Superlensing using hyperbolic metamaterials: the scalar case

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Abstract

This paper is devoted to superlensing using hyperbolic metamaterials: the possibility to image an arbitrary object using hyperbolic metamaterials without imposing any conditions on the size of the object and the wave length. To this end, two types of schemes are suggested and their analysis are given. The superlensing devices proposed are independent of the object. It is worth noting that the study of hyperbolic metamaterials is challenging due to the change of type of the modeling equations, elliptic in some regions, hyperbolic in some others.

1 Introduction

Metamaterials are smart materials engineered to have properties that have not yet been found in nature. They have recently attracted a lot of attention from the scientific community, not only because of potentially interesting applications, but also because of challenges in understanding their peculiar properties.

Negative index materials (NIMs) is an important class of metamaterials. Their study was initiated a few decades ago in the seminal paper of Veselago [31], in which he postulated the existence of such materials. New fabrication techniques now allow the construction of NIMs at scales that are interesting for applications, and have made them a very active topic of investigation. One of the interesting properties of NIMs is superlensing, i.e., the possibility to beat the Rayleigh diffraction limit [1]: no constraint between the size of the object and the wavelength is imposed.

Based on the theory of optical rays, Veselago discovered that a slab lens of index -1 could exhibit an unexpected superlensing property with no constraint on the size of the object to be imaged [31]. Later studies by Nicorovici, McPhedran, and Milton [24], Pendry [25, 26], Ramakrishna and Pendry in [29], for constant isotropic objects and dipole sources, showed similar properties for cylindrical lenses in the two dimensional quasistatic regime, for the Veselago slab and cylindrical lenses in the finite frequency regime, and for spherical lenses in the finite frequency regime. Superlensing of arbitrary inhomogeneous objects using NIMs in the acoustic

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[1] The Rayleigh diffraction limit is on the resolution of lenses made of a standard dielectric material: the size of the smallest features in the images they produce is about a half of the wavelength of the incident light.

and electromagnetic settings was established in [15, 19] for related lens designs. Other interesting properties of NIMs include cloaking using complementary media [10, 17, 23], cloaking a source via anomalous localized resonance [1, 2, 9, 13, 16, 20, 22], and cloaking an arbitrary object via anomalous localized resonance [21].

In this paper, we are concerned with another type of metamaterials: hyperbolic metamaterials (HMMs). These materials have quite promising potential applications to subwavelength imaging and focusing; see [27] for a recent interesting survey on hyperbolic materials and their applications. We focus here on their superlensing properties. The peculiar properties and the difficulties in the study of NIMs come from (can be explained by) the fact that the equations modelling their behaviors have sign changing coefficients. In contrast, the modeling of HMMs involve equations of changing type, elliptic in some regions, hyperbolic in others.

We first describe a general setting concerning HMMs and point out some of their general properties. Consider a standard medium that occupies a region Ω of $\mathbb{R}^d$ ($d = 2, 3$) with standard (elliptic) material constant A, except for a subset D in which the material is hyperbolic with material constant A^H in the quasistatic regime (the finite frequency regime is also considered in this paper and is discussed later). Thus, A^H is a symmetric hyperbolic matrix-valued function defined in D and A is a symmetric uniformly elliptic matrix-valued function defined in $\Omega \setminus D$. Since metamaterials usually contain damping (metallic) elements, it is also relevant to assume that the medium in D is lossy (some of its electromagnetic energy is dissipated as heat) and study the situation as the loss goes to 0. The loss can be taken into account by adding $-i\delta I$ to A^H, where I denotes the identity matrix, where $i^2 = -1$, and where $\delta > 0$ is a parameter meant to be small. With the loss, the medium in the whole of Ω is thus characterized by the matrix-valued function A_δ defined by

$$A_\delta = \begin{cases} A & \text{in } \Omega \setminus D, \\ A^H - i\delta I & \text{in } D. \end{cases} \tag{1.1}$$

For a given (source) function $f \in L^2(\Omega)$, the propagation of light/sound is modeled in the quasistatic regime by the equation

$$\operatorname{div}(A_\delta \nabla u_\delta) = f \text{ in } \Omega, \tag{1.2}$$

with an appropriate boundary condition on $\partial\Omega$.

Understanding the behaviour of u_δ as $\delta \to 0_+$ is a difficult question in general due to two facts. Firstly, equation (1.2) has both elliptic (in $\Omega \setminus D$) and hyperbolic (in D) characters. It is hence out of the range of the standard theory of elliptic and hyperbolic equations. Secondly, even if (1.2) is of hyperbolic character in D, the situation is far from standard since the problem in D is not an initial boundary problem. There are constraints on both the Dirichlet and Neumann boundary conditions (the transmission conditions). As a consequence, equation (1.2) is very unstable (see Section 5).

In this paper, we study superlensing using HMMs. The use of hyperbolic media in the construction of lenses was suggested by Jacob et al. in [6] and was experimentally verified by Liu et al. in [12]. The proposal of [6] concerns cylindrical lenses in which the hyperbolic material is given in standard polar coordinates by

$$A^H = a_\theta e_\theta \times e_\theta - a_r e_r \times e_r, \tag{1.3}$$

where a_θ and a_r are positive constants [2]. Denoting the inner radius and the outer radius of the cylinder respectively by r_1 and r_2, Jacob et al. argued that

$$\text{the resolution is } \frac{r_1}{r_2}\lambda, \tag{1.4}$$

where λ is the wave number. They supported their prediction by numerical simulations.

The goal of our paper is to go beyond the resolution problem to achieve superlensing using HMMs as discussed in [15, 19] in the context of NIMs, i.e., to be able to image an object without imposing restrictions on the ratio between its size and the wavelength of the incident light. We propose two constructions for superlensing, which are based on two different mechanisms, inspired by two basic properties of the one dimensional wave equation.

The first mechanism is based on the following simple observation. Let u be a smooth solution of the system

$$\begin{cases} \partial^2_{tt} u(t,x) - \partial^2_{xx} u(t,x) = 0 \text{ in } \mathbb{R}_+ \times [0, 2\pi], \\ u(t,\cdot) \text{ is } 2\pi\text{-periodic.} \end{cases} \tag{1.5}$$

Then u can be written in the form

$$u(t,x) = a_0 + b_0 t + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{\pm} a_{n,\pm} e^{i(\pm nt + nx)} \text{ in } \mathbb{R}_+ \times [0, 2\pi],$$

for some constants $a_0, b_0, a_{n,\pm} \in \mathbb{C}$. For the class of Cauchy data satisfying the condition $\int_0^{2\pi} \partial_t u(0,x)\, dx = 0$, we have

$$b_0 = 0.$$

This implies

$$u(t,\cdot) = u(t+2\pi,\cdot) \quad \text{and} \quad \partial_t u(t,\cdot) = \partial_t u(t+2\pi,\cdot) \quad \text{for all } t \geq 0, \tag{1.6}$$

and thus the values of u and $\partial_t u$ are transported without alteration over time intervals of length 2π. We speak of tuned superlensing to describe devices that achieve superlensing using this property.

In particular, we propose the following two dimensional superlensing device in the annulus $B_{r_2} \setminus B_{r_1}$:

$$A^H = \frac{1}{r} e_r \times e_r - r e_\theta \times e_\theta \text{ in } B_{r_2} \setminus B_{r_1}, \tag{1.7}$$

under the requirement that

$$r_2 - r_1 \in 2\pi\mathbb{N}_+. \tag{1.8}$$

Throughout the paper, B_r denotes the open ball in $\mathbb{R}^d$ centered at the origin and of radius r. We also use the standard notations for the polar coordinates in two dimensions and the spherical coordinates in three dimensions. With the choice of A^H in (1.7), we have

$$\operatorname{div}(A^H \nabla u) = \frac{1}{r}(\partial^2_{rr} u - \partial^2_{\theta\theta} u) \text{ in } B_{r_2} \setminus B_{r_1}.$$

[2] It seems to us that in their proposal these constants can be chosen quite freely.

Hence, if u is a solution to the equation $\operatorname{div}(A^H \nabla u) = 0$ in $B_{r_2} \setminus B_{r_1}$ then

$$\partial^2_{rr} u - \partial^2_{\theta\theta} u = 0 \text{ in } B_{r_2} \setminus B_{r_1}. \tag{1.9}$$

For the special choice of boundary conditions considered in its statement, Theorem 1 below shows that (1.8) and (1.9) imply that

$$u(r_2 x/|x|) = u(r_1 x/|x|) \quad \text{and} \quad \partial_r u(r_2 x/|x|) = \partial_r u(r_1 x/|x|). \tag{1.10}$$

This in turn implies the magnification of the medium contained inside B_{r_1} by a factor r_2/r_1 (the precise meaning of this is given in the statement of Theorem 1).

Our second class of superlensing devices is inspired by another observation concerning the one dimensional wave equation. Given $T > 0$, let u be a solution with appropriate regularity to the system

$$\begin{cases} \partial^2_{tt} u - \partial^2_{xx} u = 0 \text{ in } (-T, 0) \times [0, 2\pi], \\ -\partial^2_{tt} u + \partial^2_{xx} u = 0 \text{ in } (0, T) \times [0, 2\pi], \\ u \text{ is } 2\pi\text{-periodic w.r.t. } x, \\ u(0_+, \cdot) = u(0_-, \cdot), \, \partial_t u(0_+, \cdot) = -\partial_t u(0_-, \cdot) \text{ in } [0, 2\pi]. \end{cases} \tag{1.11}$$

Then

$$u(t, x) = u(-t, x) \text{ for } (t, x) \in (0, T) \times [0, 2\pi]. \tag{1.12}$$

Indeed, set

$$v(t, x) = u(-t, x) \quad \text{and} \quad w(t, x) = v(t, x) - u(t, x) \text{ for } (t, x) \in (0, T) \times (0, 2\pi).$$

Then

$$\begin{cases} \partial^2_{tt} w - \partial^2_{xx} w = 0 \text{ in } (0, T) \times [0, 2\pi], \\ w(\cdot, 0) = w(\cdot, 2\pi) = 0 \text{ in } (0, T), \\ w \text{ is } 2\pi\text{-periodic w.r.t. } x, \\ w(0_+, \cdot) = \partial_t w(0_+, \cdot) = 0 \text{ in } [0, 2\pi]. \end{cases}$$

Therefore, $w = 0$ in $(0, T) \times (0, 2\pi)$ by the uniqueness of the Cauchy problem for the wave equation. This implies that $u(t, x) = u(-t, x)$ for $(t, x) \in (0, T) \times (0, 2\pi)$ as claimed. In this direction, we propose the following superlensing device in $B_{r_2} \setminus B_{r_1}$ in both two and three dimensions, with $r_m = (r_1 + r_2)/2$:

$$A^H = \begin{cases} \dfrac{1}{r} e_r \otimes e_r - r e_\theta \otimes e_\theta & \text{in } B_{r_2} \setminus B_{r_m}, \\ -\dfrac{1}{r} e_r \otimes e_r + r e_\theta \otimes e_\theta & \text{in } B_{r_m} \setminus B_{r_1}, \end{cases} \quad \text{for } d = 2$$

and

$$A^H = \begin{cases} \dfrac{1}{r^2} e_r \otimes e_r - (e_\theta \otimes e_\theta + e_\varphi \otimes e_\varphi) & \text{in } B_{r_2} \setminus B_{r_m}, \\ -\dfrac{1}{r^2} e_r \otimes e_r + (e_\theta \otimes e_\theta + e_\varphi \otimes e_\varphi) & \text{in } B_{r_m} \setminus B_{r_1}, \end{cases} \quad \text{for } d = 3.$$

In a compact form, one has

$$A^H = \begin{cases} \dfrac{1}{r^{d-1}} e_r \otimes e_r - r^{3-d}(I - e_r \otimes e_r) & \text{in } B_{r_2} \setminus B_{r_m}, \\ -\dfrac{1}{r^{d-1}} e_r \otimes e_r + r^{3-d}(I - e_r \otimes e_r) & \text{in } B_{r_m} \setminus B_{r_1}. \end{cases} \tag{1.13}$$

The choice of A^H in (1.13) implies that

$$\operatorname{div}(A^H \nabla u) = \frac{1}{r^{d-1}} \Big(\partial^2_{rr} u - \Delta_{\partial B_1} u \Big) \text{ in } B_{r_2} \setminus B_{r_m},$$

and

$$\operatorname{div}(A^H \nabla u) = -\frac{1}{r^{d-1}} \Big(\partial^2_{rr} u - \Delta_{\partial B_1} u \Big) \text{ in } B_{r_m} \setminus B_{r_1},$$

where $\Delta_{\partial B_1}$ denotes the Laplace-Beltrami operator on the unit sphere of $\mathbb{R}^d$. Note also that

$$\partial_r u(r_2^-, \theta) = \partial_r u(r_2^+, \theta), \quad \partial_r u(r_m^-, \theta) = -\partial_r u(r_m^+, \theta), \quad \partial_r u(r_1^-, \theta) = -\partial_r u(r_1^+, \theta).$$

