STABILITY FOR QUANTITATIVE PHOTOACOUSTIC TOMOGRAPHY REVISITED

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Abstract. This paper is concerned with the stability issue in determining the absorption and the diffusion coefficients in quantitative photoacoustic imaging. We establish a global conditionnal Hölder stability inequality from the knowledge of two internal data obtained from optical waves, generated by two point sources in a region where the optical coefficients are known.

Mathematics subject classification : 35R30.

Key words : Elliptic equations, diffusion coefficient, absorption coefficient, stability inequality, multiwave imaging.

Contents

1. Introduction

Throughout this text $n \geq 3$ is a fixed integer. If $0 < \beta \leq 1$ we denote by $C^{0,\beta}(\mathbb{R}^n)$ the vector space of bounded continuous functions f on $\mathbb{R}^n$ satisfying

$$[f]_\beta = \sup\left\{\frac{|f(x)-f(y)|}{|x-y|^\beta};\ x,y\in\mathbb{R}^n,\ x\neq y\right\} < \infty.$$

$C^{0,\beta}(\mathbb{R}^n)$ is then a Banach space when it is endowed with its natural norm

$$\|f\|_{C^{0,\beta}(\mathbb{R}^n)} = \|f\|_{L^\infty(\mathbb{R}^n)} + [f]_\beta.$$

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Define $C^{1,\beta}(\mathbb{R}^n)$ as the vector space of functions f from $C^{0,\beta}(\mathbb{R}^n)$ so that $\partial_j f \in C^{0,\beta}(\mathbb{R}^n)$, $1 \le j \le n$. The vector space $C^{1,\beta}(\mathbb{R}^n)$ equipped with the norm

$$\|f\|_{C^{1,\beta}(\mathbb{R}^n)} = \|f\|_{C^{0,\beta}(\mathbb{R}^n)} + \sum_{j=1}^n \|\partial_j f\|_{C^{0,\beta}(\mathbb{R}^n)}$$

is a Banach space.

The data in this paper consists in $\xi_1, \xi_2 \in \mathbb{R}^n$, $\Omega \Subset \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$ of class $C^{1,1}$, $0 < \alpha < 1$, $0 < \theta < \alpha$, $\lambda > 1$ and $\kappa > 1$. For notational convenience the set of data will denoted by $\mathfrak{D}$. That is

$$\mathfrak{D} = (n, \xi_1, \xi_2, \Omega, \alpha, \theta, \lambda, \kappa).$$

Denote by $\mathcal{D}(\lambda, \kappa)$ the set of couples $(a, b) \in C^{1,1}(\mathbb{R}^n) \times C^{0,1}(\mathbb{R}^n)$ satisfying

$$\lambda^{-1} \le a \quad \text{and} \quad \|a\|_{C^{1,1}(\mathbb{R}^n)} \le \lambda, \tag{1.1}$$

$$\kappa^{-1} \le b \quad \text{and} \quad \|b\|_{C^{0,1}(\mathbb{R}^n)} \le \kappa, \tag{1.2}$$

Define further the elliptic operator $L_{a,b}$ acting as follows

$$L_{a,b}u(x) = -\mathrm{div}(a(x)\nabla u(x)) + b(x)u(x). \tag{1.3}$$

We show in Section 2 that if $(a, b) \in \mathcal{D}(\lambda, \kappa)$ then the operator $L_{a,b}$ admits a unique fundamental solution $G_{a,b}$ satisfying, where $\xi \in \mathbb{R}^n$,

$$G_{a,b}(\cdot, \xi) \in C^{2,\alpha}_{\mathrm{loc}}(\mathbb{R}^n \setminus \{\xi\}), \quad L_{a,b}G_{a,b}(\cdot, \xi) = 0 \text{ in } \mathbb{R}^n \setminus \{\xi\},$$

and, for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$u = \int_{\mathbb{R}^n} G_{a,b}(\cdot, \xi) f(\xi) d\xi$$

belongs to $H^2(\mathbb{R}^n)$ and it is the unique solution of $L_{a,b}u = f$.

We deal in the present work with the problem of reconstructing $(a, b) \in \mathcal{D}(\lambda, \kappa)$ from energies generated by two point sources located at ξ_1 and ξ_2. Precisely, if $u_j(a, b) = G_{a,b}(\cdot, \xi_j)$, $j = 1, 2$, we want to determine (a, b) from the internal measurements

$$v_j(a, b) = bu_j(a, b) \quad \text{in } \Omega, \quad j = 1, 2.$$

This inverse problem is related to photoacoustic tomography (PAI) where optical energy absorption causes thermoelastic expansion of the tissue, which in turn generates a pressure wave [21]. This acoustic signal is measured by transducers distributed on the boundary of the sample and it is used for imaging optical properties of the sample. The internal data $v_1(a, b)$ and $v_2(a, b)$ are obtained by performing a first step consisting in a linear initial to boundary inverse problem for the acoustic wave equation. Therefore the inverse problem that arises from this first inversion is to determine the diffusion coefficient a and the absorption coefficient b from the internal data $v_1(a, b)$ and $v_2(a, b)$ that are proportional to the local absorbed optical energy inside the sample. This inverse problem is known in the literature as quantitative photoacoustic tomography [1, 4, 2, 3, 8, 7, 19].

Photoacoustic imaging provides in theory images of optical contrasts and ultrasound resolution [21]. Indeed, the resolution is mainly due to the small wavelength of acoustic waves, while the contrast is somehow related to the sensitivity of optical waves to absorption and scattering properties of the medium in the diffusive regime.

Assuming that the optical waves are generated by two point sources $\delta_{\xi_i}, i = 1, 2,$ we aim to derive a stability estimate for the recovery of the optical coefficients from internal data. We point out that taking the optical wave generated by a point source outside the sample seems to be more realistic than assuming a boundary condition.

In the statement of Theorem 1.1 below $C = C(\mathfrak{D}) > 0$ and $0 < \gamma = \gamma(\mathfrak{D}) < 1$ are constants.

Theorem 1.1. *For any $(a,b),(\tilde{a},\tilde{b}) \in \mathcal{D}(\lambda,\kappa)$ satisfying $(a,b) = (\tilde{a},\tilde{b})$ on Γ, we have*

$$\|a-\tilde{a}\|_{C^{1,\alpha}(\overline{\Omega})} + \|b-\tilde{b}\|_{C^{0,\alpha}(\overline{\Omega})} \le C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\gamma}.$$

The rest of this text is organized as follows. In section 2 we construct a fundamental solution and give its regularity induced by that of the coefficients of the operator under consideration. We also establish in this section a lower bound of the local L^2-norm of the gradient of the quotient of two fundamental solutions near one of the point sources. This is the key point for establishing our stability inequality. This last result is then used in Section 3 to obtain a uniform polynomial lower bound of the local L^2-norm of the gradient in a given region. This polynomial lower bound is obtained in two steps. In the first step we derive, via a three-ball inequality for the gradient, a uniform lower bound of negative exponential type. We use then in the second step an argument based on the so-called frequency function in order to improve this lower bound. In the last section we prove our main theorem following the known method consisting in reducing the original problem to the stability of an inverse conductivity problem.

2. Fundamental solutions

2.1. **Constructing fundamental solutions.** In this subsection we construct a fundamental solution of divergence form elliptic operators. Since our construction relies on heat kernel estimates, we first recall some known results.

Consider the parabolic operator $P_{a,b}$ acting as follows

$$P_{a,b}u(x,t) = -L_{a,b}u(x,t) - \partial_t u(x,t)$$

and set

$$Q = \{(x,t,\xi,\tau) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R};\ \tau < t\}.$$

Recall that a fundamental solution of the operator $P_{a,b}$ is a function $E_{a,b} \in C^{2,1}(Q)$ verifying $P_{a,b}E = 0$ in Q and, for every $f \in C_0^\infty(\mathbb{R}^n)$,

$$\lim_{t\downarrow\tau}\int_{\mathbb{R}^n} E_{a,b}(x,t,\xi,\tau)f(\xi)d\xi = f(x), \quad x \in \mathbb{R}^n.$$

The classical results in the monographs by A. Friedman [12], O. A. Ladyzenskaja, V. A. Solonnikov and N.N Ural'ceva [18] show that $P_{a,b}$ admits a non negative fundamental solution when $(a,b) \in \mathcal{D}(\lambda,\kappa)$.

It is worth mentioning that if $a = c$, for some constant $c > 0$, and $b = 0$ then the fundamental solution $E_{c,0}$ is explicitly given by

$$E_{c,0}(x,t,\xi,\tau) = \frac{1}{[4\pi c(t-\tau)]^{n/2}}e^{-\frac{|x-\xi|^2}{4c(t-\tau)}}, \quad (x,t,\xi,\tau) \in Q.$$

Examining carefully the proof of the two-sided Gaussian bounds in [11], we see that these bounds remain valid whenever $a \in C^{1,1}(\mathbb{R}^n)$ satisfies

$$\lambda^{-1} \le a \quad \text{and} \quad \|a\|_{C^{1,1}(\mathbb{R}^n)} \le \lambda. \tag{2.1}$$

More precisely we have the following theorem in which

$$\mathcal{E}_c(x,t) = \frac{c}{t^{n/2}} e^{-\frac{|x|^2}{ct}}, \quad x \in \mathbb{R}^n,\ t > 0,\ c > 0.$$

Theorem 2.1. *There exists a constant $c = c(n,\lambda) > 1$ so that, for any $a \in C^{1,1}(\mathbb{R}^n)$ satisfying (2.1), we have*

$$\mathcal{E}_{c^{-1}}(x-\xi, t-\tau) \le E_{a,0}(x,t;\xi,\tau) \le \mathcal{E}_c(x-\xi,t-\tau), \tag{2.2}$$

for all $(x,t,\xi,\tau) \in Q$.

The relationship between $\mathcal{E}_c$ and $E_{c,0}$ is given by the formula

$$\mathcal{E}_c(x-\xi,t-\tau) = \frac{(\pi c)^{n/2-1}}{\pi} E_{c/4,0}(x,t,\xi,\tau), \quad (x,t,\xi,\tau) \in Q. \tag{2.3}$$

The following comparaison principle will be useful in the sequel.

Lemma 2.1. *Let $(a,b_1),(a,b_2) \in \mathcal{D}(\lambda,\kappa)$ so that $b_1 \le b_2$. Then $E_{a,b_2} \le E_{a,b_1}$.*

Proof. Pick $0 \le f \in C_0^\infty(\mathbb{R}^n)$. Let u be the solution of the initial value problem

$$P_{a,b_1} u(x,t) = 0 \ \in \mathbb{R}^n \times \{t > \tau\}, \quad u(x,\tau) = f.$$

We have

$$u(x,t) = \int_{\mathbb{R}^n} E_{a,b_1}(x,t;\xi,\tau) f(\xi) d\xi \ge 0. \tag{2.4}$$

On the other hand, as $P_{a,b_1}u(x,t) = 0$ can be rewritten as

$$P_{a,b_2} u(x,t) = [b_1(x) - b_2(x)] u(x,t),$$

we obtain

$$\begin{aligned} u(x,t) = &\int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,\tau) f(\xi) d\xi \\ &- \int_\tau^t \int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,s)[b_1(\xi) - b_2(\xi)] u(\xi,s) d\xi ds. \end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5), we get

$$\int_{\mathbb{R}^n} E_{a,b_2}(x,t;\xi,\tau) f(\xi) d\xi \le \int_{\mathbb{R}^n} E_{a,b_1}(x,t;\xi,\tau) f(\xi) d\xi,$$

which yields in a straightforward manner the expected inequality. □

Consider, for $(a,b) \in \mathcal{D}(\lambda,\kappa)$, the unbounded operator $A_{a,b} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined

$$A_{a,b} = -L_{a,b}, \quad D(A_{a,b}) = H^2(\mathbb{R}^n).$$

It is well known that $A_{a,b}$ generates an analytic semigroup $e^{tA_{a,b}}$. Therefore in light of [6, Theorem 4 on page 30, Theorem 18 on page 44 and the proof in the beginning of Section 1.4.2 on page 35] $k_{a,b}(t,x;\xi)$, the Schwarz kernel of $e^{tA_{a,b}}$, is Hölder continuous with respect to x and ξ, satisfies

$$|k_{a,b}(t,x,\xi)| \le e^{-\delta t} \mathcal{E}_c(x-\xi, t-\tau) \tag{2.6}$$

and, for $|h| \le \sqrt{t} + |x-\xi|$,

$$|k_{a,b}(t,x+h,\xi) - k_{a,b}(t,x,\xi)| \le e^{-\delta t}\left(\frac{|h|}{\sqrt{t}+|x-\xi|}\right)^{\eta}\mathcal{E}_c(x-\xi,t-\tau), \tag{2.7}$$

$$|k_{a,b}(t,x,\xi+h) - k_{a,b}(t,x,\xi)| \le e^{-\delta t}\left(\frac{|h|}{\sqrt{t}+|x-\xi|}\right)^{\eta}\mathcal{E}_c(x-\xi,t-\tau), \tag{2.8}$$

where $c = c(n,\lambda,\kappa) > 0$ and $\delta = \delta(n,\lambda,\kappa) > 0$ and $\eta > 0$ are constants.

From the uniqueness of solutions of the Cauchy problem

$$u'(t) = A_{a,b}u(t),\ t > 0, \quad u(0) = f \in C_0^\infty(\mathbb{R}^n) \tag{2.9}$$

we deduce in a straightforward manner that $k_{a,b}(t,x;\xi) = E_{a,b}(x,t;\xi,0)$.

Prior to giving the construction of the fundamental solution for the variable coefficients operators, we state a result for operators with constant coefficients. This result is proved in Appendix A.

