ENHANCEMENT OF ELECTROMAGNETIC FIELDS CAUSED BY INTERACTING SUBWAVELENGTH CAVITIES

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Abstract. This article is devoted to the asymptotic analysis of the electromagnetic fields scattered by a perfectly conducting plane containing two sub-wavelength rectangular cavities separated by a sub-wavelength distance. The problem is formulated through an integral equation, and a spectral analysis of the integral operator is performed. Using the generalized Rouché theorem on operator valued functions, it is possible to localize two types of resonances, symmetric and anti-symmetric, in a neighborhood of each zero of some explicit function, associated to the limiting geometry. For the symmetric modes, the fields in the cavities interact in phase, and the system of 2 cavities essentially acts as a dipole. In the anti-symmetric case, the fields oscillate in anti-phase, and the system behaves like a quadripole. Asymptotic expansions of the resonances, the far-field and the near-field are given.

Keywords: Resonances, generalized Rouché theorem, integral operator, asymptotic expansion

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1. Introduction

The interaction of light and rough metallic surfaces may give rise to fascinating phenomena, such as transmission of light through subwavelength apertures, or such as Surface Enhancement Raman Scattering [5, 12, 11, 8]. The optical excitation of resonant modes may lead to an energy concentration and localization in volumes much smaller than λ^3, where λ is the wavelength of the incident light. Potential applications are numerous, in particular for near field microscopy.

We are interested in metallic surfaces that contain parallel slits, which have been studied experimentally in [5, 12]. Our objective is to understand the role played by the rugosity of the surfaces in the creation of resonant modes. The case of planar devices with rectangular cavities is particularly interesting. On the one hand, they can be manufactured with controled precision by current lithographic processes at the appropriate scales, and are widely used in opto-electronics. On the other hand, their simple geometry allows one to develop the mathematical analysis very far. Experimental results suggest that the amplification factors of the fields depend on the width of the cavities. In [6], the case of a half plane containing a single cavity of width w was considered. The authors studied the asymptotic of the Green function as $w \to 0$ using techniques based on integral representatio [4, 6, 3], which are well adapted to such geometry. The limiting Green function turns out to be that of an infinite half plane on which a dipole is placed. The present paper extends the analysis of [6] to a two sub-wavelength cavities separated by a sub-wavelength distance, and studies the interaction between the cavities.

Let us briefly summarize our analysis and the main results. Due to geometrical considerations and the choice of a time harmonic incident field [5, 6], the scattering problem can be reduced to a Helmholtz equation. By Green formulae we reformulate the Helmholtz equation as a system of integral equations (2.14) defined on the aperture of the two cavities, and satisfied by normal derivatives of the solution u. The operator-valued matrix $\mathcal{S}_w(k)$ associated to the system depends on the width w of the cavities and the frequency k of the incident field. The kernels of the integral operators are formed by Green functions of the Helmholtz operator in the rectangular cavities and in the upper half plane. It is thus possible to derive a rigorous asymptotic expansion of $\mathcal{S}_w(k)$ with respect to w (Lemma 3.2), and to ensure the invertibility of its dominant term in suitable fractional Sobolev spaces (see Corollary 4.5). Based on Fredholm theory we then prove that the scattering resonances of the two cavities are exactly the poles of $\mathcal{S}_w^{-1}(k)$. Using the generalized Rouché Theorem for meromorphic operator-valued functions

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(see [9] and the recent monograph [3]), and the asymptotic expansion of $\mathcal{S}_w^{-1}(k)$, we first localize the resonances in the lower half complex plane, and then derive their asymptotic (Theorem 4.12). Precisely, we prove the existence of two types of resonances. The first type corresponds to a symmetrical mode with an imaginary part of order $O(w)$, and the second type corresponds to an antisymmetric mode with a smaller imaginary part, of order $O(w^3)$, which leads to a much stronger electromagnetic enhancement (see Theorem 4.13). The derived partition of the set of resonances results from near-field coupling of the cavities, and confirms the experimental results observed in [11]. Using again the generalized Rouché Theorem we derive the asymptotic expansion of the field u far from (Theorem 5.1) and close to (Theorem 5.4) the resonances. When the frequency is far from the set of resonances, the field is essentially the same as in the case of a single cavity (see [6]). However, when it is close to the resonances, the radiation pattern strongly depends on which mode is excited. Indeed, if the symmetric mode is active, then the scattered field u behaves asymptotically like that of an infinite half plane on which a dipole is placed, as in the case of single cavity. When the anti-symmetric mode is excited, the singularity is that of a quadripole. Finally, we perform an asymptotic expansion of the field inside the cavities (see Remark 5.7). In particular, we show that the field u actually concentrates in the cavities when the frequency is close to the resonances. Moreover we prove that close to the symmetrical mode, the field is oriented in the same direction in both cavities, while close to the anti-symmetrical mode it is oriented in two opposite directions.
The paper is organized as follows, we state the scattering problem in section 2, and reformulate it as an integral equation. Section 3 is devoted to the asymptotic expansion of $\mathcal{S}_w(k)$ as w goes to zero. In section 4, we derive useful qualitative properties of the operator-valued function $k \to \mathcal{S}_w(k)$, as well as the asymptotic expansion of $\mathcal{S}_w^{-1}(k)$ and the scattering resonances when w is close to zero. Based on the previous results we give in section 5, the asymptotic expansion of the scattered field in different regions of the scattering domain. Finally, in the appendix, we recall some results of Ghoberg and Sigal [9] on the operator version of the Residue theorem.

2. Problem formulation

Let $\Omega \subset \mathbb{R}^2$ be the domain defined by

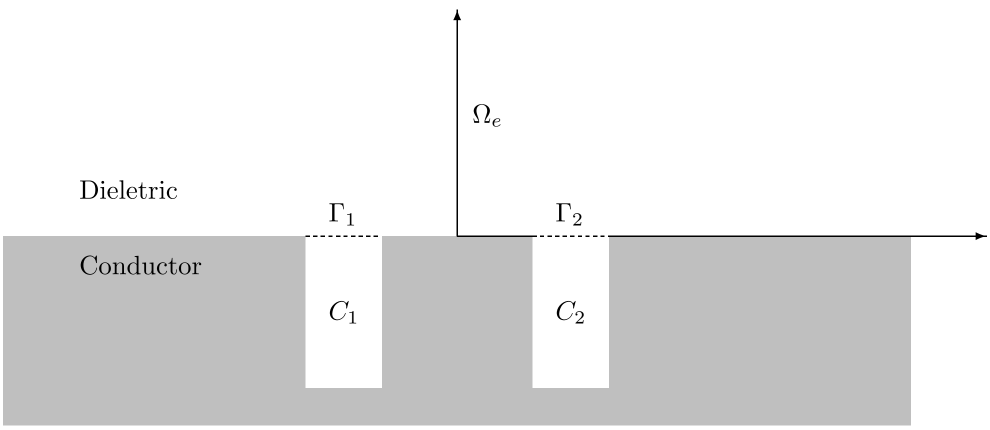
$$\Omega \quad := \quad \Omega_e \cup C_1 \cup C_2 \cup \Gamma_1 \cup \Gamma_2,$$

where Ω_e is the upper half-plane $\mathbb{R}^2_+$, $C_i,\ i = 1, 2$, are the rectangular cavities:

$$C_1 := \big(-(d+1)w, -(d-1)+w\big) \times \big(-h, 0\big), \qquad C_2 := \big((d-1)w, (d+1)\big) \times \big(-h, 0\big),$$

and Γ_i denotes the aperture of $C_i,\ i = 1, 2$, i.e.,

$$\Gamma_1 \quad := \quad \big(-(d+1)w, -(d-1)w\big) \times \{0\}, \quad \Gamma_2 \quad := \quad \big((d-1)w, (d+1)w\big) \times \{0\}.$$



The cavities C_i, $i = 1, 2$ are illuminated by a source $f(x) \in L^2(\Omega_e)$ with compact support in Ω_e. In the harmonic regime, and under the same assumptions as in [6, 12], the Maxwell equations that govern the propagation of electromagnetic fields in the scattering domain Ω can be reduced to the following Helmholtz equation

$$\begin{cases} \Delta u + k^2 u = f(x) \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \\ \lim\limits_{|x| \to +\infty} \sqrt{|x|} \left(\dfrac{\partial u}{\partial r}(x) - ik\, u(x) \right) = 0, \end{cases} \tag{2.1}$$

where $u(x)$ represents the component of the magnetic field in the transverse magnetic polarization.
It is known that the problem (2.1) has a unique solution whenever $\mathrm{Im}(k) \geq 0$ [2]. The mapping $R(k) : f \to u$ defines an operator-valued function which is holomorphic in $\mathrm{Im}(k) \geq 0$. It has a meromorphic extension to the whole complex plane, except for a countable number of poles: These values of k are the resonant frequencies. In other words, they are the values k for which (2.1) has non-trivial solutions when $f = 0$. The space of such non-trivial solutions, called characteristic functions, has finite dimension. When the pole k_j is simple, the solution operator $R(k)$ can be factorized in the form

$$R(k) \quad = \quad \frac{R_{-1,j}}{k - k_j} + R_{0,j}(k),$$

where $R_{-1,j}$ is a finite rank operator, and where $R_{0,j}(k)$ is an operator-valued function which is holomorphic near k_j [10]. The confinement of the electromagnetic fields around the cavities occurs at frequencies $k \in \mathbb{R}_+$ close to $\mathrm{Re}(k_j)$, if the imaginary part $\mathrm{Im}(k_j)$ is small enough. In this case $\frac{\|R_{-1,j}\|}{|\mathrm{Im}(k_j)|}$ represents the factor of enhancement of the fields. Experimental results (see [5] and references therein) show that when the width of the cavities is smaller than the wavelength, the resonant frequencies are close to the real axis. In [6], the resonnant frequencies to a simple cavity have been studied, as the cavity width $w \to 0$. Precisely, it is proven there that $\mathrm{Im}(k_j) = O(w \ln(w))$ as $w \to 0$. The imaginary part of the resonant frequencies also represents the lifetime of the confinement phenomena (see section 2.1 in [6]), which plays an important role in applications.
Recently, it was shown in [5] that a system of two deep identical cavities at subwavelength distance could produce resonant frequencies much closer to the real axis than those created by a simple cavity. Thus the optical excitation of such resonances can lead to a larger factor of concentration of the electromagnetic fields near the cavities. In this work, we analyse how the interplay between the fields in two cavities may cause such phenomena.

2.1. **Fractional Sobolev spaces.** Let $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R})$ the space of tempered distributions $u \in \mathcal{S}'(\mathbb{R})$ with Fourier transform $\widehat{u} \in L^2_{\mathrm{loc}}(\mathbb{R})$, and

$$\|u\|^2_{H^s(\mathbb{R})} := \int_{\mathbb{R}} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 \, d\xi < +\infty.$$

Let I be a bounded and open interval in $\mathbb{R}$. For $s \geq 0$ the sobolev space $H^s(I)$ is defined by

$$H^s(I) := \{u \in \mathcal{D}'(I) : u = U|_I \text{ for some } U \in H^s(\mathbb{R})\}.$$

It is endowed with the norm

$$\|u\|_{H^s(I)} = \inf\{\|U\|_{H^s(\mathbb{R})} : U \in H^s(\mathbb{R}), \quad U|_I = u\}.$$

It follows that $H^s(I)$ (resp. $H^s(\mathbb{R})$) equipped with the norm $\|\cdot\|_{H^s(I)}$ (resp. $\|\cdot\|_{H^s(\mathbb{R})}$) is a Hilbert space. We also denote by $\widetilde{H}^s(I)$ the closure of $\mathcal{C}^\infty_c(I)$ in $H^s(\mathbb{R})$ so that $H^{-s}(I) = [\widetilde{H}^s(I)]'$ and $\widetilde{H}^{-s}(I) = [H^s(I)]'$ (see [13, Theorem 3.30 (i)]). Moreover, by [13, Theorem 3.29 (ii)] we have that

$$\widetilde{H}^s(I) = \{u \in H^s(\mathbb{R}) : \mathrm{Supp}\, u \subset \overline{I}\},$$

and when $s \geq 0$, [13, Theorem 3.3] asserts that

$$\widetilde{H}^s(I) = \{u \in L^2(I) : \tilde{u} \in H^s(\mathbb{R})\},$$

where $\tilde{u}$ denotes the extension of u by zero outside I.

In the sequel we will mostly be concerned with the cases $s = \pm 1/2$; in particular, it is proved (see [13, Theorem 3.30 (ii)]) that $H^{1/2}(I)$ can be identified to the space of functions $u \in L^2(I)$ satisfying

$$\int_I \int_I \frac{|u(x)-u(y)|^2}{|x-y|^2}\, dx\, dy < +\infty.$$

If $\phi \in \widetilde{H}^{-1/2}(I)$ and $f \in H^{1/2}(I)$, we will denote by $\langle \phi, f\rangle$ the duality product between $\widetilde{H}^{-1/2}(I)$ and $H^{1/2}(I)$. Similarly for vector valued functions, the duality product between $[\widetilde{H}^{-1/2}(I)]^2$ and $[H^{1/2}(I)]^2$ will be denoted by $\langle \phi, f\rangle := \langle \phi_1, f_1\rangle + \langle \phi_2, f_2\rangle$ whenever $\phi = (\phi_1, \phi_2) \in [\widetilde{H}^{-1/2}(I)]^2$ and $f = (f_1, f_2) \in [H^{1/2}(I)]^2$.

2.2. **The Green functions.** Fix a source point $y \in \Omega_e$, and consider G_e the Green function of the Helmholtz operator in Ω_e with a homogeneous Neumann boundary condition and a Sommerfeld radiation condition at infinity, *i.e.*,

$$\begin{cases} \Delta G_e(\cdot\,;y) + k^2 G_e(\cdot\,;y) = \delta_y(.) \quad \text{ in } \Omega_e, \\ \dfrac{\partial G_e}{\partial \nu}(\cdot\,;y) = 0 \quad \text{ on } \partial\Omega_e, \\ \lim\limits_{|x|\to+\infty} \sqrt{|x|}\left(\dfrac{\partial G_e}{\partial r}(x;y) - ik\, G_e(x;y)\right) = 0. \end{cases} \tag{2.2}$$

It is known that

$$G_e(x;y) = -\frac{i}{4} H_0^{(1)}(k|x-y|) - \frac{i}{4} H_0^{(1)}(k|x-\tilde{y}|), \tag{2.3}$$

where $\tilde{x} = (x_1, x_2)$ is the symmetric of $x = (x_1, x_2)$ with respect to the x_1-axis and $H_0^{(1)}$ stands for the Hankel function of first kind and zero order. We introduce the function $u_e(x) \in H^1_{\rm loc}(\Omega_e)$ solution to

$$\begin{cases} \Delta u_e + k^2 u_e = f \quad \text{ in } \Omega_e, \\ \dfrac{\partial u_e}{\partial \nu} = 0 \quad \text{ on } \partial\Omega_e, \\ \lim\limits_{|x|\to+\infty} \sqrt{|x|}\left(\dfrac{\partial u_e}{\partial r}(x) - ik\, u_e(x)\right) = 0, \end{cases}$$

which may be represented as

$$u_e(x) = \int_{\Omega_e} G_e(x;y) f(y)\, dy. \tag{2.4}$$

Let G_i ($i = 1, 2$) be the Green function of the Helmholtz operator in the cavity $\mathcal{C}_i$, with a homogeneous Neumann boundary condition, *i.e.*,

$$\begin{cases} \Delta G_i(\cdot\,;y) + k^2 G_i(\cdot\,;y) = \delta_y(.) \quad \text{ in } C_i, \\ \dfrac{\partial G_i}{\partial \nu}(\cdot\,;y) = 0 \quad \text{ on } \partial C_i. \end{cases} \tag{2.5}$$

For $k^2 \neq \left(\frac{m\pi}{2w}\right)^2 - \left(\frac{n\pi}{h}\right)^2$, the Green functions G_1 and G_2 have the following spectral decomposition

$$G_1(x;y) = \frac{2}{wh} \sum_{m,n=0}^{+\infty} \frac{\cos\left(\frac{m\pi}{2}\left(\frac{x_1}{w}+d+1\right)\right)\cos\left(\frac{m\pi}{2}\left(\frac{y_1}{w}+d+1\right)\right)\cos\left(n\pi\left(\frac{x_2}{h}+1\right)\right)\cos\left(n\pi\left(\frac{y_2}{h}+1\right)\right)}{k^2 - \left(\frac{m\pi}{2w}\right)^2 - \left(\frac{n\pi}{h}\right)^2}.$$

and

$$G_2(x;y) = \frac{2}{wh} \sum_{m,n=0}^{+\infty} \frac{\cos\left(\frac{m\pi}{2}\left(\frac{x_1}{w}-d+1\right)\right)\cos\left(\frac{m\pi}{2}\left(\frac{y_1}{w}-d+1\right)\right)\cos\left(n\pi\left(\frac{x_2}{h}+1\right)\right)\cos\left(n\pi\left(\frac{y_2}{h}+1\right)\right)}{k^2 - \left(\frac{m\pi}{2w}\right)^2 - \left(\frac{n\pi}{h}\right)^2}.$$

Remark 2.1. *Since u and G_i, $i = 1, 2$ depend on w and k, we sometimes write $u(x;w)$ and $G_i(x;y;w,k)$, $i = 1, 2$ respectively in place of $u(x)$ and $G_i(x;y)$, $i = 1, 2$ to emphasize the dependence on these parameters.*