Hence, if u is an appropriate solution to the equation $\operatorname{div}(A^H \nabla u) = 0$ in $B_{r_2} \setminus B_{r_1}$, then, by taking into account the transmission conditions on ∂B_{r_m}, one has

$$\begin{cases} \partial^2_{rr} u - \Delta_{\partial B_1} u = 0 & \text{in } B_{r_2} \setminus B_{r_m}, \\ -\partial^2_{rr} u + \Delta_{\partial B_1} u = 0 & \text{in } B_{r_m} \setminus B_{r_1}, \\ u\big|_{B_{r_2} \setminus B_{r_m}} = u\big|_{B_{r_m} \setminus B_{r_1}}, \ \ \partial_r u\big|_{B_{r_2} \setminus B_{r_m}} = -\partial_r u\big|_{B_{r_m} \setminus B_{r_1}} & \text{on } \partial B_{r_m}. \end{cases} \tag{1.14}$$

As in (1.12), one derives that

$$u\big((s + r_m)\hat{x}\big) = u\big((r_m - s)\hat{x}\big) \text{ for } \hat{x} \in \partial B_1, s \in (0, r_2 - r_m);$$

which yields

$$u(r_2^- \hat{x}) = u(r_1^+ \hat{x}) \quad \text{ and } \quad \partial_r u(r_2^- \hat{x}) = -\partial_r u(r_1^+ \hat{x}) \text{ for } \hat{x} \in \partial B_1. \tag{1.15}$$

This in turn implies the magnification of the medium contained inside B_{r_1} by a factor r_2/r_1 (the precise meaning is given in Theorem 2). In contrast with the first proposal (1.7) where (1.8) is required, no condition is imposed on r_1 and r_2 for the second scheme (1.13). We call this method superlensing using HHMs via complementary property. The idea of using reflection takes roots in the work of the second author [14]. Similar ideas were used in the study properties of NIMs such as superlensing [15, 19], cloaking [17, 23], cloaking via anomalous localized resonance in [16, 20, 21, 22], and the stability of NIMs in [18]. Nevertheless, the superlensing properties of NIMs and HMMs are based on two different phenomena: the unique continuation principle for NIMs, and the uniqueness of the Cauchy problem for the wave equation for HMMs.

We now state two results that illustrate tuned superlensing and superlensing via complementary property. Suppose that an object to-be-magnified in B_{r_1} is characterized by a symmetric uniformly elliptic matrix-valued function a. Throughout the paper, to deal with sufficiently regular solutions of the wave equation, we assume that

$$a \text{ is of class } C^1 \text{ in a neighborhood of } \partial B_{r_1}. \tag{1.16}$$

Suppose that outside B_{r_2} the medium is homogeneous and the lens is characterized by a matrix-valued function A^H in $B_{r_2} \setminus B_{r_1}$. The whole system (taking loss into account) is then given by

$$A_\delta = \begin{cases} I & \text{in } \Omega \setminus B_{R_2}, \\ A^H - i\delta I & \text{in } B_{r_2} \setminus B_{r_1}, \\ a & \text{in } B_{r_1}. \end{cases} \tag{1.17}$$

Set

$$H^1_m(\Omega) := \Big\{ u \in H^1(\Omega); \int_{\partial\Omega} u = 0 \Big\}. \tag{1.18}$$

Concerning the scheme where A^H is defined by (1.7), we have

Theorem 1. *Let $d = 2$, $0 < \delta < 1$, $0 < r_1 < r_2$ with $r_2 - r_1 \in 2\pi\mathbb{N}_+$, let Ω be a smooth bounded connected open subset of $\mathbb{R}^2$, and let $f \in L^2(\Omega)$ with $\displaystyle\int_\Omega f = 0$. Assume that $B_{r_2} \subset\subset \Omega$ and $\operatorname{supp} f \subset \Omega \setminus B_{r_2}$. Let $u_\delta \in H^1_m(\Omega)$ be the unique solution to the system*

$$\begin{cases} \operatorname{div}(A_\delta \nabla u_\delta) = f & \textit{in } \Omega, \\ \partial_\nu u_\delta = 0 & \textit{on } \partial\Omega, \end{cases} \tag{1.19}$$

where A_δ is given by (1.17) with A^H defined by (1.7). We have

$$\|u_\delta\|_{H^1(\Omega)} \le C \|f\|_{L^2(\Omega)} \quad \textit{and} \quad u_\delta \to u_0 \textit{ strongly in } H^1(\Omega) \textit{ as } \delta \to 0, \tag{1.20}$$

where $u_0 \in H^1_m(\Omega)$ is the unique solution to (1.19) with $\delta = 0$ and C is a positive constant independent of f and δ. Moreover, $u_0 = \hat{u}$ in $\Omega \setminus B_{r_2}$ where $\hat{u} \in H^1_m(\Omega)$ is the unique solution to the system

$$\begin{cases} \operatorname{div}(\hat{A} \nabla \hat{u}) = f & \textit{in } \Omega, \\ \partial_\nu \hat{u} = 0 & \textit{on } \partial\Omega, \end{cases} \quad \textit{where} \quad \hat{A}(x) = \begin{cases} I & \textit{in } \Omega \setminus B_{r_2}, \\ a\big(r_1 x / r_2\big) & \textit{in } B_{r_2}. \end{cases} \tag{1.21}$$

Concerning the scheme where A^H is defined by (1.13), we establish

Theorem 2. *Let $d = 2, 3$, $0 < \delta < 1$, $0 < r_1 < r_2$, Ω be a smooth bounded connected open subset of $\mathbb{R}^d$, and let $f \in L^2(\Omega)$ with $\displaystyle\int_\Omega f = 0$. Assume that $B_{r_2} \subset\subset \Omega$ and $\operatorname{supp} f \subset \Omega \setminus B_{r_2}$. Let $u_\delta \in H^1_m(\Omega)$ be the unique solution to the system*

$$\begin{cases} \operatorname{div}(A_\delta \nabla u_\delta) = f & \textit{in } \Omega, \\ \partial_\nu u_\delta = 0 & \textit{on } \partial\Omega, \end{cases} \tag{1.22}$$

where A_δ is given by (1.17) with A^H defined by (1.13). We have

$$\|u_\delta\|_{H^1(\Omega)} \le C \|f\|_{L^2(\Omega)} \quad \textit{and} \quad u_\delta \to u_0 \textit{ strongly in } H^1(\Omega), \tag{1.23}$$

where $u_0 \in H^1_m(\Omega)$ *is the unique solution to* (1.22) *with* $\delta = 0$ *and* C *is a positive constant independent of* f *and* δ*. Moreover,* $u_0 = \hat{u}$ *in* $\Omega \setminus B_{r_2}$ *where* $\hat{u} \in H^1_m(\Omega)$ *is the unique solution to the system*

$$\begin{cases} \operatorname{div}(\hat{A}\nabla \hat{u}) = f & \text{in } \Omega, \\ \partial_\nu \hat{u} = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{where} \quad \hat{A}(x) = \begin{cases} I & \text{in } \Omega \setminus B_{r_2}, \\ \dfrac{r_1^{d-2}}{r_2^{d-2}} a\Big(\dfrac{r_1}{r_2}x\Big) & \text{in } B_{r_2}. \end{cases} \tag{1.24}$$

Some comments on Theorems 1 and 2 are in order. The well-posedness and the stability of (1.19) and (1.22) are established in Lemma 1. The existence and the uniqueness of u_0 are part of the statements of Theorems 1 and 2. Since f is arbitrary with support in $\Omega \setminus B_{r_2}$, it follows from the definition of $\hat{A}$ that the object in B_{r_1} is magnified by a factor r_2/r_1. It is worth noting that the matrix a can be an arbitrary function inside B_{r_1}, provided it is uniformly elliptic and smooth near ∂B_{r_1}. The lensing properties of the proposed devices in $B_{r_2} \setminus B_{r_1}$ are independent of the object.

The paper is organized as follows. Section 2 is devoted to tuned superlensing via HMMs. There, besides the proof of Theorem 1, we also discuss variants in two and three dimensions in the finite frequency regime (Theorems 3 and 4). Section 3 concerns superlensing using HMMs via the complementary property. In this section, we prove Theorem 2 and establish its finite frequency variant (Theorem 5). Finally, in Section 4, we construct HMMs with the required properties, as limits as $\delta \to 0$ of effective media obtained from the homogenization of composite structures, mixtures of a dielectric and a metal. The final section concerns the stability of HMMs. We show there on a simple example, that the properties of an inclusion of hyperbolic metamaterial, embedded in a matrix of dielectric material, are strongly dependent on the geometry of the inclusion. Numerical simulations of some of the results of our paper are presented in [5]. It would be interesting to analyse the corresponding problems for the full Maxwell system and to investigate other possible applications of HMMs. We plan to address these questions in future work.

2 Tuned superlensing using HMMs

In this section, we first present two lemmas on the stability of (1.2), (1.21), and (1.24) and their variants in the finite frequency regime. In the second part, we discuss a toy model which illustrates tuned superlensing with hyperbolic media. In the third part, we give the proof of Theorem 1. In the last part, we discuss its variants in the finite frequency regime.

2.1 Two useful lemmas

We first establish the following lemma which implies the well-posedness of (1.2). In what follows, for a subset D of $\mathbb{R}^d$, $\mathbb{1}_D$ denotes its characteristic function. For a function $u \in L^2(\Omega)$ and $D \subset\subset \Omega$, we set $u^+ = u_{|\Omega\setminus\overline{D}}$, $u^- = u_{|D}$. When u has a well-defined trace on ∂D, we set $[u] = u^+ - u^-$ on ∂D. We also use similar notations for $A\nabla u \cdot \nu$. We have

Lemma 1. *Let $d = 2, 3$, $k \geq 0$, $\delta_0 > 0$, $0 < \delta < \delta_0$. Let $D \subset\subset \Omega$ be two smooth bounded connected open subsets of $\mathbb{R}^d$. Let A be a bounded matrix-valued function defined in Ω such that A is uniformly elliptic in $\Omega \setminus D$, A is piecewise C^1 in Ω, and let Σ be a complex bounded function such that $\Im(\Sigma) \geq 0$. Set*

$$A_\delta(x) = A(x) - i\delta \mathbb{1}_D(x) I \text{ and } \Sigma_\delta(x) = \Sigma(x) + i\delta \mathbb{1}_D(x) \text{ in } \Omega. \tag{2.1}$$

Let $g_\delta \in [H^1(\Omega)]^*$, the dual space of $H^1(\Omega)$, be such that g_δ is square integrable near $\partial\Omega$ and in the case $k = 0$, assume in addition that $\int_\Omega g_\delta = 0$. There exists a unique solution $v_\delta \in H^1(\Omega)$ if $k > 0$ (respectively $v_\delta \in H^1_m(\Omega)$ if $k = 0$) to the system

$$\begin{cases} \operatorname{div}(A_\delta \nabla v_\delta) + k^2 \Sigma_\delta v_\delta = g_\delta & \text{in } \Omega, \\ A \nabla v_\delta \cdot \nu - ikv_\delta = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

Moreover,

$$\|v_\delta\|^2_{H^1(\Omega)} \leq \frac{C}{\delta} \left| \int_\Omega g_\delta \bar{v}_\delta \right| + \|g_\delta\|^2_{[H^1(\Omega)]^*}, \tag{2.3}$$

for some positive constant C depending only on Ω, D, and k. Consequently,

$$\|v_\delta\|_{H^1(\Omega)} \leq \frac{C}{\delta} \|g_\delta\|_{[H^1(\Omega)]^*}. \tag{2.4}$$

Proof. We only prove the result for $k > 0$. The case $k = 0$ follows similarly and is left to the reader. The proof is in the spirit of that of [20, Lemma 2.1]. The existence of v_δ can be derived from the uniqueness of v_δ by using the limiting absorption principle, see, e.g., [18]. We now establish the uniqueness of v_δ by showing that $v_\delta = 0$ if $g_\delta = 0$. Multiplying the equation of v_δ by $\bar{v}_\delta$ (the conjugate of v_δ) and integrating by parts, we obtain

$$-\int_\Omega \langle A_\delta \nabla v_\delta, \nabla v_\delta \rangle + k^2 \int_\Omega \Sigma_\delta |v_\delta|^2 + \int_{\partial\Omega} ik|v_\delta|^2 = 0.$$

Considering the imaginary part and using the definition (2.1) of A_δ and Σ_δ, we have

$$v_\delta = 0 \text{ in } D. \tag{2.5}$$

This implies $v_\delta^- = A_\delta \nabla v_\delta^- \cdot \nu = 0$ on ∂D; which yields, by the transmission conditions on ∂D,

$$v_\delta^+ = A \nabla v_\delta^+ \cdot \nu = 0 \text{ on } \partial D.$$

It follows from the unique continuation (see, e.g., [28]) that $v_\delta = 0$ also in $\Omega \setminus D$. The proof of uniqueness is complete.

