$$

where $w^+(x) = w(x, \psi_1(x) + \delta/2)$ and $w^-(x) = w(x, -\psi_2(x) - \delta/2)$. The function $x \to \psi(x) + \delta$ is in fact the vertical distance $\text{dist}(\partial D_1, \partial D_2)$ in the region where the inclusions are close to touching. If $\delta > 0$ is fixed, the norm $[\cdot]_{1/2}$ is equivalent to the norm (27) on $\dot{H}^{1/2}(\partial D^\delta)$.

Next we show that the norm (15) of $\mathfrak{H}_S$, as a subspace of $\mathcal{W}_0^{1,-1}(I\!R^2)$, is equivalent to the norm (28).

Theorem 3. *Assume that* D_i, $i = 1, 2$ *satisfy the assumptions* A1–A4. *Then, there exists constants* $C_i > 0$, $i = 1, 2$, *which only depend on* D_i^0, $i = 1, 2$ *such that*

$$C_1 \, [v]_{1/2}^2 \;\; \leq \; \int_{D'} |\nabla v(X)|^2 \, dX \;\; \leq \; C_2 \, [v]_{1/2}^2, \tag{29}$$

for all $v \in \mathfrak{H}_S$.

Proof. The proof is split in 2 steps. Firstly, we prove the right-hand inequality in (29) by constructing a suitable function $\tilde{v}$ which has the same Dirichlet boundary as v and by using the fact that v is harmonic in D' and has minimal energy. Secondly, we prove the left-hand inequality.

Step 1.
We divide the domain D' into three parts. To this end, we introduce two auxiliary functions $\psi_{1,\varepsilon}, \psi_{2,\varepsilon}$, defined on $I\!R$, which satisfy (see Figure 2) :

$$\psi_{j,\varepsilon} \;\equiv\; \psi_j, \qquad j = 1, 2, \;\; |x| \leq \varepsilon, \tag{30}$$

$$||\psi_{j,\varepsilon}||_{\mathcal{C}^{1,\beta}} \;\leq\; 2||\psi_j||_{\mathcal{C}^{1,\alpha}} \varepsilon^{\nu}, \tag{31}$$

where $\nu = \alpha - \beta > 0$. The existence of such functions follows from the $\mathcal{C}^{1,\alpha}$ regularity of ψ_1 and ψ_2, and from the fact that

$$\psi_j(0) \;=\; \psi_j'(0) \;=\; 0.$$

For instance, one can take

$$\psi_{j,\varepsilon}(x) \;\; = \;\; \begin{cases} \psi_j(x), & |x| < \varepsilon, \\ 2\psi_j(\pm\varepsilon) - \psi_j(\pm 2\varepsilon \mp x), & \varepsilon \leq \pm x \leq 2\varepsilon \\ 2\psi_j(\pm\varepsilon), & \pm x > 2\varepsilon. \end{cases}$$

We also introduce the narrow strip $Q_{\varepsilon,\delta}$ defined by

$$Q_{\varepsilon,\delta} \;\; = \;\; \left\{ X \;=\; (x_1, x_2); \; x_1 \in I\!R, \; -\psi_{2,\varepsilon}(x_1) - \frac{\delta}{2} < x_2 < \psi_{1,\varepsilon}(x_1) + \frac{\delta}{2} \right\}.$$

Denote $\partial Q^i_{\varepsilon,\delta} = \{X \;=\; (x_1, \psi_{i,\varepsilon}(x_1)); \; x_1 \in I\!R\}$, $i = 1, 2$ the top and bottom boundaries of $Q_{\varepsilon,\delta}$.

Let $v \in \mathfrak{H}_S$. We recall that $v \in \mathcal{W}_0^{1,-1}(I\!R^2)$ and is harmonic in D and in D'. Classical results in potential theory (see for instance [23]) show that there exists two functions $v_j \in \mathcal{W}^{1,-1}(I\!R^2 \setminus \overline{D}_j)$, $j = 1, 2$, such that

$$\begin{cases} \Delta v_j(X) \;=\; 0 & \text{in } I\!R^2 \setminus \overline{D_j} \\ v_j(X) \;=\; v(X) & \text{on } \partial D_j, \end{cases}$$

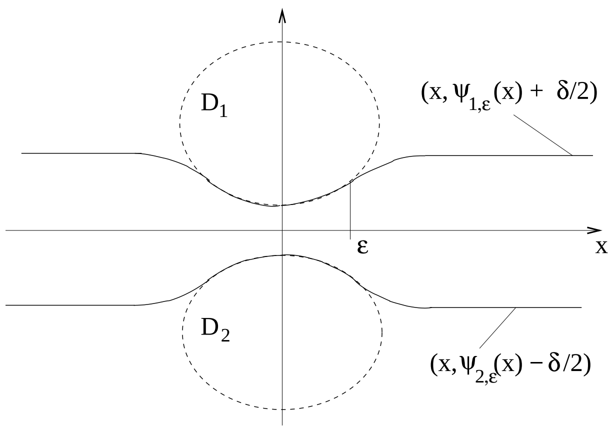

Figure 2: A possible contruction of $\psi_{j,\varepsilon}, j=1,2$: the part between $(\varepsilon, 2\varepsilon)$ is obtained by rotating the part between $(0,\varepsilon)$ around the point $(\varepsilon, \psi_j(\varepsilon))$