2.3. **Integral representation.** In this section, we derive an integral representation of $u(x)$ that is equivalent to the problem (2.1). Multiplying equation (2.1) by $G_e(x;y)$, integrating over Ω_e and using the Green formula in Ω_e implies that for every $y \in \Omega_e$

$$\begin{aligned} u_e(y) &= \int_{\Omega_e} f(x)\, G_e(x;y)\, dx = \int_{\Omega_e} \left[\Delta u(x) + k^2 u(x)\right] G_e(x;y)\, dx \qquad (2.6)\\ &= \int_{\Omega_e} \left[\Delta G_e(x;y) + k^2 G_e(x;y)\right] u(x)\, dx \\ &\quad + \int_{\partial\Omega_e} \left[\frac{\partial u}{\partial \nu}(x)\, G_e(x;y) - \frac{\partial G_e}{\partial \nu}(x;y)\, u(x)\right] d\sigma(x) \\ &= u(y) - \int_{\Gamma_1\cup\Gamma_2} \frac{\partial u}{\partial x_2}(x_1,0)\, G_e(x_1,0;y)\, dx_1, \qquad (2.7) \end{aligned}$$

where we used equations (2.2) and (2.1) in the last equality.
Similarly, multiplying (2.1) by $G_i(x;y)$, integrating over C_i and applying the Green formula in C_i together with equation (2.5), and noting that f is supported in Ω_e, leads to

$$0 \;=\; u(y) + \int_{\Gamma_i} \frac{\partial u}{\partial x_2}(x_1,0)\, G_i(x_1,0;y)\, dx_1 \quad \text{for every } y \in C_i. \qquad (2.8)$$

Since $u \in H^1_{\text{loc}}(\Omega)$ and $\Delta u = f - k^2 u \in L^2_{\text{loc}}(\Omega)$, it follows by elliptic regularity that $u \in H^2_{\text{loc}}(\Omega)$. Sobolev imbedding implies that u is a continuous function on $\Gamma_i, i = 1,2$. Hence, letting y tends to $\Gamma_1 \cup \Gamma_2$ in (2.7), letting y tends to Γ_i in (2.8), and taking the difference, we infer that for every $y_1 \in \Gamma_1$

$$\begin{aligned} &\int_{\Gamma_1} \frac{\partial u}{\partial x_2}(x_1,0)\, \left[G_1 + G_e\right](x_1,0;y_1,0)\, dx_1 \\ &\qquad\qquad + \int_{\Gamma_2} \frac{\partial u}{\partial x_2}(x_1,0) G_e(x_1,0;y_1,0)\, dx_1 = -u_e(y_1,0), \qquad (2.9) \end{aligned}$$

and that for every $y_1 \in \Gamma_2$

$$\begin{aligned} &\int_{\Gamma_2} \frac{\partial u}{\partial x_2}(x_1,0)\, \left[G_2 + G_e\right](x_1,0;y_1,0)\, dx_1 \\ &\qquad\qquad + \int_{\Gamma_1} \frac{\partial u}{\partial x_2}(x_1,0) G_e(x_1,0;y_1,0)\, dx_1 = -u_e(y_1,0). \qquad (2.10) \end{aligned}$$

Next, we rescale both previous equations. To this end, we set

$$\Gamma \;:=\; (-1,1),$$

and for $x \in \Gamma$

$$\left\{\begin{array}{lcl} \phi_1(x) &:=& \dfrac{\partial u}{\partial x_2}(wx - wd, 0), \\ \phi_2(x) &:=& \dfrac{\partial u}{\partial x_2}(wx + wd, 0), \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{lcl} g_1(x) &:=& -\dfrac{1}{w} u_e(wx - wd, 0), \\ g_2(x) &:=& -\dfrac{1}{w} u_e(wx + wd, 0), \end{array}\right.$$

From (2.9) and (2.10), we deduce that for every $y \in \Gamma$

$$\int_\Gamma \Big\{ \left[G_1 + G_e\right](wx - wd, 0; wy - wd, 0)\, \phi_1(x) + G_e(wx + wd, 0; wy - wd, 0)\, \phi_2(x) \Big\}\, dx = g_1(y) \quad (2.11)$$

and

$$\int_\Gamma \Big\{ \left[G_2 + G_e\right](wx + wd, 0; wy + wd, 0)\, \phi_2(x) + G_e(wx - wd, 0; wy + wd, 0)\, \phi_1(x) \Big\}\, dx = g_2(y). \quad (2.12)$$

Let us define the 2×2 matrix-valued kernel

$$s_w(x,y,k) := \begin{pmatrix} \left[G_1 + G_e\right](wx - wd, 0; wy - wd, 0) & G_e(wx + wd, 0; wy - wd, 0) \\ G_e(wx - wd, 0; wy + wd, 0) & \left[G_2 + G_e\right](wx + wd, 0; wy + wd, 0) \end{pmatrix}. \qquad (2.13)$$

Since $u \in H^1_{\text{loc}}(\Omega)$ and $\Delta u \in L^2_{\text{loc}}(\Omega)$, it follows that $\frac{\partial u}{\partial x_2}|_{\Gamma_i} \in [H^{1/2}(\Gamma_i)]' = \widetilde{H}^{-1/2}(\Gamma_i)$ and thus ϕ_1 and $\phi_2 \in \widetilde{H}^{-1/2}(\Gamma)$.

Similarly, since $u_e \in H^1_{\rm loc}(\Omega_E)$, we deduce that $u_e|_{\Gamma_i} \in H^{1/2}(\Gamma_i)$ and thus g_1 and $g_2 \in H^{1/2}(\Gamma)$. Consequently, it is natural to define the integral operator $\mathcal{S}_w(k) : [\widetilde{H}^{-1/2}(\Gamma)]^2 \to [H^{1/2}(\Gamma)]^2$ by

$$\mathcal{S}_w(k)\phi(x) := \int_\Gamma s_w(x,y,k)\,\phi(y)\,dy, \quad \text{for every } \phi = (\phi_1,\phi_2) \in [\widetilde{H}^{-1/2}(\Gamma)]^2. \tag{2.14}$$

3. Asymptotic expansion

In this section we derive an asymptotic expansion of the kernel s_w and of the associated integral operator $\mathcal{S}_w(k)$. This can be achieved thanks to the explicit expression (2.13) of s_w using standard tools of pseudo-differential analysis (see [15]). In the sequel, we fix $w_0 > 0$ and set $r_0 := \frac{\pi}{(2w_0)}$. In the complex plane, $D_r(a) := \{z \in \mathbb{C} : |z-a| < r\}$ denotes the disc centered at a and of of radii $r > 0$. If $a = 0$, we write $D_r := D_r(0)$ and $D_r^+ := (\mathbb{C} \setminus \mathbb{R}_-) \cap D_r$.

Lemma 3.1. *For every $(w,k) \in (0,w_0) \times D^+_{r_0}$ the kernel s_w has the following asymptotic expansion:*

$$s_w(x,y,k) = \theta_w(k) + s(x,y) + s_1(x,y)w + \sum_{n=2}^{+\infty} s_n(x,y,k)w^n + \sum_{n=2}^{+\infty} t_n(x,y,k)w^n \ln w, \tag{3.1}$$

where

$$\theta_w(k) := \begin{pmatrix} \dfrac{\alpha(k)}{w} + \delta_2 + \dfrac{1}{\pi}(\ln k + \ln w) & \delta + \dfrac{1}{\pi}(\ln k + \ln w) \\ \delta + \dfrac{1}{\pi}(\ln k + \ln w) & \dfrac{\alpha(k)}{w} + \delta_2 + \dfrac{1}{\pi}(\ln k + \ln w) \end{pmatrix}. \tag{3.2}$$

is a constant 2×2 matrix, $\alpha(k)$ is a complex function defined in (3.11),

$$\begin{aligned} s(x,y) \;&:=\; \frac{1}{\pi}\ln\left[4|x-y|\left|\sin\left(\frac{\pi}{4}(x-y)\right)\right|\left|\sin\left(\frac{\pi}{4}(x+y+2)\right)\right|\right]\mathrm{I} \\ &\quad + \begin{pmatrix} \delta_1 & \dfrac{1}{\pi}\ln|x-y+2d| \\ \dfrac{1}{\pi}\ln|x-y-2d| & \delta_1 \end{pmatrix}. \end{aligned} \tag{3.3}$$

is a matrix valued kernel independent of k, δ_i, $i = 1,2$ are constants to be fixed later (Remark(4.6)) satisfying $\delta_1 + \delta_2 = \delta$, $i = 1,2$ where δ is an universal constant defined in (3.8).

$$s_1(x,y) := -\frac{1}{h}\left(\frac{2}{3} - \frac{|x-y| + x + y + 2}{4} + \frac{(x-y)^2 + (x+y+2)^2}{8}\right)\mathrm{I}.$$

Moreover, for every $n \geq 1$, there exists functions f_n, g_n and h_n defined in (3.7) and (3.15) such that

$$\begin{aligned} s_{2n}(x,y,k) \;&:=\; \Big(f_{2n}(x-y,k) + h_{2n}(x-y,k) + h_{2n}(x+y+2,k)\Big)\mathrm{I} \\ &\quad + \begin{pmatrix} 0 & f_{2n}(x-y+2d,k) \\ f_{2n}(x-y-2d,k) & 0 \end{pmatrix}, \end{aligned} \tag{3.4}$$

$$s_{2n+1}(x,y,k) := \Big(h_{2n+1}(x-y,k) + h_{2n+1}(x+y+2,k)\Big)\mathrm{I},$$

and

$$t_{2n}(x,y,k) := \begin{pmatrix} g_{2n}(x-y,k) & g_{2n}(x-y+2d,k) \\ g_{2n}(x-y-2d,k) & g_{2n}(x-y,k) \end{pmatrix}, \quad t_{2n+1}(x,y,k) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Finally, there exists a constant $C_1 > 0$ (that only depends on w_0) such that for every $k \in D^+_{r_0}$,

$$\left\| s_1 w + \sum_{n=2}^{+\infty} s_n(\cdot,\cdot,k)w^n + \sum_{n=2}^{+\infty} t_n(\cdot,\cdot,k)w^n \ln w \right\|_{\mathcal{C}^{0,1}(\overline{\Gamma}\times\overline{\Gamma})} \leq C_1 w. \tag{3.5}$$

Proof. We consider the asymptotic expansion of the Hankel function near zero (see [1, page 360]),

$$H_0^{(1)}(z) = \left\{1 + \frac{2i}{\pi}\left[\ln\left(\frac{z}{2}\right) + \gamma\right]\right\}\left(\sum_{n=0}^{+\infty}\frac{(-z^2/4)^n}{(n!)^2}\right) - \frac{2i}{\pi}\sum_{n=1}^{+\infty}\left(\sum_{m=1}^{n}\frac{1}{m}\right)\frac{(-z^2/4)^n}{(n!)^2},$$

where γ is the Euler constant. For every $t \in \mathbb{R}$, we thus obtain

$$-\frac{i}{2}H_0^{(1)}(kw|t|) = \delta + \frac{1}{\pi}(\ln w + \ln k) + \frac{1}{\pi}\ln|t| + \sum_{n=1}^{+\infty} f_{2n}(t,k)w^{2n} + \sum_{n=1}^{+\infty} g_{2n}(t,k)w^{2n}\ln w \tag{3.6}$$

where, for each $n \in \mathbb{N}^*$,

$$\begin{cases} f_{2n}(t,k) := \dfrac{(-1/4)^n}{\pi(n!)^2}\left(-\displaystyle\sum_{m=1}^{n}\frac{1}{m} + \pi\delta + \ln|t| + \ln k\right)k^{2n}t^{2n}, \\ g_{2n}(t,k) := \dfrac{(-1/4)^n}{\pi(n!)^2}\,k^{2n}t^{2n}, \end{cases} \tag{3.7}$$

and

$$\delta := -\frac{i}{2} + \frac{1}{\pi}(\gamma - \ln 2). \tag{3.8}$$

Note that $g_{2n}(\cdot,k) \in \mathcal{C}^\infty(\mathbb{R})$ for any $k \in \mathbb{C}$, while $g_{2n}(t,\cdot)$ is analytic in $\mathbb{C}$. On the other hand, since the function $t \mapsto t^2\ln|t|$ is of class $\mathcal{C}^{1,\nu}(\mathbb{R})$ for any $\nu \in [0,1)$ (see [6]), it follows that $f_{2n}(\cdot,k) \in \mathcal{C}^{2n-1,\nu}(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R}\setminus\{0\})$ and that $f_{2n}(t,\cdot)$ is analytic in $\mathbb{C}\setminus\mathbb{R}_-$. Moreover, there exists a constant $C > 0$ (depending only on w_0) such that for every $(w,k) \in (0,w_0)\times D_{r_0}$,

$$\sum_{n=1}^{+\infty}\|f_{2n}(\cdot,k)\|_{\mathcal{C}^{0,1}([-4,4])}w^{2n-2} \le C \quad \text{and} \quad \sum_{n=1}^{+\infty}\|g_{2n}(\cdot,k)\|_{\mathcal{C}^{0,1}([-4,4])}w^{2n-2} \le C.$$

Consequently, the limits

$$f(t,k,w) := \sum_{n=1}^{+\infty} f_{2n}(t,k)w^{2n} \quad \text{and} \quad g(t,k,w) := \sum_{n=1}^{+\infty} g_{2n}(t,k)w^{2n}\ln w \tag{3.9}$$

exist and furthermore

$$\|f(\cdot,k,w)\|_{\mathcal{C}^{0,1}([-4,4])} \le Cw^2 \quad \text{and} \quad \|g(\cdot,k,w)\|_{\mathcal{C}^{0,1}([-4,4])} \le Cw^2\ln w. \tag{3.10}$$

Note that provided $w \in (0,w_0)$, the first series in (3.9) (as well as its first derivative with respect to t) is uniformly converges as a function of $(t,k) \in [-4,4]\times D_{r_0}^+$. Consequently, $f(\cdot,k,w) \in \mathcal{C}^{1,\nu}([-4,4]) \cap \mathcal{C}^\infty([-4,4]\setminus\{0\})$ for any $k \in D_{r_0}^+$, and the function $f(t,\cdot,w)$ is analytic in $D_{r_0}^+$ for every $t \in [-4,4]$. Arguing similarly for the second series in (3.9), we can show that $g(\cdot,k,w) \in \mathcal{C}^\infty([-4,4])$ for any $k \in D_{r_0}^+$, and that $g(t,\cdot,w)$ is analytic in $D_{r_0}^+$ for every $t \in [-4,4]$. As consequence, we infer from the definition of G_e, that

$$\begin{aligned} G_e(wx-wd,0;wy-wd,0) &= G_e(wx+wd,0;wy+wd,0) = -\frac{i}{2}H_0^{(1)}(kw|x-y|) \\ &= \delta + \frac{1}{\pi}(\ln w + \ln k) + \frac{1}{\pi}\ln|x-y| \\ &\quad + \sum_{n=1}^{+\infty} f_{2n}(x-y,k)w^{2n} + \sum_{n=1}^{+\infty} g_{2n}(x-y,k)w^{2n}\ln w, \end{aligned}$$

and that

$$\begin{aligned} G_e(wx\pm wd,0;wy\mp wd,0) &= -\frac{i}{2}H_0^{(1)}(kw|x-y\pm 2d|) \\ &= \delta + \frac{1}{\pi}(\ln w + \ln k) + \frac{1}{\pi}\ln|x-y\pm 2d| \\ &\quad + \sum_{n=1}^{+\infty} f_{2n}(x-y\pm 2d,k)w^{2n} + \sum_{n=1}^{+\infty} g_{2n}(x-y\pm 2d,k)w^{2n}\ln w. \end{aligned}$$

We now turn our attention to the Green functions inside the cavities. From the expression of G_1 and G_2, we have

$$\begin{aligned} G_1(wx-wd,0;wy-wd,0) &= G_2(wx+wd,0;wy+wd,0) \\ &= \frac{2}{wh}\sum_{m,n=0}^{+\infty} \frac{\cos\left(\frac{m\pi}{2}(x+1)\right)\cos\left(\frac{m\pi}{2}(y+1)\right)}{k^2-\left(\frac{m\pi}{2w}\right)^2-\left(\frac{n\pi}{h}\right)^2}. \end{aligned}$$

We define

$$\mathcal{R}_m(w,k) := \frac{2}{h}\sum_{n=0}^{+\infty}\frac{1}{k^2-\left(\frac{m\pi}{2w}\right)^2-\left(\frac{n\pi}{h}\right)^2}.$$

When $m=0$, we observe that $\mathcal{R}_0(w,k)=\alpha(k)$, where

$$\alpha(k) := \frac{1}{k}\left(\frac{1}{kh}+\cot(kh)\right), \tag{3.11}$$

while for $m\in\mathbb{N}^*$, $\mathcal{R}_m(w,k)$ has the following asymptotic expansion as w/m tends to zero:

$$\mathcal{R}_m(w,k) = -\frac{2}{\pi}\frac{w}{m} - \frac{4}{\pi^2 h}\frac{w^2}{m^2} - \sum_{n=3}^{+\infty} p_n(k)\frac{w^n}{m^n}, \tag{3.12}$$

where $p_n(k)$ are suitable nonnegative constants which depends on k in an analytic manner. Actually, one has

$$\begin{aligned} \mathcal{R}_m(w,k) &= -\frac{1}{k^2h}\sum_{n=1}^{+\infty}\left(\frac{2wk}{m\pi}\right)^{2n} \\ &\quad -\frac{2w}{m\pi}\left\{1+\sum_{n=1}^{+\infty}\frac{1}{n!}\left[\prod_{j=0}^{n-1}\left(\frac{1}{2}+j\right)\right]\left(\frac{2wk}{m\pi}\right)^{2n}\right\}\left\{\frac{2}{1-e^{-2h\sqrt{\left(\frac{m\pi}{2w}\right)^2-k^2}}}-1\right\}, \end{aligned}$$

and thus, from (3.12), there exists a constant $C'>0$ (that only depends on w_0) such that

$$\sum_{n=2}^{+\infty} p_n(k)\frac{w^n}{m^n} \le C'\sum_{n=2}^{+\infty}\left(\frac{2wk}{m\pi}\right)^n. \tag{3.13}$$