We next establish (2.3) by contradiction. Assume that there exists $(g_\delta) \subset [H^1(\Omega)]^*$ such that g_δ is square integrable near $\partial\Omega$,

$$\|v_\delta\|_{H^1(\Omega)} = 1 \text{ and } \frac{1}{\delta} \Big| \int_\Omega g_\delta \bar{v}_\delta \Big| + \|g_\delta\|^2_{[H^1(\Omega)]^*} \to 0, \tag{2.6}$$

as $\delta \to \hat{\delta} \in [0, \delta_0]$. In fact, we may assume that these properties hold for a sequence $(\delta_n) \to \hat{\delta}$. However, for the simplicity of notation, we still use δ instead of δ_n to denote an element of such a sequence. We only consider the case $\hat{\delta} = 0$; the case $\hat{\delta} > 0$ follows similarly. Without loss of generality, one may assume that (v_δ) converges to v_0 strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ for some $v_0 \in H^1(\Omega)$. Then, by (2.6),

$$\begin{cases} \operatorname{div}(A_0 \nabla v_0) + k^2 \Sigma_0 v_0 = 0 & \text{in } \Omega, \\ A \nabla v_0 \cdot \nu - ikv_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

Multiplying the equation of v_δ by $\bar{v}_\delta$ and integrating by parts, we obtain

$$-\int_\Omega \langle A_\delta \nabla v_\delta, \nabla v_\delta \rangle + k^2 \int_\Omega \Sigma_\delta |v_\delta|^2 + \int_{\partial\Omega} ik|v_\delta|^2 = \int_\Omega g_\delta \bar{v}_\delta. \tag{2.8}$$

Considering the imaginary part of (2.8) and using (2.6), we have

$$\lim_{\delta \to 0} \Big(\|\nabla v_\delta\|_{L^2(D)} + \|v_\delta\|_{L^2(D)} + \|v_\delta\|_{L^2(\partial\Omega)} \Big) = 0, \tag{2.9}$$

from which it follows by semi-continuity of the norm that $v_0 = 0$ in D. Invoking again the unique continuation principle shows that $v_0 = 0$ in $\Omega \setminus D$ as well. The weak convergence of v_δ to v_0 then implies

$$\lim_{\delta \to 0} \|v_\delta\|_{L^2(\Omega)} = 0. \tag{2.10}$$

Further, the real part of (2.8) together with (2.6), (2.10), and (2.9), yields

$$\lim_{\delta \to 0} \|\nabla v_\delta\|_{L^2(\Omega \setminus D)} = 0. \tag{2.11}$$

Combining (2.10), (2.9), and (2.11) yields

$$\lim_{\delta \to 0} \|v_\delta\|_{H^1(\Omega)} = 0 :$$

which contradicts (2.6). The proof is complete. □

Remark 1. In the case $k = 0$, the result in Lemma 1 also holds for zero Dirichlet boundary condition in which $g \in H^{-1}(\Omega)$, the dual space of $H^1_0(\Omega)$. Moreover, the constant C depends only on the ellipticity of A, and on δ_0, D, and Ω. The proof follows the same lines.

The following standard result is repeatedly used in this paper:

Lemma 2. *Let $d = 2, 3$, $k \geq 0$. Let D, V, Ω be smooth bounded connected open subsets of $\mathbb{R}^d$ such that $D \subset\subset \Omega$, $\partial D \subset V \subset \Omega$. Let A be a matrix-valued function and Σ be a complex function, both defined in Ω, such that*

A is **uniformly elliptic** *in Ω and $\Sigma \in L^\infty(\Omega)$ with $\Im(\Sigma) \geq 0$ and $\Re(\Sigma) \geq c > 0$,*

for some constant c. Assume that $A \in C^1(\Omega \setminus D)$ and $A \in C^1(V \cap \bar{D})$. Let $g \in L^2(\Omega)$ and in the case $k = 0$ assume in addition that $\int_\Omega g = 0$. There exists a unique solution $v \in H^1(\Omega)$ if $k > 0$ (respectively $v \in H^1_m(\Omega)$ if $k = 0$) to the system

$$\begin{cases} \operatorname{div}(A \nabla v) + k^2 \Sigma v = g & \text{in } \Omega, \\ A \nabla v \cdot \nu - ikv = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover,

$$\|v\|_{H^1(\Omega)} \le C\|g\|_{L^2(\Omega)} \quad \text{and} \quad \|v\|_{H^2(V\setminus D)} \le C\|g\|_{L^2(\Omega)}, \tag{2.12}$$

for some positive constant C independent of f.

Proof. The existence, the uniqueness, and the first inequality of (2.12) follow from the Fredholm theory by the uniform ellipticity of A in Ω and the boundary condition used. The second inequality of (2.12) can be obtained by Nirenberg's method of difference quotients (see, e.g., [4]) using the smoothness assumption of A and the boundedness of Σ. The details are left to the reader. □

2.2 A toy problem

In this section, we consider a toy problem for tuned superlensing using HMMs, in which the geometry is rectangular. Given three positive constants l, L and T, we define [3]

$$\mathcal{R} = [-l, L] \times [0, 2\pi], \quad \mathcal{R}_l = [-l, 0] \times [0, 2\pi], \quad \mathcal{R}_c = [0, T] \times [0, 2\pi], \quad \mathcal{R}_r = [T, L] \times [0, 2\pi].$$

Denote

$$\Gamma := \partial\mathcal{R}, \quad \Gamma_{c,0} = \{0\} \times [0, 2\pi], \quad \text{and} \quad \Gamma_{c,T} = \{T\} \times [0, 2\pi].$$

Let a be a uniformly elliptic matrix-valued function defined in $\mathcal{R}_l \cup \mathcal{R}_r$. We set

$$a_\delta = \begin{pmatrix} 1 - i\delta & 0 \\ 0 & -1 - i\delta \end{pmatrix},$$

and define

$$A_\delta = \begin{cases} a & \text{in } \mathcal{R}_l \cup \mathcal{R}_r, \\ a_\delta & \text{in } \mathcal{R}_c, \end{cases}$$

so that the superlensing device occupies the region $\mathcal{R}_c$. For $f \in L^2(\mathcal{R})$ with $\operatorname{supp} f \cap \mathcal{R}_c = \emptyset$, let $u_\delta \in H^1_0(\mathcal{R})$ be the unique solution to the equation

$$\operatorname{div}(A_\delta \nabla u_\delta) = f \text{ in } \mathcal{R}. \tag{2.13}$$

Assume that $\|u_\delta\|_{H^1(\mathcal{R})}$ is bounded as $\delta \to 0$. Then, up to a subsequence, u_δ converges weakly to some $u_0 \in H^1_0(\mathcal{R})$. It is clear that u_0 is a solution to

$$\operatorname{div}(A_0 \nabla u_0) = f \text{ in } \mathcal{R}. \tag{2.14}$$

More precisely, $u_0 \in H^1_0(\mathcal{R})$ satisfies (2.14) if and only if u_0 satisfies the elliptic-hyperbolic system

$$\operatorname{div}(a\nabla u_0) = f \text{ in } \mathcal{R}_l \cup \mathcal{R}_r \quad \text{and} \quad \partial^2_{x_1x_1} u_0 - \partial^2_{x_2x_2} u_0 = 0 \text{ in } \mathcal{R}_c,$$

and the transmission conditions

$$\begin{cases} u_0\big|_{\mathcal{R}_l} = u_0\big|_{\mathcal{R}_c} \\ \partial_{x_1} u_0\big|_{\mathcal{R}_l} = \partial_{x_1} u_0\big|_{\mathcal{R}_c}, \end{cases} \text{on } \Gamma_{c,0} \quad \text{and} \quad \begin{cases} u_0\big|_{\mathcal{R}_r} = u_0\big|_{\mathcal{R}_c} \\ \partial_{x_1} u_0\big|_{\mathcal{R}_r} = \partial_{x_1} u_0\big|_{\mathcal{R}_c}, \end{cases} \text{on } \Gamma_{c,T}.$$

[3] Letters c, l, r stand for center, left, and right.

This problem is ill-posed: in general, there is no solution in $H^1_0(\mathcal{R})$, and so, $\|u_\delta\|_{H^1(\mathcal{R})} \to +\infty$, as $\delta \to 0$ (see Section 5 for an example of such situation). Nevertheless, for some special choices of T, discussed below, the problem is well-posed and its solutions have peculiar properties.

To describe them, we introduce an "effective domain" $\mathcal{R}_T = [-l, L-T] \times [0, 2\pi]$ and

$$\hat{A}(x_1,x_2), \hat{f}(x_1,x_2) = \begin{cases} a(x_1,x_2), f(x_1,x_2) & \text{in } \mathcal{R}_l \\ a(x_1+T,x_2), f(x_1+T,x_2) & \text{in } \mathcal{R}_T \setminus \mathcal{R}_l. \end{cases}$$

In what follows, we assume that $\hat{A} \in C^2(\overline{\mathcal{R}_T})$.

Proposition 1. *Let $0 < \delta < 1$, $f \in L^2(\mathcal{R})$, and $u_\delta \in H^1_0(\mathcal{R})$ be the unique solution of (2.13). Assume that $T \in 2\pi\mathbb{N}_+$ and $\operatorname{supp} f \cap \mathcal{R}_c = \emptyset$. Then*

$$\|u_\delta\|_{H^1} \le C\|f\|_{L^2(\mathcal{R})} \quad \textit{and} \quad u_\delta \to u_0 \quad \textit{strongly in } H^1(\mathcal{R}), \tag{2.15}$$

where $u_0 \in H^1_0(\mathcal{R})$ is the unique solution of (2.13) with $\delta = 0$ and C is a positive constant independent of δ and f. We also have

$$u_0(x_1,x_2) = \begin{cases} \hat{u}(x_1,x_2) & \textit{in } \mathcal{R}_l, \\ \hat{u}(x_1-T,x_2) & \textit{in } \mathcal{R}_r, \end{cases}$$

where $\hat{u} \in H^1_0(\mathcal{R}_T)$ is the unique solution to the equation

$$\operatorname{div}(\hat{A}\nabla\hat{u}) = \hat{f} \textit{ in } \mathcal{R}_T, \tag{2.16}$$

Remark 2. It follows from Proposition 1 that u_0 can be computed as if the structure in $\mathcal{R}_c$ had disappeared. This phenomenon is similar to that in the Veselago setting: superlensing occurs.

Proof. The proof of Proposition 1 is in the spirit of the approach used by the second author in [14] to deal with negative index materials. The key point is to construct the unique solution u_0 to the limiting problem appropriately and then estimate u_δ by studying the difference $u_\delta - u_0$.