Lemma 2.2. *Let $\mu > 0$ and $\nu > 0$ be two constants. Then the fundamental solution for the operator $-\mu\Delta + \nu$ is given by $G_{\mu,\nu}(x,\xi) = \mathcal{G}_{\mu,\nu}(x-\xi)$, $x,\xi \in \mathbb{R}^n$, with*

$$\mathcal{G}_{\mu,\nu}(x) = (2\pi\mu)^{-n/2}(\sqrt{\nu\mu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}), \quad x \in \mathbb{R}^n.$$

Here $K_{n/2-1}$ is the usual modified Bessel function of second kind. Moreover the following two-sided inequality holds

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n-2}} \le \mathcal{G}_{\mu,\nu}(x) \le C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n-2}}, \quad x \in \mathbb{R}^n, \tag{2.10}$$

for some constant $C = C(n,\mu,\nu) > 1$.

The main result of this section is the following theorem.

Theorem 2.2. *Let $(a,b) \in \mathcal{D}(\lambda,\kappa)$. Then there exists a unique function $G_{a,b}$ satisfying $G_{a,b}(\cdot,\xi) \in C(\mathbb{R}^n \setminus \{\xi\})$, $\xi \in \mathbb{R}^n$, $G_{a,b}(x,\cdot) \in C(\mathbb{R}^n \setminus \{x\})$, $x \in \mathbb{R}^n$, and*
(i) $L_{a,b}G_{a,b}(\cdot,\xi) = 0$ in $\mathscr{D}'(\mathbb{R}^n \setminus \{\xi\})$, $\xi \in \mathbb{R}^n$,
(ii) for any $f \in C_0^\infty(\mathbb{R}^n)$,

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi)f(\xi)d\xi$$

belongs to $H^2(\mathbb{R}^n)$ and it is the unique solution of $L_{a,b}u = f$,
(iii) there exist two constants $c = c(n,\lambda) > 1$ and $C = C(n,\lambda,\kappa) > 1$ so that

$$C^{-1}\frac{e^{-2\sqrt{c\kappa}|x-\xi|}}{|x-\xi|^{n-2}} \le G_{a,b}(x,\xi) \le C\frac{e^{-\frac{|x-\xi|}{\sqrt{c\kappa}}}}{|x-\xi|^{n-2}}. \tag{2.11}$$

Proof. Pick $s \ge 1$ arbitrary. Applying Hölder inequality, we find

$$\int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi \le \|k_{a,b}(t,x,\cdot)\|_{L^s(\mathbb{R}^n)}\|f\|_{L^{s'}(\mathbb{R}^n)},$$

where s' is the conjugate exponent of s.

But, according to (2.6)

$$\|k_{a,b}(t,x,\cdot)\|^s_{L^s(\mathbb{R}^n)} \le \left(\frac{c}{t^{n/2}}\right)^s\int_{\mathbb{R}^n} e^{-\frac{s|x-\xi|^2}{ct}}d\xi.$$

Next, making a change of variable $\xi = (\sqrt{ct/s})\eta + x$, we get

$$\|k_{a,b}(t,x,\cdot)\|_{L^s(\mathbb{R}^n)}^s \le \left(\frac{c}{t^{n/2}}\right)^s \left(\frac{ct}{s}\right)^{n/2} \int_{\mathbb{R}^n} e^{-|\eta|^2} d\eta.$$

Hence

$$\|k_{a,b}(t,x,\cdot)\|_{L^s(\mathbb{R}^n)} \le t^{n(1/s-1)/2} C_s,$$

with

$$C_s = c\left(\frac{c}{s}\right)^{n/2} \left(\int_{\mathbb{R}^n} e^{-c|\eta|^2} d\eta\right)^{1/s}.$$

We get, by choosing $1 \le s \le \frac{n}{n-2} < \tilde{s}$,

$$\begin{aligned}
&\int_0^{+\infty}\int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi dt \\
&\qquad = \int_0^1 \int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi dt + \int_1^{+\infty}\int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)|f(\xi)|d\xi dt \\
&\qquad \le C_s\|f\|_{L^{s'}(\mathbb{R}^n)} \int_0^1 t^{\frac{n}{2}(1/s-1)}dt + C_{\tilde{s}}\|f\|_{L^{\tilde{s}'}(\mathbb{R}^n)} \int_1^{+\infty} t^{\frac{n}{2}(1/\tilde{s}-1)}dt.
\end{aligned}$$

In light of Fubini's theorem we obtain

$$\int_0^{+\infty}\int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)f(\xi)d\xi dt = \int_{\mathbb{R}^n}\left(\int_0^{+\infty} k_{a,b}(t,x,\xi)dt\right) f(\xi)d\xi. \tag{2.12}$$

Define $G_{a,b}$ as follows

$$G_{a,b}(x,\xi) = \int_0^{+\infty} k_{a,b}(t,x,\xi)dt.$$

Then (2.12) takes the form

$$\int_0^{+\infty}\int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)f(\xi)d\xi dt = \int_{\mathbb{R}^n} G_{a,b}(x,\xi)f(\xi)d\xi. \tag{2.13}$$

Noting that $A_{a,b}$ is invertible, we obtain

$$\begin{aligned}
-A_{a,b}^{-1}f(x) &= \left(\int_0^{+\infty} e^{tA_{a,b}} f dt\right)(x) \\
&= \int_0^{+\infty}\int_{\mathbb{R}^n} k_{a,b}(t,x,\xi)f(\xi)d\xi dt, \quad x \in \mathbb{R}^n.
\end{aligned}$$

This and (2.13) entail

$$-A_{a,b}^{-1}f(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi)f(\xi)d\xi, \quad x \in \mathbb{R}^n.$$

In other words, u defined by

$$u(x) = \int_{\mathbb{R}^n} G_{a,b}(x,\xi)f(\xi)d\xi, \quad x \in \mathbb{R}^n,$$

belongs to $H^2(\mathbb{R}^n)$ and satisfies $L_{a,b}u = f$.

Noting that, for $x \neq \xi$,

$$\int_0^{+\infty} \frac{1}{t^{n/2}} e^{-\frac{|x-\xi|^2}{ct}} dt = \left(c^{n/2-1}\int_0^{+\infty} \tau^{n/2-2}e^{-\tau}d\tau\right)\frac{1}{|x-\xi|^{n-2}},$$

we get in light of (2.7)

$$|G_{a,b}(x+h,\xi)-G_{a,b}(x,\xi)| \le \frac{C}{|x-\xi|^{n+2+\eta}}|h|^{\eta}, \quad x\neq \xi,\ |h|\le |x-\xi|,$$

where $C=C(n,\lambda,\kappa)$ is a constant. In particular, $G_{a,b}(\cdot,\xi)\in C(\mathbb{R}^n\setminus\{\xi\})$. Similarly, using (2.8) instead of (2.7), we obtain $G_{a,b}(x,\cdot)\in C(\mathbb{R}^n\setminus\{x\})$. More specifically we have

$$|G_{a,b}(x,\xi+h)-G_{a,b}(x,\xi)| \le \frac{C}{|x-\xi|^{n+2+\eta}}|h|^{\eta}, \quad x\neq \xi,\ |h|\le |x-\xi|. \tag{2.14}$$

Take $\xi\in\mathbb{R}^n$ and $\omega\Subset\mathbb{R}^n\setminus\xi$, and pick $g\in C_0^\infty(\omega)$. Then set

$$w_{a,b}(y)=\int_\omega G_{a,b}(x,y)g(x)dx, \quad y\in B(\xi,\mathrm{dist}(\xi,\overline{\omega})/2).$$

It follows from (2.14) that, for $y\in B(\xi,\mathrm{dist}(\xi,\overline{\omega}))$ and $|h|<\mathrm{dist}(y,\overline{\omega})$, we have

$$|w_{a,b}(y+h)-w_{a,b}(y)|\le \frac{C}{\mathrm{dist}(y,\overline{\omega})^{n+2+\eta}}|h|^{\eta}.$$

Therefore $w_{a,b}\in C(B(\xi,\mathrm{dist}(\xi,\overline{\omega})/2)$.

Let $\mathcal{M}(\mathbb{R}^n)$ be the space of bounded measures on $\mathbb{R}^n$. Pick a sequence (f_n) of a positive functions of $C_0^\infty(\mathbb{R}^n)$ converging in $\mathcal{M}(\mathbb{R}^n)$ to δ_ξ and let $u_n=-A_{a,b}^{-1}f_n$. In consequence, according to Fubini's theorem, we have

$$\begin{aligned}\int_\omega u_n(x)g(x)dx &= \int_\omega\int_{\mathbb{R}^n} G_{a,b}(x,y)g(x)f_n(y)dy\\ &= \int_{\mathbb{R}^n} w_{a,b}(y)f_n(y)dy \longrightarrow w_{a,b}(\xi)=\int_\omega G_{a,b}(x,\xi)g(x)dx,\end{aligned}$$

where we used that $\mathrm{supp} f_n\subset B(\xi,\mathrm{dist}(\xi,\overline{\omega})/2)$, provided that n is sufficiently large. That is we proved that u_n converges to $G_{a,b}(\cdot,\xi)$ weakly in $L^2_{\mathrm{loc}}(\mathbb{R}^n\setminus\{\xi\})$ (think to the fact that $C_0^\infty(\omega)$ is dense in $L^2(\omega)$).

Now, as $L_{a,b}u_n=f_n$, we find $L_{a,b}G_{a,b}(\cdot,\xi)=0$ in $\mathbb{R}^n\setminus\{\xi\}$ in the distributional sense.

We note that the uniqueness of $G_{a,b}$ follows from that of u.

As $\kappa^{-1}\le b\le\kappa$ we deduce from Lemma 2.1 that

$$E_{a,\kappa}(x,t,\xi,0)\le E_{a,b}(x,t,\xi,0)\le E_{a,\kappa^{-1}}(x,t,\xi,0).$$

But a simple change of variable shows that

$$E_{a,\kappa^{-1}}(x,t,\xi,0)=e^{-\kappa^{-1}t}E_{a,0}(x,t,\xi,0) \tag{2.15}$$

and

$$E_{a,\kappa}(x,t,\xi,0)=e^{-\kappa t}E_{a,0}(x,t,\xi,0). \tag{2.16}$$

Therefore, from Theorem 2.1 and identity (2.3), there exists a constant $c=c(n,\lambda)>1$ so that

$$\begin{aligned}e^{-\kappa t}\frac{(\pi c^{-1})^{n/2-1}}{\pi}E_{c^{-1}/4,0}(x,t,\xi,0)&\le E_{a,b}(x,t,\xi,0)\\ &\le e^{-\kappa^{-1}t}\frac{(\pi c)^{n/2-1}}{\pi}E_{c/4,0}(x,t,\xi,0),\end{aligned}$$

which, combined with identities (2.15) and (2.16), gives

$$\frac{(\pi c^{-1})^{n/2-1}}{\pi}E_{c^{-1}/4,\kappa}(x,t,\xi,0) \le E_{a,b}(x,t,\xi,0) \le \frac{(\pi c)^{n/2-1}}{\pi}E_{c/4,\kappa^{-1}}(x,t,\xi,0).$$

From the uniqueness of $G_{a,b}$, we obtain by integrating over $(0,+\infty)$ each member of the above inequalities

$$\frac{(\pi c^{-1})^{n/2-1}}{\pi}G_{c^{-1}/4,\kappa}(x,\xi) \le G_{a,b}(x,\xi) \le \frac{(\pi c)^{n/2-1}}{\pi}G_{c/4,\kappa^{-1}}(x,\xi).$$

These two-sided inequalities together with (2.10) yield in a straightforward manner (2.11). □

The function $G_{a,b}$ given by the previous theorem is usually called a fundamental solution of the operator $L_{a,b}$.

2.2. **Regularity of fundamental solutions.** Let $\xi \in \mathbb{R}^n$ and $\mathcal{O} \Subset \mathcal{O}' \Subset \mathbb{R}^n \setminus \{\xi\}$ with $\mathcal{O}'$ of class $C^{1,1}$. As $G_{a,b}(\cdot,\xi) \in C(\partial\mathcal{O}')$, we get from [15, Theorem 6.18, page 106] (interior Hölder regularity) that $G_{a,b}(\cdot,\xi)$ belongs to $C^{2,\alpha}(\mathcal{O})$.

Proposition 2.1. *There exist $C = C(n,\lambda,\kappa,\alpha)$ and $\varkappa = \varkappa(\alpha) > 2$ so that, for any $\xi \in \mathbb{R}^n$ and $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi\}$, we have*

$$\|G_{a,b}(\cdot,\xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le C\Lambda(\mathbf{d}+\varrho)^{\varkappa}\max\left(\varrho^{-(2+\alpha)},1\right)\varrho^{-n+2}. \tag{2.17}$$

Here $\varrho = dist\left(\xi,\overline{\mathcal{O}}\right)$, $\mathbf{d} = diam(\mathcal{O})$ and

$$\Lambda(h) = [1+h(1+h)(1+h^{\alpha})]\lambda, \quad h>0.$$

The proof of this proposition is based the following lemma consisting in an adaptation of the usual interior Schauder estimates. The proof of this technical lemma will be given in Appendix A.

Lemma 2.3. *There exists two constants $C = C(n,\alpha)$ and $\varkappa = \varkappa(\alpha) > 1$ with the property that, for any bounded subset $\mathcal{Q}$ of $\mathbb{R}^n$, $\delta > 0$ so that $\mathcal{Q}_\delta = \{x \in \mathcal{Q};\ dist(x,\partial\mathcal{Q}) > \delta\} \neq \emptyset$, $w \in C^{2,\alpha}(\mathcal{Q}) \cap C\left(\overline{\mathcal{Q}}\right)$ satisfying $L_{a,b}w = 0$ in $\mathcal{Q}$ and $\mathcal{Q}' \subset \mathcal{Q}_\delta$, we have*

$$\|w\|_{C^{2,\alpha}\left(\overline{\mathcal{Q}'}\right)} \le C\max\left(\delta^{-(2+\alpha)},1\right)\Lambda(\mathbf{d})^{\varkappa}\|w\|_{C\left(\overline{\mathcal{Q}}\right)}, \tag{2.18}$$

where Λ is as in Proposition 2.1 and $\mathbf{d} = diam(\mathcal{Q})$.