In addition, there exists a constant $C>0$, that only depends on the shape of each inclusion, such that

$$\|v_j\|^2_{\mathcal{W}^{1,-1}(I\!R^2\setminus\overline{D}_j)} \leq C\|v\|^2_{H^{\frac{1}{2}}(\partial D)}, \tag{32}$$

from which one also obtain

$$\int_{I\!R^2\setminus\overline{D_j}} |\nabla v_j|^2\, dX \quad \leq \quad C\|v\|^2_{H^{\frac{1}{2}}(\partial D)}. \tag{33}$$

We now fix $0<r_0$ large enough such that the ball $B_{r_0}(0)$ contains the two inclusions. Let $\chi(X)\in C^2_0(I\!R^2)$ and $r_1>r_0$ such that

$$\chi(X) \quad = \quad \begin{cases} 1 \qquad \text{for } X\in B_{r_0}(0), \\ \|\chi\|_{C^1(I\!R^2)}\leq 1. \\ \text{supp}(\chi)\subset S_{r_1}:=[-r_1,r_1]^2. \end{cases}$$

We set $v_{ij}(x_1) \;=\; v_i|_{\partial Q^j_{\varepsilon,\delta}}(x_1, \psi_{j,\varepsilon}(x_1)+\delta/2),\; i,j=1,2$. The trace Thereom implies that $v_{ij}\in H^{\frac{1}{2}}(I\!R)$ and that there exists a constant C, that only depends on the shape of each inclusion, such that

$$\|\chi v_{ij}\|^2_{H^{\frac{1}{2}}(I\!R)} \leq C\|v_i\|^2_{\mathcal{W}^{1,-1}(I\!R^2\setminus\overline{D_i})} \qquad i,j=1,2.$$

For X in the strip $Q_{\varepsilon,\delta}$, we set

$$L(X) \quad = \quad \frac{x_2+\psi_{2,\varepsilon}(x_1)+\delta/2}{\psi_{1,\varepsilon}(x_1)+\psi_{2,\varepsilon}(x_1)+\delta},$$

and we define

$$\tilde{v}(X) \;=\; \chi(X)\begin{cases} v_1(X) & X \in I\!R^2_+ \setminus \left(\overline{Q_{\varepsilon,\delta}} \cup \overline{D_1}\right), \\ \left(v_1(X) - v_2(X)\right) L(X) + v_2(X) & X \in Q_{\varepsilon,\delta}, \\ v_2(X) & X \in I\!R^2_- \setminus \left(\overline{Q_{\varepsilon,\delta}} \cup \overline{D_2}\right). \end{cases}$$

Since $L(X) \in W^{1,\infty}(Q_{\varepsilon,\delta})$ and takes the values 1 and 0 respectively on $\partial Q^1_{\varepsilon,\delta}$ and $\partial Q^2_{\varepsilon,\delta}$, the function $\tilde{v} \in \mathcal{W}^{1,-1}_0(I\!R^2)$ and coincides with v on ∂D. Proposition 1 implies that

$$\begin{aligned} \int_{I\!R^2\setminus\overline{D}} |\nabla v|^2\, dX &\le \int_{I\!R^2\setminus\overline{D}} |\nabla \tilde{v}|^2\, dX \\ &\le \int_{I\!R^2\setminus\overline{D_1}} |\nabla(\chi v_1)|^2\, dX + \int_{I\!R^2\setminus\overline{D_2}} |\nabla(\chi v_2)|^2\, dX \\ &\quad + \int_{Q_{\varepsilon,\delta}\cap S_{r_1}} |\nabla \tilde{v}|^2\, dX. \end{aligned} \tag{34}$$

The estimate (32) combined with fact that $\|\chi\|_{C^1(I\!R^2)} \le 1$ give the desired bound for the first two terms on the right hand side. We now focus on the last term. To simplify the notation we denote $Q_{r_1,\varepsilon,\delta} := Q_{\varepsilon,\delta} \cap S_{r_1}$. A straightforward computation shows that

$$\begin{aligned} \int_{Q_{r_1,\varepsilon,\delta}} |\nabla \tilde{v}|^2\, dX \;\le\; C\Bigg(&\int_{Q_{r_1,\varepsilon,\delta}} |\nabla v_1|^2 + |\nabla v_2|^2\, dX + \int_{Q_{r_1\varepsilon,\delta}} v_1^2 + v_2^2\, dX \\ &+ \int_{Q_{r_1,\varepsilon,\delta}} (v_1 - v_2)^2\, |\nabla L|^2\, dX\Bigg). \end{aligned}$$