We first construct a solution $u_0 \in H^1_0(\mathcal{R})$ to (2.13) with $\delta = 0$. Since $\hat{A} \in C^2(\overline{\mathcal{R}_T})$ and since $f \in L^2(\mathcal{R})$, the regularity theory for elliptic equations (see, e.g., [11, 3.2.1.2]) implies that $\hat{u} \in H^2(\mathcal{R}_T)$ and

$$\|\hat{u}\|_{H^2(\mathcal{R}_T)} \le C\|f\|_{L^2(\mathcal{R})}. \tag{2.17}$$

Here and in what follows in this proof, C denotes a positive constant independent of f and δ. It follows that $\hat{u}(0,x_2) \in H^1(\Gamma_{c,0})$ and $\partial_1\hat{u}(0,x_2) \in L^2(\Gamma_{c,0})$. Interpretting x_1 and x_2 as respectively time and space variables in the rectangle $\mathcal{R}_c$, we seek a solution $v \in C\big([0,T]; H^1_0(0,2\pi)\big) \cap C^1([0,T]; L^2(0,2\pi))$ of the wave equation

$$\partial^2_{x_1x_1}v - \partial^2_{x_2x_2}v = 0 \quad \text{in } \mathcal{R}_c, \tag{2.18}$$

with zero boundary condition, i.e., $v = 0$ on $\Gamma \cap \partial\Omega_c$, and with the following initial conditions

$$v(0,x_2) = \hat{u}(0,x_2) \quad \text{and} \quad \partial_{x_1}v(0,x_2) = \partial_{x_1}\hat{u}\big|_{\mathcal{R}_l}(0,x_2).$$

Existence and uniqueness of v follow from the standard theory of the wave equation by taking into account the regularity information in (2.17). We also have, for $0 \le x_1 \le T$,

$$\int_0^{2\pi} |\partial_{x_1} v(x_1,x_2)|^2 + |\partial_{x_2} v(x_1,x_2)|^2\,dx_2 = \int_0^{2\pi} |\partial_{x_1} v(0,x_2)|^2 + |\partial_{x_2} v(0,x_2)|^2\,dx_2$$
$$= \int_0^{2\pi} |\partial_{x_1} \hat{u}\big|_{\mathcal{R}_l}(0,x_2)|^2 + |\partial_{x_2} \hat{u}(0,x_2)|^2\,dx_2. \tag{2.19}$$

Furthermore, as v vanishes on $\Gamma \cap \partial\Omega_c$, it can be represented in $\mathcal{R}_c$ as

$$v(x_1,x_2) = \sum_{n=1}^{\infty} \sin(nx_2)\big[a_n\cos(nx_1) + b_n\sin(nx_1)\big], \tag{2.20}$$

where $a_n, b_n \in \mathbb{R}$ are determined by the initial conditions satisfied by v at $x_1 = 0$. Since $T \in 2\pi\mathbb{N}$, it follows from this representation that

$$v(0,\cdot) = v(T,\cdot) \quad \text{and} \quad \partial_{x_1} v(0,\cdot) = \partial_{x_1} v(T,\cdot) \text{ in } [0,2\pi], \tag{2.21}$$

for any initial conditions, and hence for any f with $\operatorname{supp} f \cap \mathcal{R}_c = \emptyset$. Define

$$u_0(x_1,x_2) = \begin{cases} \hat{u}(x_1,x_2) & \text{in } \mathcal{R}_l, \\ v(x_1,x_2) & \text{in } \mathcal{R}_c, \\ \hat{u}(x_1 - T, x_2) & \text{in } \mathcal{R}_r. \end{cases} \tag{2.22}$$

It follows from (2.16), (2.18), and (2.21) that $u_0 \in H^1_0(\Omega)$ and that it is a solution to (2.13) with $\delta = 0$. Moreover, by (2.17) and (2.19),

$$\|u_0\|_{H^1(\mathcal{R})} \le C\|f\|_{L^2(\mathcal{R})}. \tag{2.23}$$

We next establish the uniqueness of u_0. Let $w_0 \in H^1_0(\Omega)$ be a solution to (2.13) with $\delta = 0$. Since w_0 can be represented as in (2.20) in $\mathcal{R}_c$, we obtain

$$w_0(0,\cdot) = w_0(T,\cdot) \quad \text{and} \quad \partial_{x_1} w_0(0,\cdot) = \partial_{x_1} w_0(T,\cdot) \text{ in } [0,2\pi].$$

We can thus define for (x_1,x_2) in $\mathcal{R}_T$

$$\hat{w}(x_1,x_2) = \begin{cases} w_0(x_1,x_2) & \text{in } \mathcal{R}_l, \\ w_0(x_1 - T, x_2) & \text{otherwise}, \end{cases}$$

which is a solution to (2.16). By uniqueness for this elliptic equation, it follows that $\hat{w} \equiv \hat{u}$ in $\mathcal{R}_T$, and (2.22) shows that $w_0 \equiv u_0$ in $\mathcal{R}$.

Finally, we establish (2.15). Define

$$v_\delta = u_\delta - u_0 \;\text{ in } \mathcal{R}. \tag{2.24}$$

We have

$$\begin{aligned}\operatorname{div}(A_\delta \nabla v_\delta) &= \operatorname{div}(A_\delta \nabla u_\delta) - \operatorname{div}(A_\delta \nabla u_0)\\ &= \operatorname{div}(A_\delta \nabla u_\delta) - \operatorname{div}(A_0 \nabla u_0) + \operatorname{div}(A_0 \nabla u_0) - \operatorname{div}(A_\delta \nabla u_0) \text{ in } \mathcal{R}.\end{aligned}$$

It follows that $v_\delta \in H^1_0(\Omega)$ is the solution to

$$\operatorname{div}(A_\delta \nabla v_\delta) = \operatorname{div}(i\delta \mathbb{1}_{\mathcal{R}_c} \nabla u_0) \text{ in } \mathcal{R}. \tag{2.25}$$

As in (2.4) in Lemma 1, we obtain from (2.23) that

$$\|v_\delta\|_{H^1(\mathcal{R})} \le \frac{C}{\delta} \|\delta \nabla u_0\|_{L^2(\mathcal{R}_c)} \le C \|f\|_{L^2(\mathcal{R})}; \tag{2.26}$$

which implies the first inequality of (2.15). It follows that a subsequence of v_δ converges weakly to some v_0, solution to (2.14) with $f = 0$. Uniqueness shows that $v_0 = 0$, and that the whole sequence v_δ converges weakly to 0. As in (2.3) in Lemma 1, we deduce from (2.23), (2.25), and (2.26) that

$$\|u_\delta - u_0\|^2_{H^1(\mathcal{R})} = \|v_\delta\|^2_{H^1(\mathcal{R})} \le C \left| \int_{\mathcal{R}_c} i \nabla u_0 \nabla v_\delta \right| \to 0, \tag{2.27}$$

as v_δ converges weakly to 0 in $H^1(\mathcal{R})$. The proof is complete. □

2.3 Tuned superlensing using HMMs in the quasistatic regime. Proof of Theorem 1

The proof is in the spirit of Proposition 1: the main idea is to construct u_0 and then estimate $u_\delta - u_0$. We have

$$\|\hat{u}\|_{H^1(\Omega)} \le C \|f\|_{L^2(\Omega)}. \tag{2.28}$$

Using (1.16) and applying Lemma 2, we derive that $\hat{u} \in H^2(\Omega \setminus B_{r_2})$ and

$$\|\hat{u}\|_{H^2(\Omega \setminus B_{r_2})} \le C \|f\|_{L^2(\Omega)}. \tag{2.29}$$

Define a function v in $B_{r_2} \setminus B_{r_1}$ by

$$\partial^2_{rr} v - \partial^2_{\theta\theta} v = 0, \quad v \text{ is periodic with respect to } \theta, \tag{2.30}$$

and

$$v(r_2, \theta) = \hat{u}(r_2, \theta) \quad \text{ and } \quad \partial_r v(r_2, \theta) = r_2 \partial_r \hat{u}^+(r_2, \theta) \text{ for } \theta \in [0, 2\pi]. \tag{2.31}$$

By considering (2.30) as a Cauchy problem for the wave equation with periodic boundary conditions, in which r and θ are seen as a time and a space variable respectively, the standard theory shows that there exists a unique such $v(r, \theta) \in C\big([r_1, r_2]; H^1_{per}(0, 2\pi)\big) \cap C^1([r_1, r_2]; L^2(0, 2\pi))$. We also have, for $r_1 \le r \le r_2$,

$$\begin{aligned}\int_0^{2\pi} |\partial_r v(r, \theta)|^2 + |\partial_\theta v(r, \theta)|^2 \, d\theta &= \int_0^{2\pi} |\partial_r v(r_2, \theta)|^2 + |\partial_\theta v(r_2, \theta)|^2 \, d\theta \\ &= \int_0^{2\pi} r_2^2 |\partial_r \hat{u}^+(r_2, \theta)|^2 + |\partial_\theta \hat{u}(r_2, \theta)|^2 \, d\theta;\end{aligned} \tag{2.32}$$

which yields, by (2.29),

$$\|v\|_{H^1(B_{r_2}\setminus B_{r_1})} \le C\|f\|_{L^2(\Omega)}. \tag{2.33}$$

Moreover, v can be represented in the form

$$v(r,\theta) = a_0 + b_0 r + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{\pm} a_{n,\pm} e^{i(nr\pm n\theta)} \text{ in } B_{r_2}\setminus B_{r_1}, \tag{2.34}$$

where $a_0, b_0, a_{n,\pm} \in \mathbb{C}$. Since $\hat{u}^+$ is harmonic in $\Omega\setminus B_{r_2}$, we have

$$\begin{aligned} b_0 &= \frac{1}{2\pi}\int_0^{2\pi} \partial_r v(r_2,\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi} r_2 \partial_r \hat{u}^+(r_2,\theta)\, d\theta \\ &= \frac{1}{2\pi}\int_{\partial B_{r_2}} \partial_r \hat{u}^+(x)\, dx = \frac{1}{2\pi}\int_{\partial\Omega} \partial_\nu \hat{u}(x)\, dx = 0. \end{aligned}$$

Since $r_2 - r_1 \in 2\pi\mathbb{N}_+$, it follows that

$$v(r_1,\theta) = v(r_2,\theta) \quad \text{and} \quad \partial_r v(r_1,\theta) = \partial_r v(r_2,\theta) \text{ for } \theta\in[0,2\pi]. \tag{2.35}$$

Set

$$u_0 = \begin{cases} \hat{u} & \text{in } \Omega\setminus B_{r_2}, \\ v & \text{in } B_{r_2}\setminus B_{r_1}, \\ \hat{u}\big(r_2\cdot/r_1\big) & \text{in } B_{r_1}. \end{cases} \tag{2.36}$$

It follows from (2.28), (2.31), (2.33), and (2.35) that $u_0 \in H^1(\Omega)$ and

$$\|u_0\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{2.37}$$

We also have

$$\operatorname{div}(A_0\nabla u_0) = f \text{ in } \Omega\setminus(\partial B_{r_1}\cup\partial B_{r_2}). \tag{2.38}$$

On the other hand, from (1.7), (2.31) and the definition of A_0, we have

$$[A_0\nabla u_0\cdot e_r] = \partial_r u_0^+ - \frac{1}{r_2}\partial_r u_0^- = \partial_r u_0^+ - \frac{1}{r_2}\partial_r v = 0 \text{ on } \partial B_{r_2} \tag{2.39}$$

and from (2.35) and the definition of $\hat{A}$, we obtain

$$\begin{aligned} [A_0\nabla u_0\cdot e_r](x) &= \frac{1}{r_1}\partial_r u_0^+(x) - A_0\nabla u_0^-\cdot e_r(x) = \frac{1}{r_1}\partial_r v(x) - \frac{r_2}{r_1}a(x)\nabla\hat{u}^-(r_2x/r_1)\cdot e_r \\ &= \frac{1}{r_1}\partial_r v(r_2x/r_1) - \frac{r_2}{r_1}\hat{A}(r_2x/r_1)\nabla\hat{u}^+(r_2x/r_1)\cdot e_r = 0 \text{ on } \partial B_{r_1}. \end{aligned} \tag{2.40}$$

A combination of (2.38), (2.39), and (2.40) yields that

$$\operatorname{div}(A_0\nabla u_0) = f \text{ in } \Omega;$$

which implies that u_0 is a solution to (1.19) with $\delta = 0$.

We next establish the uniqueness of u_0. Let $w_0 \in H^1(\Omega)$ be a solution to (1.19) with $\delta = 0$. Since w_0 can be represented as in (2.34) in $B_{r_2} \setminus B_{r_1}$, we have

$$w_0(r_1, \theta) = w_0(r_2, \theta) \quad \text{ and } \quad \partial_r w_0(r_1, \theta) = \partial_r w_0(r_2, \theta) \text{ for } \theta \in [0, 2\pi]. \tag{2.41}$$

Define

$$\hat{w}(x) = \begin{cases} w_0(x) & \text{in } \Omega \setminus B_{r_2}, \\ w_0\big(r_1 x / r_2\big) & \text{in } B_{r_2}. \end{cases}$$

It follows from (2.41) that $\hat{w} \in H^1(\Omega)$. One can verify that $\hat{w}$ is a solution of (1.21). Hence $\hat{w} = \hat{u}$; which yields $w_0 = u_0$.