Proof of Proposition 2.1. We get, by applying Lemma 2.3 with $\mathcal{Q}' = \mathcal{O}$, $\delta = \varrho/2$ and $\mathcal{Q} = \left\{x \in \mathbb{R}^n;\ \text{dist}\left(x,\overline{\mathcal{O}}\right) < \varrho/2\right\}$,

$$\|G_{a,b}(\cdot,\xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le C\Lambda(\mathbf{d}+\varrho)^{\varkappa}\max\left(\delta^{-(2+\alpha)},1\right)\|G_{a,b}(\cdot,\xi)\|_{C\left(\overline{\mathcal{Q}}\right)}.$$

This and (2.11) yield

$$\|G_{a,b}(\cdot,\xi)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le C\Lambda(\mathbf{d}+\varrho)^{\varkappa}\max\left(\delta^{-(2+\alpha)},1\right)\varrho^{-n+2}e^{-\varrho/\sqrt{c\kappa}}, \tag{2.19}$$

with $C = C(n,\lambda,\kappa,\alpha)$ and $c = c(n,\lambda)$. It is then clear that (2.19) implies (2.17). □

The preceding proposition together with Lemma A.2 enable us to state the following corollary.

Corollary 2.1. *There exist* $C = C(n, \lambda, \kappa, \alpha, \theta)$ *and* $\varkappa = \varkappa(\alpha) > 1$ *so that, for any* $\xi \in \mathbb{R}^n$ *and* $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi\}$, *we have*

$$
\begin{aligned}
(2.20)\qquad &\|G_{a,b}(\cdot,\xi)\|_{H^{2+\theta}(\mathcal{O})}\\
&\le C\Lambda(\mathbf{d}+\varrho)^{\varkappa}\max\left(\mathbf{d}^{n/2},\mathbf{d}^{n/2+\alpha-\theta}\right)\max\left(\varrho^{-(2+\alpha)},1\right)\varrho^{-n+2},
\end{aligned}
$$

where $\varrho = \mathrm{dist}\left(\xi, \overline{\mathcal{O}}\right)$, $\mathbf{d} = \mathrm{diam}(\mathcal{O})$.

Corollary 2.2. *There exist* $C = C(n, \lambda, \kappa, \alpha)$ *and* $c = c(n, \lambda, \kappa, \alpha)$ *so that, for any* $\xi_1, \xi_2 \in \mathbb{R}^n$ *and* $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$, *we have*

$$
(2.21)\qquad \left\|\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4,
$$

where $\varrho_- = \min\left(\mathrm{dist}\left(\xi_1, \mathcal{O}\right), \mathrm{dist}\left(\xi_2, \mathcal{O}\right)\right)$ *and* $\varrho_+ = \max\left(\mathrm{dist}\left(\xi_1, \mathcal{O}\right), \mathrm{dist}\left(\xi_2, \mathcal{O}\right)\right)$.

Proof. In this proof $C = C(n, \lambda, \kappa, \alpha)$, $c = c(n, \lambda, \kappa, \alpha)$ and $\varkappa = \varkappa(\alpha) > 2$ are generic constants.

From Proposition 2.1, we have

$$
(2.22)\qquad \|G_{a,b}(\cdot,\xi_j)\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le C\Lambda(\mathbf{d}+\varrho_+)^{\varkappa}\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2},\quad j=1,2.
$$

Let $C_0 \ge 1$ end $c_0 \ge 1$ be the constants in (2.11) and fix $0 < \delta_0 \le 1$. Then the first inequality in (2.11) gives

$$
\frac{1}{G_{a,b}(\cdot,\xi_1)} \le C_0\left(\mathbf{d}+\varrho_+\right)^{n-2}e^{2\sqrt{c_0\kappa}(\mathbf{d}+\varrho_+)}.
$$

This inequality together with Lemma A.1 in Appendix A yield

$$
(2.23)\qquad \left\|\frac{1}{G_{a,b}(\cdot,\xi_1)}\right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\|G_{a,b}(\cdot,\xi_1)\|_{C^{2,\alpha}(\overline{\mathcal{O}})}\right)^3.
$$

Then in light of (2.22) and (2.23), we get from the interpolation inequality in [15, Lemma 6.35, page 135]

$$
\left\|\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le K_{\mathcal{O}}Ce^{c\mathbf{d}}\left(1+(1+\mathbf{d})^{\varkappa}\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4,
$$

for some constant $K_{\mathcal{O}}$, and hence

$$
\left\|\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right\|_{C^{2,\alpha}(\overline{\mathcal{O}})} \le K_{\mathcal{O}}Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4.
$$

The expected inequality follows by noting that $K_{\mathcal{O}}$ can be dominated by a universal constant multiplied by $|B|$, for some ball B of radius $2\mathbf{d}$ so that $\mathcal{O} \Subset B$. The reason in that the interpolation constant for an arbitrary ball of radius R is equal to R^n multiplied by the interpolation constant of the unit ball. □

This corollary combined with Lemma A.2 yields the following result.

Corollary 2.3. *There exist* $C = C(n, \lambda, \kappa, \alpha, \theta)$ *and* $c = c(n, \lambda, \kappa, \alpha, \theta)$ *so that, for any* $\xi_1, \xi_2 \in \mathbb{R}^n$ *and* $\mathcal{O} \Subset \mathbb{R}^n \setminus \{\xi_1, \xi_2\}$, *we have*

$$
(2.24)\qquad \left\|\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right\|_{H^{2+\theta}(\mathcal{O})} \le Ce^{c(\mathbf{d}+\varrho_+)}\left(1+\max\left(\varrho_-^{-(2+\alpha)},1\right)\varrho_-^{-n+2}\right)^4.
$$

Here $\varrho_\pm$ *is the same as in Corollary 2.2.*

2.3. Gradient estimate of the quotient of two fundamental solutions.

Lemma 2.4. *There exist* $x^*\in B(\xi_2,|\xi_1-\xi_2|/2)\setminus\{\xi_2\}$, $C=(n,\lambda,\kappa,|\xi_1-\xi_2|)>0$ *and* $\rho=\rho(n,\lambda,\kappa,|\xi_1-\xi_2|)>0$ *so that* $\overline{B}(x^*,\rho)\subset B(\xi_2,|\xi_1-\xi_2|/2)\setminus\{\xi_2\}$ *and*

$$C\le \left\|\nabla\left(\frac{G_{a,b}(\cdot,\xi_2)}{G_{a,b}(\cdot,\xi_1)}\right)\right\|_{L^2(B(x^*,\rho))}.$$

Proof. We set for notational convenience $w=G_{a,b}(\cdot,\xi_2)/G_{a,b}(\cdot,\xi_1)$. In light of Theorem 2.2, we obtain by straightforward computations the following two-sided inequality

$$\frac{C^{-1}}{|x-\xi_2|^{n-2}}\le w(x)\le \frac{C}{|x-\xi_2|^{n-2}},\quad x\in B(\xi_2,|\xi_1-\xi_2|/2)\setminus\{\xi_2\}. \tag{2.25}$$

Here and until the rest of this proof $C=C(n,\lambda,\kappa,|\xi_1-\xi_2|)$ is a generic constant.

Set $\tilde t=|\xi_1-\xi_2|/4$ and define

$$\varphi(t,\theta)=w(\xi_2+t\theta),\quad (t,\theta)\in(0,\tilde t]\times\mathbb{S}^{n-1}.$$

According to Proposition 2.1, $\varphi\in C^{2,\alpha}_{\rm loc}((0,\tilde t]\times\mathbb{S}^{n-1})$ and hence

$$\varphi(\tilde t,\theta)-\varphi(t,\theta)=\int_t^{\tilde t}\nabla w(\xi_2+s\theta)\cdot\theta ds,$$

which in turn gives

$$\begin{aligned}|\varphi(\tilde t,\theta)-\varphi(t,\theta)|^2&\le(\tilde t-t)\int_t^{\tilde t}|\nabla w(\xi_2+s\theta)|^2ds\\&\le \tilde t\int_t^{\tilde t}|\nabla w(\xi_2+s\theta)|^2ds\\&\le \tilde t\int_t^{\tilde t}\frac{s^{n-1}}{t^{n-1}}|\nabla w(\xi_2+s\theta)|^2ds,\quad (t,\theta)\in(0,\tilde t]\times\mathbb{S}^{n-1}.\end{aligned}$$

Whence, where $t\in(0,\tilde t]$,

$$t^{n-1}\int_{\mathbb{S}^{n-1}}|\varphi(\tilde t,\theta)-\varphi(t,\theta)|^2d\theta\le \tilde t\int_{\mathscr{C}_t}|\nabla w(x)|^2dx. \tag{2.26}$$

Here

$$\mathscr{C}_t=\left\{x\in\mathbb{R}^n:\ t<|x-\xi_2|<\tilde t\right\}.$$

On the other hand inequalities (2.25) imply, where $(t,\theta)\in(0,\tilde t]\times\mathbb{S}^{n-1}$,

$$\frac{C^{-1}}{t^{n-2}}\le\varphi(t,\theta)\le\frac{C}{t^{n-2}}.$$

Let us then choose $t_0\le\tilde t$ sufficiently small in such a way that

$$\frac{C^{-1}}{t^{n-2}}-\frac{C}{\tilde t^{n-2}}>0,\quad t\in(0,t_0].$$

Therefore

$$\left(\frac{C^{-1}}{t^{n-2}}-\frac{C}{\tilde t^{n-2}}\right)^2\le|\varphi(\tilde t,\theta)-\varphi(t,\theta)|^2 \tag{2.27}$$

if $(t,\theta)\in(0,t_0]\times\mathbb{S}^{n-1}$.

We then obtain by combining inequalities (2.26) and (2.27)

$$|\mathbb{S}^{n-1}|\left(\frac{C^{-1}}{t^{n-2}}-\frac{C}{\tilde{t}^{n-2}}\right)^2 \le \tilde{t}\int_{\mathscr{C}_t}|\nabla w(x)|^2\,dx, \quad t\in(0,t_0].$$

We have in particular

$$C\le \int_{\mathscr{C}_{t_0}}|\nabla w(x)|^2\,dx.$$

Let $\rho=t_0/4$. Then it is straightforward to check that, for any $x\in\overline{\mathscr{C}_{t_0}}$,

$$\overline{B}(x,\rho)\subset\{y\in\mathbb{R}^n;\ 3t_0/4\le|y-\xi_2|\le 5\tilde{t}/4\}\subset B(\xi_2,|\xi_1-\xi_2|/2)\setminus\{\xi_2\}.$$

Since $\overline{\mathscr{C}_{t_0}}$ is compact, we find a positive integer $N=N(\lambda,\kappa,|\xi_1-\xi_2|)$ and $x_j\in\overline{\mathscr{C}_{t_0}}$, $j=1,\cdots,N$, so that

$$\overline{\mathscr{C}_{t_0}}\subset\bigcup_{j=1}^N B(x_j,\rho).$$

Hence

$$C\le\int_{\cup_{j=1}^N B(x_j,\rho)}|\nabla w(x)|^2\,dx.$$

Pick then $x^*\in\{x_j,\ 1\le j\le N\}$ in such a way that

$$\int_{B(x^*,\rho)}|\nabla w(x)|^2\,dx=\max_{1\le j\le N}\int_{B(x_j,\rho)}|\nabla w(x)|^2\,dx.$$

Therefore

$$C\le\int_{B(x^*,\rho)}|\nabla w(x)|^2\,dx.$$

This finishes the proof. □

3. Uniform lower bound for the gradient

Let $\mathcal{O}$ be a Lipschitz bounded domain of $\mathbb{R}^n$ and $\sigma\in C^{0,1}(\overline{\mathcal{O}})$ satisfying

$$\varkappa^{-1}\le\sigma \quad\text{and}\quad \|\sigma\|_{C^{0,1}(\overline{\mathcal{O}})}\le\varkappa, \tag{3.1}$$

for some fixed constant $\varkappa>1$.

In this section we prove a polynomial lower bound of the local L^2-norm of the gradient of solutions of

$$L_\sigma u=\operatorname{div}(\sigma\nabla u)=0 \quad\text{in }\mathcal{O}.$$

In a first step we establish, via a three-ball inequality for the gradient, a uniform lower bound of negative exponential type. We use then in a second step an argument based on the so-called frequency function in order to improve this lower bound.

3.1. **Preliminary lower bound.** We need hereafter the following three-ball inequality for the gradient.

Theorem 3.1. *Let $0 < k < \ell < m$ be real. There exist two constants $C = C(n, \varkappa, k, \ell, m) > 0$ and $0 < \gamma = \gamma(n, \varkappa, k, \ell, m) < 1$ so that, for any $v \in H^1(\mathcal{O})$ satisfying $L_\sigma v = 0$, $y \in \mathcal{O}$ and $0 < r < dist(y, \partial\mathcal{O})/m$, we have*

$$C\|\nabla v\|_{L^2(B(y,\ell r))} \leq \|\nabla v\|^{\gamma}_{L^2(B(y,kr))}\|\nabla v\|^{1-\gamma}_{L^2(B(y,mr))}.$$

A proof of this theorem can be found in [9] or [10].

Define the geometric distance d_g^D on the bounded domain D of $\mathbb{R}^n$ by

$$d_g^D(x, y) = \inf\left\{\ell(\psi);\ \psi : [0, 1] \to D \text{ Lipschitz path joining } x \text{ to } y\right\},$$

where

$$\ell(\psi) = \int_0^1 |\dot{\psi}(t)|dt$$

is the length of ψ.

Note that according to Rademacher's theorem any Lipschitz continuous function $\psi : [0, 1] \to D$ is almost everywhere differentiable with $|\dot{\psi}(t)| \leq k$ a.e. $t \in [0, 1]$, where k is the Lipschitz constant of ψ.