Note that the previous series converges when $m\in\mathbb{N}^*$, $w<w_0$ and $k\in D_{r_0}$ (recall that $r_0=\pi/(2w_0)$). Using (3.12) and the definition of G_1 and G_2, we immediately deduce that

$$\begin{aligned} G_1(wx-wd,0;wy-wd,0) &= G_2(wx+wd,0;wy+wd,0) \\ &= \frac{\alpha(k)}{w}-\frac{1}{\pi}\sum_{m=1}^{+\infty}\frac{\cos\left(\frac{m\pi}{2}(x-y)\right)+\cos\left(\frac{m\pi}{2}(x+y+2)\right)}{m} \\ &\quad -\frac{2}{\pi^2 h}w\sum_{m=1}^{+\infty}\frac{\cos\left(\frac{m\pi}{2}(x-y)\right)+\cos\left(\frac{m\pi}{2}(x+y+2)\right)}{m^2} \\ &\quad -\sum_{n=2}^{+\infty}p_n(k)w^n\left(\sum_{m=1}^{+\infty}\frac{\cos\left(\frac{m\pi}{2}(x-y)\right)+\cos\left(\frac{m\pi}{2}(x+y+2)\right)}{m^{n+1}}\right). \end{aligned}$$

As a consequence,

$$\begin{aligned} G_1(wx-wd,0;wy-wd,0) &= G_2(wx+wd,0;wy+wd,0) \\ &= \frac{\alpha(k)}{w}+\frac{1}{\pi}\ln\left[4\left|\sin\left(\frac{\pi}{4}(x-y)\right)\right|\left|\sin\left(\frac{\pi}{4}(x+y+2)\right)\right|\right] \\ &\quad -\frac{w}{h}\left(\frac{2}{3}-\frac{|x-y|+x+y+2}{4}+\frac{(x-y)^2+(x+y+2)^2}{8}\right) \\ &\quad +\sum_{n=2}^{+\infty}[h_n(x-y,k)+h_n(x+y+2,k)]w^n, \end{aligned} \tag{3.14}$$

where we defined

$$h_n(t,k) := -\sum_{m=1}^{+\infty} p_n(k) \frac{\cos\left(\frac{m\pi t}{2}\right)}{m^{n+1}}. \tag{3.15}$$

From [6], we know that for each $k \in \mathbb{C}$, the function $h_2(\cdot, k) \in \mathcal{C}^{1,\nu}(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$ for any $\nu \in [0,1)$, and thus by iteration, the functions $h_n(\cdot, k) \in \mathcal{C}^{n-1,\nu}(\mathbb{R}) \cap \mathcal{C}^\infty(\mathbb{R} \setminus \{0\})$ for every $n \geq 2$. Moreover the function $h_n(t, \cdot)$ is analytic on $\mathbb{C}$ for every $t \in \mathbb{R}$. Using (3.13), we observe that for $w < w_0$ and $k \in D_{r_0}$, then

$$\begin{aligned}
\sum_{n=2}^{+\infty} \|h_n(\cdot, k)\|_{\mathcal{C}^{0,1}([-4,4])} w^{n-2} &\leq \frac{1}{w^2} \sum_{m=1}^{+\infty} \sum_{n=2}^{+\infty} p_n(k) \frac{w^n}{m^n} \leq \frac{C'}{w^2} \sum_{m=1}^{+\infty} \sum_{n=2}^{+\infty} \left(\frac{2wk}{m\pi}\right)^n \\
&\leq \frac{C'\pi^2}{6w^2} \sum_{n=2}^{+\infty} \left(\frac{2wk}{\pi}\right)^n \leq C'',
\end{aligned}$$

for some constant $C'' > 0$ depending only on w_0. Hence we deduce that the limit

$$h(t,k,w) := \sum_{n=2}^{+\infty} h_n(t,k) w^n, \tag{3.16}$$

exists and that

$$\|h(\cdot, k, w)\|_{\mathcal{C}^{0,1}([-4,4])} \leq C'' w^2. \tag{3.17}$$

Since the series (3.16) is uniformly converging as a function of $(t,k) \in [-4,4] \times D_{r_0}$, it follows that $h(\cdot, k, w) \in \mathcal{C}^{1,\nu}([-4,4]) \cap \mathcal{C}^\infty([-4,4] \setminus \{0\})$ for every $k \in D_{r_0}$ and $h(t, \cdot, w)$ is analytic in D_{r_0} for every $t \in [-4,4]$.

We define $\theta_w(k)$, s, s_1, s_n and t_n (for $n \geq 2$) as in the statement of Lemma 3.1, and let

$$\rho_w(x,y,k) := s_1(x,y)w + \sum_{n=2}^{+\infty} s_n(x,y,k) w^n + \sum_{n=2}^{+\infty} t_n(x,y,k) w^n \ln w.$$

Thanks to (3.9) and (3.16), we deduce that for every $w \in (0, w_0)$, both previous series are uniformly converging in $\overline{\Gamma} \times \overline{\Gamma} \times D_{r_0}^+$ as functions of (x,y,k). Consequently $\rho_w(\cdot,\cdot,k) \in \mathcal{C}^{0,1}(\overline{\Gamma} \times \overline{\Gamma})$ for every $k \in D_{r_0}^+$ and $\rho_w(x,y,\cdot)$ is analytic in $D_{r_0}^+$ for every $(x,y) \in \overline{\Gamma} \times \overline{\Gamma}$. Finally, s_w expands as announced in (3.1), and from (3.10) and (3.17) we immediately deduce (3.5). $\square$

For every $\phi \in [\mathcal{C}_c^\infty(\Gamma)]^2$, define the following integral operators

$$\left\{\begin{aligned}
\mathbf{\Theta}_w(k)\phi(x) &:= \theta_w(k) \int_\Gamma \phi(y)\, dy, \\
\mathcal{S}\phi(x) &:= \int_\Gamma s(x,y)\, \phi(y)\, dy, \\
\mathcal{S}_1\phi(x) &:= \int_\Gamma s_1(x,y)\, \phi(y)\, dy, \\
\mathcal{S}_n(k)\phi(x) &:= \int_\Gamma s_n(x,y,k)\, \phi(y)\, dy, \\
\mathcal{T}_n(k)\phi(x) &:= \int_\Gamma t_n(x,y,k)\, \phi(y)\, dy.
\end{aligned}\right.$$

We now deduce from Lemma 3.1 an asymptotic expansion of the integral operator $\mathcal{S}_w(k)$.

Lemma 3.2. *For every $(w,k) \in (0,w_0) \times D_{r_0}^+$, the operator $\mathcal{S}_w(k)$ admits the following asymptotic expansion:*

$$\mathcal{S}_w(k) = \mathbf{\Theta}_w(k) + \mathcal{S} + \mathcal{S}_1 w + \sum_{n=2}^{+\infty} \mathcal{S}_n(k) w^n + \sum_{n=2}^{+\infty} \mathcal{T}_n(k) w^n \ln w,$$

where $\mathcal{S}_1$, $\mathcal{S}_n(k)$ and $\mathcal{T}_n(k)$ are compact from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$ and

$$\left\| \mathcal{S}_1 w + \sum_{n=2}^{+\infty} \mathcal{S}_n(k) w^n + \sum_{n=2}^{+\infty} \mathcal{T}_n(k) w^n \ln w \right\| \le C_2 w, \tag{3.18}$$

for some constant $C_2 > 0$ depending only on w_0.

Proof. Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R};[0,1])$ be a cut-off function satisfying $\chi(t) = 1$ for every $t \in \Gamma$. For any integer $n \ge 2$ and every $(x,y) \in \mathbb{R} \times \mathbb{R}$, define

$$\begin{cases} \tilde{s}_n(x,y,k) := \chi(x)\chi(y) s_n(x,y,k), \\ \tilde{t}_n(x,y,k) := \chi(x)\chi(y) t_n(x,y,k). \end{cases}$$

Denoting by σ_n and τ_n the symbols of $\tilde{s}_n$ and $\tilde{t}_n$ respectively, it can be seen that σ_n and τ_n belong to the class of symbols $S^{-n}_{1,0}$ and thus, thanks to standard pseudo-differential analysis results [15, Chapter II], it follows that the associated integral operators $\tilde{\mathcal{S}}_n(k)$ and $\tilde{\mathcal{T}}_n(k)$ defined, for every $\phi \in [\mathcal{C}_c^\infty(\mathbb{R})]^2$, by

$$\tilde{\mathcal{S}}_n(k)\phi(x) := \int_{\mathbb{R}} \tilde{s}_n(x,y,k)\,\phi(y)\,dy \quad \text{and} \quad \tilde{\mathcal{T}}_n(k)\phi(x) := \int_{\mathbb{R}} \tilde{t}_n(x,y,k)\,\phi(y)\,dy$$

are bounded from $[H^{-1/2}(\mathbb{R})]^2$ to $[H^{n-1/2}(\mathbb{R})]^2$. As a consequence, the operators $\mathcal{S}_n(k)$ and $\mathcal{T}_n(k)$ defined above are bounded from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{n-1/2}(\Gamma)]^2$, and using the compact imbedding of $H^{n-1/2}(\Gamma)$ into $H^{1/2}(\Gamma)$ for $n \ge 2$, it follows that they are actually compact from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$. From [6] we also know that the integral operator $\mathcal{S}_1$ is compact from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$. Moreover, thanks to (3.1) and the Dominated Convergence Theorem, the operator $\mathcal{S}_w(k)$ admits the expected asymptotic expansion for $(w,k) \in (0,w_0) \times D^+_{r_0}$. Finally, according to (3.5), the operator

$$\mathcal{R}_w(k) := \mathcal{S}_1 w + \sum_{n=2}^{+\infty} \mathcal{S}_n(k) w^n + \sum_{n=2}^{+\infty} \mathcal{T}_n(k) w^n \ln w,$$

is an integral operator with kernel $\rho_w(x,y,k)$ and we have $\|\mathcal{R}_w(k)\| \le C_2 w$ where $C_2 > 0$ is a constant depending on w_0 and C_1. □

4. Asymptotic of $\mathcal{S}_w^{-1}(k)$ and of the resonances

Let $\widetilde{S}$ be the integral operator form $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$ associated to the kernel

$$\frac{1}{\pi} \ln \left[|x-y| \left| \sin\left(\frac{\pi}{4}(x-y)\right)\right| \left| \sin\left(\frac{\pi}{4}(x+y+2)\right)\right| \right] \mathrm{I}.$$

The following result is proved in [6](see Lemma 5.1).

Lemma 4.1. *The operator $\widetilde{S}$ is invertible from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$.*

Theorem 3.2 implies that for every $(w,k) \in (0,w_0) \times D^+_{r_0}$, we have $\mathcal{S}_w(k) = \mathbf{\Theta}_w(k) + \mathcal{S} + \mathcal{R}_w(k)$ where

$$\mathbf{\Theta}_w(k) = \theta_w(k) \begin{pmatrix} \langle \cdot, e_1 \rangle \\ \langle \cdot, e_2 \rangle \end{pmatrix},$$

the matrix $\theta_w(k)$ is defined in (3.2), and $e_1 = (1,0)$, $e_2 = (0,1)$.

Theorem 4.2. *The operator valued function $k \mapsto \mathcal{S}_w(k)$ is finitely meromorphic and of Fredholm type in $\mathbb{C} \setminus \mathbb{R}_-$, its poles are $\{\pm\big((\frac{n\pi}{h})^2 + (\frac{m\pi}{2w})^2\big)^{\frac{1}{2}} :\ n, m \in \mathbb{N}\}$, and the operator $\mathcal{S}_w(k)$ is invertible from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$ for $\mathrm{Im}(k) \ge 0$. Moreover the operator valued function $k \mapsto \mathcal{S}_w^{-1}(k)$ is finitely meromorphic in $\mathbb{C} \setminus \mathbb{R}_-$ and its poles are exactly the resonances of the open cavities.*

Proof. By the expression (2.13) of the kernel s_w, it is clear that the poles of $\mathcal{S}_w(k)$ are exactly $\{\pm\big((\frac{n\pi}{h})^2 + (\frac{m\pi}{2w})^2\big)^{\frac{1}{2}} :\ n, m \in \mathbb{N}\}$. Then expanding the operator $\mathcal{S}_w(k)$ as a Laurent series around each of these poles implies that the range of operator that multiplies $\frac{1}{k\pm\left((\frac{n\pi}{h})^2+(\frac{m\pi}{2w})^2\right)^{\frac{1}{2}}}$ is of dimension two. The range of that operator is actually the vector space spanned by the function

$$\cos\left(\frac{m\pi}{2}(x+1)\right)(e_1+e_2), \quad x\in\Gamma.$$

This implies that $\mathcal{S}_w(k)$ is finitely meromorphic in $\mathbb{C}\setminus\mathbb{R}_-$.

Now, according to Lemma 3.2, the only term independent of k in the previous Laurent series, is the operator

$$\mathcal{A}_0 := \begin{pmatrix} \delta_2+\frac{\ln w}{\pi} & \delta+\frac{\ln w}{\pi} \\ \delta+\frac{\ln w}{\pi} & \delta_2+\frac{\ln w}{\pi} \end{pmatrix}\langle .,1\rangle + \mathcal{S}+\mathcal{S}_1 w.$$

We remark that the following kernel

$$\begin{pmatrix} \delta_1+\ln(4) & \dfrac{1}{\pi}\ln|x-y+2d| \\ \dfrac{1}{\pi}\ln|x-y-2d| & \delta_1+\ln(4) \end{pmatrix}$$

belongs to $C^\infty(\overline{\Gamma}\times\overline{\Gamma})$ and so the corresponding integral operator is compact from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$. It follows that $\mathcal{S}$ is a compact perturbation of $\tilde{S}$ which is invertible from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$ (Lemma 4.1). Since $\mathcal{S}_1$ is compact (Lemma 3.2) the operator $\mathcal{A}_0$ is also a compact perturbation of $\tilde{S}$ and so it is a Fredholm operator of index zero. Consequently, the operator valued function $k\mapsto\mathcal{S}_w(k)$ is of Fredholm type of index zero as well (see the appendix). The invertibility of the operator $\mathcal{S}_w(k)$ for $\mathrm{Im}(k)\geq 0$ is a consequence of the fact that the equation (2.1) admits a unique solution for such frequencies k. Finally, we deduce from the Steinberg Theorem (Theorem 5.10) that the operator valued function $k\mapsto\mathcal{S}_w^{-1}(k)$ is finitely meromorphic in $\mathbb{C}\setminus\mathbb{R}_-$ and its poles are exactly the resonances of the open cavities. ☐

Let $\widehat{S}: [\widetilde{H}^{-1/2}(\Gamma)]^2\to[H^{1/2}(\Gamma)]^2$ be the integral operator with kernel

$$\frac{1}{\pi}\ln\left[4|x-y|\left|\sin\left(\frac{\pi}{4}(x-y)\right)\right|\left|\sin\left(\frac{\pi}{4}(x+y+2)\right)\right|\right]\mathrm{I}+\begin{pmatrix} 0 & \dfrac{1}{\pi}\ln|x-y+2d| \\ \dfrac{1}{\pi}\ln|x-y-2d| & 0 \end{pmatrix}.$$

Let $\widetilde{H}_0^{-1/2}(\Gamma)$ denotes the space of functions φ in $\widetilde{H}^{-1/2}(\Gamma)$ satisfying $\int_\Gamma\varphi(x)dx = 0$.

Lemma 4.3. *$\widehat{S}$ is Fredholm of index zero on $[\widetilde{H}^{-1/2}(\Gamma)]^2$. In addition, $\widehat{S}$ is coercive on $[\widetilde{H}_0^{-1/2}(\Gamma)]^2$, i.e.,*

$$-\langle\varphi,\widehat{S}\varphi\rangle\geq C\|\varphi\|^2_{[\widetilde{H}^{-1/2}(\Gamma)]^2}, \qquad \forall\varphi\in[\widetilde{H}_0^{-1/2}(\Gamma)]^2, \tag{4.1}$$

where $C>0$ is a fixed constant.