We next establish the inequality in (1.20). Set

$$v_\delta = u_\delta - u_0 \text{ in } \Omega. \tag{2.42}$$

Then $v_\delta \in H^1(\Omega)$ and satisfies

$$\operatorname{div}(A_\delta \nabla v_\delta) = \operatorname{div}(i\delta \mathbb{1}_{B_{r_2} \setminus B_{r_1}} \nabla u_0) \text{ in } \Omega$$

and

$$\partial_\nu v_\delta = 0 \text{ on } \partial\Omega.$$

Applying (2.4) of Lemma 1, we obtain from (2.37) that

$$\|v_\delta\|_{H^1(\Omega)} \le C \|\nabla u_0\|_{L^2(\Omega)},$$

which implies the inequality in (1.20). The same argument as that in Proposition 1 shows that $v_\delta \to 0$ weakly in $H^1(\Omega)$. Applying (2.3) of Lemma 1, we derive from (2.37) that

$$\|u_\delta - u_0\|^2_{H^1(\Omega)} = \|v_\delta\|^2_{H^1(\Omega)} \le C \left| \int_{B_{r_2} \setminus B_{r_1}} i \nabla u_0 \nabla v_\delta \right| \to 0 \text{ as } \delta \to 0,$$

which completes the proof. □

2.4 Tuned superlensing using HMMs in the finite frequency regime

In this section we consider variants of Theorem 2 in the finite frequency regime. Assume that the region B_{r_1} to be magnified is characterized by a pair (a, σ) of a matrix-valued function a and a complex function σ such that a satisfies the standard conditions mentioned in the introduction (a is uniformly elliptic in B_{r_1} and (1.16) holds) and σ satisfies the following standard conditions

$$\sigma \in L^\infty(B_{r_1}), \text{ with } \Im(\sigma) \ge 0 \text{ and } \Re(\sigma) \ge c > 0, \tag{2.43}$$

for some constant c. Assume that the lens without loss is characterized by a pair (A^H, Σ^H) in $B_{r_2} \setminus B_{r_1}$. Taking loss into account, the overall medium is characterized by

$$A_\delta, \Sigma_\delta = \begin{cases} I, 1 & \text{in } \Omega \setminus B_{r_2}, \\ A^H - i\delta I, \Sigma^H + i\delta & \text{in } B_{r_2} \setminus B_{r_1}, \\ a, \sigma & \text{in } B_{r_1}, \end{cases} \tag{2.44}$$

Given a (source) function $f \in L^2(\Omega)$ and given a frequency $k > 0$, standard arguments show that there is a unique solution $u_\delta \in H^1(\Omega)$ to the system

$$\begin{cases} \operatorname{div}(A_\delta \nabla u_\delta) + k^2 \Sigma_\delta u_\delta = f & \text{in } \Omega, \\ \partial_\nu u_\delta - iku_\delta = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.45}$$

We first consider the three dimensional finite frequency case. The superlens in $B_{r_2} \setminus B_{r_1}$ is defined by

$$(A^H, \Sigma^H) = \left(\frac{1}{r^2} e_r \otimes e_r - (e_\theta \otimes e_\theta + e_\varphi \otimes e_\varphi), \frac{1}{4k^2 r^2} \right) \text{ in } B_{r_2} \setminus B_{r_1}. \tag{2.46}$$

Note that Σ^H also depends on k. We have

Theorem 3. *Let $d = 3$, $k > 0$, $0 < \delta < 1$, and let Ω be a smooth bounded connected open subset of $\mathbb{R}^3$ and let $0 < r_1 < r_2$ be such that $r_2 - r_1 \in 4\pi\mathbb{N}_+$ and $B_{r_2} \subset\subset \Omega$. Let $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \setminus B_{r_2}$ and let $u_\delta \in H^1(\Omega)$ be the unique solution of* (2.45) *where (A^H, Σ^H) is given by* (2.46)*. We have*

$$\|u_\delta\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \quad \textit{and} \quad u_\delta \to u_0 \quad \textit{strongly in } H^1(\Omega), \tag{2.47}$$

where $u_0 \in H^1(\Omega)$ is the unique solution to (2.45) *with $\delta = 0$ and C is a positive constant independent of f and δ. Moreover, $u_0 = \hat{u}$ in $\Omega \setminus B_{r_2}$ where $\hat{u}$ is the unique solution to the system*

$$\begin{cases} \operatorname{div}(\hat{A} \nabla \hat{u}) + k^2 \hat{\Sigma} \hat{u} = f & \textit{in } \Omega, \\ \partial_\nu \hat{u} - ik\hat{u} = 0 & \textit{on } \partial\Omega, \end{cases} \quad \textit{where } \hat{A}, \hat{\Sigma} = \begin{cases} I, 1 & \textit{in } \Omega \setminus B_{r_2}, \\ \dfrac{r_1}{r_2} a\Big(\dfrac{r_1}{r_2} \cdot\Big), \dfrac{r_1^3}{r_2^3} \sigma\Big(\dfrac{r_1}{r_2} \cdot\Big) & \textit{in } B_{r_2}. \end{cases} \tag{2.48}$$

From the definition of (A^H, Σ^H) in (2.46), one derives that if u is a solution to the equation $\operatorname{div}(A^H \nabla u) + k^2 \Sigma^H u = 0$ in $B_{r_2} \setminus B_{r_1}$ then

$$\partial^2_{rr} u - \Delta_{\partial B_1} u + \frac{1}{4} u = 0 \text{ in } B_{r_2} \setminus B_{r_1}.$$

This equation plays a similar role as the wave equation in (1.9).

Proof. We have, by Lemma 2, that

$$\|\hat{u}\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \quad \text{and} \quad \|\hat{u}\|_{H^2(\Omega \setminus B_{r_2})} \le C\|f\|_{L^2(\Omega)}.$$

Set

$$u_0 = \begin{cases} \hat{u} & \text{in } \Omega \setminus B_{r_2}, \\ v & \text{in } B_{r_2} \setminus B_{r_1}, \\ \hat{u}(r_2 \cdot / r_1) & \text{in } B_{r_1}, \end{cases} \tag{2.49}$$

where $v \in H^1(B_{r_2} \setminus B_{r_1})$ is the unique solution of

$$\partial^2_{rr} v - \Delta_{\partial B_1} v + \frac{1}{4} v = 0 \text{ in } B_{r_2} \setminus B_{r_1}, \tag{2.50}$$

$$v = \hat{u} \text{ on } \partial B_{r_2} \quad \text{ and } \quad \partial_r v = r_2^2 \partial_r \hat{u}^+ \text{ on } \partial B_{r_2}. \tag{2.51}$$

For $n \geq 0$ and $-n \leq m \leq n$, let Y_n^m denote the spherical harmonic function of degree n and of order m, which satisfies

$$\Delta_{\partial B_1} Y_n^m + n(n+1) Y_n^m = 0 \quad \text{on } \partial B_1.$$

Since the family $\big(Y_n^m\big)$ is dense in $L^2(\partial B_1)$, v can be represented in the form

$$v(x) \;=\; \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \sum_{\pm} a_{nm,\pm} e^{\pm i \lambda_n r} Y_m^n(\hat{x}), \qquad x \in B_{r_2} \setminus B_{r_1}, \tag{2.52}$$

where $\lambda_n = (n + 1/2)$, $r = |x|$ and $\hat{x} = \frac{x}{|x|}$. Note that the 0-order term in (2.50) has been chosen in $B_{r_2} \setminus B_{r_1}$ so that the dispersion relation writes

$$\lambda_n^2 = n(n+1) + \frac{1}{4} = (n + \frac{1}{2})^2,$$

which implies that all the terms $e^{\pm i \lambda_n r}$ in (2.52), and thus v, are 4π-periodic functions of r. Since $r_2 - r_1 \in 4\pi \mathbb{N}_+$, it follows that

$$v(r_1 \hat{x}) = v(r_2 \hat{x}) \quad \text{ and } \quad \partial_r v(r_1 \hat{x}) = \partial_r v(r_2 \hat{x}) \quad \text{for } \hat{x} \in \partial B_1. \tag{2.53}$$

We have, by (2.51),

$$[A_0 \nabla u_0 \cdot e_r] = \partial_r \hat{u}^+ - \frac{1}{r_2^2} \partial_r v = 0 \text{ on } \partial B_{r_2} \tag{2.54}$$

and, by (2.51) and (2.53),

$$\begin{aligned}
[A_0 \nabla u_0 \cdot e_r](x) &= \frac{1}{r_1^2} \partial_r v(x) - \frac{r_2}{r_1} a(x) \nabla \hat{u}^-(r_2 x / r_1) \cdot e_r \\
&= \frac{1}{r_1^2} \partial_r v(r_2 x / r_1) - \frac{r_2^2}{r_1^2} \hat{A}(r_2 x / r_1) \nabla \hat{u}^-(r_2 x / r_1) \cdot e_r \\
&= \frac{1}{r_1^2} \partial_r v(r_2 x / r_1) - \frac{r_2^2}{r_1^2} \partial_r \hat{u}^+(r_2 x / r_1) = 0 \text{ on } \partial B_{r_1}.
\end{aligned} \tag{2.55}$$

As in the proof of Theorem 1, one can check that u_0 is the unique solution of (2.45) with $\delta = 0$ where (A^H, Σ^H) is given by (2.46). Moreover,

$$\|u_\delta - u_0\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}$$

and

$$u_\delta \to u_0 \text{ in } H^1(\Omega).$$

The proof of these facts is the same as in the proof of Theorem 1. □

We next deal with a variant of Theorem 1 in the two dimensional finite frequency regime. Set

$$(A^H, \Sigma^H) = (\frac{1}{r} e_r \otimes e_r - r e_\theta \otimes e_\theta, 0) \text{ in } B_{r_2} \setminus B_{r_1}. \tag{2.56}$$

The following theorem describes the superlensing property of the device defined by (2.56).

Theorem 4. *Let $d = 2$, $k > 0$, $0 < \delta < 1$, and let Ω be a smooth bounded connected open subset of $\mathbb{R}^2$. Let $0 < r_1 < r_2$ be such that $r_2 - r_1 \in 2\pi\mathbb{N}_+$ and $B_{r_2} \subset\subset \Omega$. Let $f \in L^2(\Omega)$ with $\operatorname{supp} f \subset \Omega \setminus B_{r_2}$ and let $u_\delta \in H^1(\Omega)$ be the unique solution of (2.45) where (A^H, Σ^H) is given by (2.56). We have*

$$\|u_\delta\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \quad \text{and} \quad u_\delta \to u_0 \quad \text{strongly in } H^1(\Omega) \text{ as } \delta \to 0, \tag{2.57}$$

where $u_0 \in H^1(\Omega)$ is the unique solution to (2.45) with $\delta = 0$ and C is a positive constant independent of f and δ. Moreover, $u_0 = \hat{u}$ in $\Omega \setminus B_{r_2}$, where $\hat{u} \in H^1(\Omega \setminus B_{r_2})$ is the unique solution to the system

$$\begin{cases} \operatorname{div}(\hat{A}\nabla \hat{u}) + k^2 \hat{\Sigma} \hat{u} = f & \text{in } \Omega, \\ \partial_\nu \hat{u} - ik\hat{u} = 0 & \text{on } \partial\Omega, \\ [\hat{A}\nabla \hat{u} \cdot \nu] = 0 & \text{on } \partial B_{r_2}, \\ [\hat{u}] = c \displaystyle\int_{\partial B_{r_2}} \hat{A}\nabla \hat{u} \cdot \nu & \text{on } \partial B_{r_2}, \end{cases} \tag{2.58}$$

where

$$\hat{A}(x), \hat{\Sigma}(x) = \begin{cases} I, 1 & \text{in } \Omega \setminus B_{r_2}, \\ a\Big(\frac{r_1}{r_2}x\Big), \frac{r_1^2}{r_2^2}\sigma\Big(\frac{r_1}{r_2}x\Big) & \text{in } B_{r_2}, \end{cases} \quad \text{and} \quad c = \frac{r_2 - r_1}{2\pi r_2}.$$

Since f is arbitrary with support in $\Omega \setminus B_{r_2}$, it follows from the definition of $(\hat{A}, \hat{\Sigma})$ that the object in B_{r_1} is magnified by a factor r_2/r_1.