Lemma 3.1. *Let D be a bounded Lipschitz domain of $\mathbb{R}^n$. Then $d_g^D \in L^\infty(D \times D)$ and there exists a constant $\mathfrak{c}_D > 0$ so that*

$$|x - y| \leq d_g^D(x, y) \leq \mathfrak{c}_D|x - y|, \quad x, y \in D. \tag{3.2}$$

We refer to [20, Lemma A3] for a proof.

In this subsection we use the following notations

$$\mathcal{O}^\delta = \{x \in \mathcal{O};\ \mathrm{dist}(x, \partial\mathcal{O}) > \delta\}$$

and

$$\chi(\mathcal{O}) = \sup\{\delta > 0;\ \mathcal{O}^\delta \neq \emptyset\}.$$

Define

$$\mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta) = \big\{u \in H^1(\mathcal{O});\ L_\sigma u = 0 \text{ in } \mathcal{O}, \\ \|\nabla u\|_{L^2(\mathcal{O})} \leq M,\ \|\nabla u\|_{L^2(B(x_0,\delta))} \geq \eta\big\}, \tag{3.3}$$

with $\delta \in (0, \chi(\mathcal{O})/3)$, $x_0 \in \mathcal{O}^{3\delta}$, $\eta > 0$ and $M \geq 1$ satisfying $\eta < M$.

Lemma 3.2. *There exist two constants $c = c(n, \varkappa) \geq 1$ and $0 < \gamma = \gamma(n, \varkappa) < 1$ so that, for any $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta)$ and $x \in \mathcal{O}^{3\delta}$, we have*

$$e^{-[\ln(cM/\eta)/\gamma]e^{[2n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}} \leq \|\nabla u\|_{L^2(B(x,\delta))}, \tag{3.4}$$

with $\mathfrak{c} = \mathfrak{c}_{\mathcal{O}}$ is as in Lemma 3.1.

Proof. Pick $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta)$. Let $x \in \mathcal{O}^{3\delta}$ and $\psi : [0, 1] \to \mathcal{O}$ be a Lipschitz path joining $x = \psi(0)$ to $x_0 = \psi(1)$, so that $\ell(\psi) \leq 2d_g(x_0, x)$. Here and henceforth, for simplicity convenience, we use $d_g(x_0, x)$ instead of $d_g^{\mathcal{O}}(x_0, x)$.

Let $t_0 = 0$ and $t_{k+1} = \inf\{t \in [t_k, 1];\ \psi(t) \notin B(\psi(t_k), \delta)\}$, $k \geq 0$. We claim that there exists an integer $N \geq 1$ verifying $\psi(1) \in B(\psi(t_N), \delta)$. If not, we would have $\psi(1) \notin B(\psi(t_k), \delta)$ for any $k \geq 0$. As the sequence (t_k) is non decreasing and bounded from above by 1, it converges to $\hat{t} \leq 1$. In particular, there exists an integer $k_0 \geq 1$ so that $\psi(t_k) \in B\left(\psi(\hat{t}), \delta/2\right)$, $k \geq k_0$. But this contradicts the fact that $|\psi(t_{k+1}) - \psi(t_k)| \geq \delta$, $k \geq 0$.

Let us check that $N \le N_0$ where $N_0 = N_0(n, |x-x_0|, \mathfrak{c}, \delta)$. Pick $1 \le j \le n$ so that

$$\max_{1\le i\le n} |\psi_i(t_{k+1}) - \psi_i(t_k)| = |\psi_j(t_{k+1}) - \psi_j(t_k)|,$$

where ψ_i is the ith component of ψ. Then

$$\delta \le n\,|\psi_j(t_{k+1}) - \psi_j(t_k)| = n\left|\int_{t_k}^{t_{k+1}} \dot\psi_j(t)dt\right| \le n\int_{t_k}^{t_{k+1}} |\dot\psi(t)|dt.$$

Consequently, where $t_{N+1} = 1$,

$$(N+1)\delta \le n\sum_{k=0}^{N}\int_{t_k}^{t_{k+1}} |\dot\psi(t)|dt = n\ell(\psi) \le 2nd_g(x_0,x) \le 2n\mathfrak{c}|x-x_0|.$$

Therefore

$$N \le N_0 = \left[\frac{2n\mathfrak{c}|x-x_0|}{\delta}\right].$$

Let $y_0 = x$ and $y_k = \psi(t_k)$, $1 \le k \le N$. If $|z - y_{k+1}| < \delta$, then $|z-y_k| \le |z-y_{k+1}| + |y_{k+1}-y_k| < 2\delta$. In other words $B(y_{k+1},\delta) \subset B(y_k, 2\delta)$.

We get from Theorem 3.1

$$\|\nabla u\|_{L^2(B(y_j,2\delta))} \le C\|\nabla u\|^{1-\gamma}_{L^2(B(y_j,3\delta))}\|\nabla u\|^{\gamma}_{L^2(B(y_j,\delta))}, \quad 0 \le j \le N, \tag{3.5}$$

for some constants $C = C(n,\varkappa) > 0$ and $0 < \gamma = \gamma(n,\varkappa) < 1$.

Set $I_j = \|\nabla u\|_{L^2(B(y_j,\delta))}$, $0 \le j \le N$ and $I_{N+1} = \|\nabla u\|_{L^2(B(x_0,\delta))}$. Since $B(y_{j+1},\delta) \subset B(y_j,2\delta)$, $1 \le j \le N-1$, estimate (3.5) implies

$$I_{j+1} \le CM^{1-\gamma}I_j^{\gamma},\ \ 0 \le j \le N. \tag{3.6}$$

Let $C_1 = C^{1+\gamma+\ldots+\gamma^{N+1}}$ and $\beta = \gamma^{N+1}$. Then by a simple induction argument estimate (3.6) yields

$$I_{N+1} \le C_1 M^{1-\beta} I_0^{\beta}. \tag{3.7}$$

Without loss of generality, we assume in the sequel that $C \ge 1$ in (3.6). Using that $N \le N_0$, we have

$$\beta \ge \beta_0 = s^{N_0+1},$$
$$C_1 \le C^{\frac{1}{1-s}},$$
$$\left(\frac{I_0}{M}\right)^{\beta} \le \left(\frac{I_0}{M}\right)^{\beta_0}.$$

These estimates in (3.7) give

$$\frac{I_{N+1}}{M} \le C^{\frac{1}{1-\gamma}}\left(\frac{I_0}{M}\right)^{\gamma^{N_0+1}},$$

from which we deduce that

$$\|\nabla u\|_{L^2(B(x_0,\delta))} \le C^{\frac{1}{1-\gamma}} M^{1-\gamma^{N_0+1}}\|\nabla u\|^{\gamma^{N_0+1}}_{L^2(B(x,\delta))}.$$

But $M \ge 1$. Whence

$$\eta \le \|\nabla u\|_{L^2(B(x_0,\delta))} \le C^{\frac{1}{1-\gamma}} M\|\nabla u\|^{\gamma^{N_0+1}}_{L^2(B(x,\delta))}.$$

The expected inequality follows readily from this last estimate. □

3.2. **An estimate for the frequency function.** Some tools in the present section are borrowed from [13, 14, 17]. Let $u \in H^1(\mathcal{O})$ and $\sigma \in C^{0,1}(\overline{\mathcal{O}})$ satisfying the bounds (3.1). We recall that the usual frequency function, relative to the operator L_σ, associated to u is defined by

$$N(u)(x_0,r)=\frac{rD(u)(x_0,r)}{H(u)(x_0,r)},$$

provided that $B(x_0,r)\Subset\mathcal{O}$, with

$$D(u)(x_0,r)=\int_{B(x_0,r)}\sigma(x)|\nabla u(x)|^2dx,$$
$$H(u)(x_0,r)=\int_{\partial B(x_0,r)}\sigma(x)u^2(x)dS(x).$$

Define also

$$K(u)(x_0,r)=\int_{B(x,r)}\sigma(x)u(x)^2dx.$$

Prior to studying the properties of the frequency function, we prove some preliminary results.

Fix $u\in H^2(\mathcal{O})$ so that $L_\sigma u=0$ in $\mathcal{O}$ and, for simplicity convenience, we drop in the sequel the dependence on u of N, D, H and K.

Lemma 3.3. *For $x_0\in\mathcal{O}^\delta$ and $0<r<\delta$, we have*

$$\partial_r H(x_0,r)=\frac{n-1}{r}H(x_0,r)+\tilde{H}(x_0,r)+2D(x_0,r), \tag{3.8}$$
$$\partial_r D(x_0,r)=\frac{n-2}{r}D(x_0,r)+\tilde{D}(x_0,r)+2\hat{H}(x_0,r). \tag{3.9}$$

Here

$$\tilde{H}(x_0,r)=\int_{\partial B(x_0,r)}u^2\nabla\sigma(x)\cdot\nu(x)dS(x),$$
$$\hat{H}(x_0,r)=\int_{\partial B(x_0,r)}\sigma(x)(\partial_\nu u(x))^2dS(x),$$
$$\tilde{D}(x_0,r)=\int_{B(x_0,r)}|\nabla u(x)|^2\nabla\sigma(x)\cdot(x-x_0)dx.$$

Proof. Pick $x_0\in\mathcal{O}^\delta$ and $0<r<\delta$. A simple change of variable yields

$$H(x_0,r)=\int_{B(0,1)}\sigma(x_0+ry)u^2(x_0+ry)r^{n-1}dS(y).$$

Hence

$$\begin{aligned}\partial_r H(x_0,r) &= \frac{n-1}{r}H(x_0,r) + \int_{B(0,1)} \nabla(\sigma u^2)(x_0+ry)\cdot y r^{n-1} dS(y)\\ &= \frac{n-1}{r}H(x_0,r) + \int_{B(0,1)} u^2\nabla\sigma(x_0+ry)\cdot y r^{n-1} dS(y)\\ &\qquad\qquad + \int_{\partial B(0,1)} \sigma\nabla(u^2)(x_0+ry)\cdot y r^{n-1} dS(y)\\ &= \frac{n-1}{r}H(x_0,r) + \int_{\partial B(x_0,r)} u^2\nabla\sigma(x)\cdot\nu(x) dS(x)\\ &\qquad + \int_{\partial B(x_0,r)} \sigma(x)\nabla(u^2)(x)\cdot\nu(x) dS(x)\\ &= \frac{n-1}{r}H(x_0,r) + \tilde{H}(x_0,r) + \int_{\partial B(x_0,r)} \sigma\nabla(u^2)(x)\cdot\nu(x) dS(x).\end{aligned}$$

Identity (3.8) will follow if we prove

$$2D(x_0,r) = \int_{\partial B(x_0,r)} \sigma\nabla(u^2)(x)\cdot\nu(x) dS(x).$$

To this end, we observe that $\mathrm{div}(\sigma\nabla u) = 0$ implies

$$\mathrm{div}(\sigma\nabla(u^2)) = 2u\mathrm{div}(\sigma\nabla u) + 2\sigma|\nabla u|^2 = 2\sigma|\nabla u|^2.$$

We then get by applying the divergence theorem

$$\begin{aligned}2D(x_0,r) &= \int_{B(x_0,r)} \mathrm{div}(\sigma(x)\nabla(u^2)(x))dx\\ &= \int_{\partial B(x_0,r)} \sigma(x)\nabla(u^2)(x)\cdot\nu(x) dS(x).\end{aligned} \tag{3.10}$$

By a change of variable we have

$$D(x_0,r) = \int_0^r \int_{\partial B(0,1)} \sigma(x_0+ty)|\nabla u(x_0+ty)|^2 t^{n-1} dS(y)dt.$$

Hence

$$\begin{aligned}\partial_r D(x_0,r) &= \int_{\partial B(0,1)} \sigma(x_0+ry)|\nabla u(x_0+ty)|^2 r^{n-1} dS(y)\\ &= \int_{\partial B(x_0,r)} \sigma(x)|\nabla u(x)|^2 dS(x)\\ &= \frac{1}{r}\int_{\partial B(x_0,r)} \sigma(x)|\nabla u(x)|^2 (x-x_0)\cdot\nu(x) dS(x).\end{aligned}$$

An application of the divergence theorem then gives

$$\partial_r D(x_0,r) = \frac{1}{r}\int_{B(x_0,r)} \mathrm{div}(\sigma(x)|\nabla u(x)|^2(x-x_0))dx.$$

Therefore

$$\partial_r D(x_0,r) = \frac{1}{r}\int_{B(x_0,r)} |\nabla u(x)|^2 \mathrm{div}(\sigma(x)(x-x_0))dx + \frac{1}{r}\int_{B(x_0,r)} \sigma(x)(x-x_0)\cdot\nabla(|\nabla u(x)|^2)dx$$

implying

$$\partial_r D(x_0,r) = \frac{n}{r}D(x_0,r) + \frac{1}{r}\tilde{D}(x_0,r) + \frac{1}{r}\int_{B(x_0,r)} \sigma(x)(x-x_0)\cdot\nabla(|\nabla u(x)|^2)dx. \tag{3.11}$$

On the other hand,

$$\begin{aligned}
&\int_{B(x_0,r)} \sigma(x)(x_j-x_{0,j})\partial_j(\partial_i u(x))^2 dx \\
&\quad = 2\int_{B(x_0,r)} \sigma(x)(x_j-x_{0,j})\partial_{ij}^2 u\partial_i u(x)dx \\
&\quad = -2\int_{B(x_0,r)} \partial_i\left[\partial_i u(x)\sigma(x)(x_j-x_{0,j})\right]\partial_j u(x)dx \\
&\qquad\qquad + 2\int_{\partial B(x_0,r)} \sigma(x)\partial_i u(x)(x_j-x_{0,j})\partial_j u(x)\nu_i(x)dS(x) \\
&\quad = -2\int_{B(x_0,r)} \partial_{ii}^2 u(x)\sigma(x)(x_j-x_{0,j})\partial_j u(x)dx \\
&\qquad - 2\int_{B(x_0,r)} \partial_i u(x)\partial_j u(x)\partial_i\left[\sigma(x)(x_j-x_{0,j})\right]dx \\
&\qquad\qquad + 2\int_{\partial B(x_0,r)} \sigma(x)\partial_i u(x)(x_j-x_{0,j})\partial_j u(x)\nu_i(x)dS(x).
\end{aligned}$$