Here we have used the facts that $\|L(X)\|_{C^0(I\!R^2)}, \|\chi\|_{C^1(I\!R^2)} \le 1$. We infer from (32) that the first two integrals of the right hand side of the last inequality are bounded by $C\|v\|_{H^{1/2}(\partial D)}$. A simple calculation yields

$$|\nabla L(X)| \le \frac{C}{\psi_\varepsilon(x_1) + \delta} \qquad X \in Q_{\varepsilon,\delta},$$

where $\psi_\varepsilon(x_1) = \psi_{1,\varepsilon}(x_1) + \psi_{2,\varepsilon}(x_1)$. We note that for $x_1 > \varepsilon$, the width of $Q_{\varepsilon,\delta}$, $\psi_\varepsilon(x) + \delta$ is bounded below independently of δ, and therefore

$$\int_{Q_{r_1,\varepsilon,\delta}} (v_1 - v_2)^2\, |\nabla L|^2 \;\le\; C\int_{Q_{r_1,\varepsilon,\delta}} \frac{(v_1 - v_2)^2}{(\psi_\varepsilon(x_1) + \delta)^2} \;=:\; CI$$

For fixed x_1, let $J(x_1)$ denote the interval $]-\psi_{2,\varepsilon}(x_1)-\delta/2, \psi_{1,\varepsilon}(x_1)+\delta/2[$. Since the function $x_2+\psi_{2,\varepsilon}(x_1)+\delta/2$ vanishes on $\partial Q^2_{\varepsilon,\delta}$, an integration by parts shows that

$$\begin{aligned}\int_{J(x_1)} (v_1-v_2)^2\, dx_2 &= (\psi_\varepsilon(x_1)+\delta)\,(v_{11}-v_{21})^2\,(x_1)\\ &\quad -\int_{J(x_1)} \partial_{x_2}\left[(v_1-v_2)^2\right](x_2+\psi_{2,\varepsilon}(x_1)+\delta/2)\,dx_2.\end{aligned}$$

It follows that

$$\begin{aligned} I &\leq \int_{-r_1}^{r_1} \frac{(v_{11}-v_{21})^2\,(x_1)}{\psi_\varepsilon(x_1)+\delta}dx_1 \;+\; 2\int_{Q_{r_1,\varepsilon,\delta}} \frac{|v_1-v_2|\;|\partial_{x_2}(v_1-v_2)|}{\psi_\varepsilon(x_1)+\delta}dX,\\ &\leq \int_{-r_1}^{r_1} \frac{(v_{11}-v_{21})^2\,(x_1)}{\psi_\varepsilon(x_1)+\delta}dx_1 \;+\; 4\int_{Q_{\varepsilon,\delta}} \left(|\nabla v_1|^2+|\nabla v_2|^2\right)dX + I/2,\end{aligned}$$

from which we deduce

$$I \;\leq\; 8\left(\int_{-r_1}^{r_1} \frac{(v_{11}-v_{21})^2\,(x_1)}{\psi_\varepsilon(x_1)+\delta}dx_1 \;+\; \int_{\mathbb{R}^2\setminus\overline{D_1}} |\nabla v_1|^2\,dX \;+\; \int_{\mathbb{R}^2\setminus\overline{D_2}} |\nabla v_2|^2\,dX\right).$$

On the other hand, we can write

$$v_{21}(x_1)-v_{22}(x_1) \;=\; \int_{J(x_1)} \partial_{x_2} v_2(x_1,x_2)dx_2,$$

so that

$$\frac{(v_{21}-v_{22})^2\,(x_1)}{\psi_\varepsilon(x_1)+\delta} \;\leq\; \int_{J(x_1)} |\partial_{x_2} v_2(x_1,x_2)|^2\,dx_2,$$

and consequently

$$I \;\leq\; C\left(\int_{-r_1}^{r_1} \frac{(v_{11}-v_{22})^2\,(x_1)}{\psi_\varepsilon(x_1)+\delta}dx_1 + \int_{\mathbb{R}^2\setminus\overline{D_1}} |\nabla v_1|^2\,dX \;+\; \int_{\mathbb{R}^2\setminus\overline{D_2}} |\nabla v_2|^2\,dX\right),$$

where $C>0$ is a constant that does not depend on δ. Further, since ψ_ε is bounded away from 0 independently of δ for $|x_1|>\varepsilon$, and invoking the trace Theorem and (32) we see that

$$\begin{aligned}\int_{-r_1}^{r_1} \frac{(v_{11}-v_{22})^2\,(x_1)}{\psi_\varepsilon(x_1)+\delta}dx_1 &\leq \int_{-\varepsilon}^{\varepsilon} \frac{(v_{11}-v_{22})^2\,(x_1)}{\psi_\varepsilon(x_1)+\delta}dx_1\\ &\quad + C\left(||v_1||^2_{\mathcal{W}^{-1,1}(\mathbb{R}^2\setminus\overline{D_1})} + ||v_2||^2_{\mathcal{W}^{-1,1}(\mathbb{R}^2\setminus\overline{D_1})}\right)\\ &\leq \int_{-\varepsilon}^{\varepsilon} \frac{(v_{11}-v_{22})^2\,(x_1)}{\psi_\varepsilon(x_1)+\delta}dx_1 \;+\; C\,||v||^2_{H^{1/2}(\partial D)},\end{aligned}$$

where C depends on ε, but is independent of δ. Recalling (34), we finally obtain

$$\int_{D'} |\nabla v|^2 \quad \leq \quad C\left(||v||^2_{H^{1/2}(\partial D)} + \int_{-\varepsilon}^{\varepsilon} \frac{(v_{11}-v_{22})^2}{\psi_\varepsilon(x_1)+\delta}\, dx_1 \right),$$

where C is independent of δ and v. One can check that this inequality also holds when v is replaced by $v+c$, where c is any constant. Taking the minimum over $c \in I\!R$ proves the right-hand inequality in (29).