Proof of Theorem 4. The proof is in the spirit of Theorem 1. The main difference is the fact that in the representation (2.64) below, the term b_0 does not vanish in general. The solution to the wave equation in the lens $B_{r_2} \setminus B_{r_1}$ is thus the sum of a periodic function and a linear term (in r). The constant c in the second transmission condition of (2.58) accounts precisely for the latter. The well-posedness of (2.58) is established in Lemma 3 below. From this Lemma it follows that

$$\|\hat{u}\|_{H^1(\Omega \setminus \partial B_{r_2})} \le C\|f\|_{L^2(\Omega)}. \tag{2.59}$$

Applying Lemma 3, we derive that $u \in H^2(\Omega \setminus B_{r_2})$ and

$$\|\hat{u}\|_{H^2(\Omega \setminus B_{r_2})} \le C\|f\|_{L^2(\Omega)}. \tag{2.60}$$

Let v defined in $H^1(B_{r_2} \setminus B_{r_1})$ be the unique solution of

$$\partial^2_{rr} v - \partial^2_{\theta\theta} v = 0, \quad v \text{ is periodic with respect to } \theta, \tag{2.61}$$

and

$$v(r_2,\theta) = \hat{u}(r_2,\theta) \quad \text{ and } \quad \partial_r v(r_2,\theta) = r_2 \partial_r \hat{u}^+(r_2,\theta) \text{ for } \theta \in [0, 2\pi]. \tag{2.62}$$

As in (2.33), we have

$$\|v\|_{H^1(B_{r_2}\setminus B_{r_1})} \le C\|f\|_{L^2(\Omega)}. \tag{2.63}$$

Moreover, v can be represented in the form

$$v(r,\theta) = a_0 + b_0 r + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \sum_{\pm} a_{n,\pm} e^{i(nr \pm n\theta)} \text{ in } B_{r_2} \setminus B_{r_1}, \tag{2.64}$$

where $a_0, b_0, a_{n,\pm} \in \mathbb{C}$. Since $r_2 - r_1 \in 2\pi\mathbb{N}_+$, it follows that, for $\theta \in [0, 2\pi]$,

$$v(r_2,\theta) - v(r_1,\theta) = b_0(r_2 - r_1), \quad \text{ and } \quad \partial_r v(r_1,\theta) = \partial_r v(r_2,\theta). \tag{2.65}$$

It is clear that

$$\int_{\partial B_{r_2}} \partial_r v(x)\, dx = 2\pi b_0 r_2.$$

Set

$$u_0 = \begin{cases} \hat{u} & \text{ in } \Omega \setminus B_{r_2}, \\ v & \text{ in } B_{r_2} \setminus B_{r_1}, \\ \hat{u}\big(r_2 \cdot / r_1\big) & \text{ in } B_{r_1}. \end{cases} \tag{2.66}$$

We have, by (2.62),

$$[u_0] = \hat{u}^+ - v = 0 \text{ on } \partial B_{r_2}$$

and the definition of c together with (2.62) and (2.65) yields

$$\begin{aligned} [u_0](r_1,\theta) &= v(r_1,\theta) - \hat{u}^-(r_2,\theta) \\ &= v(r_2,\theta) - b_0(r_2 - r_1) - \Big(\hat{u}^+(r_2,\theta) - 2\pi b_0 r_2 \frac{r_2 - r_1}{2\pi r_2}\Big) = 0 \text{ for } \theta \in [0, 2\pi]. \end{aligned}$$

We derive from (2.59) and (2.63) that

$$\|u_0\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{2.67}$$

We also have

$$\operatorname{div}(A_0 \nabla u_0) + k^2 \Sigma_0 u_0 = f \text{ in } \Omega \setminus (\partial B_{r_1} \cup \partial B_{r_2}). \tag{2.68}$$

As in (2.39) and (2.40) in the proof of Theorem 1, we have

$$[A_0 \nabla u_0 \cdot e_r] = 0 \text{ on } \partial B_{r_2} \quad \text{ and } \quad [A_0 \nabla u_0 \cdot e_r] = 0 \text{ on } \partial B_{r_1}. \tag{2.69}$$

A combination of (2.68) and (2.69) yields that

$$\operatorname{div}(A_0 \nabla u_0) + k^2 \Sigma_0 u_0 = f \text{ in } \Omega;$$

which implies that u_0 is a solution to (2.45) with $\delta = 0$.

The proof of the uniqueness of u_0 and the convergence of u_δ to u_0 in $H^1(\Omega)$ are the same as in the proof of Theorem 1. □

The following lemma is used in the proof of Theorem 4.

Lemma 3. *Let $d = 2, 3$, and $k > 0$. Let D, V, Ω be smooth bounded connected open subsets of $\mathbb{R}^d$ such that $D \subset\subset \Omega$, $\partial D \subset V \subset \Omega$. Let A be a bounded, piecewise C^1, matrix-valued function defined in Ω which is assumed to be* **uniformly elliptic** *in Ω and let Σ be a bounded complex-valued function, such that $Im(\Sigma) \geq 0$ in Ω. Assume that $A \in C^1(\Omega \setminus D)$ and $A \in C^1(V \cap \bar{D})$. Let $g \in L^2(\Omega)$ and $c \in \mathbb{R}$. There exists a unique solution $v \in H^1(\Omega \setminus \partial D)$ to the system*

$$\begin{cases} \operatorname{div}(A\nabla v) + k^2 \Sigma v = g & \text{in } \Omega \setminus \partial D, \\ A\nabla v \cdot \nu - ikv = 0 & \text{on } \partial\Omega, \\ [A\nabla v \cdot \nu] = 0 & \text{on } \partial D, \\ [v] = c \displaystyle\int_{\partial D} A\nabla v \cdot \nu & \text{on } \partial D. \end{cases} \tag{2.70}$$

Moreover,

$$\|v\|_{H^1(\Omega\setminus\partial D)} \leq C\|g\|_{L^2(\Omega)} \quad \text{and} \quad \|v\|_{H^2(V\setminus D)} \leq C\|g\|_{L^2(\Omega)}, \tag{2.71}$$

for some positive constant C independent of g.

Proof. The existence of v can be derived from the uniqueness of v by using the limiting absorption principle. We now establish the uniqueness for (2.70). Let $v \in H^1(\Omega \setminus \partial D)$ be a solution to (2.70) with $g = 0$. Multiplying the equation by $\overline{v}$, integrating over $\Omega \setminus D$ and over D, yields

$$\int_\Omega \left(A\nabla v \cdot \nabla \overline{v} - k^2 \Sigma |v|^2\right) + c\Big|\int_{\partial D} A\nabla v \cdot \nu\Big|^2 - ik\int_{\partial\Omega} |v|^2 = 0. \tag{2.72}$$

Taking the imaginary part, we obtain that $v = 0$ on $\partial\Omega$. The boundary condition in (2.70) then implies $A\nabla v \cdot \nu = 0$ on $\partial\Omega$. It thus follows from the unique continuation principle that $v = 0$ in $\Omega \setminus \overline{D}$, and in particular $v^+ = A\nabla v^+ \cdot \nu = 0$ on ∂D. From the transmission conditions of v on ∂D in (2.70), it follows that $v^- = A\nabla v^- \cdot \nu = 0$ on ∂D as well. We conclude from the unique continuation principle that $v \equiv 0$ in D. The proof of uniqueness is complete.

We next establish the first inequality of (2.71) by contradiction. Assume that there exists a sequence $g_n \in L^2(\Omega)$ which is square integrable near $\partial\Omega$, and an associated sequence of solutions $(v_n) \subset H^1(\Omega \setminus \partial D)$ to (2.70) such that

$$\lim_{n\to+\infty} ||g_n||_{[H^1(\Omega)]^*} = 0 \quad \text{and} \quad ||v_n||_{H^1(\Omega\setminus\partial D)} = 1. \tag{2.73}$$

Extracting a subsequence, we may assume that v_n converges weakly in $H^1(\Omega \setminus \partial D)$ and strongly in $L^2(\Omega)$ to some $v \in H^1(\Omega \setminus \partial D)$ which is a solution to (2.70) with right-hand side 0. By uniqueness, $v = 0$ in Ω and thus v_n converges to 0 weakly in $H^1(\Omega \setminus \partial D)$ and strongly in $L^2(\Omega)$. Similar to (2.72), we have

$$\int_\Omega \left(A\nabla v_n \cdot \nabla \overline{v}_n - k^2 \Sigma |v_n|^2\right) + c\Big|\int_{\partial D} A\nabla v_n \cdot \nu\Big|^2 - ik\int_{\partial\Omega} |v_n|^2 = \int_\Omega g_n \overline{v_n},$$

By considering the real part, using (2.73), and noting that

$$c\int_{\partial D} A\nabla v_n \cdot \nu = [v_n] \text{ on } \partial D \quad \text{and} \quad [v_n] \to 0 \text{ in } L^2(\partial D),$$

we derive that

$$\int_{\partial\Omega} A\nabla v_n \cdot \nabla \overline{v}_n \to 0 \text{ as } n \to +\infty.$$

Hence $v_n \to 0$ in $H^1(\Omega \setminus \partial D)$. This contradicts (2.73).

The second inequality of (2.71) can be obtained by Nirenberg's method of difference quotients (see, e.g., [4]) using the smoothness assumption of A and the boundedness of Σ. The details are left to the reader. $\square$

3 Superlenses using HMMs via complementary property

In this section, we consider a lens with coefficients (A^H, Σ^H) in $B_{r_2} \setminus B_{r_1}$ in the finite frequency regime of the form

$$(A^H, \Sigma^H) = \begin{cases} \Big(\dfrac{1}{r^{d-1}} e_r \otimes e_r - r^{3-d}(I - e_r \otimes e_r), \dfrac{1}{r^2}\Big) & \text{in } B_{r_2} \setminus B_{r_m}, \\ \Big(-\dfrac{1}{r^{d-1}} e_r \otimes e_r + r^{3-d}(I - e_r \otimes e_r), -\dfrac{1}{r^2}\Big) & \text{in } B_{r_m} \setminus B_{r_1}, \end{cases} \tag{3.1}$$

where

$$r_m = (r_1 + r_2)/2.$$

It will be clear below, that the choice $\Sigma^H = 1/r^2$ in $B_{r_2} \setminus B_{r_m}$ and $-1/r^2$ in $B_{r_m} \setminus B_{r_1}$ is just a matter of simplifying the presentation. Any real-valued pair $(\tilde{\sigma}_1/r^2, \tilde{\sigma}_2/r^2) \in L^\infty(B_{r_m} \setminus B_{r_1}) \times L^\infty(B_{r_2} \setminus B_{r_m})$ which satisfies

$$\tilde{\sigma}_2(x) = -\tilde{\sigma}_1\big((|x| - r_m)x/|x|\big)$$

is admissible. The superlensing property of the device (3.1) is given by the following theorem:

Theorem 5. *Let $d = 2, 3$, $k > 0$, Ω be a smooth bounded connected open subset of $\mathbb{R}^d$, and let $f \in L^2(\Omega)$. Fix $0 < r_1 < r_2$ and assume that $B_{r_2} \subset\subset \Omega$ and* $\operatorname{supp} f \subset \Omega \setminus B_{r_2}$. *Let $u_\delta \in H^1(\Omega)$ $(0 < \delta < 1)$ be the unique solution to (2.45) where (A^H, Σ^H) is given by (3.1). We have*

$$\|u_\delta\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)} \quad \textit{and} \quad u_\delta \to u_0 \quad \textit{strongly in } H^1(\Omega), \tag{3.2}$$

where $u_0 \in H^1(\Omega)$ is the unique solution of (2.45) where (A^H, Σ^H) is given by (3.1) corresponding to $\delta = 0$ and C is a positive constant independent of f and δ. Moreover, $u_0 = \hat{u}$ in $\Omega \setminus B_{r_2}$, where $\hat{u}_\delta$ is the unique solution to the system

$$\begin{cases} \operatorname{div}(\hat{A}\nabla \hat{u}) + k^2 \hat{\Sigma} \hat{u} = f & \textit{in } \Omega, \\ \partial_\nu \hat{u} - ik\hat{u} = 0 & \textit{on } \partial\Omega, \end{cases} \tag{3.3}$$

where

$$\hat{A}(x), \hat{\Sigma}(x) = \begin{cases} I, 1 & \textit{in } \Omega \setminus B_{r_2}, \\ \dfrac{r_1^{d-2}}{r_2^{d-2}} a\Big(\dfrac{r_1}{r_2} x\Big), \dfrac{r_1^d}{r_2^d} \sigma\Big(\dfrac{r_1}{r_2} x\Big) & \textit{in } B_{r_2}. \end{cases}$$

Since f is arbitrary with support in $\Omega \setminus B_{r_2}$, it follows from the definition of $\hat{A}$ that the object in B_{r_1} is magnified by a factor r_2/r_1. We emphasize again that no condition is imposed on $r_2 - r_1$.