Thus, taking into account that $\sigma\Delta u = -\nabla\sigma\cdot\nabla u$,

$$\int_{B(x_0,r)} \sigma(x)(x-x_0)\cdot\nabla(|\nabla u(x)|^2)dx = -2\int_{B(x_0,r)} \sigma(x)|\nabla u(x)|^2 dx + 2r\int_{\partial B(x_0,r)} \sigma(x)(\partial_\nu u(x))^2 dS(x).$$

This identity in (3.11) yields

$$\partial_r D(x_0,r) = \frac{n-2}{r}D(x_0,r) + \frac{1}{r}\tilde{D}(x_0,r) + 2\hat{H}(x_0,r).$$

That is we proved (3.9). □

Lemma 3.4. *We have*

$$K(x_0,r) \le \frac{\delta^n e^{\delta\varkappa^2}}{n}H(x_0,r), \quad x_0\in\mathcal{O}^\delta,\ 0<r<\delta.$$

Proof. Since

$$H(x_0,r) = \frac{1}{r}\int_{\partial B(x_0,r)} \sigma(x)u^2(x)(x-x_0)\cdot\nu(x)dS(x),$$

we find by applying the divergence theorem

$$H(x_0,r)=\frac{1}{r}\int_{B(x_0,r)}\operatorname{div}\left(\sigma(x)u^2(x)(x-x_0)\right)dx. \tag{3.12}$$

Hence

$$\begin{aligned}H'(x_0,r)&=-\frac{1}{r}H(x_0,r)+\frac{1}{r}\int_{\partial B(x_0,r)}\operatorname{div}\left(\sigma(x)u^2(x)(x-x_0)\right)dS(x)\\&=\frac{n-1}{r}H(r)+\int_{\partial B(x_0,r)}\partial_\nu\sigma(x)u^2(x)dS(x)\\&\qquad+2\int_{\partial B(x_0,r)}\sigma(x)\partial_\nu u(x)u(x)dS(x).\end{aligned}$$

But

$$\begin{aligned}&\int_{\partial B(x_0,r)}\sigma(x)\partial_\nu u(x)u(x)dS(x)\\&\qquad=\int_{B(x_0,r)}\operatorname{div}(\sigma(x)\nabla u(x))u+\int_{B(x_0,r)}\sigma(x)|\nabla u|^2dx\\&\qquad=\int_{B(x_0,r)}\sigma(x)|\nabla u(x)|^2dx=D(x_0,r).\end{aligned}$$

Therefore

$$\begin{aligned}H'(x_0,r)&=\frac{n-1}{r}H(x_0,r)+2D(x_0,r)+\int_{\partial B(x_0,r)}\partial_\nu\sigma(x)u^2(x)dS(x)\\&\geq\int_{\partial B(x_0,r)}\partial_\nu\sigma(x)u^2(x)dS(x)\\&\geq\int_{\partial B(x_0,r)}\frac{\partial_\nu\sigma(x)}{\sigma(x)}\sigma(x)u^2(x)dS(x)\geq-\varkappa^2H(x_0,r),\end{aligned}$$

where we used that $H(x_0,r)\geq 0$ and $D(x_0,r)\geq 0$.

Consequently $r\to e^{r\varkappa^2}H(x_0,r)$ is non decreasing and then

$$\begin{aligned}\int_0^r H(x_0,t)t^{n-1}dt&\leq\int_0^r e^{t\varkappa^2}H(x_0,t)t^{n-1}dt\\&\leq\int_0^r e^{r\varkappa^2}H(x_0,r)t^{n-1}dt\leq\frac{r^n}{n}e^{r\varkappa^2}H(x_0,r).\end{aligned}$$

As

$$K(x_0,r)=\int_0^r H(x_0,t)t^{n-1}dt,$$

we end up getting

$$K(x_0,t)\leq\frac{\delta^n e^{\delta\varkappa^2}}{n}H(x_0,r).$$

This completes the proof. □

Now straightforward computations yield, for $x_0\in\mathcal{O}^\delta$ and $0<r<\delta$,

$$\frac{\partial_r N(x_0,r)}{N(x_0,r)}=\frac{1}{r}+\frac{\partial_r D(x_0,r)}{D(x_0,r)}-\frac{\partial_r H(x_0,r)}{H(x_0,r)}. \tag{3.13}$$

Lemma 3.5. *For $x_0 \in \mathcal{O}^\delta$ and $0 < r < \delta$, we have*

$$N(x_0, r) \leq e^{\mu\delta} N(x_0, \delta),$$

with $\mu = \varkappa^2 (1 + \chi(\mathcal{O}))$.

Proof. We have from formulas (3.8) and (3.9) and identity (3.13)

$$\frac{\partial_r N(x_0,r)}{N(x_0,r)} = \frac{\tilde{D}(x_0,r)}{D(x_0,r)} - \frac{\tilde{H}(x_0,r)}{H(x_0,r)} + 2\frac{\hat{H}(x_0,r)}{D(x_0,r)} - 2\frac{D(x_0,r)}{H(x_0,r)} \tag{3.14}$$
$$= \frac{\tilde{D}(x_0,r)}{D(x_0,r)} - \frac{\tilde{H}(x_0,r)}{H(x_0,r)} + 2\frac{\hat{H}(x_0,r)H(x_0,r) - D(x_0,r)^2}{D(x_0,r)H(x_0,r)}.$$

But from (3.10) we have

$$D(x_0, r) = \int_{\partial B(x_0,r)} \sigma(x) u(x) \partial_\nu u(x) dS(x).$$

Then we find by applying Cauchy-Schwarz's inequality

$$D(x_0,r)^2 \leq \left(\int_{\partial B(x_0,r)} \sigma(x) u^2(x) dS(x) \right) \left(\int_{\partial B(x_0,r)} \sigma(x) (\partial_\nu u)^2(x) dS(x) \right).$$

That is

$$D^2(x_0,r) \leq H(x_0,r)\hat{H}(x_0,r). \tag{3.15}$$

This and (3.14) lead

$$\frac{\partial_r N(x_0,r)}{N(x_0,r)} \geq \frac{\tilde{D}(x_0,r)}{D(x_0,r)} - \frac{\tilde{H}(x_0,r)}{H(x_0,r)}. \tag{3.16}$$

On the other hand

$$\left|\tilde{H}(x_0,r)\right| \leq \varkappa \|\nabla a\|_\infty H(x_0,r) \leq \varkappa^2 H(x_0,r), \tag{3.17}$$

and similarly

$$\left|\tilde{D}(x_0,r)\right| \leq \varkappa^2 \delta D(x_0,r). \tag{3.18}$$

In light of (3.16), (3.17) and (3.18), we derive

$$\frac{\partial_r N(x_0,r)}{N(x_0,r)} \geq -\mu,$$

that is to say

$$\partial_r (e^{\mu r} N(x_0,r)) \geq 0.$$

Consequently

$$N(x_0,r) \leq e^{\mu(\delta - r)} N(x_0,\delta) \leq e^{\mu\delta} N(x_0,\delta),$$

as expected. □

3.3. **Polynomial lower bound.**

Lemma 3.6. *There exist two constants $c = c(n,\varkappa) > 0$ and $0 < \gamma = \gamma(n,\varkappa) < 1$ so that if*

$$\mathcal{C}_0(h) = Me^{c(1+\mathbf{d})\delta + [2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|\mathfrak{c}]h}}, \quad h > 0,$$

then

$$\|N(u)(x,\cdot)\|_{L^\infty(0,\delta)} \le \mathcal{C}_0(|x-x_0|/\delta),$$

for any $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$, where $\mathfrak{c} = \mathfrak{c}_{\mathcal{O}}$ is as in Lemma 3.1.

Proof. Pick $x \in \mathcal{O}^\delta$. Then from Lemma 3.2

$$\|\nabla u\|_{L^2(B(x,\delta/3))} \ge e^{-[\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}},$$

for some constant $c = c(n,\varkappa)$ and $0 < \gamma = \gamma(n,\varkappa)) < 1$.

On the other hand, we establish in a quite classical manner the following Caccioppoli's inequality

$$\|\nabla u\|^2_{L^2(B(x,\delta/3))} \le \frac{\varpi\varkappa^2(1+\mathbf{d})}{\delta^2}\|u\|^2_{L^2(B(x,\delta))},$$

where ϖ is a universal constant. Therefore

$$\|u\|^2_{L^2(B(x,\delta))} \ge \tilde{\mathcal{C}}_0(|x-x_0|/\delta), \tag{3.19}$$

where

$$\tilde{\mathcal{C}}_0(h) = \frac{\delta^2}{\varpi\varkappa^2(1+\mathbf{d})}e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|\mathfrak{c}]h}}, \quad h > 0. \tag{3.20}$$

Since $K(u)(x,\delta) \ge \varkappa^{-1}\|u\|^2_{L^2(B(x,\delta))}$, we find

$$K(u)(x,\delta) \ge \frac{\delta^2}{\varpi\varkappa^3(1+\mathbf{d})}e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}}. \tag{3.21}$$

In light of Lemma 3.4, we derive from (3.21)

$$H(u)(x,\delta) \ge \frac{\delta^{-n+2}e^{-\varkappa^2\delta}}{n\varpi\varkappa^3(1+\mathbf{d})}e^{-[2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}}. \tag{3.22}$$

In light of Lemma 3.5, we get

$$N(x,r) \le \delta\varkappa e^{\varkappa^2(1+\mathbf{d})\delta}\frac{\|\nabla u\|_{L^2(\mathcal{O})}}{H(u)(x,\delta)}, \quad 0 < r < \delta,$$

This inequality and (3.22) give, where $c = c(n,\varkappa)$ is a constant,

$$N(x,r) \le Me^{c(1+\mathbf{d})\delta + [2\ln(cM/\eta)/\gamma]e^{[6n|\ln\gamma|]\mathfrak{c}|x-x_0|/\delta}}, \quad 0 < r < \delta,$$

which is the expected inequality. □

Proposition 3.1. *Let $\mathcal{C}_0$ be as in Lemma 3.6, $\tilde{\mathcal{C}}_0$ as in (3.20) and set*

$$\mathcal{C}_1(h) = \mathcal{C}_0(h) + n - 1, \quad h > 0, \tag{3.23}$$

$$\tilde{\mathcal{C}}_2(h) = \delta^{-n+1}e^{-\varkappa^2\delta}\tilde{\mathcal{C}}_0(h), \quad h > 0. \tag{3.24}$$

If $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ then

$$\tilde{\mathcal{C}}_2(|x-x_0|/\delta)\left(\frac{r}{\delta}\right)^{\mathcal{C}_0(|x-x_0|/\delta)+n-1} \le \|u\|^2_{L^2(B(x,r))}, \quad x \in \mathcal{O}^\delta,\ 0 < r < \delta.$$

Proof. Observing that, where $H = H(u)$,

$$\partial_r \left(\ln \frac{H(x,r)}{r^{n-1}} \right) = \frac{\partial_r H(x,r)}{H(x,r)} - \frac{n-1}{r},$$

we get from Lemma 3.6, (3.8) and the fact that $|\tilde{H}(x,r)| \leq \varkappa^2 H(x,r)$,

$$\partial_r \left(\ln \frac{H(x,r)}{r^{n-1}} \right) \leq \varkappa^2 + \frac{N(x,r)}{r} \leq \varkappa^2 + \frac{\mathcal{C}_0(|x-x_0|/\delta)}{r}, \quad 0 < r < \delta,$$

Thus

$$\int_{sr}^{s\delta} \partial_t \left(\ln \frac{H(x,t)}{t^{n-1}} \right) dt = \ln \frac{H(x,s\delta) r^{n-1}}{H(x,sr)\delta^{n-1}} \leq \varkappa^2 (\delta - r)s + \mathcal{C}_0(|x-x_0|/\delta) \ln \frac{\delta}{r},$$

for $0 < s < 1$ and $0 < r < \delta$.

Hence

$$H(x,s\delta) \leq e^{\varkappa^2 \delta} \left(\frac{\delta}{r} \right)^{\mathcal{C}_0(|x-x_0|/\delta)+n-1} H(x,sr),$$

and then

$$\begin{aligned} \|u\|^2_{L^2(B(x,\delta))} &= \delta^{n-1} \int_0^1 H(x,s\delta) s^{n-1} ds \\ &\leq e^{\varkappa^2 \delta} \left(\frac{\delta}{r} \right)^{\mathcal{C}_0(|x-x_0|/\delta)+n-1} r^{n-1} \int_0^1 H(x,rs) s^{n-1} ds \\ &\leq \delta^{n-1} e^{\varkappa^2 \delta} \left(\frac{\delta}{r} \right)^{\mathcal{C}_0(|x-x_0|/\delta)+n-1} \|u\|^2_{L^2(B(x,r))}. \end{aligned}$$

Combined with (3.19) this estimate yields in a straightforward manner

$$\delta^{-n+1} e^{-\varkappa^2 \delta} \tilde{\mathcal{C}}_0(|x-x_0|/\delta) \left(\frac{r}{\delta} \right)^{\mathcal{C}_0(|x-x_0|/\delta)+n-1} \leq \|u\|^2_{L^2(B(x,r))}.$$

This is the expected inequality. □

For a bounded domain D, we denote the first non zero eigenvalue of the Laplace-Neumann operator on D by $\mu_2(D)$. Since $\mu_2(B(x_0,r)) = \mu_2(B(0,1))/r^2$, we get by applying Poincaré-Wirtinger's inequality

$$\begin{aligned} \|w - \{w\}\|^2_{L^2(B(x,r))} &\leq \frac{1}{\mu_2(B(x,r))} \|\nabla w\|^2_{L^2(B(x,r))} \\ &\leq \frac{r^2}{\mu_2(B(0,1))} \|\nabla w\|^2_{L^2(B(x,r))}, \end{aligned} \tag{3.25}$$

for any $w \in H^1(B(x,r))$, where $\{w\} = \frac{1}{|B(x,r)|} \int_{B(x,r)} w(x)dx$.