Proof. The operator $\widehat{S}$ is a compact perturbation of $\widetilde{S}$ and so is Fredholm of index zero on $[\widetilde{H}^{-1/2}(\Gamma)]^2$. Let $\varphi=(\varphi_1,\varphi_2)\in[\widetilde{H}^{-1/2}(\Gamma)]^2$ and define

$$\begin{aligned} \varphi_1(x;w) &:= \frac{1}{w}\varphi_1(\frac{x}{w}+d) \qquad x\in\Gamma_1, \\ \varphi_2(x;w) &:= \frac{1}{w}\varphi_2(\frac{x}{w}-d) \qquad x\in\Gamma_2. \end{aligned}$$

We consider v solution to the following Helmholtz equation

$$\begin{cases} \Delta v+k^2v = 0 \quad \text{in } \Omega_e\cup C_1\cup C_2, \\ \partial_{x_2}v|_+ = \partial_{x_2}v|_- = \varphi_1(.;w) \text{ on } \Gamma_1 \\ \partial_{x_2}v|_+ = \partial_{x_2}v|_- = \varphi_2(.;w) \text{ on } \Gamma_2, \\ \dfrac{\partial v}{\partial\nu}=0 \quad \text{on } \partial\Omega, \\ \lim\limits_{|x|\to+\infty}\sqrt{|x|}\left(\dfrac{\partial v}{\partial r}(x)-ik\,v(x)\right)=0. \end{cases} \tag{4.2}$$

Multiplying v in the equation (4.2) by $G_e(x,y)$ and integrating by parts we obtain

$$v(x) \quad = \quad -\sum_{i=1}^{2}\int_{\Gamma_i} G_e(x,y)\varphi_i(y;w)d\sigma(y), \quad x \in \Omega_e. \tag{4.3}$$

The Green formula inside the cavities yields

$$v(x) \quad = \quad \int_{\Gamma_i} G_i(x,y)\varphi_i(y;w)d\sigma(y), \quad x \in C_i, \qquad i = 1,2. \tag{4.4}$$

Taking the difference between the traces of the equations (4.3) and (4.4) on both sides of Γ_i, we obtain

$$[v(x)] \quad = \quad -\sum_{i=1}^{2}\int_{\Gamma_i} \big(G_e(x,y) + G_i(x,y)\big)\varphi_i(y;w)d\sigma(y), \quad x \in \Gamma_1 \cup \Gamma_2, \tag{4.5}$$

where $[v(x)] = v|_+ - v|_-$. Now, we fix $k = i = \sqrt{-1}$. Due to the explicit expression of G_e (equation (2.3)) one can easily check that when $k = i$, the function $v(x)$ belongs to $H^1(\Omega_e)$. Thus, multiplying the equation (4.2) by $\overline{v}$ and integrating over Ω_e leads to

$$\int_{\Omega_e} |\nabla v(x)|^2 dx + \int_{\Omega_e} |v(x)|^2 dx \quad = \quad \sum_{i=1}^{2}\int_{\Gamma_i} \varphi_i(x;w)\overline{v}|_+(x)d\sigma(x). \tag{4.6}$$

Similarly, multiplying the equation (4.2) by $\overline{v}$ and integrating over C_i implies

$$\int_{C_i} |\nabla v(x)|^2 dx + \int_{C_i} |v(x)|^2 dx \quad = \quad -\int_{\Gamma_i} \varphi_i(x;w)\overline{v}|_-(x)d\sigma(x). \tag{4.7}$$

Adding the equations (4.8) and (4.7) we obtain

$$\int_{\Omega} |\nabla v(x)|^2 dx + \int_{\Omega} |v(x)|^2 dx \quad = \quad \sum_{i=1}^{2}\int_{\Gamma_i} \varphi_i(x;w)\overline{[v(x)]}d\sigma(x). \tag{4.8}$$

Replacing $[v(x)]$ by its expression in (4.5) and rescaling the right hand term in the last equation with respect to w, we find

$$\int_{\Omega} |\nabla v(x)|^2 dx + \int_{\Omega} |v(x)|^2 dx \quad = \quad -\langle \varphi, \mathcal{S}_w(i)\varphi\rangle.$$

On the other hand we have (see [13])

$$C\|\varphi\|^2_{[\widetilde{H}^{-1/2}(\Gamma)]^2} \; = \; C\|\partial_{x_2} v\|^2_{H^{-1/2}(\partial\Omega_e)} \leq \int_{\Omega_e} |\nabla v(x)|^2 dx + \int_{\Omega_e} |v(x)|^2 dx,$$

where $C > 0$ is a constant independent of w. Combining the last two equalities yield

$$-\langle \varphi, \mathcal{S}_w(i)\varphi\rangle \geq C\|\varphi\|^2_{[\widetilde{H}^{-1/2}(\Gamma)]^2}. \tag{4.9}$$

Since $\varphi \in [\widetilde{H}^{-1/2}(\Gamma)]^2$, Lemma 3.2 implies

$$S_w(i)\varphi \to \widehat{S}\varphi, \qquad \text{as} \quad w \to 0.$$

Taking the limit of the equation (4.9) as w tends to zero we obtain the desired result. $\square$

Lemma 4.4. *Let $T : [\widetilde{H}^{-1/2}(\Gamma)]^2 \times \mathbb{R}^2 \to [H^{1/2}(\Gamma)]^2 \times \mathbb{R}^2$ be the integral operator defined by*

$$T(\psi, a) \quad := \quad \big(\widehat{S}\psi - a, \int_{\Gamma} \psi(x)dx\big).$$

Then, T is invertible.

Proof. Since $\widehat{S}$ is Fredholm of index zero T is also a Fredholm operator with zero index. Thus, we only need to prove injectivity. In fact, if $\widehat{S}\psi - a = 0$ and $\int_\Gamma \psi(x)dx = 0$, then $\langle \psi, \widehat{S}\psi\rangle = 0$. The inequality (4.1) implies that $\psi = 0$ and in turn $a = 0$. Thus T is invertible. $\square$

Let $\psi_e = (\psi_1, \psi_2)$ and $a_e = (a_1, a_2)$ be the unique solution to the following system

$$T(\psi_e, a_e) = (0, e_1),$$

Let $\tilde{\psi}_e = (\psi_2(-x), \psi_1(-x))$ and $\tilde{a}_e = (a_2, a_1)$. By taking into account the symmetries of the kernel of $\widehat{S}$ we obtain

$$T(\tilde{\psi}_e, \tilde{a}_e) = (0, e_2).$$

Theorem 4.5. *Let δ_1 be a fixed real constant such that $\delta_1 \neq -a_1 \pm a_2$. Then, $\mathcal{S}$ is invertible from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$.*

Proof. It follows from the asymptotic in Lemma 3.1 that $\mathcal{S} = \delta_1 \begin{pmatrix} \langle \cdot, e_1 \rangle \\ \langle \cdot, e_2 \rangle \end{pmatrix} + \hat{S}$. Lemma 4.4 implies that $\mathcal{S}$ is of Fredholm type with index zero. Thus, we only need to prove injectivity. Assume that $\mathcal{S}\varphi = 0$ with $\varphi \in [\widetilde{H}^{-1/2}(\Gamma)]^2$. Let

$$\varphi_0 := \varphi - \langle \varphi, e_1 \rangle \psi_e - \langle \varphi, e_2 \rangle \tilde{\psi}_e.$$

Then, φ_0 belongs to $[\widetilde{H}_0^{-1/2}(\Gamma)]^2$ and satisfies

$$\widehat{S}\varphi_0 = -\begin{pmatrix} \delta_1 + a_1 & a_2 \\ a_2 & \delta_1 + a_1 \end{pmatrix} \int_\Gamma \varphi(x) dx.$$

Consequently $\langle \varphi_0, \widehat{S}\varphi_0 \rangle = 0$. We deduce from Lemma 4.3 that $\varphi_0 = 0$ and hence φ is also zero. $\square$

Remark 4.6. *We assume throughout the paper that $\delta_1 \in \mathbb{R}$ is fixed such that $\delta_1 \neq -a_1 \pm a_2$. The constant δ_2 is immediately determined by the relation $\delta_1 + \delta_2 = \delta$, where δ is defined by* (3.8)*. Finally, the constants δ_i, $i = 1, 2$ are independent of w.*

Define $\mathcal{L}_w(k) := \mathcal{S} + \mathcal{R}_w(k)$. By (3.18) and Corollary 4.5, the operator $\mathcal{L}_w(k)$ is invertible from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$ for w small enough (depending only on w_0), and it admits the following asymptotic expansion:

$$\begin{aligned}
\mathcal{L}_w^{-1}(k) &= \mathcal{S}^{-1} \sum_{m=0}^{+\infty} \left(-\mathcal{R}_w(k)\mathcal{S}^{-1}\right)^m \\
&= \mathcal{S}^{-1} - \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}w - \mathcal{S}^{-1}\mathcal{T}_2(k)\mathcal{S}^{-1}w^2 \ln w + \left[\mathcal{S}^{-1}\mathcal{S}_1^2\mathcal{S}^{-2} - \mathcal{S}^{-1}\mathcal{S}_2(k)\mathcal{S}^{-1}\right]w^2 \\
&\quad + 2\,\mathcal{S}^{-1}\mathcal{S}_1\mathcal{T}_2(k)\mathcal{S}^{-2}w^3 \ln w + \left[2\,\mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}_2(k)\mathcal{S}^{-2} - \mathcal{S}^{-1}\mathcal{S}_1^3\mathcal{S}^{-3} - \mathcal{S}^{-1}\mathcal{S}_3(k)\mathcal{S}^{-1}\right]w^3 \\
&\quad + \sum_{\substack{n \geq 4 \\ m \geq 0}} \mathcal{L}_{mn}(k) w^n (\ln w)^m,
\end{aligned} \tag{4.10}$$

where each $\mathcal{L}_{mn}(k)$ is a compact operator from $[H^{1/2}(\Gamma)]^2$ to $[\widetilde{H}^{-1/2}(\Gamma)]^2$, holomorphic with respect to $k \in D_{r_0}^+$. Moreover, since $\rho_w(x, y, \cdot)$ is analytic in $D_{r_0}^+$ for every $(x, y) \in \overline{\Gamma} \times \overline{\Gamma}$, it follows that the kernel of $\mathcal{L}_w(k)$ is analytic with respect to $k \in D_{r_0}^+$, and that the operator valued function $k \mapsto \mathcal{L}_w(k)$ is holomorphic in the domain $D_{r_0}^+$. As a consequence of the Steinberg Theorem (Theorem 5.10), we deduce that $k \mapsto \mathcal{L}_w^{-1}(k)$ is holomorphic as well in the same domain $D_{r_0}^+$.

Remark 4.7. Let ϕ and $\boldsymbol{\psi} \in [\widetilde{H}^{-1/2}(\Gamma)]^2$ be such that $\mathcal{L}_w(k)\phi = e_1$ and $\mathcal{L}_w(k)\boldsymbol{\psi} = e_2$. Using the the expression of the kernel of $\mathcal{L}_w(k)$ together with the fact that for each $n \geq 1$ the function $h_n(\cdot, k)$ in (3.15) is 4-periodic, we infer that $\phi_1(x) = \psi_2(-x)$ and consequently

$$\langle \mathcal{L}_w^{-1}(k)e_1, e_1 \rangle = \langle \phi, e_1 \rangle = \langle \phi_1, 1 \rangle = \langle \psi_2, 1 \rangle = \langle \boldsymbol{\psi}, e_2 \rangle = \langle \mathcal{L}_w^{-1}(k)e_2, e_2 \rangle.$$

Moreover, since $\mathcal{L}_w(k)$ is a self adjoint operator from $[\widetilde{H}^{-1/2}(\Gamma)]^2$ to $[H^{1/2}(\Gamma)]^2$, it follows that

$$\langle \mathcal{L}_w^{-1}(k)e_1, e_2 \rangle = \langle \mathcal{L}_w^{-1}(k)e_2, e_1 \rangle.$$

Remark 4.8. We define the matrix

$$Q_w(k) := \begin{pmatrix} \langle e_1, \mathcal{L}_w^{-1}(k)e_1\rangle & \langle e_1, \mathcal{L}_w^{-1}(k)e_2\rangle \\ \langle e_1, \mathcal{L}_w^{-1}(k)e_2\rangle & \langle e_1, \mathcal{L}_w^{-1}(k)e_1\rangle \end{pmatrix}. \tag{4.11}$$

By the previous argument, the mapping $k \mapsto Q_w(k)$ is holomorphic, and using the asymptotic expansion (4.10) of $\mathcal{L}_w^{-1}(k)$, one can also expand $Q_w(k)$ as

$$Q_w(k) = Q_0 - Q_1 w - Q_{12}(k)w^2 \ln w + Q_2(k)w^2 + Q_{13}(k)w^3 \ln w + Q_3(k)w^3 + \sum_{\substack{n\geq 4\\ m\geq 0}} Q_{mn}(k)w^n(\ln w)^m,$$

where

$$Q_0 = \begin{pmatrix} \langle e_1, \mathcal{S}^{-1}e_1\rangle & \langle e_1, \mathcal{S}^{-1}e_2\rangle \\ \langle e_1, \mathcal{S}^{-1}e_2\rangle & \langle e_1, \mathcal{S}^{-1}e_1\rangle \end{pmatrix}, \quad Q_1 = \begin{pmatrix} \langle e_1, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}e_1\rangle & \langle e_1, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}e_2\rangle \\ \langle e_1, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}e_2\rangle & \langle e_1, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}e_1\rangle \end{pmatrix}$$

and $Q_{12}(k)$, $Q_2(k)$, $Q_{13}(k)$, $Q_3(k)$ and $Q_{mn}(k)$ (for $n \geq 4$, $m \geq 0$) are 2×2 matrices which are holomorphic with respect to k.
Denote $\varphi_i := \mathcal{S}^{-1}e_i$, $i = 1, 2$. A straight forward computation implies that

$$\begin{pmatrix} \langle \psi_e, \mathcal{S}\varphi_i\rangle \\ \langle \tilde{\psi}_e, \mathcal{S}\varphi_i\rangle \end{pmatrix} = \begin{pmatrix} \delta_1 + a_1 & a_2 \\ a_2 & \delta_1 + a_1 \end{pmatrix} \int_\Gamma \varphi_i(x)dx \;=\; e_i \qquad i = 1, 2.$$

Note that thanks to Remark 4.6 the matrix on the right hand side is invertible and we have

$$\int_\Gamma \varphi_i(x)dx \;=\; \lambda_0 \begin{pmatrix} \delta_1 + a_1 & -a_2 \\ -a_2 & \delta_1 + a_1 \end{pmatrix} e_i \qquad i = 1, 2,$$

where $\lambda_0 = \left((\delta_1 + a_1)^2 - a_2^2\right)^{-1}$. It follows that

$$\det Q_0 \;=\; \lambda_0 \neq 0,$$

and thus, the matrix Q_0 is invertible. In the sequel, we also consider the following quantities

$$q_0^{\pm} := \left\langle e_1, \mathcal{S}^{-1}(e_1 \pm e_2)\right\rangle, \quad q_1^{\pm} := \left\langle e_1, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}(e_1 \pm e_2)\right\rangle,$$
$$q_{12}^{\pm}(k) := \left\langle e_1, \mathcal{S}^{-1}\mathcal{T}_2(k)\mathcal{S}^{-1}(e_1 \pm e_2)\right\rangle,$$
$$q_2^{\pm}(k) := \left\langle e_1, (\mathcal{S}^{-1}\mathcal{S}_1^2\mathcal{S}^{-2} - \mathcal{S}^{-1}\mathcal{S}_2(k)\mathcal{S}^{-1})(e_1 \pm e_2)\right\rangle,$$
$$q_{13}^{\pm}(k) := 2\left\langle e_1, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{T}_2(k)\mathcal{S}^{-2}(e_1 \pm e_2)\right\rangle,$$

and

$$q_3^{\pm}(k) := \left\langle e_1, (2\,\mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}_2(k)\mathcal{S}^{-2} - \mathcal{S}^{-1}\mathcal{S}_1^3\mathcal{S}^{-3} - \mathcal{S}^{-1}\mathcal{S}_3(k)\mathcal{S}^{-1})(e_1 \pm e_2)\right\rangle.$$

Note that according to the definition of the kernels of the operators $\mathcal{S}$, $\mathcal{S}_1$, $\mathcal{S}_n(k)$ and $\mathcal{T}_n(k)$ for $n \geq 2$, all above quantities are real numbers except for $q_2^{\pm}(k)$ and $q_3^{\pm}(k)$ (see (3.4) and (3.7)).

We now derive an explicit expression for the inverse of $\mathcal{S}_w(k)$.

Proposition 4.9. *For $w \in (0, w_0)$ small enough and $k \in D_{r_0}^+$,*

$$\mathcal{S}_w^{-1}(k) = \mathcal{L}_w^{-1}(k) - \mathcal{L}_w^{-1}(k)\theta_w(k)F_w^{-1}(k)\begin{pmatrix} \langle \cdot, \mathcal{L}_w^{-1}(k)e_1\rangle \\ \langle \cdot, \mathcal{L}_w^{-1}(k)e_2\rangle \end{pmatrix}, \tag{4.12}$$

where $F_w(k) := \mathrm{I} + \theta_w(k)Q_w(k)$, $\theta_w(k)$ is given by (3.2), and $Q_w(k)$ is defined in (4.11). Moreover, the resonances of the open cavities coincide with the poles of the matrix valued function $k \mapsto \theta_w(k)F_w^{-1}(k)$.