Proof. Again, the proof mimics that of Theorem 1. We have

$$\|\hat{u}\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}, \tag{3.4}$$

and, by (1.16) and Lemma 2,

$$\|\hat{u}\|_{H^2(\Omega\setminus B_{r_2})} \le C\|f\|_{L^2(\Omega)}. \tag{3.5}$$

Define v in $B_{r_2} \setminus B_{r_m}$ as follows

$$\partial^2_{rr} v - \Delta_{\partial B_1} v + k^2 v = 0 \text{ in } B_{r_2} \setminus B_{r_m} \tag{3.6}$$

and, on ∂B_{r_2},

$$v = \hat{u} \quad \text{ and } \quad \partial_r v = r_2^{d-1} \partial_r \hat{u}^+. \tag{3.7}$$

We consider (3.6) and (3.7) as a Cauchy problem for the wave equation defined on the manifold ∂B_1 for which r plays the role of the time variable. By the standard theory for the wave equation, there exists a unique such $v \in C\big([r_m, r_2]; H^1(\partial B_1)\big) \cap C^1([r_m, r_2]; L^2(\partial B_1))$. We also have

$$\begin{aligned}\int_{\partial B_1} & |\partial_r v(r,\xi)|^2 + |\nabla_{\partial B_1} v(r,\xi)|^2 + k^2 |v(r,\xi)|^2 \, d\xi \\ &= \int_{\partial B_1} |\partial_r v(r_2,\xi)|^2 + |\nabla_{\partial B_1} v(r_2,\xi)|^2 + k^2 |v(r_2,\xi)|^2 \, d\xi \\ &= \int_{\partial B_1} r_2^{2(d-1)} |\partial_r \hat{u}^+(r_2,\xi)|^2 + |\nabla_{\partial B_1} \hat{u}(r_2,\xi)|^2 + k^2 |\hat{u}(r_2,\xi)|^2 \, d\xi.\end{aligned} \tag{3.8}$$

It follows that $v \in H^1(B_{r_2} \setminus B_{r_m})$ and

$$\|v\|_{H^1(B_{r_2}\setminus B_{r_m})} \le C\|f\|_{L^2(\Omega)}. \tag{3.9}$$

Let $v_R \in H^1(B_{r_m} \setminus B_{r_1})$ be the reflection of v through ∂B_{r_m}, i.e.,

$$v_R(x) = v\big((r_m - |x|)x/|x|\big) \text{ in } B_{r_m} \setminus B_{r_1}. \tag{3.10}$$

Define

$$u_0 = \begin{cases} \hat{u} & \text{in } \Omega \setminus B_{r_2}, \\ v & \text{in } B_{r_2} \setminus B_{r_m}, \\ v_R & \text{in } B_{r_m} \setminus B_{r_1}, \\ \hat{u}(r_2 \cdot / r_1) & \text{in } B_{r_1}. \end{cases}$$

Then $u_0 \in H^1(\Omega)$ and

$$\operatorname{div}(A_0 \nabla u_0) + k^2 \Sigma_0 u_0 = f \text{ in } \Omega \setminus (\partial B_{r_1} \cup \partial B_{r_2}). \tag{3.11}$$

On the other hand, from the definition of u_0 and v, we have

$$[A_0 \nabla u_0 \cdot e_r] = \partial_r \hat{u}^+ - \frac{1}{r_2^{d-1}} \partial_r v = 0 \text{ on } \partial B_{r_2}, \tag{3.12}$$

The properties of the reflection and the definition of A^H garantee that the transmission conditions also hold on ∂B_{r_m}, and from the definition of $\hat{A}$ and (3.7), we obtain

$$\begin{aligned}
[A_0 \nabla u_0 \cdot e_r](x) &= -\frac{1}{r_1^{d-1}} \partial_r v_R(x) - \frac{r_2}{r_1} a(x) \nabla \hat{u}^-(r_2 x/r_1) \cdot e_r \\
&= \frac{1}{r_1^{d-1}} \partial_r v(r_2 x/r_1) - \frac{r_2^{d-1}}{r_1^{d-1}} \hat{A}(r_2 x/r_1) \nabla \hat{u}^-(r_2 x/r_1) \cdot e_r \\
&= \frac{1}{r_1^{d-1}} \partial_r v(r_2 x/r_1) - \frac{r_2^{d-1}}{r_1^{d-1}} \partial_r \hat{u}^+(r_2 x/r_1) = 0 \text{ on } \partial B_{r_1}.
\end{aligned} \tag{3.13}$$

A combination of (3.11), (3.12), and (3.13) yields that $u_0 \in H^1(\Omega)$ and satisfies

$$\operatorname{div}(A_0 \nabla u_0) + k^2 \Sigma_0 u_0 = f \text{ in } \Omega;$$

which implies that u_0 is a solution for $\delta = 0$. We also obtain from (3.4), (3.5), (3.8), and (3.9) that

$$\|u_0\|_{H^1(\Omega)} \le C \|f\|_{L^2(\Omega)}. \tag{3.14}$$

We next establish the uniqueness of u_0. Let $w_0 \in H^1(\Omega)$ be a solution for $\delta = 0$. Note that w_0 is fully determined in $B_{r_2} \setminus B_{r_m}$ from the Cauchy data $w_0(r_2\hat{x}), \partial_r w_0(r_2\hat{x}), \hat{x} \in \partial B_1$. Given the form of the coefficients A^H, w must also have the symmetry

$$w_0(x) = w_0\big((r_m - |x|)x/|x|\big) \text{ in } B_{r_m} \setminus B_{r_1}.$$

It follows that for $\hat{x} \in \partial B_1$

$$w_0(r_2\hat{x}) \;=\; w_0(r_1\hat{x}) \quad \text{and} \quad \partial_r w_0(r_2\hat{x}) \;=\; \partial_r w_0(r_1\hat{x}).$$

Thus the function $\hat{w}$ defined by

$$\hat{w}(x) \quad = \quad \begin{cases} w_0(x) & x \in \Omega \setminus B_{r_2}, \\ w_0(r_1 x/r_2) & x \in B_{r_2}, \end{cases}$$

is a solution to (3.3). By uniqueness for this elliptic equation, $\hat{w}_0 = \hat{u}$, which in turn implies that $w_0 = u_0$ and uniqueness of u_0 follows.

Finally, we establish (3.2). Set

$$v_\delta = u_\delta - u_0 \text{ in } \Omega. \tag{3.15}$$

It is easy to see that $v_\delta \in H^1_0(\Omega)$ and that it satisfies

$$\operatorname{div}(A_\delta \nabla v_\delta) + k^2 \Sigma_\delta v_\delta = \operatorname{div}(i\delta \mathbb{1}_{B_{r_2} \setminus B_{r_1}} \nabla u_0) - i\delta k^2 \mathbb{1}_{B_{r_2} \setminus B_{r_1}} u_0 \text{ in } \Omega.$$

Applying (2.4) of Lemma 1, we derive from (3.14) that

$$\|v_\delta\|_{H^1(\Omega)} \le C\|\nabla u_0\|_{L^2(\Omega)}, \tag{3.16}$$

which implies the uniform bound in (3.2) and, as in the proof of Theorem 1, that v_δ converges weakly to 0 in $H^1(\Omega)$. Applying (2.3) of Lemma 1 and using (3.14) and (3.16), we obtain

$$\|u_\delta - u_0\|^2_{H^1(\Omega)} = \|v_\delta\|^2_{H^1(\Omega)} \le C\left\{\left|\int_{B_{r_2}\setminus B_{r_1}} \nabla u_0 \nabla \overline{v_\delta}\right| + \left|\int_{B_{r_2}\setminus B_{r_1}} u_0 \overline{v_\delta}\right|\right\} \to 0,$$

since v_δ converges weakly to 0, which completes the proof. □

Proof of Theorem 2. The proof of Theorem 2 is similar to the above proof and is left to the reader. □

4 Constructing hyperbolic metamaterials

In this section, we show how one can design the type of hyperbolic media used in the previous sections, by homogenization of layered materials. We restrict ourselves to superlensing using HMMs via complementary property in the three dimensional quasistatic case, in order to build a medium A^H_δ that satisfies, as $\delta \to 0$,

$$A^H_\delta \quad \to \quad A^H \;=\; \begin{cases} \dfrac{1}{r^2} e_r \otimes e_r - (I - e_r \otimes e_r) & \text{in } B_{r_2} \setminus B_{r_m}, \\ -\dfrac{1}{r^2} e_r \otimes e_r + (I - e_r \otimes e_r) & \text{in } B_{r_m} \setminus B_{r_1}, \end{cases} \tag{4.1}$$

such as that considered in (1.13). Recall that $r_m = (r_1 + r_2)/2$. The argument can easily be adapted to tuned superlensing using HMMs in two dimensions and to superlensing using HMMs via complementary property in two dimensions and to the finite frequency regime. Our approach follows the arguments developped by Murat and Tartar [8] for the homogenization of laminated composites.

For a fixed $\delta > 0$, let $\theta = 1/2$ and let χ denote the characteristic function of the interval $(0, 1/2)$. For $\varepsilon > 0$, set, for $x \in B_{r_2} \setminus B_{r_m}$,

$$\begin{aligned} b_{1,\varepsilon,\delta}(x) &= \frac{1}{r^2}\left[(-1 - i\delta)\chi(r/\varepsilon) + \big(1 - \chi(r/\varepsilon)\big)/3\right], \\ b_{2,\varepsilon,\delta}(x) &= (-3 - i\delta)\chi(r/\varepsilon) + \big(1 - \chi(r/\varepsilon)\big), \end{aligned}$$

and, for $x \in B_{r_m} \setminus B_{r_1}$,

$$\begin{aligned} b_{1,\varepsilon,\delta}(x) &= \frac{1}{r^2}\Big[(-1/3 - i\delta)\chi(r/\varepsilon) + \big(1 - \chi(r/\varepsilon)\big)\Big], \\ b_{2,\varepsilon,\delta}(x) &= (-1 - i\delta)\chi(r/\varepsilon) + 3\big(1 - \chi(r/\varepsilon)\big). \end{aligned}$$

Note that since periodic functions converge weakly* to their average in L^∞, one can easily compute the L^∞ weak-* limits

$$b_{1,H,\delta} := \left(w* - \lim_{\varepsilon\to 0}(b_{1,\varepsilon,\delta})^{-1}\right)^{-1} \quad \text{and} \quad b_{2,H,\delta} := w* - \lim_{\varepsilon\to 0} b_{2,\varepsilon,\delta}, \tag{4.2}$$

and in particular we have in $B_{r_2}\setminus B_{r_m}$

$$\begin{cases} b_{1,H,\delta}(x) &=& \dfrac{2(1+i\delta)}{r^2(2+3i\delta)} &=& \left(1 - i\delta/2 + O(\delta^2)\right)/r^2, \\ b_{2,H,\delta}(x) &=& (-1-i\delta/2), \end{cases} \tag{4.3}$$

and in $B_{r_m}\setminus B_{r_1}$

$$\begin{cases} b_{1,H,\delta}(x) &=& \dfrac{-2/3-2i\delta}{r^2(2/3-i\delta)} &=& -1-9i\delta/2+O(\delta^2), \\ b_{2,H,\delta}(x) &=& (1-i\delta/2). \end{cases} \tag{4.4}$$

Set

$$a_{\varepsilon,\delta}(x) = b_{1,\varepsilon,\delta}(r)e_r\otimes e_r + b_{2,\varepsilon,\delta}(r)\left(e_\theta\otimes e_\theta + e_\varphi\otimes e_\varphi\right). \tag{4.5}$$

Let a be a uniformly elliptic matrix-valued function and define

$$A_{\varepsilon,\delta}(x) = \begin{cases} I & \text{in } \Omega\setminus B_{r_2}, \\ a_{\varepsilon,\delta} & \text{in } B_{r_2}\setminus B_{r_1}, \\ a & \text{in } B_{r_1}, \end{cases} \tag{4.6}$$

and

$$A^H_\delta(x) = \begin{cases} I & \text{in } \Omega\setminus B_{r_2}, \\ b_{1,H,\delta}e_r\otimes e_r + b_{2,H,\delta}\left(e_\theta\otimes e_\theta + e_\varphi\otimes e_\varphi\right) & \text{in } B_{r_2}\setminus B_{r_1}, \\ a & \text{in } B_{r_1}. \end{cases} \tag{4.7}$$

We have

Proposition 2. *Let $0<r_1<r_2$, and let Ω be a smooth bounded connected open subset of $\mathbb{R}^3$ such that $B_{r_2}\subset\subset\Omega$. Given $f\in L^2(\Omega)$ with $\operatorname{supp} f\cap B_{r_2}=\emptyset$, let $u_{\varepsilon,\delta}\in H^1_0(\Omega)$ be the unique solution to*

$$\operatorname{div}(A_{\varepsilon,\delta}\nabla u_{\varepsilon,\delta}) = f \text{ in } \Omega,$$

where $A_{\varepsilon,\delta}$ is given by (4.6). Then, as $\varepsilon\to 0$, $u_{\varepsilon,\delta}$ converges weakly in $H^1(\Omega)$ to $u_{H,\delta}\in H^1_0(\Omega)$ the unique solution of the equation

$$\operatorname{div}(A^H_\delta\nabla u_{H,\delta}) = f \text{ in } \Omega,$$

where A^H_δ is defined by (4.7).