Noting that $\mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ is invariant under the transformation $u \to u - \{u\}$, we can state the following consequence of Proposition 3.1

Corollary 3.1. *With the notations of Proposition 3.1, if* $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ *then*

$$\mathcal{C}_2(|x-x_0|/\delta) \left(\frac{r}{\delta} \right)^{\mathcal{C}_1(|x-x_0|/\delta)} \leq \|\nabla u\|^2_{L^2(B(x,r))}, \quad x \in \mathcal{O}^\delta,\ 0 < r < \delta,$$

with

$$\mathcal{C}_2(h) = \mu_2(B(0,1)) \delta^{-2} \tilde{\mathcal{C}}_2(h), \quad h > 0, \tag{3.26}$$

with $\tilde{\mathcal{C}}_2$ *as in Proposition 3.1.*

It is important to remark that the argument we used to obtain Corollary 3.1 from Proposition 3.1 is no longer valid if we substitute L_σ by L_σ plus a multiplication operator by a function σ_0.

The following consequence of the preceding corollary will be useful in the proof of Theorem 1.1.

Lemma 3.7. *Let $\omega \Subset \mathcal{O}$ and set $\delta = dist(\omega, \partial\mathcal{O})$. Let $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ and $f \in C^{0,\alpha}(\mathcal{O})$. Then we have*

$$\|f\|_{L^\infty(\omega)} \le \hat{\mathcal{C}}_3 \|f\|^{1-\hat{\mu}}_{C^{0,\alpha}(\overline{\mathcal{O}})} \|f|\nabla u|^2\|^{\hat{\mu}}_{L^1(\mathcal{O})}, \tag{3.27}$$

with

$$\hat{\mu} = \frac{\alpha}{\max_{x\in\overline{\mathcal{O}}} \mathcal{C}_1(|x-x_0|/\delta) + \alpha},$$
$$\hat{\mathcal{C}}_3 = \max\left(2\delta^\alpha (\max\left(1, (\hat{\mathcal{C}}_2\delta^\alpha)^{-1}\right), \max\left(1, M^2\right)(\hat{\mathcal{C}}_2\delta^\alpha)^{-1}\right),$$

where $\hat{\mathcal{C}}_2 = \max_{x\in\overline{\mathcal{O}}} \mathcal{C}_2(|x-x_0|/\delta)$ with $\mathcal{C}_2$ is as in Corollary 3.1.

Proof. By homogeneity it is enough to consider those functions $f \in C^{0,\alpha}(\mathcal{O})$ satisfying $\|f\|_{C^{0,\alpha}(\mathcal{O})} = 1$.

Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be respectively as in (3.23) and (3.26). Let $u \in \mathscr{S}(\mathcal{O}, x_0, M, \eta, \delta/3)$ and $f \in C^{0,\alpha}(\mathcal{O})$ satisfying $\|f\|_{C^{0,\alpha}(\mathcal{O})} = 1$. Pick then $x \in \overline{\omega}$. From Corollary 3.1, we have

$$\mathcal{C}_2(|x-x_0|/\delta)\left(\frac{r}{\delta}\right)^{\mathcal{C}_1(|x-x_0|/\delta)} \le \|\nabla u\|_{L^2(B(x,r))}, \quad 0<r<\delta. \tag{3.28}$$

On the other hand, it is straightforward to check that

$$|f(x)| \le |f(y)| + r^\alpha, \quad y \in B(x,r).$$

Whence

$$|f(x)| \int_{B(x,r)} |\nabla u(y)|^2 dy \le \int_{B(x,r)} |f(y)||\nabla u(y)|^2 dy + r^\alpha \int_{B(x,r)} |\nabla u(y)|^2 dy.$$

That is we have

$$|f(x)|\|\nabla u\|^2_{L^2(B(x,r)} \le \|f|\nabla u|^2\|_{L^1(B(x,r))} + r^\alpha \|\nabla u\|^2_{L^2(B(x,r))}.$$

Since u is non constant, $\|\nabla u\|^2_{L^2(B(x,r))} \ne 0$ for any $0 < r < \delta$ by the unique continuation property. Therefore

$$|f(x)| \le \frac{\|f|\nabla u|^2\|_{L^1(B(x,r))}}{\|\nabla u\|^2_{L^2(B(x,r))}} + r^\alpha, \quad 0<r<\delta.$$

This and (3.28) entail

$$|f(x)| \le \mathcal{C}_2(|x-x_0|/\delta)^{-1}\left(\frac{\delta}{r}\right)^{\mathcal{C}_1(|x-x_0|)} \|f|\nabla u|^2\|_{L^1(B(x,r))} + r^\alpha, \quad 0<r<\delta.$$

Hence

$$|f(x)| \le \mathcal{C}_2(|x-x_0|/\delta)^{-1}\left(\frac{1}{s}\right)^{\mathcal{C}_1(|x-x_0|)} \|f|\nabla u|^2\|_{L^1(\mathcal{O})} + \delta^\alpha s^\alpha, \quad 0<s<1.$$

In consequence

$$\|f\|_{L^\infty(\omega)} \le \hat{\mathcal{C}}_2 \left(\frac{1}{s}\right)^{\hat{\alpha}} \|f|\nabla u|^2\|_{L^1(\mathcal{O})} + \delta^\alpha s^\alpha, \quad 0 < s < 1,$$

where $\hat{\alpha} = \max_{x\in\overline{\mathcal{O}}} \mathcal{C}_1(|x-x_0|/\delta)$.

This inequality leads to the expected one using a very classical argument.

□

4. Proof Theorem 1.1

Pick $(a,b), (\tilde{a},\tilde{b}) \in \mathcal{D}(\lambda,\kappa)$ and let $u_j = G_{a,b}(\cdot,\xi_j)$ and $\tilde{u}_j = G_{\tilde{a},\tilde{b}}(\cdot,\xi_j)$, $j=1,2$. By simple computations we can check that $w = u_2/u_1$ is the solution of the equation

$$\mathrm{div}(\sigma\nabla w) = 0 \quad \text{in } \mathbb{R}^n \setminus \{\xi_1,\xi_2\},$$

with

$$\sigma = au_1^2 = \frac{av_1^2}{b^2}.$$

Similarly, $\tilde{w} = \tilde{u}_2/\tilde{u}_1$ is the solution of the equation

$$\mathrm{div}(\tilde{\sigma}\nabla \tilde{w}) = 0 \quad \text{in } \mathbb{R}^n \setminus \{\xi_1,\xi_2\},$$

with

$$\tilde{\sigma} = \tilde{a}\tilde{u}_1^2 = \frac{\tilde{a}\tilde{u}_1^2}{\tilde{b}^2}.$$

We know from Lemma 2.4 that there exist $x^* \in B(\xi_2, |\xi_1-\xi_2|/2) \setminus \{\xi_2\}$, $\eta_0 = (n,\lambda,\kappa,|\xi_1-\xi_2|) > 0$ and $\rho = \rho(n,\lambda,\kappa,|\xi_1-\xi_2|) > 0$ so that $\overline{B}(x^*,\rho) \subset B(\xi_2, |\xi_1-\xi_2|/2) \setminus \{\xi_2\}$ and

$$\eta_0 \le \|\nabla w\|_{L^2(B(x^*,\rho))}. \tag{4.1}$$

Fix then a bounded domain $\mathcal{Q}$ of $\mathbb{R}^n \setminus \{\xi_1,\xi_2\}$ is such a way that $\Omega \cup B(x^*,\rho) \Subset \mathcal{Q}$, and set

$$\delta = \mathrm{dist}(\Omega \cup B(x^*,\rho), \partial\mathcal{Q}).$$

In the rest of this proof $\mathbf{d} = \mathrm{diam}(\mathcal{Q})$.

According to Corollary 2.3

$$\|\nabla w\|_{L^2(\mathcal{Q})} \le M = Ce^{c(\mathbf{d}+\varrho_+)} \left(1 + \max\left(\varrho_-^{-(2+\alpha)}, 1\right) \varrho_-^{-n+2}\right)^4, \tag{4.2}$$

with $C = C(n,\lambda,\kappa,\alpha,\theta)$ and $c = c(n,\lambda,\kappa,\alpha,\theta)$, $\varrho_- = \min\left(\mathrm{dist}\left(\xi_1,\mathcal{Q}\right), \mathrm{dist}\left(\xi_2,\mathcal{Q}\right)\right)$ and $\varrho_+ = \max\left(\mathrm{dist}\left(\xi_1,\mathcal{Q}\right), \mathrm{dist}\left(\xi_2,\mathcal{Q}\right)\right)$.

Now, since

$$\|\sigma\|_{C^{0,1}(\overline{Q})} \le \|a\|_{C^{0,1}(\overline{Q})} \|u_1\|^2_{C^{0,1}(\overline{Q})},$$

we get, similarly to the end of the proof of Corollary 2.3, from [15, Lemma 6.35, page 135]

$$\|\sigma\|_{C^{0,1}(\overline{Q})} \le C\|a\|_{C^{0,1}(\overline{Q})} \|u_1\|^2_{C^{2,\alpha}(\overline{Q})}.$$

Here $C = C(n,\lambda,\kappa,\mathbf{d},\xi_1,\xi_2)$.

This inequality together with Proposition 2.1 yield

$$\|\sigma\|_{C^{0,1}(\overline{Q})} \le C, \tag{4.3}$$

where $C = C(n,\lambda,\kappa,\mathbf{d},\xi_1,\xi_2)$.

On the other hand, we have from (2.11)

$$C^{-1}\min_{x\in\overline{\mathcal{Q}}}\frac{e^{-2\sqrt{c\kappa}|x-\xi_1|}}{|x-\xi_1|^{n-2}}\le u_1,\quad \text{in } \overline{\mathcal{Q}}, \tag{4.4}$$

with $c=c(n,\lambda)$ and $C=C(n,\lambda,\kappa)$.

We get by combining (4.3) and (4.4) that there exists $\varkappa=\varkappa(n,\lambda,\kappa,\alpha,\Omega,\xi_1,\xi_2)>1$ so that

$$\varkappa^{-1}\le\sigma\quad\text{and}\quad \|\sigma\|_{C^{0,1}(\overline{Q})}\le\varkappa.$$

Next, if $\rho\le\delta/3$ then (4.1) implies obviously

$$\eta_0\le\|\nabla w\|_{L^2(B(x_0,\delta/3))}, \tag{4.5}$$

with η as in (4.1).

When $\rho>\delta/3$ we can use the three-ball inequality in Theorem 3.1 in order to get

$$\tilde{C}\|\nabla w\|_{L^2(B(x^*,\rho))}\le\|\nabla w\|_{L^2(B(x_0,\delta/3))}^{s}\|\nabla w\|_{L^2(B(x^*,\rho+\delta/3))}^{1-s},$$

where $\tilde{C}=\tilde{C}(n,\lambda,\kappa,\Omega,\xi_1,\xi_2)$ and $0<s=s(n,\lambda,\kappa,\Omega,\xi_1,\xi_2)<1$. Whence

$$(\tilde{C}\eta_0)^{1/s}M^{(s-1)/s}\le\|\nabla w\|_{L^2(B(x_0,\delta/3))}. \tag{4.6}$$

In light of (4.2), (4.5) and (4.6), we can infer that, for some $\eta=\eta(n,\lambda,\kappa,\Omega,\xi_1,\xi_2)$, $w\in\mathscr{S}(\mathcal{Q},x^*,M,\eta,\delta/3)$, where M is as in (4.2) and $\mathscr{S}(\mathcal{Q},x^*,M,\eta,\delta/3)$ is defined in (3.3).

Lemma 4.1. *We have*

$$C\|(\sigma-\tilde{\sigma})|\nabla w|^2\|_{L^1(\Omega)}\le\|w-\tilde{w}\|_{L^2(\Omega)}^{\theta/(2+\theta)}+\|\sigma-\tilde{\sigma}\|_{L^\infty(\Gamma)}, \tag{4.7}$$

with $C=C(n,\lambda,\kappa,\Omega,\alpha,\theta,\xi_1,\xi_2)>0$.