Proof. Let $\phi \in [\widetilde{H}^{-1/2}(\Gamma)]^2$ and $g \in [H^{1/2}(\Gamma)]^2$ be such that $\mathcal{S}_w(k)\phi = (\boldsymbol{\Theta}_w(k) + \mathcal{L}_w(k))\phi = g$. Applying the operator $\mathcal{L}_w^{-1}(k)$ on the left we obtain that

$$\phi = \mathcal{L}_w^{-1}(k)\boldsymbol{g} - \mathcal{L}_w^{-1}(k)\theta_w(k)\begin{pmatrix} \langle \phi, e_1\rangle \\ \langle \phi, e_2\rangle \end{pmatrix}. \tag{4.13}$$

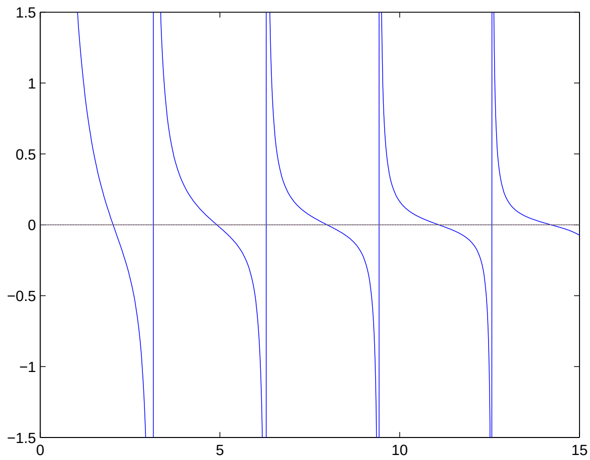


FIGURE 1. The function $k \mapsto \alpha(k)$

In view of (4.13), in order to deduce the expression of $\mathcal{S}_w^{-1}(k)$ we need to compute $\langle \phi, e_1\rangle$ and $\langle \phi, e_2\rangle$ in terms of g. Taking the scalar product of (4.13) with $e_i \in [H^{1/2}(\Gamma)]^2$ ($i = 1$ and 2) and using the fact that $\mathcal{L}_w^{-1}(k)$ is a self-adjoint operator, we find

$$\left\langle \theta_w(k) \begin{pmatrix} \langle \phi, e_1\rangle \\ \langle \phi, e_2\rangle \end{pmatrix}, \mathcal{L}_w^{-1}(k) e_i \right\rangle + \langle \phi, e_i\rangle = \langle g, \mathcal{L}_w^{-1}(k) e_i\rangle.$$

Decomposing the matrix $\theta_w(k)$ on the basis $\{e_i \otimes e_j\}_{i,j=1,2}$ we obtain the following the system

$$\Big(\mathrm{I} + \theta_w(k) Q_w(k)\Big) \begin{pmatrix} \langle \phi, e_1\rangle \\ \langle \phi, e_2\rangle \end{pmatrix} = \begin{pmatrix} \langle g, \mathcal{L}_w^{-1}(k) e_1\rangle \\ \langle g, \mathcal{L}_w^{-1}(k) e_2\rangle \end{pmatrix},$$

where $Q_w(k)$ is defined in (4.11). This system is invertible provided that $\det F_w(k) \neq 0$, in which case we deduce

$$\begin{pmatrix} \langle \phi, e_1\rangle \\ \langle \phi, e_2\rangle \end{pmatrix} = F_w^{-1}(k) \begin{pmatrix} \langle g, \mathcal{L}_w^{-1}(k) e_1\rangle \\ \langle g, \mathcal{L}_w^{-1}(k) e_2\rangle \end{pmatrix}, \tag{4.14}$$

and the anounced expression (4.12) follows from (4.13) and (4.14). Finally, since the operator valued function $k \mapsto \mathcal{L}_w^{-1}(k)$ is holomorphic in $D_{r_0}^+$, it follows directly from (4.12) that the poles of $\mathcal{S}_w^{-1}(k)$ coincide with those of the matrix valued function $k \mapsto \theta_w(k) F_w^{-1}(k)$. $\square$

Let $\mathcal{Z} := \{k_\ell\}_{\ell \in \mathbb{N}}$ be the set of zeros of the function α defined in (3.11), satisfying $k_\ell < k_{\ell+1}$ for all $\ell \in \mathbb{N}$. Let $\mathcal{P} := \{\ell\pi/h\}_{\ell \in \mathbb{N}^*}$ denote the poles of α. It is easily seen that each zero has multiplicity one. Moreover, since for each $\ell \in \mathbb{N}$, k_ℓ is an isolated point, there exists $r_\ell > 0$ such that

$$\big(\mathcal{Z} \cup \mathcal{P}\big) \cap D_{r_\ell}(k_\ell) = \{k_\ell\}.$$

We also introduce the integer $\ell_0 :=\ \max\{\ell \in \mathbb{N} : |k_\ell| < r_0\}$.

Next, we prove that the resonances are exactly the poles of the function $k \mapsto F_w^{-1}(k)$, and then we localize two resonances inside each disk $D_{r_\ell}(k_\ell)$, using the generalized Rouché Theorem (Theorem 5.11) and the following technical lemma, which allows one to replace the operator $\mathcal{L}_w(k)$ by the matrix $F_w(k)$ in the contour integral that appears in the statement of the Rouché Theorem.

Lemma 4.10. *For every disk $D \subset\subset D_{r_0}^+$ and every function $f : \overline{D} \to \mathbb{R}$, we have*

$$\operatorname{tr} \int_{\partial D} \mathcal{S}_w^{-1}(k) \frac{d\mathcal{S}_w(k)}{dk} f(k)\, dk = \operatorname{tr} \int_{\partial D} F_w^{-1}(k) \frac{dF_w(k)}{dk} f(k)\, dk.$$

Proof. Since the operator $\frac{d\mathcal{L}_w(k)}{dk}$ is self adjoint, we have that

$$\begin{aligned}\mathcal{S}_w^{-1}(k)\frac{d\mathcal{S}_w(k)}{dk} &= \mathcal{L}_w^{-1}(k)\frac{d\mathcal{L}_w(k)}{dk}\\ &+\mathcal{L}_w^{-1}(k)\theta_w(k)\Big(\mathrm{I}+\theta_w(k)Q_w(k)\Big)^{-1}\left[\Big(\mathrm{I}+\theta_w(k)Q_w(k)\Big)\theta_w^{-1}(k)\frac{d\theta_w(k)}{dk}\begin{pmatrix}\langle\cdot,e_1\rangle\\ \langle\cdot,e_2\rangle\end{pmatrix}\right.\\ &\left.-\begin{pmatrix}\langle\cdot,\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle\cdot,\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\frac{d\theta_w(k)}{dk}\begin{pmatrix}\langle\cdot,e_1\rangle\\ \langle\cdot,e_2\rangle\end{pmatrix}-\begin{pmatrix}\langle\cdot,\frac{d\mathcal{L}_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle\cdot,\frac{d\mathcal{L}_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\right].\end{aligned}$$

Using now the fact that the matrix $\frac{d\theta_w(k)}{dk}$ is symmetric, we get that

$$\begin{aligned}&\begin{pmatrix}\langle\cdot,\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle\cdot,\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\frac{d\theta_w(k)}{dk}\begin{pmatrix}\langle\cdot,e_1\rangle\\ \langle\cdot,e_2\rangle\end{pmatrix}\\ &\qquad=\begin{pmatrix}\langle e_1,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle & \langle e_2,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle e_1,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle & \langle e_2,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\begin{pmatrix}\langle\cdot,e_1\rangle\\ \langle\cdot,e_2\rangle\end{pmatrix}.\end{aligned}$$

Hence, multiplying by $f(k)$, integrating over ∂D and taking the trace we infer that

$$\begin{aligned}\mathrm{tr}\int_{\partial D}\mathcal{S}_w^{-1}(k)\frac{d\mathcal{S}_w(k)}{dk}f(k)\,dk &= \mathrm{tr}\int_{\partial D}\mathcal{L}_w^{-1}(k)\frac{d\mathcal{L}_w(k)}{dk}f(k)\,dk\\ &+\mathrm{tr}\int_{\partial D}\theta_w(k)\Big(\mathrm{I}+\theta_w(k)Q_w(k)\Big)^{-1}\left\{\left[\Big(\mathrm{I}+\theta_w(k)Q_w(k)\Big)\theta_w^{-1}(k)\frac{d\theta_w(k)}{dk}\right.\right.\\ &\left.-\begin{pmatrix}\langle e_1,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle & \langle e_2,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle e_1,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle & \langle e_2,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\right]\begin{pmatrix}\langle\cdot,\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle\cdot,\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\\ &\left.-\begin{pmatrix}\langle\cdot,\mathcal{L}_w^{-1}(k)\frac{d\mathcal{L}_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle\cdot,\mathcal{L}_w^{-1}(k)\frac{d\mathcal{L}_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\right\}f(k)\,dk.\end{aligned}$$

Using Remark 4.7, we observe that

$$\theta_w(k)Q_w(k)\theta_w^{-1}(k)\frac{d\theta_w(k)}{dk}=\begin{pmatrix}\langle e_1,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle & \langle e_2,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle e_1,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle & \langle e_2,\frac{d\theta_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix},$$

and consequently according to the symmetry of the matrix $\theta_w^{-1}(k)$ we deduce

$$\begin{aligned}\mathrm{tr}\int_{\partial D}\mathcal{S}_w^{-1}(k)\frac{d\mathcal{S}_w(k)}{dk}f(k)\,dk &= \mathrm{tr}\int_{\partial D}\mathcal{L}_w^{-1}(k)\frac{d\mathcal{L}_w(k)}{dk}f(k)\,dk\\ &+\mathrm{tr}\int_{\partial D}\theta_w(k)\Big(\mathrm{I}+\theta_w(k)Q_w(k)\Big)^{-1}\left\{\theta_w^{-1}(k)\frac{d\theta_w(k)}{dk}\begin{pmatrix}\langle\cdot,\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle\cdot,\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\right.\\ &\left.-\begin{pmatrix}\langle\cdot,\mathcal{L}_w^{-1}(k)\frac{d\mathcal{L}_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_1\rangle\\ \langle\cdot,\mathcal{L}_w^{-1}(k)\frac{d\mathcal{L}_w(k)}{dk}\mathcal{L}_w^{-1}(k)e_2\rangle\end{pmatrix}\right\}f(k)\,dk.\end{aligned}$$

From the relations

$$\left\{\begin{array}{rcl}\theta_w(k)\big(\mathrm{I}+\theta_w(k)Q_w(k)\big)^{-1}\theta_w^{-1}(k) &=& \big(\mathrm{I}+\theta_w(k)Q_w(k)\big)^{-1},\\ \theta_w(k)\big(\mathrm{I}+\theta_w(k)Q_w(k)\big)^{-1} &=& \big(\mathrm{I}+\theta_w(k)Q_w(k)\big)^{-1}\theta_w(k),\end{array}\right.$$

we infer

$$\operatorname{tr} \int_{\partial D} \mathcal{S}_w^{-1}(k) \frac{d\mathcal{S}_w(k)}{dk} f(k)\, dk = \operatorname{tr} \int_{\partial D} \mathcal{L}_w^{-1}(k) \frac{d\mathcal{L}_w(k)}{dk} f(k)\, dk$$

$$+\operatorname{tr} \int_{\partial D} \Big(\mathrm{I} + \theta_w(k) Q_w(k)\Big)^{-1} \left\{ \frac{d\theta_w(k)}{dk} \begin{pmatrix} \langle \cdot, \mathcal{L}_w^{-1}(k) e_1 \rangle \\ \langle \cdot, \mathcal{L}_w^{-1}(k) e_2 \rangle \end{pmatrix} \right.$$

$$\left. +\theta_w(k) \frac{d}{dk} \begin{pmatrix} \langle \cdot, \mathcal{L}_w^{-1}(k) e_1 \rangle \\ \langle \cdot, \mathcal{L}_w^{-1}(k) e_2 \rangle \end{pmatrix} \right\} f(k)\, dk,$$

so that we finally obtain

$$\begin{aligned} \operatorname{tr} \int_{\partial D} \mathcal{S}_w^{-1}(k) \frac{d\mathcal{S}_w(k)}{dk} f(k)\, dk \quad &= \quad \operatorname{tr} \int_{\partial D} \mathcal{L}_w^{-1}(k) \frac{d\mathcal{L}_w(k)}{dk} f(k)\, dk \\ &\quad +\operatorname{tr} \int_{\partial D} F_w^{-1}(k) \frac{dF_w(k)}{dk} f(k)\, dk. \end{aligned}$$

On the other hand, since the operator valued function $k \mapsto \mathcal{L}_w(k)$ is holomorphic in $D_{r_0}^+$, it follows from the generalized Rouché Theorem (Theorem 5.11) that

$$\operatorname{tr} \int_{\partial D} \mathcal{L}_w^{-1}(k)\, \frac{d\mathcal{L}_w(k)}{dk}\, f(k)\, dk = 0,$$

and the proof is complete. □

By (3.2), the poles of $k \mapsto \theta_w(k)$ are exactly those of $k \mapsto \alpha(k)$, *i.e.*, $\mathcal{P} := \{\ell\pi/h\}_{\ell \in \mathbb{N}^*}$. Applying Lemma 4.10 with $D = D_{r_\ell}(\ell\pi/h)$ (for $\ell \in \mathbb{N}^*$) and $f = 1$, we deduce that the poles of $k \mapsto \mathcal{S}_w^{-1}(k)$ are exactly those of $k \mapsto F_w^{-1}(k)$.

Corollary 4.11. *The resonances of the open cavities are the poles of the matrix valued function* $k \mapsto F_w^{-1}(k)$.

We now prove the existence of two resonances close to each k_ℓ, for every $\ell \in \mathbb{N}$.

Theorem 4.12. *Let* $w \in (0, w_0)$ *and* $k_\ell \in D_{r_0}^+$ *be a fixed zero of the function* α. *Then, there exists two resonances of the open cavities* $k_{\ell,w}^+$ *and* $k_{\ell,w}^-$ *inside the disk* $D_{r_\ell}(k_\ell)$ *which are respectively the zeroes of the functions*

$$\lambda_w^+(k) := 1 + \left[\frac{\alpha(k)}{w} + \frac{2}{\pi}(\ln k + \ln w) + 2\delta - \delta_1 \right] \langle e_1, \mathcal{L}_w^{-1}(k)(e_1 + e_2) \rangle, \tag{4.15}$$

and

$$\lambda_w^-(k) := 1 + \left(\frac{\alpha(k)}{w} - \delta_1 \right) \langle e_1, \mathcal{L}_w^{-1}(k)(e_1 - e_2) \rangle. \tag{4.16}$$

Proof. Fix $\ell \in \mathbb{N}$. From the definition of $F_w(k)$, $\theta_w(k)$ and $Q_w(k)$, we infer that for every $k \in \partial D_{r_\ell}(k_\ell)$,

$$F_w(k) = \frac{1}{w}\big[\alpha(k) Q_0 + \Sigma_w(k)\big],$$

$$\frac{dF_w(k)}{dk} = \frac{1}{w}\big[\alpha'(k) Q_0 + \Sigma'_w(k)\big],$$

and

$$F_w^{-1}(k) = \frac{w}{\alpha(k)}\, Q_0^{-1}\big[\mathrm{I} + \Upsilon_w(k)\big],$$

where $\Sigma_w(k)$ and $\Upsilon_w(k)$ are 2×2 matrices which are holomorphic on $\partial D_{r_\ell}(k_\ell)$ and such that

$$\sup_{k \in \partial D_{r_\ell}(k_\ell)} |\Sigma_w(k)| \to 0, \qquad \sup_{k \in \partial D_{r_\ell}(k_\ell)} |\Sigma'_w(k)| \to 0 \quad \text{and} \quad \sup_{k \in \partial D_{r_\ell}(k_\ell)} |\Upsilon_w(k)| \to 0$$

as $w \to 0$. Therefore, we see that

$$F_w^{-1}(k) \frac{dF_w(k)}{dk} \xrightarrow[w \to 0]{} \frac{\alpha'(k)}{\alpha(k)}\, \mathrm{I}$$

uniformly with respect to $k \in \partial D_{r_\ell}(k_\ell)$, and thus

$$\lim_{w\to 0} \operatorname{tr} \int_{\partial D_{r_\ell}(k_\ell)} F_w^{-1}(k)\frac{dF_w(k)}{dk}\,dk = 2\int_{\partial D_{r_\ell}(k_\ell)} \frac{\alpha'(k)}{\alpha(k)}\,dk.$$

On the other hand, by our choice of the radius r_ℓ, we know that the function α admits exactly one zero and no pole inside the disk $D_{r_\ell}(k_\ell)$. Applying the Residue Theorem (see *e.g.* [14]) we deduce that

$$\frac{1}{2i\pi}\int_{\partial D_{r_\ell}(k_\ell)} \frac{\alpha'(k)}{\alpha(k)}\,dk = 1,$$

and thus for w small enough (depending on ℓ) one has

$$\frac{1}{2i\pi} \operatorname{tr} \int_{\partial D_{r_\ell}(k_\ell)} F_w^{-1}(k)\frac{dF_w(k)}{dk}\,dk = 2.$$

in viex of Lemma 4.10 (with $D = D_{r_\ell}(k_\ell)$ and $f = 1$) it follows that

$$\frac{1}{2i\pi} \operatorname{tr} \int_{\partial D_{r_\ell}(k_\ell)} \mathcal{S}_w^{-1}(k)\frac{d\mathcal{S}_w(k)}{dk}\,dk = 2,$$

and thus the operator $\mathcal{S}_w(k)$ admits two characteristic values $k^+_{\ell,w}$ and $k^-_{\ell,w}$ in $D_{r_\ell}(k_\ell)$.

We know from Corollary 4.11 that the resonances $k^\pm_{\ell,w}$ are the poles of the function $F_w^{-1}(k)$ in $D_{r_\ell}(k_\ell)$. A simple computation shows that the eigenvalues of $F_w(k)$ are given by the functions $\lambda_w^\pm(k)$ defined in (4.15) and (4.16), and their associated eigenvectors are $\frac{e_1\pm e_2}{\sqrt{2}}$. Thus, one can write

$$F_w^{-1}(k) = \frac{(e_1+e_2)\otimes(e_1+e_2)}{2\lambda_w^+(k)} + \frac{(e_1-e_2)\otimes(e_1-e_2)}{2\lambda_w^-(k)}, \tag{4.17}$$

and consequently, the poles of $F_w^{-1}(k)$ in $D_{r_\ell}(k_\ell)$ are exactly the zeros of $\lambda_w^\pm(k)$ in $D_{r_\ell}(k_\ell)$. □

We next establish an asymptotic expansion of the resonances of the open cavities as their width w tends to zero.