Remark 3. Materials given in (4.5) could in principle be fabricated as a laminated composite containing anisotropic metallic phases with a conductivity described by a Drude model. Also note that the imaginary part of A_δ^H has the form $-i\delta M$, where M is a diagonal, positive definite matrix, and is not strictly equal to $-i\delta I$ as in the hypotheses of Theorem 2. Nevertheless, its results hold for this case as well.

Proof. For notational ease, we drop the dependance on δ in the notation. By Lemma 1 (see also Remark 1), there exists a unique solution $u_\varepsilon \in H_0^1(\Omega)$ to

$$\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = f \text{ in } \Omega, \tag{4.8}$$

which further satisfies $||u_\varepsilon||_{H^1(\Omega)} \leq C\,||f||_{L^2(\Omega)}$, with C independent of ε (it may depend on δ though). We may thus assume, that up to a subsequence, u_ε converges weakly in $H^1(\Omega)$ to some $u_H \in H^1(\Omega)$. Standard results in homogenization [8] show that $u_H \in H_0^1(\Omega)$ solves an equation of the same type as (4.8):

$$\mathrm{div}(A^H \nabla u_H) = f \text{ in } \Omega, \tag{4.9}$$

where the tensor of homogenized coefficients A_H has the form

$$A_H(x) \quad = \quad \begin{cases} I & \text{for } x \in \Omega \setminus B_{r_2}, \\ a_H(x) & \text{for } x \in B_{r_2} \setminus B_{r_1}, \\ a(x) & \text{for } x \in B_{r_1}. \end{cases}$$

To identify the tensor a_H, set

$$\sigma_{1,\varepsilon} = r^2 b_{1,\varepsilon} \partial_r u_\varepsilon \quad \text{in } B_{r_2} \setminus B_{r_1}. \tag{4.10}$$

Using spherical coordinates in $B_{r_2} \setminus B_{r_1}$, we have

$$\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = \frac{1}{r^2}\partial_r(r^2 b_{1,\varepsilon}\partial_r u_\varepsilon) + \frac{b_{2,\varepsilon}}{r^2}\Delta_{\partial B_1} u_\varepsilon \text{ in } B_{r_2} \setminus B_{r_1},$$

where $\Delta_{\partial B_1}$ denotes the Laplace-Beltrami operator on ∂B_1. This implies, since $\mathrm{supp}\, f \cap B_{r_2} = \emptyset$,

$$\partial_r \sigma_{1,\varepsilon} = -\Delta_{\partial B_1}\big(b_{2,\varepsilon}(r) u_\varepsilon\big) \text{ in } B_{r_2} \setminus B_{r_1},$$

since $b_{2,\varepsilon}$ only depends on r for a fixed ε. Consequently, $\sigma_{1,\varepsilon}$ and $\partial_r \sigma_{1,\varepsilon}$ are uniformly bounded with respect to ε in $L^2\big(r_1, r_2, L^2(\partial B_1)\big)$ and in $L^2\big(r_1, r_2, H^{-1}(\partial B_1)\big)$ respectively. Invoking Aubin compactness theorem as in [8], we infer that up to a subsequence, $\sigma_{1,\varepsilon}$ converges strongly in $L^2\big(r_1, r_2, H^{-1}(\partial B_1)\big)$ to some limit $\sigma_{1,H} \in L^2(B_{r_2} \setminus B_{r_1})$. Rewriting (4.10) as

$$\big(r^2 b_{1,\varepsilon}\big)^{-1} \sigma_{1,\varepsilon} = \partial_r u_\varepsilon,$$

and letting $\varepsilon \to 0$, yields

$$\begin{aligned} \sigma_{1,H} \quad &= \quad \big(w* - \lim(r^2 b_{1,\varepsilon})^{-1}\big)^{-1} \partial_r u_H \\ &= \quad \frac{r^2}{w* - \lim(b_{1,\varepsilon})^{-1}} \partial_r u_H. \end{aligned}$$

On the other hand, since $u_\varepsilon \to u_H$ strongly in $L^2(\Omega)$, it follows that $b_{2,\varepsilon}(r)u_\varepsilon \to w*-\lim b_{2,\varepsilon}(r)u_H$ in L^2. We derive that

$$\partial_r\big(r^2 b_{1,H}\partial_r u_H\big) + \Delta_{\partial B_1}\big(b_{2,H}u_H\big) = 0 \text{ in } B_{r_2}\setminus B_{r_1}, \tag{4.11}$$

where $b_{1,H} = \big(w*-\lim(b_{1,\varepsilon})^{-1}\big)^{-1}$ and $b_{2,H} = w*-\lim b_{2,\varepsilon}$. We can then identify

$$a_H = b_{1,H} e_r\otimes e_r + b_{2,H}\left(e_\theta\otimes e_\theta + e_\varphi\otimes e_\varphi\right),$$

which, given (4.3–4.4), has the form considered in (3.1).

Since periodic functions weakly-* converge to their average in L^∞ one easily checks that in fact the whole sequence u_ε converges to the unique H^1_0-solution to (4.11). □

5 Stability of HMMs

Both the mechanisms for superlensing, that we propose in this paper, rely on the ability to transport the Cauchy data without alteration (or barely) from one interface of the lens to the other. In this section, we investigate the sensitivity of these results to the constraints on the design, namely to the conditions $r_2 - r_1 \in 2\pi\mathbb{N}^+$ or $r_m = \dfrac{r_2+r_1}{2}$ that are assumed in the previous sections. To this end, let $l>0$ and $L>0$, and consider

$$\mathcal{R} = (-l,L)\times(0,2\pi), \quad \mathcal{R}_l = (-l,0)\times(0,2\pi), \quad \mathcal{R}_L = (0,L)\times(0,2\pi).$$

We also set $\Gamma = \partial\mathcal{R}$, and

$$\Gamma^-_{lat} = \{-l\}\times(0,2\pi), \quad \Gamma^+_{lat} = \{L\}\times(0,2\pi), \quad \Gamma_{tb} = \big((-l,L)\times\{2\pi\}\big)\cup\big((-l,L)\times\{2\pi\}\big).$$

Let A_δ denote the conductivity defined in $\mathcal{R}$ by

$$A_\delta(x) = \begin{cases} I & x\in\mathcal{R}_l,\\ \begin{pmatrix} 1-i\delta & 0\\ 0 & -1-i\delta\end{pmatrix} & x\in\mathcal{R}_L.\end{cases}$$

Let $f\in H^{1/2}_{0,0}(\Gamma^-_{lat})$ [4] and for $\delta>0$ denote $u_\delta\in H^1(\mathcal{R})$ the unique solution to

$$\begin{cases} \operatorname{div}(A_\delta\nabla u_\delta) = 0 & \text{in } \mathcal{R},\\ u_\delta = 0 & \text{on } \Gamma_{tb},\\ u_\delta = f & \text{on } \Gamma^-_{lat},\\ u_\delta = 0 & \text{on } \Gamma^+_{lat}.\end{cases} \tag{5.1}$$

This configuration corresponds to that of Section 2.2, where only the left half of the domain (cut through the middle of the hyperlens) is considered. Note that one could equally study the configuration where a homogeneous Neumann boundary condition is imposed on Γ^+_{lat}.

[4] The closure of $C^\infty_c(\Gamma^-_{lat})$ in $H^{1/2}(\Gamma^-_{lat})$.

The transmission conditions on $x_1 = 0$ read

$$u_\delta(0^-, x_2) \;=\; u_\delta(0^+, x_2) \qquad \text{and} \qquad \partial_{x_1} u_\delta(0^-, x_2) \;=\; \partial_{x_1} u_\delta(0^+, x_2), \qquad 0 < x_2 < 2\pi.$$

Using the same arguments as in Section 2, one can show that if there exists a solution $u_0 \in H^1(\mathcal{R})$ of (5.1) with $\delta = 0$ then the problem is stable in the sense that (u_δ) remains bounded in $H^1(\mathcal{R})$. Otherwise, there exists a sequence $(\delta_n) \to 0$ such that $\|u_{\delta_n}\|_{H^1(\mathcal{R})} \to +\infty$ as $n \to +\infty$. We now compute such a possible solution u_0. If u_0 solves (5.1) with $\delta = 0$, then it must have the form

$$u_0(x_1, x_2) \;=\; \begin{cases} \sum_{n\geq 1} sin(nx_2)\,(a_n e^{nx_1} + b_n e^{-nx_1}) & -l < x < 0 \\ \sum_{n\geq 1} sin(nx_2)\,(\alpha_n \cos(nx_1) + \beta_n \sin(nx_1)) & 0 < x < L, \end{cases}$$

where $a_n, b_n, \alpha_n, \beta_n \in \mathbb{R}$. Assume that the Dirichelt data on Γ^-_{lat} decomposes as

$$f(-l, x_2) \;=\; \sum_{n=1}^{\infty} f_n \sin(nx_2),$$

for some $f_n \in \mathbb{C}$. Expressing the transmission on $x_1 = 0$, and the boundary conditions on $\Gamma^\pm_{lat}$ yields 4×4 homogeneous linear systems

$$\begin{pmatrix} e^{-nl} & e^{nl} & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & -1 \\ 0 & 0 & \cos(nL) & \sin(nL) \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ \alpha_n \\ \beta_n \end{pmatrix} \;=\; \begin{pmatrix} f_n \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad n \geq 1,$$

with determinants

$$d_n \;:=\; e^{-nl}\left[\cos(nL) - \sin(nL)\right] - e^{nl}\left[\cos(nL) + \sin(nL)\right].$$

Under the condition

$$d_n \neq 0 \text{ for } n \geq 1$$

we can solve for the coefficients

$$\begin{pmatrix} a_n \\ b_n \\ \alpha_n \\ \beta_n \end{pmatrix} \;=\; \frac{1}{d_n} \begin{pmatrix} f_n\left[\cos(nL) - \sin(nL)\right] \\ -f_n\left[\cos(nL) + \sin(nL)\right] \\ -2f_n \sin(nL) \\ 2f_n \cos(nL) \end{pmatrix},$$

and construct a formal solution to (5.1) when $\delta = 0$. The requirement that $u_0 \in H^1(\mathcal{R})$ however imposes conditions on the growth of the d_n's.

Assume that L/π is irrational and Diophantine of class $r \in \mathbb{N}_+$, i.e., there exists $\varepsilon > 0$ such that

$$\forall\,(p, q) \in \mathbb{Z} \times \mathbb{Z}^* \quad \left|\frac{L}{\pi} - \frac{p}{q}\right| \;>\; \frac{\varepsilon}{q^r}.$$

Let $p \in \mathbb{N}$ be such that $\pi p + \pi/4 < nL < \pi(p+1) + \pi/4$. Then one has for n large enough

$$\begin{aligned}
|d_n| &\geq e^{nl}\Big|\cos(nL) + \sin(nL)\Big| - 2e^{-nl} \\
&= e^{nl}\left|\cos(nL) + \sin(nL) - \left(\cos\left(\frac{3\pi}{4} + \pi p\right) + \sin\left(\frac{3\pi}{4} + \pi p\right)\right)\right| - 2e^{-nl} \\
&\geq e^{nl}\frac{2\sqrt{2}}{\pi}\left|nL - \left(\frac{3\pi}{4} + \pi p\right)\right| - 2e^{-nl} \\
&\geq e^{nl} 2\sqrt{2} n \left|\frac{L}{\pi} - \frac{3+p}{4n}\right| - 2e^{-nl} \\
&\geq e^{nl} 2\sqrt{2}\frac{n\varepsilon}{(4n)^r} - 2e^{-nl} \; > \; cn,
\end{aligned}$$

for some $c > 0$. It follows that

$$\sum_{n\geq 1}(1+n^2)(a_n^2 + b_n^2 + \alpha_n^2 + \beta_n^2) \quad < \quad +\infty,$$

and there exists a solution $u_0 \in H^1(\mathcal{R})$ to (5.1).

Assume now that $L = \dfrac{4p+3}{4q}$ for some $p, q \in \mathbb{N}, q \neq 0$. Then $\cos(nL) + \sin(nL)$ vanishes for an infinite number of n's, for which $d_n = O(e^{-nl})$. One can then construct examples of data f such that $\sum_{n=1}^{\infty}(1+n^2)(a_n^2 + b_n^2 + \alpha_n^2 + \beta_n^2)$ is not converging. In this case, there is no solution in $H^1(\Omega)$ to (5.1).

Given the dense character of Diophantine numbers, we see that, as the dissipation parameter tends to 0, the solution operator is clearly *not continous* with respect to the geometry of the HMMs region (see also [3, 7] for related questions concerning the Dirichlet problem for the wave equation).

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