Proof. Clearly, if $\zeta=\sigma-\tilde{\sigma}$ and $u=w-\tilde{w}$, then

$$\mathrm{div}(\tilde{\sigma}\nabla u)=\mathrm{div}(\zeta\nabla w).$$

Recall that sgn_0 is the sign function defined on $\mathbb{R}$ by: $\mathrm{sgn}_0(t)=-1$ if $t<1$, $\mathrm{sgn}_0(0)=0$ and $\mathrm{sgn}_0(t)=1$ if $t>0$. Since

$$\begin{aligned}\mathrm{div}(|\zeta|\nabla w)&=\nabla|\zeta|\cdot\nabla w+|\zeta|\Delta w\\&=\mathrm{sgn}_0(\zeta)\nabla\zeta\cdot\nabla w+\mathrm{sgn}_0(\zeta)\zeta\Delta w\\&=\mathrm{sgn}_0(\zeta)\mathrm{div}(\zeta\nabla w)=\mathrm{sgn}_0(\zeta)\mathrm{div}(\tilde{\sigma}\nabla u),\end{aligned}$$

we get by integrating by parts

$$\begin{aligned}\int_\Omega|\zeta||\nabla w|^2dx&=-\int_\Omega\mathrm{div}(|\zeta|\nabla w)wdx+\int_\Gamma|\zeta|w\partial_\nu wdS(x)\\&=-\int_\Omega\mathrm{sgn}_0(\zeta)\mathrm{div}(\tilde{\sigma}\nabla u)wdx+\int_\Gamma|\zeta|w\partial_\nu wdS(x).\end{aligned} \tag{4.8}$$

Thus

$$\int_\Omega|\zeta||\nabla w|^2dx\le C\left(\|u\|_{H^2(\Omega)}+\|\zeta\|_{L^\infty(\Gamma)}\right).$$

This, the following interpolation inequality

$$\|u\|_{H^2(\Omega)}\le c_\Omega\|u\|_{L^2(\Omega)}^{\theta/(2+\theta)}\|u\|_{H^{2+\theta}(\Omega)}^{2/(2+\theta)}$$

and Corollary 2.3 give (4.7). □

We have from (3.27) in Lemma 3.7

$$\|\tilde{\sigma}-\sigma\|_{C(\overline{\Omega})} \le \hat{\mathcal{C}}_3 \|\tilde{\sigma}-\sigma\|^{1-\hat{\mu}}_{C^{0,\alpha}(\overline{\Omega})} \|(\sigma-\tilde{\sigma})|\nabla w|^2\|^{\hat{\mu}}_{L^1(\Omega)},$$

from which we obtain

$$\|\tilde{\sigma}-\sigma\|_{C(\overline{\Omega})} \le \hat{\mathcal{C}}_3 \max\left(1, \|\tilde{\sigma}-\sigma\|_{C^{0,\alpha}(\overline{\Omega})}\right) \|(\sigma-\tilde{\sigma})|\nabla w|^2\|^{\hat{\mu}}_{L^1(\Omega)}.$$

Combined with Proposition 2.1, this inequality gives

$$\|\tilde{\sigma}-\sigma\|_{C(\overline{\Omega})} \le C \|(\sigma-\tilde{\sigma})|\nabla w|^2\|^{\hat{\mu}}_{L^1(\Omega)}.$$

Here and henceforward, $C = C(n, \lambda, \kappa, \Omega, \alpha, \theta, \xi_1, \xi_2) > 0$ is a generic constant.

Therefore, we obtain in light of Lemma 4.1

$$\|\tilde{\sigma}-\sigma\|_{C(\overline{\Omega})} \le C \left(\|w-\tilde{w}\|^{\theta/(2+\theta)}_{L^2(\Omega)} + \|\sigma-\tilde{\sigma}\|_{C(\Gamma)} \right)^{\hat{\mu}}.$$

Since $\tilde{a} = a$ and $\tilde{b} = b$ on Γ and regarding the regularity of u_i and $\tilde{u}_i$, $i = 1, 2$, we finally get

$$\|\tilde{\sigma}-\sigma\|_{C(\overline{\Omega})} \le C \left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2-\tilde{v}_2\|_{C(\overline{\Omega})} \right)^{\hat{\mu}_0}, \tag{4.9}$$

with

$$\hat{\mu}_0 = \frac{\theta\hat{\mu}}{2+\theta}.$$

The following lemma will be used in sequel.

Lemma 4.2. *There exist two constants* $0 < \hat{\mu}_1 = \hat{\mu}_1(n, \Omega, \lambda, \kappa, \alpha, \theta, \xi_1, \xi_2) < 1$ *and* $C = C(n, \Omega, \lambda, \kappa, \alpha, \theta, \xi_1, \xi_2) > 0$ *so that*

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\alpha}(\overline{\Omega})} \le C \left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})} + \|v_2-\tilde{v}_2\|_{C(\overline{\Omega})} \right)^{\hat{\mu}_1}. \tag{4.10}$$

Proof. In this proof $C = C(n, \Omega, \lambda, \kappa, \alpha, \theta, \xi_1, \xi_2) > 0$ is a generic constant.

It is not hard to check that

$$-\operatorname{div}(\sigma\nabla u_1^{-1}) = v_1 \quad \text{in } \Omega,$$
$$-\operatorname{div}(\tilde{\sigma}\nabla \tilde{u}_1^{-1}) = \tilde{v}_1 \quad \text{in } \Omega.$$

Hence

$$-\operatorname{div}(\sigma\nabla(u_1^{-1}-\tilde{u}_1^{-1})) = (v_1-\tilde{v}_1) + \operatorname{div}((\sigma-\tilde{\sigma})\nabla\tilde{u}_1^{-1}) \quad \text{in } \Omega.$$

By the usual Hölder a priori estimate (see [15, Theorem 6.6, page 98])

$$C\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\alpha}(\overline{\Omega})} \le \|v_1-\tilde{v}_1\|_{C^{0,\alpha}(\overline{\Omega})} + \|\operatorname{div}((\sigma-\tilde{\sigma})\nabla\tilde{u}_1^{-1})\|_{C^{0,\alpha}(\overline{\Omega})} + \|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{0,\alpha}(\Gamma)}.$$

Consequently

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\alpha}(\overline{\Omega})} \le C \left(\|v_1-\tilde{v}_1\|_{C^{0,\alpha}(\overline{\Omega})} + \|\sigma-\tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})} \right), \tag{4.11}$$

where we used that

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{0,\alpha}(\Gamma)} = \|b(v_1^{-1}-\tilde{v}_1^{-1})\|_{C^{0,\alpha}(\Gamma)}.$$

On the other hand, since

$$\|\sigma-\tilde{\sigma}\|_{C^{1,1}(\overline{\Omega})} \le C, \quad \|v_1-\tilde{v}_1\|_{C^{1,\alpha}(\overline{\Omega})} \le C$$

and Ω is $C^{1,1}$, we get again from the interpolation inequality in [15, Lemma 6.35, page 135]

$$\|\sigma-\tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})}\le C\|\sigma-\tilde{\sigma}\|_{C(\overline{\Omega})}^{\tau},\quad \|v_1-\tilde{v}_1\|_{C^{0,\alpha}(\overline{\Omega})}\le C\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}^{\tau}, \tag{4.12}$$

where $0<\tau=\tau(\Omega,\alpha)<1$ is a constant.

Inequality (4.15) in (4.11) yields

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\alpha}(\overline{\Omega})}\le C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}^{\tau}+\|\sigma-\tilde{\sigma}\|_{C(\overline{\Omega})}^{\tau}\right). \tag{4.13}$$

On the other hand we have from (4.9)

$$\|\tilde{\sigma}-\sigma\|_{C(\overline{\Omega})}\le C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\hat{\mu}_0}. \tag{4.14}$$

Whence, we get in light of inequalities (4.13) and (4.14), where $\hat{\mu}_1=\tau\hat{\mu}_0$,

$$\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{2,\alpha}(\overline{\Omega})}\le C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\hat{\mu}_1}.$$

This is the expected inequality. □

Also, since

$$\|\sigma-\tilde{\sigma}\|_{C^{1,1}(\overline{\Omega})}\le C,\quad \|v_1-\tilde{v}_1\|_{C^{2,\alpha}(\overline{\Omega})}\le C,$$

we can proceed as in the preceding proof to get

$$\|\sigma-\tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})}\le C\|\sigma-\tilde{\sigma}\|_{C(\overline{\Omega})}^{\tau},\quad \|v_1-\tilde{v}_1\|_{C^{1,\alpha}(\overline{\Omega})}\le C\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}^{\tau}, \tag{4.15}$$

the constant $0<\tau=\tau(\Omega,\alpha)<1$.

But

$$\begin{aligned} a-\tilde{a}=\sigma u_1^{-2}-\tilde{\sigma}\tilde{u}_1{}^{-2}&=(\sigma-\tilde{\sigma})u_1^{-2}+\tilde{\sigma}(u_1^{-2}-\tilde{u}_1^{-2})\\ &=(\sigma-\tilde{\sigma})u_1^{-2}+\tilde{\sigma}(u_1^{-1}+\tilde{u}_1^{-1})(u_1^{-1}-\tilde{u}_1^{-1}). \end{aligned}$$

Hence

$$\|a-\tilde{a}\|_{C^{1,\alpha}(\overline{\Omega})}\le C\left(\|u_1^{-1}-\tilde{u}_1^{-1}\|_{C^{1,\alpha}(\overline{\Omega})}+\|\sigma-\tilde{\sigma}\|_{C^{1,\alpha}(\overline{\Omega})}\right). \tag{4.16}$$

This inequality together with (4.9), (4.10) and (4.15) entail

$$\|a-\tilde{a}\|_{C^{1,\beta}(\overline{\Omega})}\le C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\hat{\mu}_1}. \tag{4.17}$$

We can proceed similarly for $b-\tilde{b}$. Since

$$b-\tilde{b}=v_1u_1^{-1}-\tilde{v}_1\tilde{u}_1^{-1}=(v_1-\tilde{v}_1)u_1^{-1}+\tilde{v}_1(u_1^{-1}-\tilde{u}_1^{-1}),$$

we have

$$\|b-\tilde{b}\|_{C^{0,\beta}(\overline{\Omega})}\le C\left(\|v_1-\tilde{v}_1\|_{C(\overline{\Omega})}+\|v_2-\tilde{v}_2\|_{C(\overline{\Omega})}\right)^{\hat{\mu}_1}. \tag{4.18}$$

The expected inequality follows by putting together (4.17) and (4.18).

Appendix A. Proof of technical lemmas

Proof of Lemma 2.2. In this proof $C = C(n, \mu, \nu) > 1$ is a generic constant.

For a given constant $\nu > 0$, it is well known that $G_{1,\nu}$, the fundamental solution of the operator $-\Delta + \nu$, is given by $G_{1,\nu}(x, \xi) = \mathcal{G}_{1,\nu}(x - \xi)$, $x, \xi \in \mathbb{R}^n$, with

$$\mathcal{G}_{1,\nu}(x) = (2\pi)^{-n/2}(\sqrt{\nu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|).$$

In the particular case $n = 3$, we have $K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z}$ and therefore

$$\mathcal{G}_{1,\nu}(x) = \frac{e^{-\sqrt{\nu}|x|}}{4\pi|x|},$$

in dimension three.

Let $f \in C_0^\infty(\mathbb{R}^n)$, $\mu > 0$, and $\nu > 0$ be two constants, and denote by u the solution of the equation

$$(-\mu\Delta + \nu)u = f \quad \text{in } \mathbb{R}^n.$$

Then

$$u(x) = \int_{\mathbb{R}^n} G_{\mu,\nu}(x, \xi)f(\xi)d\xi, \quad x \in \mathbb{R}^n. \tag{A.1}$$

We remark that $v(x) = u(\sqrt{\mu}x)$, $x \in \mathbb{R}^n$ satisfies $(-\Delta + \nu)v = f(\sqrt{\mu}\,\cdot)$. Whence

$$\begin{aligned} u(\sqrt{\mu}x) = v(x) &= \int_{\mathbb{R}^n} \mathcal{G}_{1,\kappa}(x - \xi)f(\sqrt{\mu}\xi)d\xi \\ &= \mu^{-n/2}\int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}(x - \xi/\sqrt{\mu})f(\xi)d\xi, \quad x \in \mathbb{R}^n. \end{aligned}$$

Hence

$$u(x) = \mu^{-n/2}\int_{\mathbb{R}^n} \mathcal{G}_{1,\nu}((x - \xi)/\sqrt{\mu})f(\xi)d\xi, \quad x \in \mathbb{R}^n. \tag{A.2}$$

Comparing (A.1) and (A.2) we find

$$G_{\mu,\nu}(x, \xi) = \mu^{-n/2}\mathcal{G}_{1,\nu}((x - \xi)/\sqrt{\mu}), \quad x, \xi \in \mathbb{R}^n.$$

Consequently $G_{\mu,\nu}(x, \xi) = \mathcal{G}_{\mu,\nu}(x - \xi)$ with

$$\mathcal{G}_{\mu,\nu}(x) = (2\pi\mu)^{-n/2}(\sqrt{\nu\mu}/|x|)^{n/2-1}K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}), \quad x \in \mathbb{R}^n. \tag{A.3}$$

By the usual asymptotic formula for modified Bessel functions of the second kind (see for instance [5, 9.7.2, page 378]) we have, when $|x| \to \infty$,

$$K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) = \left(\frac{\pi\sqrt{\mu}}{2\sqrt{\nu}|x|}\right)^{1/2} e^{-\sqrt{\nu}|x|/\sqrt{\mu}}\left(1 + O(1/|x|)\right),$$

where $O(1/|x|)$ only depends on n, μ and ν.

Consequently, there exits $R = R(n, \mu, \nu) > 0$ so that

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} \le K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \le C\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}}, \quad |x| \ge R. \tag{A.4}$$

Substituting if necessary R by $\max(R, 1)$, we have

$$\frac{1}{|x|^{n/2-1}} \le \frac{1}{|x|^{1/2}}, \quad |x| \ge R. \tag{A.5}$$

Moreover, we have

$$\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} = \left[|x|^{(n-3)/2}e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}\right]\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R.$$

Since the function $x \to |x|^{(n-3)/2}e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}$ is bounded in $\mathbb{R}^n$, we deduce

$$\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{1/2}} \leq C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R. \tag{A.6}$$

Using (A.5) and (A.6) in (A.4) in order to obtain

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n/2-1}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\mu})}}{|x|^{n/2-1}}, \quad |x| \geq R. \tag{A.7}$$

We now establish a similar estimate when $|x| \to 0$. To this end we recall that according to formula [5, 9.6.9, page 375] we have

$$K_{n/2-1}(\rho) \sim \frac{1}{2}\Gamma(n/2-1)\left(\frac{2}{\rho}\right)^{n/2-1} \quad \text{as } \rho \to 0,$$

from which we deduce in a straightforward manner that there exists $0 < r \leq R$ so that

$$C^{-1}\frac{e^{-\sqrt{\nu}|x|/\sqrt{\mu}}}{|x|^{n/2-1}} \leq K_{n/2-1}(\sqrt{\nu}|x|/\sqrt{\mu}) \leq C\frac{e^{-\sqrt{\nu}|x|/(2\sqrt{\nu})}}{|x|^{n/2-1}}, \quad |x| \leq r. \tag{A.8}$$

The expected two sided inequality (2.10) follows by combining (A.4), (A.7) and (A.8). □

Proof of Lemma 2.3. Let $\mathcal{Q}$ be an open subset of $\mathbb{R}^n$, set $d = \mathrm{diam}(\mathcal{Q})$, $d_x = \mathrm{dist}(x, \partial\mathcal{Q})$ and $d_{x,y} = \min(d_x, d_y)$.