Theorem 4.13. *For every $\ell \in \mathbb{N}$, the resonances $k^\pm_{\ell,w}$ have the following asymptotic expansion:*

$$\begin{aligned}
k^+_{\ell,w} &= k_\ell - \frac{2w\ln w}{\pi\alpha'(k_\ell)} - \frac{1}{\alpha'(k_\ell)}\left[\frac{1}{q_0^+} + 2\left(\frac{\ln k_\ell}{\pi} + \delta - \frac{\delta_1}{2}\right)\right] w + o(w),\\
k^-_{\ell,w} &= k_\ell - \frac{1-\delta_1 q_0^-}{q_0^-\,\alpha'(k_\ell)}w - \frac{1}{(q_0^-)^2\alpha'(k_\ell)}\left[\frac{(1-\delta_1 q_0^-)^2\alpha''(k_\ell)}{2\alpha'(k_\ell)^2} + q_1^-\right] w^2 + \frac{q_{12}^-(k_\ell)}{(q_0^-)^2\alpha'(k_\ell)}\,w^3\ln w\\
&\quad + \frac{1}{(q_0^-)^2\alpha'(k_\ell)}\Bigg[q_2^-(k_\ell) - \frac{(q_1^-)^2}{q_0^-} - \frac{q_1^-(1-\delta_1 q_0^-)\alpha''(k_\ell)}{q_0^-\,\alpha'(k_\ell)^2}\\
&\qquad - \frac{(1-\delta_1 q_0^-)^3}{2q_0^-\,\alpha'(k_\ell)^3}\left(\frac{\alpha''(k_\ell)^2}{\alpha'(k_\ell)} - \frac{\alpha^{(3)}(k_\ell)}{3}\right)\Bigg] w^3 + o(w^3).
\end{aligned}$$

In particular, we have

$$\begin{aligned}
\operatorname{Im}(k^+_{\ell,w}) &= \frac{w}{\alpha'(k_\ell)} + o(w),\\
\operatorname{Im}(k^-_{\ell,w}) &= \frac{\operatorname{Im}(q_2^-(k_\ell))}{(q_0^-)^2\,\alpha'(k_\ell)}\,w^3 + o(w^3).
\end{aligned}$$

Proof. Fix $\ell \in \mathbb{N}$. Since $k^\pm_{\ell,w}$ is a simple zero of the function $\lambda_w^\pm(k)$ in $D_{r_\ell}(k_\ell)$, we deduce from the Residue Theorem that

$$k^\pm_{\ell,w} - k_\ell = \frac{1}{2i\pi}\int_{\partial D_{r_\ell}(k_\ell)} \frac{\frac{d\lambda_w^\pm}{dk}(k)}{\lambda_w^\pm(k)}\,(k-k_\ell)\,dk.$$

We now derive an asymptotic expansion of the functions $\frac{d\lambda_w^\pm}{dk}(k)/\lambda_w^\pm(k)$ in terms of w. We start with $\lambda_w^+(k)$. From (4.15) and (4.10) we have that for every $k \in \partial D_{r_\ell}(k_\ell)$,

$$\frac{d\lambda_w^+}{dk}(k) = \frac{\alpha'(k)}{w}\left\{q_0^+ + \left[\frac{2q_0^+}{k\pi\alpha'(k)} - q_1^+\right] w + \varepsilon_{w,1}^+(k)\right\}$$

and

$$\frac{1}{\lambda_w^+(k)} = \frac{w}{\alpha(k)q_0^+}\left\{1 - \frac{2w\ln w}{\pi\alpha(k)} - \left[\frac{2}{\alpha(k)}\left(\frac{\ln k}{\pi} + \delta - \frac{\delta_1}{2}\right) + \frac{1}{q_0^+}\left(\frac{1}{\alpha(k)} - q_1^+\right)\right] w + \varepsilon_{w,2}^+(k)\right\},$$

for some functions $\varepsilon_{w,1}^+(k)$ and $\varepsilon_{w,2}^+(k)$ holomorphic on $\partial D_{r_\ell}(k_\ell)$, such that

$$\sup_{k\in\partial D_{r_\ell}(k_\ell)} w^{-1}|\varepsilon_{w,1}^+(k)| \xrightarrow[w\to 0]{} 0 \quad \text{and} \quad \sup_{k\in\partial D_{r_\ell}(k_\ell)} w^{-1}|\varepsilon_{w,2}^+(k)| \xrightarrow[w\to 0]{} 0.$$

Hence, we obtain that

$$\frac{\frac{d\lambda_w^+}{dk}(k)}{\lambda_w^+(k)} = \frac{\alpha'(k)}{\alpha(k)}\left\{1 - \frac{2w\ln w}{\pi\alpha(k)} + \left[\frac{2}{\pi k\alpha'(k)} - \frac{2}{\alpha(k)}\left(\frac{\ln k}{\pi} + \delta - \frac{\delta_1}{2}\right) - \frac{1}{\alpha(k)q_0^+}\right] w + \eta_w^+(k)\right\}, \quad (4.18)$$

where $\eta_w^+(k)$ is a holomorphic function on $\partial D_{r_\ell}(k_\ell)$ such that

$$\sup_{k\in\partial D_{r_\ell}(k_\ell)} w^{-1}|\eta_w^+(k)| \xrightarrow[w\to 0]{} 0.$$

Since k_ℓ is a simple zero of the function α, we deduce from the Residue Theorem that for every holomorphic function f in $\overline{D_{r_\ell}(k_\ell)}$, then

$$\frac{1}{2i\pi}\int_{\partial D_{r_\ell}(k_\ell)} (k-k_\ell)\frac{\alpha'(k)}{\alpha(k)} f(k)\, dk = 0, \quad (4.19)$$

and

$$\frac{1}{2i\pi}\int_{\partial D_{r_\ell}(k_\ell)} (k-k_\ell)\frac{\alpha'(k)}{\alpha(k)^2} f(k)\, dk = \frac{f(k_\ell)}{\alpha'(k_\ell)}. \quad (4.20)$$

Thanks to (4.20) we get the asymptotic expansion of $k_{\ell,w}^+$ multiplying (4.18) by $k-k_\ell$ and integrating over $\partial D_{r_\ell}(k_\ell)$. The imaginary part of $k_{\ell,w}^+$ is obtained using the expression (3.8) of δ.

We now treat the other resonance $k_{\ell,w}^-$. We will observe *a posteriori* that the imaginary part of $k_{\ell,w}^-$ scales (at least) like w^3. For this reason, we expand the integrand $\frac{d\lambda_w^-}{dk}(k)/\lambda_w^-(k)$ up to third order. From (4.16) and (4.10) we have for $k \in \partial D_{r_\ell}(k_\ell)$,

$$\begin{aligned}
\frac{d\lambda_w^-}{dk}(k) \quad &= \quad \frac{\alpha'(k)q_0^-}{w}\left\{1 - \frac{q_1^-}{q_0^-}w + \frac{1}{q_0^-}\left[q_{12}^-(k) + \frac{\alpha(k)}{\alpha'(k)}\frac{dq_{12}^-}{dk}(k)\right] w^2\ln w\right. \\
&+ \frac{1}{q_0^-}\left[q_2^-(k) + \frac{\alpha(k)}{\alpha'(k)}\frac{dq_2^-}{dk}(k)\right] w^2 \\
&+ \frac{1}{q_0^-}\left[q_{13}^-(k) + \frac{\alpha(k)}{\alpha'(k)}\frac{dq_{13}^-}{dk}(k) - \frac{\delta_1}{\alpha'(k)}\frac{dq_{12}^-}{dk}(k)\right] w^3\ln w \\
&+ \left.\frac{1}{q_0^-}\left[q_3^-(k) + \frac{\alpha(k)}{\alpha'(k)}\frac{dq_3^-}{dk}(k) - \frac{\delta_1}{\alpha'(k)}\frac{dq_2^-}{dk}(k)\right] w^3 + \varepsilon_{w,1}^-(k)\right\}
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{\lambda_w^-(k)} &= \frac{w}{\alpha(k)q_0^-}\Bigg\{1-\frac{1}{q_0^-}\left(\frac{1-\delta_1 q_0^-}{\alpha(k)}-q_1^-\right)w-\frac{q_{12}^-(k)}{q_0^-}\,w^2\ln w\\
&+\frac{1}{q_0^-}\left[\frac{1}{q_0^-}\left(\frac{1-\delta_1 q_0^-}{\alpha(k)}-q_1^-\right)^2-\left(q_2^-(k)+\frac{\delta_1 q_1^-}{\alpha(k)}\right)\right]w^2\\
&+\frac{1}{q_0^-}\left[\frac{2q_{12}^-(k)}{q_0^-}\left(\frac{1-\delta_1 q_0^-}{\alpha(k)}-q_1^-\right)-\left(q_{13}^-(k)-\frac{\delta_1 q_{12}^-}{\alpha(k)}\right)\right]w^3\ln w\\
&+\frac{1}{q_0^-}\Bigg[\frac{2}{q_0^-}\left(\frac{1-\delta_1 q_0^-}{\alpha(k)}-q_1^-\right)\left(q_2^-(k)+\frac{\delta_1 q_1^-}{\alpha(k)}\right)-\frac{1}{(q_0^-)^2}\left(\frac{1-\delta_1 q_0^-}{\alpha(k)}-q_1^-\right)^3\\
&\qquad-\left(q_3^-(k)-\frac{\delta_1 q_2^-}{\alpha(k)}\right)\Bigg]w^3+\varepsilon_{w,2}^-(k)\Bigg\},
\end{aligned}$$

for some holomorphic functions $\varepsilon_{w,1}^-(k)$ and $\varepsilon_{w,2}^-(k)$ defined on $\partial D_{r_\ell}(k_\ell)$, such that

$$\sup_{k\in\partial D_{r_\ell}(k_\ell)} w^{-3}|\varepsilon_{w,1}^-(k)| \xrightarrow[w\to 0]{} 0 \quad\text{and}\quad \sup_{k\in\partial D_{r_\ell}(k_\ell)} w^{-3}|\varepsilon_{w,2}^-(k)| \xrightarrow[w\to 0]{} 0.$$

Hence, we obtain that

$$\begin{aligned}
\frac{\frac{d\lambda_w^-}{dk}(k)}{\lambda_w^-(k)} &= \frac{\alpha'(k)}{\alpha(k)}\Bigg\{1-\frac{1-\delta_1 q_0^-}{q_0^-\alpha(k)}w+\frac{\alpha(k)}{q_0^-\alpha'(k)}\frac{dq_{12}^-}{dk}(k)w^2\ln w\\
&+\frac{1}{q_0^+}\left[\frac{\alpha(k)}{\alpha'(k)}\frac{dq_2^-}{dk}(k)-\frac{\delta_1 q_1^-}{\alpha(k)}+\frac{1-\delta_1 q_0^-}{q_0^-\alpha(k)}\left(\frac{1-\delta_1 q_0^-}{\alpha(k)}-q_1^-\right)\right]w^2\\
&+\frac{1}{q_0^+}\left[\frac{q_{12}^-(k)}{q_0^-\alpha(k)}+\frac{\alpha(k)}{\alpha'(k)}\left(\frac{dq_{13}^-}{dk}(k)-\frac{dq_{12}^-}{dk}(k)\frac{1}{q_0^-}\left(\frac{1}{\alpha(k)}-q_1^-\right)\right)\right]w^3\ln w\\
&+\frac{1}{q_0^-}\Bigg[\frac{\alpha(k)}{\alpha'(k)}\frac{dq_3^-}{dk}(k)-\frac{\alpha(k)}{\alpha'(k)q_0^-}\frac{dq_2^-}{dk}(k)\left(\frac{1}{\alpha(k)}-q_1^-\right)+\frac{q_2^-(k)}{\alpha(k)q_0^-}+\frac{\delta_1(q_1^-)^2}{q_0^-\alpha(k)}\\
&+\left(\frac{1-\delta_1 q_0^-}{\alpha(k)}-q_1^-\right)\left(\frac{q_1^-(1+\delta_1 q_0^-)}{(q_0^-)^2\alpha(k)}-\frac{(1-\delta_1 q_0^-)^2}{(q_0^-)^2\alpha(k)^2}\right)\Bigg]w^3+\eta_w^-(k).\Bigg\},
\end{aligned}\tag{4.21}$$

where again $\eta_w^-(k)$ is a holomorphic function on $\partial D_{r_\ell}(k_\ell)$ such that

$$\sup_{k\in\partial D_{r_\ell}(k_\ell)} w^{-3}|\eta_w^-(k)| \xrightarrow[w\to 0]{} 0.$$

Since k_ℓ is a simple zero of the function α, we deduce from the Residue Theorem that for every holomorphic function f in $\overline{D_{r_\ell}(k_\ell)}$, then

$$\frac{1}{2i\pi}\int_{\partial D_{r_\ell}(k_\ell)}(k-k_\ell)\frac{\alpha'(k)}{\alpha(k)^3}\,f(k)\,dk=-\frac{\alpha''(k_\ell)}{2\alpha'(k_\ell)^3}\,f(k_\ell),\tag{4.22}$$

and

$$\frac{1}{2i\pi}\int_{\partial D_{r_\ell}(k_\ell)}(k-k_\ell)\frac{\alpha'(k)}{\alpha(k)^4}\,f(k)\,dk=\left(\frac{\alpha''(k_\ell)^2}{2\alpha'(k_\ell)^5}-\frac{\alpha^{(3)}(k_\ell)}{6\alpha'(k_\ell)^4}\right)f(k_\ell).\tag{4.23}$$

Thanks to (4.20), (4.23) we get the asymptotic expansion of $k_{\ell,w}^-$ multiplying (4.21) by $k-k_\ell$ and integrating over $\partial D_{r_\ell}(k_\ell)$. The imaginary part of $k_{\ell,w}^-$ is obtained noticing that the only complex number in the expression of $k_{\ell,w}^-$ is $q_2^-(k_\ell)$ (see Remark 4.8). □

Remark 4.14. Using the definition of $q_2^-(k_\ell)$ in Remark 4.8 together with (3.4), (3.7) and (3.15), we infer that

$$\mathrm{Im}(q_2^-(k_\ell))=-\frac{\langle\mathcal{S}^{-1}(e_1-e_2),\mathcal{S}_2(k_\ell)\mathcal{S}^{-1}(e_1-e_2)\rangle}{2},$$

where we also used the fact that $\mathcal{S}^{-1}$ is self adjoint. Moreover since the imaginary part of $\mathcal{S}_2(k_\ell)$ is an integral operator with kernel

$$(x,y)\mapsto \frac{k_\ell^2}{8}\begin{pmatrix} (x-y)^2 & (x-y+2d)^2 \\ (x-y-2d)^2 & (x-y)^2 \end{pmatrix},$$

we deduce, expanding the squares, that

$$\mathrm{Im}(q_2^-(k_\ell)) = \frac{k_\ell^2}{8}\Big(\langle xe_1+xe_2, \mathcal{S}^{-1}(e_1-e_2)\rangle - 2q_0^- d\Big)^2.$$

Since in view of (3.3) the operator $\mathcal{S}$ depends on d, it is possible to prove that the mapping $d\mapsto \langle xe_1+xe_2,\mathcal{S}^{-1}(e_1-e_2)\rangle-2q_0^- d$ is analytic in $(1,+\infty)$. Hence $\mathrm{Im}(q_2^-(k_\ell))$ vanishes for at most countably many isolated points. Thus, either we choose $d>1$ so that $\mathrm{Im}(q_2^-(k_\ell))\neq 0$ or one can perform a higher order asymptotic expansion to get a non zero coefficient that multiplies some power of w. Indeed, there exists $n\in\mathbb{N}$ $(n\geq 3)$ such that $\mathrm{Im}(k_{\ell,w}^-) = c_\ell^{(n)} w^n + o(w^n)$ with $c_\ell^{(n)}\neq 0$ because, otherwise, the resonance $k_{\ell,w}^-$ would be purely real. In any cases from the application point of view, what matters is that $\mathrm{Im}(k_l^-) < \mathrm{Im}(k_l^+)$ as w tends to zero.

To conclude this section, we examine to behavior of the operator $\mathcal{S}_w^{-1}(k)$ in two different regions of the complex domain $D_{r_0}^+$. We first focus in the resonance zone where the contribution of the singular part is large with respect to the regular part. Close the resonances, we give an asymptotic expansion of $\mathcal{S}_w^{-1}(k)$ as a Laurent series, as well as an asymptotic expansion in terms of w.