Step 2. Here we prove the left-hand side inequality in (29). Recall that the semi-norm

$$\int_{I\!R^2} |\nabla v|^2 \, dX,$$

is a norm on $\mathcal{W}_0^{1,-1}(I\!R^2)$ [23]. Thus, we deduce from the trace Theorem that there exists a constant $C>0$ that only depends on the shape of each inclusion, such that

$$C\|v\|^2_{\dot{H}^{\frac{1}{2}}(\partial D)} \leq \int_{I\!R^2} |\nabla v|^2 \, dX.$$

We define

$$v_0(X) \quad = \quad v(X) - (\psi(x_1)+\delta)^{-1} \int_{J(x_1)} v(x_1,x_2) dx_2.$$

Since $v_0(x_1,\cdot)$ has 0 vertical average, the Poincaré inequality yields

$$\begin{aligned}\int_{Q_{\varepsilon,\varepsilon,\delta}} (\psi(x_1)+\delta)^{-2} |v_0(X)|^2 \, dX \quad &\leq \quad C\int_{Q_{\varepsilon,\varepsilon,\delta}} |\partial_{x_2}(v_0(X))|^2 \, dX \\ &\leq \quad C\int_{Q_{\varepsilon,\varepsilon,\delta}} |\nabla v(X)|^2 \, dX, \qquad (35)\end{aligned}$$

where C is a constant independent of δ. A simple calculation shows that

$$\begin{aligned}&\int_{-\varepsilon}^{\varepsilon} (\psi(x_1)+\delta)^{-1} \left(v_0^2(x_1,\psi_1(x_1)+\delta/2) + v_0^2(x_1,-\psi_2(x_1)-\delta/2)\right) dx_1 \\ &= \int_{Q_{\varepsilon,\varepsilon,\delta}} (\psi(x_1)+\delta)^{-2}\, \partial_{x_2}\left((2x_2+\psi_2(x_1)-\psi_1(x_1))v_0^2(X)\right) dX \\ &\leq 2\int_{Q_{\varepsilon,\varepsilon,\delta}} (\psi(x_1)+\delta)^{-2}\left(v_0^2 + (\psi(x_1)+\delta)\, v_0 \partial_{x_2} v_0\right) dX \\ &\leq 3\int_{Q_{\varepsilon,\varepsilon,\delta}} (\partial_{x_2} v_0)^2 + (\psi(x_1)+\delta)^{-2}\, v_0^2 dX. \qquad (36)\end{aligned}$$

It follows that

$$\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} (\psi(x_1)+\delta)^{-1} \left(v(x_1,\psi_1(x_1)+\delta/2) - v(x_1,-\psi_2(x_1)-\delta/2)\right)^2 dx_1 \\
&\quad = \int_{-\varepsilon}^{\varepsilon} (\psi(x_1)+\delta)^{-1} \left(v_0(x_1,\psi_1(x_1)+\delta/2) - v_0(x_1,-\psi_2(x_1)-\delta/2)\right)^2 dx_1 \\
&\quad \le 2\int_{-\varepsilon}^{\varepsilon} (\psi(x_1)+\delta)^{-1} \left(v_0^2(x_1,\psi_1(x_1)+\delta/2) + v_0^2(x_1,-\psi_2(x_1)-\delta/2)\right) dx_1.
\end{aligned}$$

Combining (35), (36) with this last estimate, we find

$$\begin{aligned}
&\int_{-\varepsilon}^{\varepsilon} (\psi(x_1)+\delta)^{-1} \left(v(x_1,\psi_1(x_1)+\delta/2) - v(x_1,-\psi_2(x_1)-\delta/2)\right)^2 dx_1 \\
&\quad \le C\int_{D'} |\nabla v|^2\, dX,
\end{aligned}$$

which concludes the proof. ∎

Remark 2. *Applying Theorem 3 to the functions $w_{10}(X)$ and $w_{20}(X)$ defined in the proof of Lemma 1 we find*

$$\int_{\mathbb{R}^2} |\nabla w_{10}(X)|^2\, dX = \int_{\mathbb{R}^2} |\nabla w_{20}(X)|^2\, dX \sim \delta^{\frac{1-m}{m}} \qquad as \quad \delta \to 0$$

This result was obtained in [9] using a variational approach.