We introduce the following weighted Hölder semi-norms and Hölder norms, where $\sigma \in \mathbb{R}$, $0 < \gamma \leq 1$, and k is non-negative integer,

$$[w]^{(\sigma)}_{k,0;\mathcal{Q}} = [w]^{(\sigma)}_{k,\mathcal{Q}} = \sup_{x\in\mathcal{Q},\ |\beta|=k} d_x^{k+\sigma}|\partial^\beta w(x)|,$$

$$[w]^{(\sigma)}_{k,\gamma;\mathcal{Q}} = \sup_{x,y\in\mathcal{Q},\ |\beta|=k} d_{x,y}^{k+\gamma+\sigma}\frac{|\partial^\beta w(y) - \partial^\beta w(x)|}{|y-x|^\alpha},$$

$$|w|^{(\sigma)}_{k;\mathcal{Q}} = \sum_{j=0}^{k}[w]^{(\sigma)}_{j;\mathcal{Q}},$$

$$|w|^{(\sigma)}_{k,\gamma;\mathcal{Q}} = |w|^{(\sigma)}_{k;\mathcal{Q}} + [w]^{(\sigma)}_{k,\gamma;\mathcal{Q}}.$$

In term of these notations, we have

$$|a|^{(0)}_{0,\alpha;\mathcal{Q}} = \sup_{x\in\mathcal{Q}}|a(x)| + \sup_{x,y\in\mathcal{Q}} d^\alpha_{x,y}\frac{|a(y)-a(x)|}{|y-x|^\alpha} \leq (1+\mathbf{d})\lambda,$$

$$|\partial_j a|^{(1)}_{0,\alpha;\mathcal{Q}} = \sup_{x\in\mathcal{Q}} d_x|\partial_j a(x)| + \sup_{x,y\in\mathcal{O}} d^{1+\alpha}_{x,y}\frac{|\partial_j a(y)-\partial_j a(x)|}{|y-x|^\alpha} \leq \mathbf{d}(1+\mathbf{d}^\alpha)\lambda,$$

$$|b|^{(2)}_{0,\alpha;\mathcal{Q}} = \sup_{x\in\mathcal{O}} d_x^2|b(x)| + \sup_{x,y\in\mathcal{Q}} d^{2+\gamma}_{x,y}\frac{|b(y)-b(x)|}{|y-x|^\alpha} \leq \mathbf{d}^2(1+\mathbf{d}^\alpha)\lambda.$$

In consequence

$$|a|^{(0)}_{0,\alpha;\mathcal{Q}} + |\partial_j a|^{(1)}_{0,\alpha;\mathcal{Q}} + |b|^{(2)}_{0,\alpha;\mathcal{Q}} \leq \Lambda(\mathbf{d}) = [1 + (\mathbf{d}+\mathbf{d}^2)(1+\mathbf{d}^\alpha)]\lambda. \tag{A.9}$$

Following [15] we define also

$$[w]^*_{k,0;\mathcal{Q}} = [w]^*_{k,\mathcal{O}} = \sup_{x\in\mathcal{Q},\ |\beta|=k} d_x^k|\partial^\beta w(x)|,$$

$$[w]^*_{k,\gamma;\mathcal{Q}} = \sup_{x,y\in\mathcal{Q},\ |\beta|=k} d_{x,y}^{k+\alpha}\frac{|\partial^\beta w(y)-\partial^\beta w(x)|}{|y-x|^\gamma},$$

$$|w|^*_{k;\mathcal{Q}} = \sum_{j=0}^k [w]^*_{j;\mathcal{Q}},$$

$$|w|^*_{k,\gamma;\mathcal{Q}} = |w|^*_{k;\mathcal{Q}} + [w]^*_{k,\gamma;\mathcal{O}}.$$

From [15, Lemma 6.32, page 130] and its proof we have the following interpolation inequalities: suppose that j and k, non negative integers, and $0\le \beta,\gamma\le 1$ are so that $j+\beta<k+\gamma$. Then there exist $C=C(n,\alpha,\beta)>0$ and $\kappa=\kappa(\alpha,\beta)$ so that, for any $w\in C^{k,\alpha}(\mathcal{Q})$ and $\epsilon>0$, we have

$$[w]^*_{j,\beta;\mathcal{Q}} \le C\epsilon^{-\kappa}|w|_{0;\mathcal{Q}} + \epsilon[w]^*_{k,\gamma;\mathcal{Q}}, \tag{A.10}$$

$$|w|^*_{j,\beta;\mathcal{Q}} \le C\epsilon^{-\kappa}|w|_{0;\mathcal{Q}} + \epsilon[w]^*_{k,\gamma;\mathcal{Q}}. \tag{A.11}$$

Here $|w|_{0;\mathcal{Q}} = \sup_{x\in\mathcal{Q}}|w(x)|$.

Checking carefully the proof of interior Schauder estimates in [15, Theorem 6.2, page 90], we get, taking into account inequalities (A.9)-(A.11), the following result: there exist a constant $C=C(n)>0$ and $\kappa=\kappa(\alpha)$ so that, for any $\mu\le 1/2$ and $w\in C^{k,\alpha}(\mathcal{Q})$ satisfying $L_{a,b}w=0$ in $\mathcal{Q}$, we have

$$[w]^*_{2,\alpha,\mathcal{Q}} \le C\Lambda(\mathbf{d})\left(\mu^{-\kappa}|w|_{0;\mathcal{Q}} + \mu^\alpha[w]^*_{2,\alpha,\mathcal{Q}}\right). \tag{A.12}$$

Substituting in (A.12) C by $\max(C,2^{\alpha-1})$, we may assume that in (A.12), $C=C(n,\alpha)\ge 2^{\alpha-1}$. Bearing in mind that $\Lambda(\mathbf{d})>1$, we can take in (A.12), $\mu=(2C\Lambda(\mathbf{d}))^{-1/\alpha}$. We find

$$[w]^*_{2,\alpha,\mathcal{Q}} \le C\Lambda(\mathbf{d})^\varkappa|w|_{0;\mathcal{Q}}, \tag{A.13}$$

for some constants $C=C(n,\alpha)>0$ and $\varkappa=\varkappa(\alpha)>1$.

Using again interpolation inequalities (A.10) and (A.11), we deduce that

$$|w|^*_{2,\alpha,\mathcal{Q}} \le C\Lambda(\mathbf{d})^\varkappa|w|_{0;\mathcal{Q}}. \tag{A.14}$$

Let $\delta>0$ be so that $\mathcal{Q}_\delta=\{x\in\mathcal{Q};\ \mathrm{dist}(x,\partial\mathcal{Q})>\delta\}$ is nonempty. If $\mathcal{Q}'$ is an open subset of $\mathcal{Q}_\delta$ then (A.14) yields in a straightforward manner

$$\|w\|_{C^{2,\alpha}\left(\overline{\mathcal{Q}'}\right)} \le C\max\left(\delta^{-(2+\alpha)},1\right)\Lambda(\mathbf{d})^\varkappa|w|_{0;\mathcal{Q}}.$$

This is the expected inequality. □

Lemma A.1. *Let K be a compact subset of $\mathbb{R}^n$ and $f\in C^{2,\alpha}(K)$ satisfying $\min_K|f|\ge c_->0$. Then*

$$\|1/f\|_{C^{2,\alpha}(K)} \le Cc_+^4\left(1+\|f\|_{C^{2,\alpha}(K)}\right)^3, \tag{A.15}$$

where $c_+=\max(1,c_-^{-1})$ and $C=C(\mathrm{diam}(K))$ is a constant.

Proof. Let $x,y\in K$. Using $|1/f|_{0;K}\le c_+$ and the following identities

$$\frac{1}{f^2(y)}-\frac{1}{f^2(x)}=\left(\frac{1}{f(x)f^2(y)}+\frac{1}{f(x)^2f(y)}\right)(f(x)-f(y)),$$
$$\frac{1}{f^3(y)}-\frac{1}{f^3(x)}=\left(\frac{1}{f(x)f^3(y)}+\frac{1}{f^2(x)f^2(y)}+\frac{1}{f(x)^3f(y)}\right)(f(x)-f(y)),$$

we easily get

$$[1/f^j]_{\alpha;K}\le 3c_+^4[f]_{\alpha;K},\quad j=2,3. \tag{A.16}$$

Also, we have

$$\frac{\partial_i f(y)\partial_j f(x)}{f^3(y)}-\frac{\partial_i f(y)\partial_j f(x)}{f^3(x)}=\frac{\partial_i f(y)}{f^3(y)}(\partial_j f(y)-\partial_j f(x))$$
$$+\frac{\partial_j f(x)}{f^3(y)}(\partial_i f(y)-\partial_i f(x))+\left(\frac{1}{f^3(y)}-\frac{1}{f^3(x)}\right)(\partial_i f(y)\partial_j f(x)).$$

In light of (A.16), this identity yields

$$[\partial_i f\partial_j f/f^3]_{\alpha;K}\le c_+^4\left([\partial_i f]_{\alpha;K}|\partial_j f|_{0;K}\right.$$
$$\left.+[\partial_j f]_{\alpha;K}|\partial_i f|_{0;K}+[f]_{\alpha;K}|\partial_i f|_{0;K}|\partial_j f|_{0;K}\right). \tag{A.17}$$

On the other hand, since

$$\frac{\partial_{ij}^2 f(y)}{f^2(y)}-\frac{\partial_{ij}^2 f(x)}{f^2(x)}=\frac{1}{f^2(y)}(\partial_{ij}^2 f(y)-\partial_{ij}^2 f(x))+\left(\frac{1}{f^2(y)}-\frac{1}{f^2(y)}\right)\partial_{ij}^2 f(x),$$

we find, by using again (A.16),

$$[\partial_{ij}^2 f/f^2]_{\alpha;K}\le 3c_+^4\left([\partial_{ij}^2 f]_{\alpha;K}+[f]_{\alpha;K}|\partial_{ij}^2 f|_{0,K}\right). \tag{A.18}$$

Inequalities (A.17), (A.18), the identity $\partial_{ij}^2(1/f)=2\partial_i f\partial_j f/f^3-\partial_{ij}^2 f/f^2$ and the interpolation inequality [15, Lemma 6.35, page 135] (by proceeding as in Corollary 2.2) imply

$$[\partial_{ij}^2(1/f)]_{\alpha,K}\le Cc_+^4\left(1+\|f\|_{C^{2,\alpha}(K)}\right)^3, \tag{A.19}$$

with $C=C(\mathrm{diam}(K))$ is a constant.

The other terms for $1/f$ appearing in the norms $\|\cdot\|_{C^{2,\alpha}(K)}$ can be estimated similarly to the semi-norm in (A.19). Inequality (A.15) then follows. □

Lemma A.2. *$C^{2,\alpha}(\overline{\mathcal{O}})$ is continuously embedded in $H^{2+\theta}(\mathcal{O})$. Furthermore, there exists $C=C(n,\alpha-\theta)$ so that, for any $w\in C^{2,\alpha}(\overline{\mathcal{O}})$, we have*

$$\|w\|_{H^{2+\theta}(\mathcal{O})}\le C\max\left(\mathbf{d}^{n/2},\mathbf{d}^{n/2+\alpha-\theta}\right)\|w\|_{C^{2,\alpha}(\overline{\mathcal{O}})}, \tag{A.20}$$

where $\mathbf{d}=\mathrm{diam}(\mathcal{O})$.

Proof. Let $w\in C^{2,\alpha}(\overline{\mathcal{O}})$ and, for fixed $1\le i,j\le n$, set $g=\partial_{ij}^2 w$. Then

$$\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{|g(x)-g(y)|^2}{|x-y|^{n+2\theta}}dxdy\le [g]_{\alpha;\mathcal{O}}^2\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{1}{|x-y|^{n-2(\alpha-\theta)}}dxdy.$$

In light of [10, Lemma A4, page 246], this inequality yields

$$\int_{\mathcal{O}}\int_{\mathcal{O}}\frac{|g(x)-g(y)|^2}{|x-y|^{n+2\theta}}dxdy\le\frac{|\mathbb{S}^{n-1}||\mathcal{O}|\mathbf{d}^{2(\alpha-\theta)}}{2(\alpha-\theta)}[g]_{\alpha;\mathcal{O}}^2,$$

But $|\mathcal{O}| \le |B(0,\mathbf{d})|$. Hence

$$\int_{\mathcal{O}}\int_{\mathcal{O}} \frac{|g(x)-g(y)|^2}{|x-y|^{n+2\theta}}dxdy \le \frac{|\mathbb{S}^{n-1}|^2\mathbf{d}^{n+2(\alpha-\theta)}}{2(\alpha-\theta)}[g]^2_{\alpha;\mathcal{O}}. \tag{A.21}$$

Using (A.21) and the inequality

$$\|h\|^2_{L^2(\mathcal{O})} \le |\mathbb{S}^{n-1}|\mathbf{d}^n|h|_{0,\mathcal{O}}, \quad h \in C(\overline{\mathcal{O}}),$$

we get from the definition of the norm of H^s-spaces in [16, formula (1.3.2.2), page 17]

$$\|w\|_{H^{2+\theta}(\mathcal{O})} \le C\max\left(\mathbf{d}^{n/2}, \mathbf{d}^{n/2+\alpha-\theta}\right)\|w\|_{C^{2,\alpha}(\overline{\mathcal{O}})},$$

for some constant $C = C(n, \alpha-\theta)$. This is the expected inequality □

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