Theorem 4.15. *Let $k\in D_{r_0}^+$ be a frequency close to the resonances. Then there exists a holomorphic operator $\mathcal{H}_w(k) : [H^{1/2}(\Gamma)]^2 \to [\widetilde{H}^{-1/2}(\Gamma)]^2$ and finite dimensional operators $\mathbf{\Delta}_{\ell,w}^{\pm} : [H^{1/2}(\Gamma)]^2 \to [\widetilde{H}^{-1/2}(\Gamma)]^2$ (for $\ell = 0,\dots,\ell_0$) such that*

$$\mathcal{S}_w^{-1}(k) = \sum_{\ell=0}^{\ell_0}\frac{\mathbf{\Delta}_{\ell,w}^+}{k-k_{\ell,w}^+} + \sum_{\ell=0}^{\ell_0}\frac{\mathbf{\Delta}_{\ell,w}^-}{k-k_{\ell,w}^-} + \mathcal{H}_w(k), \tag{4.24}$$

where

$$\mathbf{\Delta}_{\ell,w}^{\pm} = -\frac{\Big\langle\cdot\,,\mathcal{L}_w^{-1}\big(k_{\ell,w}^{\pm}\big)(e_1\pm e_2)\Big\rangle}{2\frac{d\lambda_w^{\pm}}{dk}\big(k_{\ell,w}^{\pm}\big)}\,\mathcal{L}_w^{-1}(k_{\ell,w}^{\pm})\Big[\theta_w\big(k_{\ell,w}^{\pm}\big)(e_1\pm e_2)\Big]. \tag{4.25}$$

Moreover, the following asymptotic expansions hold:

$$\mathbf{\Delta}_{\ell,w}^{\pm} \;=\; \frac{\big\langle\cdot\,,\mathcal{S}^{-1}(e_1\pm e_2)\big\rangle}{2(q_0^{\pm})^2\alpha'(k_\ell)}\,\mathcal{S}^{-1}(e_1\pm e_2)\,w + \mathbf{\Lambda}_{\ell,w}^{\pm},$$

where $\mathbf{\Lambda}_{\ell,w}^{\pm}$ are finite dimensional operators from $[H^{1/2}(\Gamma)]^2$ to $[\widetilde{H}^{-1/2}(\Gamma)]^2$ such that

$$w^{-1}\|\mathbf{\Lambda}_{\ell,w}^{\pm}\| \xrightarrow[w\to 0]{} 0.$$

Proof. According to Theorem 4.2, Proposition 4.9 and Theorem 4.12, it is clear that $\mathcal{S}_w^{-1}(k)$ expands as (4.24) close to the resonances, and that for each $\ell = 0,\dots,\ell_0$,

$$\mathbf{\Delta}_{\ell,w}^{\pm} = \lim_{k\to k_{\ell,w}^{\pm}}(k-k_{\ell,w}^{\pm})\mathcal{S}_w^{-1}(k).$$

Using the expression of $\mathcal{S}_w^{-1}(k)$ in (4.12) and the fact that $\mathcal{L}_w^{-1}(k)$ and $\theta_w(k)$ are holomorphic in a neighborhood of $k_{\ell,w}^{\pm}$, we infer that

$$\begin{aligned}\mathbf{\Delta}_{\ell,w}^{\pm} &= -\mathcal{L}_w^{-1}\big(k_{\ell,w}^{\pm}\big)\theta_w\big(k_{\ell,w}^{\pm}\big)(e_1\pm e_2)\Big\langle\cdot,\mathcal{L}_w^{-1}\big(k_{\ell,w}^{\pm}\big)(e_1\pm e_2)\Big\rangle\lim_{k\to k_{\ell,w}^{\pm}}\frac{(k-k_{\ell,w}^{\pm})}{2\lambda_w^{\pm}(k)}\\ &= -\frac{\Big\langle\cdot\,,\mathcal{L}_w^{-1}\big(k_{\ell,w}^{\pm}\big)(e_1\pm e_2)\Big\rangle}{2\frac{d\lambda_w^{\pm}}{dk}\big(k_{\ell,w}^{\pm}\big)}\,\mathcal{L}_w^{-1}(k_{\ell,w}^{\pm})\Big[\theta_w\big(k_{\ell,w}^{\pm}\big)(e_1\pm e_2)\Big],\end{aligned}$$

where we used (4.17) for $F_w^{-1}(k)$ in the first equality.

We now derive the asymptotic expansion of $\mathbf{\Delta}^-_{\ell,w}$. According to (4.10) we have that

$$\mathcal{L}_w^{-1}\big(k^\pm_{\ell,w}\big) = \mathcal{S}^{-1} - \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}w + \tilde{\mathbf{\Lambda}}^\pm_{\ell,w}, \tag{4.26}$$

where $\tilde{\mathbf{\Lambda}}^\pm_{\ell,w}$ is a finite dimensional operator from $[H^{1/2}(\Gamma)]^2$ to $[\widetilde{H}^{-1/2}(\Gamma)]^2$ such that

$$w^{-1}\|\tilde{\mathbf{\Lambda}}^\pm_\ell(w)\| \xrightarrow[w\to 0]{} 0.$$

From (3.2), we have that

$$\theta_w\big(k^-_{\ell,w}\big)(e_1-e_2) = \left(\frac{\alpha\big(k^-_{\ell,w}\big)}{w} - \delta_1\right)(e_1-e_2),$$

and we use Taylor expansion of $k\mapsto\alpha(k)$ around k_ℓ together with Theorem 4.13 to obtain

$$\theta_w\big(k^-_{\ell,w}\big)(e_1-e_2) = \frac{1}{w}\left\{-\frac{w}{q_0^-} + o(w)\right\}(e_1-e_2). \tag{4.27}$$

Moreover, from (4.16) and (4.26), we deduce that

$$\frac{d\lambda^-_w}{dk}\big(k^-_{\ell,w}\big) = \frac{\alpha'\big(k^-_{\ell,w}\big)q_0^-}{w}\left(1 - \frac{q_1^-}{q_0^-}\,w + o(w)\right), \tag{4.28}$$

and using again a Taylor expansion of $k\mapsto\alpha'(k)$ around k_ℓ together with Theorem 4.9 leads to

$$\frac{d\lambda^-_w}{dk}\big(k^-_{\ell,w}\big) = \frac{\alpha'(k_\ell)q_0^-}{w}\left\{1 - \left[\frac{q_1^-}{q_0^-} + \frac{(1-\delta_1 q_0^-)a''(k_\ell)}{q_0^-\,\alpha'(k_\ell)^2}\right]w + o(w)\right\}. \tag{4.29}$$

Hence, gathering (4.26), (4.27), (4.28) and (4.29), we obtain the expected asymptotic of $\mathbf{\Delta}^-_{\ell,w}$.

We proceed similarly for $\mathbf{\Delta}^+_{\ell,w}$. Indeed, from (3.2) we have

$$\theta_w\big(k^+_{\ell,w}\big)(e_1+e_2) = \left\{\frac{\alpha\big(k^+_{\ell,w}\big)}{w} + \frac{2}{\pi}\left(\ln k^+_{\ell,w} + \ln w\right) + 2\delta - \delta_1\right\}(e_1+e_2),$$

and thus, using a Taylor expansion of $k\mapsto\alpha(k)$ and $k\mapsto\ln k$ around k_ℓ together with Theorem 4.9, we get that

$$\theta_w\big(k^+_{\ell,w}\big)(e_1+e_2) = \frac{1}{w}\left\{-\frac{w}{q_0^+} + o(w)\right\}(e_1+e_2). \tag{4.30}$$

Moreover, from (4.15), we deduce

$$\frac{d\lambda^+_w}{dk}\big(k^+_{\ell,w}\big) = \frac{1}{w}\left\{\alpha'\big(k^+_{\ell,w}\big)q_0^+ + \left[\frac{2q_0^+}{k^+_{\ell,w}\pi} - q_1^+\alpha'\big(k^+_{\ell,w}\big)\right]w + o(w)\right\}, \tag{4.31}$$

and a Taylor expansion of $k\mapsto\alpha'(k)$ and $k\mapsto 1/k$ around k_ℓ together with Theorem 4.9 leadsto

$$\begin{aligned}\frac{d\lambda^+_w}{dk}\big(k^+_{\ell,w}\big) \quad = \quad & \frac{\alpha'(k_\ell)q_0^+}{w}\left\{1 - \frac{2\alpha''(k_\ell)}{\pi\alpha'(k_\ell)^2}\,w\ln w\right.\\ & \left. - \left[\frac{\alpha''(k_\ell)}{\alpha'(k_\ell)^2}\left(\frac{1}{q_0^+} + 2\left(\frac{\ln k_\ell}{\pi} + \delta - \frac{\delta_1}{2}\right)\right) - \frac{2}{\pi k_\ell\alpha'(k_\ell)} + \frac{q_1^+}{q_0^+}\right]w + o(w)\right\}.\end{aligned} \tag{4.32}$$

Hence, gathering (4.26), (4.30), (4.31) and (4.32), we obtain the expected asymptotic of $\mathbf{\Delta}^+_{\ell,w}$. □

In the non resonance zone, the contribution of the singular part is negligible. The following result gives an asymptotic expansion of $\mathcal{S}_w^{-1}(k)$ in terms of w in this region $D^+_{r_0}\setminus\mathcal{Z}$, *i.e.*, when $\alpha(k)\neq 0$.

Theorem 4.16. *If $k \in D_{r_0}^+ \setminus \mathcal{Z}$, then*

$$\begin{aligned}
\mathcal{S}_w^{-1}(k) &= \mathcal{S}^{-1} - \frac{\mathcal{S}^{-1}(e_1+e_2)}{2q_0^+}\langle \cdot\,, \mathcal{S}^{-1}(e_1+e_2)\rangle - \frac{\mathcal{S}^{-1}(e_1-e_2)}{2q_0^-}\langle \cdot\,, \mathcal{S}^{-1}(e_1-e_2)\rangle \\
&+ \Bigg\{ -\mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1} + \left[\frac{\mathcal{S}^{-1}(e_1+e_2)}{2(q_0^+)^2}\left(\frac{1}{\alpha(k)} - q_1^+\right) + \frac{\mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}(e_1+e_2)}{2q_0^+}\right]\langle \cdot\,, \mathcal{S}^{-1}(e_1+e_2)\rangle \\
&+ \left[\frac{\mathcal{S}^{-1}(e_1-e_2)}{2(q_0^-)^2}\left(\frac{1}{\alpha(k)} - q_1^-\right) + \frac{\mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}(e_1-e_2)}{2q_0^-}\right]\langle \cdot\,, \mathcal{S}^{-1}(e_1-e_2)\rangle \\
&+ \frac{\mathcal{S}^{-1}(e_1+e_2)}{2q_0^+}\langle \cdot\,, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}(e_1+e_2)\rangle + \frac{\mathcal{S}^{-1}(e_1-e_2)}{2q_0^-}\langle \cdot\,, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}(e_1-e_2)\rangle \Bigg\} w + \mathbf{x}i_w(k),
\end{aligned}$$

where $\mathbf{x}i_w(k)$ is a holomorphic operator from $[H^{1/2}(\Gamma)]^2$ to $[\widetilde{H}^{-1/2}(\Gamma)]^2$ such that

$$w^{-1}\|\mathbf{x}i_w(k)\| \xrightarrow[w\to 0]{} 0 \quad \textit{for every } k \in D_{r_0}^+ \setminus \mathcal{Z}.$$

Proof. Fix $k \in D_{r_0}^+ \setminus \mathcal{Z}$. We use the expression (4.12) for $\mathcal{S}_w^{-1}(k)$ and (4.17) for $F_w^{-1}(k)$ to prove that

$$\begin{aligned}
\mathcal{S}_w^{-1}(k) &= \mathcal{L}_w^{-1}(k) - \frac{\langle \cdot, \mathcal{L}_w^{-1}(k)(e_1-e_2)\rangle}{2\lambda_w^-(k)}\mathcal{L}_w^{-1}(k)\theta_w(k)(e_1-e_2) \\
&\quad - \frac{\langle \cdot, \mathcal{L}_w^{-1}(k)(e_1+e_2)\rangle}{2\lambda_w^+(k)}\mathcal{L}_w^{-1}(k)\theta_w(k)(e_1+e_2).
\end{aligned}$$

Since $\alpha(k) \neq 0$, we get that

$$\begin{aligned}
\frac{\langle \cdot, \mathcal{L}_w^{-1}(k)(e_1-e_2)\rangle}{2\lambda_w^-(k)} &= \frac{w}{2\alpha(k)q_0^-}\Bigg\{\langle \cdot, \mathcal{S}^{-1}(e_1-e_2)\rangle + \Bigg[\frac{1}{q_0^-}\left(q_1^- - \frac{1-\delta_1 q_0^-}{\alpha(k)}\right)\langle \cdot, \mathcal{S}^{-1}(e_1-e_2)\rangle \\
&\quad - \langle \cdot, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}(e_1-e_2)\rangle\Bigg] w + \mathbf{x}i_w^-(k)\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\langle \cdot, \mathcal{L}_w^{-1}(k)(e_1+e_2)\rangle}{2\lambda_w^+(k)} &= \frac{w}{2\alpha(k)q_0^+}\Bigg\{\langle \cdot, \mathcal{S}^{-1}(e_1+e_2)\rangle - \frac{2}{\pi\alpha(k)}\langle \cdot, \mathcal{S}^{-1}(e_1+e_2)\rangle w\ln w \\
&\quad - \Bigg[\left(\frac{1}{q_0^+}\left(\frac{1}{\alpha(k)} - q_1^+\right) + \frac{2}{\alpha(k)}\left(\frac{\ln k}{\pi} + \delta - \frac{\delta_1}{2}\right)\right)\langle \cdot, \mathcal{S}^{-1}(e_1+e_2)\rangle \\
&\quad + \langle \cdot, \mathcal{S}^{-1}\mathcal{S}_1\mathcal{S}^{-1}(e_1+e_2)\rangle\Bigg] w + \mathbf{x}i_w^+(k)\Bigg\}.
\end{aligned}$$

where $\mathbf{x}i_w^{\pm}(k)$ are holomorphic operators from $[H^{1/2}(\Gamma)]^2$ to $[\widetilde{H}^{-1/2}(\Gamma)]^2$ such that

$$w^{-1}\|\mathbf{x}i_w^{\pm}(k)\| \xrightarrow[w\to 0]{} 0 \quad \text{for every } k \in D_{r_0}^+ \setminus \mathcal{Z}.$$

Formula (3.2) for $\theta_w(k)$ and the asymptotic expansion (4.10) of $\mathcal{L}_w^{-1}(k)$ to complete the proof of the theorem. □

5. Asymptotic of the field u

We first derive an asymptotic expansion of the field $u(x;w)$ far from the resonances, *i.e.*, when $\alpha(k) \neq 0$. We shall distinguish the far field and the near field.

Theorem 5.1. *Let $k \in D_{r_0}^+ \setminus \mathcal{Z}$. Then for every $y \in \Omega_e$, one has*

$$u(y) = u_e(y) + \frac{iu_e(0)}{\alpha(k)}H_0^{(1)}(k|y|)w + \rho_w(y),$$

for some function $\rho_w \in H^1_{\rm loc}(\Omega_e)$ such that $w^{-1}\rho_w \to 0$ in $H^1_{\rm loc}(\Omega_e)$. Moreover, if $y = (wy_1 \pm wd, wy_2)$ for some $y_1 \in \Gamma$ and $y_2 > 0$, then

$$\begin{aligned} u(wy_1 \pm wd, wy_2) &= u_e(wy_1 \pm wd, wy_2) - \frac{2u_e(0)}{\pi\alpha(k)} w \ln w \\ &\quad - \frac{2u_e(0)}{\pi\alpha(k)} \left[\delta + \frac{\ln k}{\pi} + \frac{\ln |(y_1 \pm d, y_2)|}{\pi} \right] w + \overline{\rho}_w(y_1, y_2), \end{aligned}$$

where $\overline{\rho}_w \in H^1(B_1^+)$ satisfies $w^{-1}\overline{\rho}_w \to 0$ in $H^1(B_1^+)$.