We recall that if $\beta_n^{\delta,\pm}$ with $n \ge 1$ is an eigenvalue of T_δ and if $w_n^{\delta,\pm}(X)$ is the associated eigenfunction, we have

$$\beta_n^{\delta,\pm} \quad = \quad \frac{\int_D \left|\nabla w_n^\delta(X)\right|^2 dX}{\int_{\mathbb{R}^2} \left|\nabla w_n^\delta(X)\right|^2 dX}.$$

As a consequence of Theorem 3, we next derive explicit lower and upper bounds for the L^2 norm of the gradient of w_n^δ, inside and outside the inclusions, in terms of $\left[w_n^\delta\right]_{1/2}$, $\|w_n^\delta\|_{\dot H^{1/2}(\partial D)}$ and $\|w_n^\delta\|_{\dot H^{1/2}(\partial D_j)}$, $j=1,2$.

Corollary 1. *There exists a constant $C>0$, that only depends on the shape of each inclusion, such that for any $w \in \mathfrak{H}_S \setminus \{0\}$*

$$\frac{1}{C}\, \frac{\sum_{j=1}^2 \|w\|^2_{\dot H^{1/2}(\partial D_j)}}{[w]^2_{1/2}} \quad \le \quad \frac{\int_D |\nabla w(X)|^2 dX}{\int_{\mathbb{R}^2} |\nabla w(X)|^2 dX} \quad \le \quad C\, \frac{\sum_{j=1}^2 \|w\|^2_{\dot H^{1/2}(\partial D_j)}}{[w]^2_{1/2}} \tag{37}$$

Proof. It is well known that

$$\int_{D_j} |\nabla w(X)|^2 \, dX \quad \sim \quad \|w\|^2_{\dot{H}^{1/2}(\partial D_j)},$$

for all harmonic functions $w \in H^1(D_j), j = 1, 2$. Combining this estimate with Theorem 3 yields the result. ■

4.2 A uniform bound on the energy.

In this paragraph, we derive an estimate of the total energy of $u - H$. For technical reasons, we assume until the end of the paragraph that $k > 1$ or equivalently that $\lambda = \frac{k+1}{2(k-1)} > 1/2$. A similar resut can be obtained in the case $0 < k < 1$ (i.e. $\lambda < -1/2$) considering the harmonic conjugates.

Proposition 5 shows that $u - H$ has the following decomposition in $\mathfrak{H}_S$:

$$u(X) \quad = \quad H(X) + \sum_{n \geq 1} u_n^{\pm} w_n^{\delta,\pm}(X), \qquad X \in I\!R^2,$$

where

$$u_n^{\pm} \quad = \quad \frac{\beta_n^{\delta,\pm}}{\frac{1}{1-k} - \beta_n^{\delta,\pm}} \frac{\int_D \nabla H(X) \nabla w_n^{\delta,\pm}(X) dX}{\int_D \left| \nabla w_n^{\delta,\pm}(X) \right|^2 dX}.$$

Let $\widetilde{H}$ be the harmonic extension in $\mathcal{W}^{-1,1}(I\!R^2)$ of $H|_D$ to D'. In other words $\widetilde{H}$ coincides with H on D and satisfies

$$\begin{cases} \Delta \widetilde{H}(X) \; = \; 0 \qquad \text{in } D', \\ \widetilde{H} \in \mathcal{W}^{-1,1}(D'). \end{cases}$$

Proposition 2 shows that there exist $\phi_H \in H^{-\frac{1}{2}}(\partial D)$ and $a_H \in I\!R$ such that

$$\widetilde{H}(X) \quad = \quad S[\phi_H](X) + a_H \qquad X \in I\!R^2$$

$$\sum_{j=1}^{2} \int_{\partial D_j} \phi_H d\sigma_X \quad = \quad 0.$$

Since $\widetilde{H}(X) - a_H \in \mathcal{W}_0^{-1,1}(I\!R^2)$, we see from (18) that

$$\int_D \nabla H(X) \nabla w_n^{\delta,\pm}(X) dX \quad = \quad \beta_n^{\delta,\pm} \int_{I\!R^2} \nabla \widetilde{H}(X) \nabla w_n^{\delta,\pm}(X) dX \qquad n \geq 1,$$

and consequently, we have

$$u_n^{\pm} \;=\; \frac{\beta_n^{\delta,\pm}}{\frac{1}{1-k}-\beta_n^{\delta,\pm}} \frac{\int_{\mathbb{R}^2} \nabla \widetilde{H}(X) \nabla w_n^{\delta,\pm}(X) dX}{\int_{\mathbb{R}^2} \left|\nabla w_n^{\delta,\pm}(X)\right|^2 dX}.$$

The orthogonality of the eigenfunctions implies that

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla(u-H)|^2 &= \sum_{n\geq 1} (u_n^{\pm})^2 \int_{\mathbb{R}^2} \left|\nabla w_n^{\delta,\pm}\right|^2 \\ &\leq \sum_{n\geq 1} \frac{\left(\int_{\mathbb{R}^2} \nabla \widetilde{H}(X) \nabla w_n^{\delta,\pm}(X) dX\right)^2}{\int_{\mathbb{R}^2} \left|\nabla w_n^{\delta,\pm}(X)\right|^2 dX}, \end{aligned} \tag{38}$$

since due to our assumption $k > 1$ we see that

$$\left| \frac{\beta_n^{\delta,\pm}}{\frac{1}{1-k}-\beta_n^{\delta,\pm}} \right| \;\leq\; 1.$$

Further, since $\widetilde{H} - a_H \in \mathfrak{H}_S$, it has a decomposition in the basis of eigenfunctions of the form

$$\int_{\mathbb{R}^2} \left|\nabla \widetilde{H}\right|^2 dX \;=\; h_0^2 + \sum_{n\geq 1} \frac{\left(\int_{\mathbb{R}^2} \nabla \widetilde{H}(X) \nabla w_n^{\delta,\pm}(X) dX\right)^2}{\int_{\mathbb{R}^2} \left|\nabla w_n^{\delta,\pm}(X)\right|^2 dX},$$

where

$$h_0^2 \;=\; \frac{\left(\int_{\mathbb{R}^2\setminus\overline{D}} \nabla \widetilde{H}(X) \nabla w_0(X) dX\right)^2}{\int_{\mathbb{R}^2\setminus\overline{D}} |\nabla w_0(X)|^2 dX},$$

so that in view of (38), we obtain

$$\int_{\mathbb{R}^2} |\nabla(u-H)|^2 \;\leq\; \int_{\mathbb{R}^2} \left|\nabla \widetilde{H}\right|^2 dX. \tag{39}$$

Hence, we have the following

Proposition 6. *Let u be the solution to* (14)*. Then*

$$\int_{\mathbb{R}^2} |\nabla(u-H)|^2 \;\leq\; C\left(\|H\|^2_{\dot{H}^{\frac{1}{2}}(\partial D)} + \|\partial_{x_2} H\|^2_{L^\infty(B_\varepsilon(0))} \right), \tag{40}$$

where C is a constant independent of δ.

Proof. Since $\widetilde{H} - a_H \in \mathfrak{H}_S$ we have by Theorem 3

$$\int_{\mathbb{R}^2} \left|\nabla \widetilde{H}\right|^2 dX \;\; \leq \;\; C\,[\widetilde{H} - a_H]_{\frac{1}{2}}^2 + \int_D |\nabla H|^2 dX \;\leq\; C\,[H]_{\frac{1}{2}}^2 + \|H\|_{\dot{H}^{\frac{1}{2}}(\partial D)},$$

for some C independent of δ. A direct calculation shows that

$$[H]_{\frac{1}{2}}^2 \;\; \leq \;\; C\left(\|H\|_{\dot{H}^{\frac{1}{2}}(\partial D)}^2 + \|\partial_{x_2} H\|_{L^\infty(B_\varepsilon(0))}^2\right).$$

Combining these estimates with inequality (39) we get the desired result. ∎

We recall that if $\beta_n^{\delta,\pm}$ with $n \neq 0$ is an eigenvalue with associated eigenfunction $w_n^{\delta,\pm}(X)$, we have

$$\beta_n^{\delta,\pm} \;\; = \;\; \frac{\int_D \left|\nabla w_n^{\delta,\pm}(X)\right|^2 dX}{\int_{\mathbb{R}^2} \left|\nabla w_n^{\delta,,\pm}(X)\right|^2 dX}.$$

Next, we derive explicit lower and upper bounds on the L^2 norm of the gradient of $w_n^{\delta,\pm}$, inside and outside the inclusions, in terms of $\left|w_n^{\delta,\pm}\right|_{1/2}$ and $\|w_n^{\delta,\pm}\|_{\dot{H}^{1/2}(\partial D_j^\delta)}$, $j = 1, 2$.

Corollary 2. *There exists a constant $C > 0$ that only depends on the inclusions D_j, such that for any $w \in \mathfrak{H}_S \setminus 0$,*

$$\frac{1}{C}\frac{\sum_{j=1}^2 \|w\|_{\dot{H}^{1/2}(\partial D^\delta)_j}^2}{|w|_{1/2}^2} \leq \frac{\int_D |\nabla W(X)|^2\, dX}{\int_{\mathbb{R}^2} |\nabla w(X)|^2\, dX} \leq C\frac{\sum_{j=1}^2 \|w\|_{\dot{H}^{1/2}(\partial D_j^\delta)}^2}{|w|_{1/2}^2}, \quad (41)$$

Proof. It is well known that

$$\int_{D_j} |\nabla w(X)|^2\, dX \sim \|w\|_{\dot{H}^{1/2}(\partial D_j)}^2,$$

for any $w \in H^1(D_j)$, j=1, 2, which, combined with Theorem 3, ends the proof. ∎

4.3 Proof of Theorem 1.

We infer from Corollary 1 and Lemma 2 that

$$\frac{1}{C}\, b_n^{\delta,\pm} \;\leq\; \beta_n^{\delta,\pm} \;\leq\; C\, b_n^{\delta,\pm} \qquad \forall\, n \geq 1, \tag{42}$$

where

$$\begin{aligned} b_n^{\delta,-} \;&:=\; \min_{\substack{F_n \subset \dot H^{1/2} \\ \dim(F_n)=n}} \;\max_{w\in F_n\setminus\{0\}} \frac{\sum_{j=1}^{2}\|w\|^2_{\dot H^{1/2}(\partial D_j)}}{\|w\|^2_{\dot H^{1/2}(\partial D^\delta)} + \int_{-\varepsilon}^{\varepsilon}\frac{|w^+(x)-w^-(x)|^2}{|x|^m+\delta}dx}, \\ b_n^{\delta,+} \;&:=\; \max_{\substack{F_n \subset \dot H^{1/2} \\ \dim(F_n)=n+1}} \;\min_{w\in F_n\setminus\{0\}} \frac{\sum_{j=1}^{2}\|w\|^2_{\dot H^{1/2}(\partial D_j)}}{\|w\|^2_{\dot H^{1/2}(\partial D^\delta)} + \int_{-\varepsilon}^{\varepsilon}\frac{|w^+(x)-w^-(x)|^2}{|x|^m+\delta}dx}. \end{aligned} \tag{43}$$

The min-max principle rewritten in the form above highlights the relationship between geometry of the contact between D_1 and D_2 and the eigenvalues $\left(\beta_n^{\delta\pm}\right)_{n\geq 1}$. Let us denote by $\sigma_n^{\delta,\pm}$ and $s_n^{\delta,\pm}$ the eigenvalues of T_δ and the numbers defined by (43) in the special case when D_1 and D_2 are the discs of radius 1, centered at the points $(0, 1+\delta/2)$ and $(0, -1-\delta/2)$. For this configuration, the exterior domain D' can be mapped conformally onto an annulus, via the mapping $z = x_1 + ix_2 \to \frac{z-a}{z+a}$ with $a = \sqrt{\delta(2-\delta)}$. One can then compute explicitely the eigenvalues [10]

$$\sigma_n^{\delta,\pm} \;=\; \frac{1}{2} \mp \frac{\rho^{2n}}{2}, \qquad \text{with } \rho := \frac{a-\delta}{a+\delta}. \tag{44}$$

As for the lower and upper bounds corresponding to (43), we obtain the following:

Corollary 3. *Assume that D_j, $j = 1, 2$ are the two discs $B_1(X_j)$, $j = 1, 2$ of radius 1, centered at centered at $X_1 = (0, 1+\delta/2)$ and $X_2 = (-1, -\delta/2)$. Then*

$$\left\{ \begin{array}{lcl} s_n^{\delta,+} &=& n\sqrt{2\delta} + o(\sqrt{\delta}) \\ s_n^{\delta,-} &=& 1 - n\sqrt{2\delta} + o(\sqrt{2\delta}). \end{array} \right. \tag{45}$$

In the rest of the section, we compare the eigenvalues $\left(\beta_n^{\delta,\pm}\right)_{n\geq 1}$ with the eigenvalues of the configuration where the inclusions are discs.

Since the inclusions are smooth and strictly convex around the point 0, there exists a diffeomorphism $\Pi : \partial D_1^0 \to \partial D_2^0$ such that

$$\Pi\,(x_1, \psi_1(x_1)) \;=\; (x_1, -\psi_2(x_1)) \qquad \text{for} \quad x_1 \in (-\varepsilon, \varepsilon).$$

Let Γ denote a diffeomorphism from the unit circle $\partial B_1(0)$ onto ∂D_1^0, such that $\Gamma(0,-1) = (0, \delta/2)$. Set $X_1^\delta = (0, 1 + \frac{\delta^{2/m}}{2}), X_2^\delta = (0, -1 - \frac{\delta^{2/m}}{2})$, let Θ denote the symmetry with respect to x_1-axis in the plane, and let τ_α denote the translation $X \in I\!R^2 \to X + (0, \alpha)$. Then $\Upsilon_1 = \tau_{\frac{\delta}{2}} \circ \Gamma \circ \tau_{\frac{-\delta^{2/m}}{2}}$ maps $\partial B_1(X_1^\delta)$ into ∂D_1 and satisfies $\Upsilon_1(0, \delta^{2/m}) = (0, \delta/2)$. Similarly, $\Upsilon_2 = \tau_{\frac{-\delta}{2}} \Pi \circ \Gamma \circ \tau_{\frac{-\delta^{2/m}}{2}} \circ \Theta$ is a diffeomorphism that maps $\partial B_1(X_2^\delta)$ onto ∂D_2. In addition the two diffeomorphisms: $\Upsilon_j,\ j = 1, 2$, satisfy

$$\Upsilon_1^{-1}\,(x_1, \psi_1(x_1)) \;=\; \Theta \circ \Upsilon_2^{-1}\,(x_1, -\psi_2(x_1)) \qquad \text{for} \quad x_1 \in (-\varepsilon, \varepsilon). \tag{46}$$

This parametrization of the inclusions leaves the Sobolev norms essentially unchanged, i.e., there exists a constant C, independent of δ such that for any $v \in H^{1/2}(\partial D_j)$

$$\frac{1}{C}\,\|v\|_{\dot H^{1/2}(\partial D_j)} \;\le\; \|v \circ \Upsilon_j\|_{\dot H^{1/2}(\partial B_1(X_j^\delta))} \;\le\; C\,\|v\|_{\dot H^{1/2}(\partial D_j)}.$$

In the sequel, we focus on the asymptotics of $\beta_n^{\delta,+}$. The results for the $\beta_n^{\delta,-}$ follow from the symmetry $\beta_n^{\delta,-} = 1 - \beta_n^{\delta,+}$. We note that the following inequality holds

$$\frac{1}{|x|^m + \delta} \;\le\; \frac{2\delta^{-1+\frac{2}{m}}}{|x|^2 + \delta^{2/m}}.$$

Hence, there exists a constant $C > 0$ independent of δ, such that

$$\begin{aligned}
&\frac{\sum_{j=1}^2 \|w\|^2_{\dot H^{1/2}(\partial D_j)}}{\|w\|^2_{\dot H^{1/2}(\partial D)} + \int_{-\varepsilon}^{\varepsilon} \frac{|w^+(x) - w^-(x)|^2}{|x|^m + \delta} dx} \\
&\ge\; C\delta^{1-\frac{2}{m}}\, \frac{\sum_{j=1}^2 \|w\|^2_{\dot H^{1/2}(\partial D_j)}}{\|w\|^2_{\dot H^{1/2}(\partial D)} + \int_{-\varepsilon}^{\varepsilon} \frac{|w^+(x) - w^-(x)|^2}{|x|^2 + \delta^{\frac{2}{m}}} dx} \\
&\ge\; C\delta^{1-\frac{2}{m}}\, \frac{\sum_{j=1}^2 \|w \circ \Upsilon_j\|^2_{\dot H^{1/2}(\partial B_1(X_j^\delta))}}{\|w \circ \Upsilon\|^2_{\dot H^{1/2}(\cup_j \partial B_1(X_j^\delta))} + \int_{-\varepsilon}^{\varepsilon} \frac{|w\circ\Upsilon^+(x) - w\circ\Upsilon^-(x)|^2}{|x|^2 + \delta^{\frac{2}{m}}} dx},
\end{aligned}$$

for all $w \in \dot{H}^{1/2}(\partial D^\delta) \setminus \{0\}$. Noticing that the last term on the right-hand side is in fact the quantity to be minimized in the case of discs separated by a distance $\delta^{\frac{2}{m}}$, and in view of (42), we obtain the following lower bound for the $b_n^{\delta,+}$

$$C\delta^{1-\frac{2}{m}} s_n^{\delta^{\frac{2}{m}},+} \quad \le \quad b_n^{\delta,+}, \qquad n \ge 1. \tag{47}$$

Recalling (45) we obtain the expression advertised in (25).

We next seek an upper bound for $b_n^{\delta,+}$.

Lemma 3. *Let $f^j = \chi(X)\left(\cos(jx_1) - 1, \cos(jx_1) + 1\right) \in H^{1/2}(\partial D_1) \times H^{1/2}(\partial D_2)$ for $j \ge 1$, where χ is a smooth cut-off function such that $\chi(X) = 1$ for $-\varepsilon \le x_1 \le \varepsilon$. Then $F_n = \{f_j, j = 0, \cdots, n\}$ satisfies*

$$\max_{w \in F_n} \frac{\sum_{j=1}^2 \|w\|^2_{\dot{H}^{1/2}(\partial D_j)}}{\|w\|^2_{\dot{H}^{1/2}(\partial D^\delta)} + \int_{-\varepsilon}^{\varepsilon} \frac{|w^+(x) - w^-(x)|^2}{|x|^m + \delta} dx} \le C \frac{n\delta^{1-\frac{1}{m}}}{n\delta^{1-\frac{1}{m}} + 1},$$

where $C > 0$ is a constant that does not depend on δ.

Proof. On the one hand, a simple calculation yields

$$\|f_j\|_{\dot{H}^{1/2}(\partial D^\delta)} = \sum_l \|\chi(X)(\cos(jx_1) + (-1)^l)\|_{\dot{H}^{1/2}(\partial D_l)} \le Cj \qquad j = 1, \cdots, n,$$

where $C > 0$ is independent of δ. On the other hand,

$$\int_{-\varepsilon}^{\varepsilon} \frac{\left|f_j^+(x) - f_j^-(x)\right|^2}{|x|^m + \delta} dx = \int_{-\varepsilon}^{\varepsilon} \frac{2}{|x|^m + \delta} dx \sim C\delta^{\frac{1-m}{m}}.$$

where $C > 0$ is a constant that does not depend on δ.

■

The upper bound in (25) is then a direct consequence of Corollary 1 as we see from the above Lemma that

$$b_n^{\delta,+} \quad \le \quad C\,\frac{n\delta^{1-\frac{1}{m}}}{n\delta^{1-\frac{1}{m}} + 1}. \tag{48}$$

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