Proof. Let $y \in \Omega_e$ be fixed. One has

$$\begin{aligned} G_e(wt, 0; y) &= -\frac{i}{2} H_0^{(1)}(k|y - (wt, 0)|) \\ &= -\frac{i}{2} H_0^{(1)}(k|y|) - \sum_{n=1}^{+\infty} \eta_n(y)(wt)^n, \end{aligned} \tag{5.1}$$

uniformly with respect to $t \in [-4, 4]$, for some functions $\eta_n(y)$. Using (2.4) we deduce

$$\begin{aligned} u_e(wt, 0) &= \int_{\Omega_e} G_e(wt, 0; z) f(z)\, dz \\ &= -\frac{i}{2} \int_{\Omega_e} H_0^{(1)}(k|z|) f(z)\, dz - \sum_{n=1}^{+\infty} \overline{\eta}_n (wt)^n, \end{aligned} \tag{5.2}$$

uniformly with respect to $t \in [-4, 4]$, where the coefficients $\overline{\eta}_n$ are defined by

$$\overline{\eta}_n := \int_{\Omega_e} \eta_n(y) f(y)\, dy.$$

Using (2.7) together with the integral equations (2.9)-(2.10), and after a change of variables we get

$$u(y) = u_e(y) - \left\langle \begin{pmatrix} G_e(w \cdot -wd, 0; y) \\ G_e(w \cdot +wd, 0; y) \end{pmatrix}, \mathcal{S}_w^{-1}(k) \begin{pmatrix} u_e(w \cdot -wd, 0) \\ u_e(w \cdot +wd, 0) \end{pmatrix} \right\rangle. \tag{5.3}$$

From Theorem 4.16 we have

$$\mathcal{S}_w^{-1}(k)(e_1 \pm e_2) = \frac{\mathcal{S}^{-1}(e_1 \pm e_2)}{q_0^{\pm}\alpha(k)} w + o(w) \quad \text{in } [\widetilde{H}^{-1/2}(\Gamma)]^2,$$

and thus the integral equations (2.9) and (2.10) imply

$$\begin{aligned} \mathcal{S}_w^{-1}(k) \begin{pmatrix} u_e(w \cdot -wd, 0) \\ u_e(w \cdot +wd, 0) \end{pmatrix} &= -\frac{i}{2} \left(\int_{\Omega_E} H_0^{(1)}(k|z|) f(z)\, dz \right) \frac{\mathcal{S}^{-1}(e_1 + e_2)}{q_0^+ \alpha(k)} w \\ &\quad + \overline{\eta}_1 d \frac{\mathcal{S}^{-1}(e_1 - e_2)}{q_0^- \alpha(k)} w - \overline{\eta}_1 \mathcal{S}^{-1}(xe_1 + xe_2) w \\ &\quad + \overline{\eta}_1 \frac{\mathcal{S}^{-1}(e_1 + e_2)}{2q_0^+} \langle (xe_1 + xe_2), \mathcal{S}^{-1}(e_1 + e_2) \rangle w \\ &\quad + \overline{\eta}_1 \frac{\mathcal{S}^{-1}(e_1 - e_2)}{2q_0^-} \langle (xe_1 + xe_2), \mathcal{S}^{-1}(e_1 - e_2) \rangle w + o(w) \end{aligned} \tag{5.4}$$

in $[\widetilde{H}^{-1/2}(\Gamma)]^2$. We now use (5.1) together with (5.4) to deduce

$$\begin{aligned}
u(y) &= u_e(y) \\
&+\left(-\frac{i}{2}H_0^{(1)}(k|y|)+O(w)\right)\left\{\frac{i}{2}\left(\int_{\Omega_E} H_0^{(1)}(k|z|)f(z)\,dz\right)\frac{\langle \mathcal{S}^{-1}(e_1+e_2),e_1+e_2\rangle}{q_0^+\alpha(k)}w\right. \\
&+\overline{\eta}_1\langle \mathcal{S}^{-1}(xe_1+xe_2),e_1+e_2\rangle w-\overline{\eta}_1 d\frac{\langle \mathcal{S}^{-1}(e_1-e_2),e_1+e_2\rangle}{q_0^-\alpha(k)}w \\
&-\overline{\eta}_1\frac{\langle \mathcal{S}^{-1}(e_1+e_2),e_1+e_2\rangle}{2q_0^+}\langle (xe_1+xe_2),\mathcal{S}^{-1}(e_1+e_2)\rangle w \\
&\left.-\overline{\eta}_1\frac{\langle \mathcal{S}^{-1}(e_1-e_2),e_1+e_2\rangle}{2q_0^-}\langle (xe_1+xe_2),\mathcal{S}^{-1}(e_1-e_2)\rangle w+o(w)\right\}.
\end{aligned}$$

In view of the symmetry of $\mathcal{S}^{-1}$, we have that $\langle \mathcal{S}^{-1}(e_1-e_2),e_1+e_2\rangle=0$, and consequently,

$$u(y)=u_e(y)+\frac{H_0^{(1)}(k|y|)}{2\alpha(k)}\left(\int_{\Omega_e} H_0^{(1)}(k|z|)f(z)\,dz\right)w+\rho_w(y), \tag{5.5}$$

where $w^{-1}\rho_w(y)\to 0$ pointwise in Ω_e. Next, we derive the estimate in $H^1_{\mathrm{loc}}(\Omega_e)$. To this end we introduce the rectangle $Q_R:=(-R,R)\times(0,R)$ with $R>2$. Now, consider a test function $\varphi\in\mathcal{C}_c^\infty(Q_R)$ that we identify with a distribution in $\widetilde{H}^{-1}(Q_R)=[H^1(Q_R)]'$ (see [13, Theorem 3.30 (i)]), where $Q_R:=(-R,R)\times(0,R)$. We also denote by $v_\varphi\in H^1_{\mathrm{loc}}(\Omega_e)$ the unique radiating solution of the Helmholtz equation:

$$\begin{cases}
\Delta v_\varphi+k^2v_\varphi = \varphi \quad \text{in } \Omega_e,\\
\dfrac{\partial v_\varphi}{\partial x_2} = 0 \quad \text{on } \partial\Omega_e,\\
\lim\limits_{|x|\to+\infty}\sqrt{|x|}\left(\dfrac{\partial v_\varphi}{\partial r}(x)-ik\,v_\varphi(x)\right) = 0,
\end{cases}$$

Consider the duality product of the integral equation (5.3) with φ in the duality pairing $(\widetilde{H}^{-1},H^1)$. In view of the trace Theorem, the fact that the mapping $\varphi\mapsto v_\varphi$ is linear and continuous from the space $\{\psi\in\widetilde{H}^{-1}(\Omega_e):\mathrm{supp}(\psi)\text{ is compact}\}$ to $H^1_{\mathrm{loc}}(\Omega_e)$, and after similar computations we obtain

$$\sup_{\varphi\in\mathcal{C}_c^\infty(Q_R)}\frac{\langle w^{-1}\rho_w,\varphi\rangle_{\widetilde{H}^{-1}(Q_R),H^1(Q_R)}}{\|\varphi\|_{\widetilde{H}^{-1}(Q_R)}}\to 0.$$

By definition of $\widetilde{H}^{-1}(Q_R)$, the space $\mathcal{C}_c^\infty(Q_R)$ is dense in $\widetilde{H}^{-1}(Q_R)$ for the $H^{-1}(\mathbb{R}^2)$ topology (see [13, page 77]). Finally we obtain that

$$w^{-1}\|\rho_w\|_{H^1(Q_R)}=\sup_{\varphi\in\widetilde{H}^{-1}(Q_R)}\frac{\langle w^{-1}\rho_w,\varphi\rangle_{\widetilde{H}^{-1}(Q_R),H^1(Q_R)}}{\|\varphi\|_{\widetilde{H}^{-1}(Q_R)}}\to 0,$$

which proves the first part of the theorem.

We now treat the asymptotic of the field u close to the cavities. By (3.6), we know that for each $y\in\Omega_e$, we have that

$$H_0^{(1)}(k|(wy_1\pm wd,wy_2)|)=\frac{2i}{\pi}\ln w+2i\left[\delta+\frac{\ln k}{\pi}+\frac{\ln|(y_1\pm d,y_2)|}{\pi}\right]+o(w). \tag{5.6}$$

This estimate holds pointwise and in $H^1(B_1^+)$ as well. Then we replace (5.6) in (5.5) and it completes the proof of the theorem. □

Remark 5.2. When the frequency $k\in D_{r_0}^+\setminus\mathcal{Z}$ is far from the resonances, the field u^w behaves in a very similar way than in the case of one cavity studied in [6]. The asymptotic response is that of two independent cavities without interaction.

Remark 5.3. By (5.1), it is clear that $\eta_1(y) = -\frac{i}{2}D_{x_1}\left[H_0^{(1)}(k|y|)\right]$ in the sense of distributions. Moreover, since the Hankel function $H_0^{(1)}$ has a logarithmic singularity in zero, it follows that the map $y \mapsto H_0^{(1)}(k|y|)$ belongs to $W^{1,p}_{\rm loc}(\Omega_e)$ for any $p \in [1,2)$. Hence, if f is more regular, *e.g.* $f \in W^{1,p'}(\Omega_e)$, with $p' = p/(p-1)$, with compact support in Ω_e, then u_e is more regular as well (in particular $u_e \in H^3_{\rm loc}(\Omega_e)$) and $\frac{\partial u_e}{\partial x_1}$ is a continuous function. It can be extended by continuity at the origin by setting

$$\frac{\partial u_e}{\partial x_1}(0) := -\int_{\Omega_e} \eta_1(z) f(z)\, dz,$$

We now derive an asymptotic expansion of the field u close to the resonances $k^{\pm}_{\ell,w}$, distinguishing again two regions of the plane: the far and the near fields.

Theorem 5.4. *Let let $k \in D^+_{r_0}$ be a frequency close to the resonances, then for every $y \in \Omega_e$, one has*

$$u(y) = u_e(y) + \sum_{\ell=0}^{\ell_0} \frac{\kappa^+_{\ell,w}(y)}{k - k^+_{\ell,w}} + \sum_{\ell=0}^{\ell_0} \frac{\kappa^-_{\ell,w}(y)}{k - k^-_{\ell,w}} + U^w(k,y), \tag{5.7}$$

where $k \mapsto U^w(k,y)$ is a holomorphic function in $D^+_{r_0}$ for every $y \in \Omega_e$, and $U^w(k,\cdot) \in H^1_{\rm loc}(\Omega_e)$ for every $k \in D^+_{r_0}$. Moreover,

$$\kappa^+_{\ell,w}(y) = \frac{iu_e(0)}{\alpha'(k_\ell)} H_0^{(1)}(k_\ell|y|)w + \rho^+_{\ell,w}(y),$$

and

$$\kappa^-_{\ell,w}(y) = \frac{i}{4\alpha'(k_\ell)}\frac{\partial u_e}{\partial x_1}(0) D_{x_1}[H_0^{(1)}(k_\ell|y|)] \left(\frac{\langle xe_1 + xe_2, \mathcal{S}^{-1}(e_1 - e_2)\rangle}{q_0^-} - 2d\right)^2 w^3 + \rho^-_{\ell,w}(y),$$

for some $\rho^{\pm}_{\ell,w}(y) \in H^1_{\rm loc}(\Omega_e)$ such that $w^{-1}\rho^+_{\ell,w} \to 0$ and $w^{-3}\rho^-_{\ell,w} \to 0$ in $H^1_{\rm loc}(\Omega_e)$; If further $y = (wy_1 \pm wd, wy_2)$ for some $y_1 \in \Gamma$ and $y_2 > 0$, then

$$\begin{aligned}\kappa^+_{\ell,w}(wy_1 \pm wd, wy_2) \quad &= \quad -\frac{2u_e(0)}{\pi\alpha'(k_\ell)} w \ln w \\ &\quad -\frac{2u_e(0)}{\alpha'(k_\ell)}\left(\delta + \frac{\ln k_\ell}{\pi} + \frac{\ln|(y_1 \pm d, y_2)|}{\pi}\right) w + \overline{\rho}^+_{\ell,w}(y_1,y_2),\end{aligned}$$

$$\begin{aligned}&\kappa^-_{\ell,w}(wy_1 \pm wd, wy_2) \\ &\quad = -\frac{\langle \Gamma^{\pm}(\cdot, y_1, y_2), \mathcal{S}^{-1}(e_1 - e_2)\rangle}{2\pi q_0^- \alpha'(k_\ell)} \frac{\partial u_e}{\partial x_1}(0)\left(2d - \frac{\langle xe_2 + xe_2, \mathcal{S}^{-1}(e_1 - e_2)\rangle}{q_0^-}\right) w^2 \\ &\qquad + \overline{\rho}^-_{\ell,w}(y_1,y_2),\end{aligned}$$

where $\Gamma^{\pm}$ are defined in (5.8)-(5.9), and $\overline{\rho}^{\pm}_{\ell,w} \in H^1(B^+_1)$ satisfy $w^{-1}\overline{\rho}^+_{\ell,w} \to 0$ and $w^{-3}\overline{\rho}^-_{\ell,w} \to 0$ in $H^1(B^+_1)$.

Proof. From the integral equation (2.7) and the asympotic expansion result close to the resonances stated in Theorem 4.15, we deduce that expression (5.7) holds with

$$\kappa^{\pm}_{\ell,w}(y) := \left\langle \begin{pmatrix} G_e(w\cdot -wd, 0; y) \\ G_e(w\cdot +wd, 0; y) \end{pmatrix}, \boldsymbol{\Delta}^{\pm}_{\ell,w} \begin{pmatrix} u_e(w\cdot -wd, 0) \\ u_e(w\cdot +wd, 0) \end{pmatrix} \right\rangle,$$

and

$$U^w(,k,y) := \left\langle \begin{pmatrix} G_e(w\cdot -wd, 0; y) \\ G_e(w\cdot +wd, 0; y) \end{pmatrix}, \mathcal{H}_w(k) \begin{pmatrix} u_e(w\cdot -wd, 0) \\ u_e(w\cdot +wd, 0) \end{pmatrix} \right\rangle.$$

From (5.1) and (5.2) together with the asymptotic of $\boldsymbol{\Delta}^+_{\ell,w}$, we obtain the expected expression for $\kappa^+_{\ell,w}(y)$. Using the asymptotic expansion of $H_0^{(1)}(k|(wy_1 \pm wd, wy_2)|)$ in (5.6) we derive the asymptotic of $\kappa^+_{\ell,w}(wy_1 \pm wd, wy_2)$.

We now treat $\kappa^-_{\ell,w}(y)$. By (5.1) and (5.2), one has for every $y \in \Omega_E$,

$$\begin{aligned}\begin{pmatrix} G_e(w\cdot -wd,0;y) \\ G_e(w\cdot +wd,0;y)\end{pmatrix} &= -\frac{i}{2}H_0^{(1)}(k|y|)(e_1+e_2) \\ &+\Big[-\eta_1(y)(xe_1+xe_2)+\eta_1(y)d(e_1-e_2)\Big]w \\ &+\Big[2\eta_2(y)d(xe_1-xe_2)-\eta_2(y)(x^2e_1+x^2e_2)-\eta_2(y)d^2(e_1+e_2)\Big]w^2 \\ &+o(w^2) \quad \text{in } [H^{1/2}(\Gamma)]^2,\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} u_e(w\cdot -wd,0) \\ u_e(w\cdot +wd,0)\end{pmatrix} &= -\frac{i}{2}\left(\int_{\Omega_e} H_0^{(1)}(k|z|)f(z)\,dz\right)(e_1+e_2) \\ &+\Big[-\overline{\eta}_1(xe_1+xe_2)+\overline{\eta}_1 d(e_1-e_2)\Big]w \\ &+\Big[2\overline{\eta}_2 d(xe_1-xe_2)-\overline{\eta}_2(x^2e_1+x^2e_2)-\overline{\eta}_2 d^2(e_1+e_2)\Big]w^2 \\ &+o(w^2) \quad \text{in } [H^{1/2}(\Gamma)]^2.\end{aligned}$$

Using the symmetric structure of the problem (see Remark 4.7) we infer that

$$\begin{cases} \langle e_1+e_2, \mathcal{L}_w^{-1}(k^-_{\ell,w})(e_1-e_2)\rangle = 0, \\ \langle xe_1-xe_2, \mathcal{L}_w^{-1}(k^-_{\ell,w})(e_1-e_2)\rangle = 0, \\ \langle x^2e_1+x^2e_2, \mathcal{L}_w^{-1}(k^-_{\ell,w})(e_1-e_2)\rangle = 0,\end{cases}$$

hence by (4.25),

$$\mathbf{\Delta}^-_{\ell,w}(e_1+e_2) = \mathbf{\Delta}^-_{\ell,w}(xe_1-xe_2) = \mathbf{\Delta}^-_{\ell,w}(x^2e_1+x^2e_2) = 0.$$

Since

$$\mathcal{L}_w^{-1}(k^-_{\ell,w})\theta_w(k^-_{\ell,w})(e_1-e_2) = \frac{\alpha(k^-_{\ell,w})}{w}\mathcal{L}_w^{-1}(k^-_{\ell,w})(e_1-e_2),$$

we obtain using again (4.25) that

$$\begin{aligned}\langle e_1+e_2, \mathbf{\Delta}^-_{\ell,w}(e_1-e_2)\rangle &= \langle xe_1-xe_2, \mathbf{\Delta}^-_{\ell,w}(e_1-e_2)\rangle \\ &= \langle x^2e_1+x^2e_2, \mathbf{\Delta}^-_{\ell,w}(e_1-e_2)\rangle = 0,\end{aligned}$$

and

$$\begin{aligned}\langle e_1+e_2, \mathbf{\Delta}^-_{\ell,w}(xe_1+xe_2)\rangle &= \langle xe_1-xe_2, \mathbf{\Delta}^-_{\ell,w}(xe_1+xe_2)\rangle \\ &= \langle x^2e_1+x^2e_2, \mathbf{\Delta}^-_{\ell,w}(xe_1+xe_2)\rangle = 0.\end{aligned}$$

Consequently, since $\mathbf{\Delta}^-_{\ell,w}$ is of order w (see Theorem 4.15) we have

$$\kappa^-_{\ell,w}(y) = \overline{\eta}_1\,\eta_1(y)\left\langle -(xe_1+xe_2)+d(e_1-e_2), \mathbf{\Delta}^-_{\ell,w}\big[-(xe_1+xe_2)+d(e_1-e_2)\big]\right\rangle w^2 + o(w^3)$$

and we obtain the desired result using the asymptotic expansion of $\mathbf{\Delta}^-_{\ell,w}$ in Theorem 4.15 together with Remark 5.3.

Using now the asymptotic expansion

$$G_e(wt,0;wy_1 \pm wd, wy_2) = \frac{\ln w}{\pi} + \left[\delta + \frac{\ln k}{\pi} + \frac{\ln|(y_1 \pm d - t, y_2)|}{\pi}\right] + o(w),$$

and defining

$$\Gamma^-(x_1,y_1,y_2) := \begin{pmatrix} \ln|(y_1-x_1,y_2)| \\ \ln|(y_1-x_1+2d,y_2)|\end{pmatrix}, \tag{5.8}$$

and

$$\Gamma^+(x_1,y_1,y_2) := \begin{pmatrix} \ln|(y_1-x_1,y_2)| \\ \ln|(y_1-x_1-2d,y_2)|\end{pmatrix}, \tag{5.9}$$

we derive the asymptotic of $\kappa^-_{\ell,w}(wy_1 \pm wd, wy_2)$.

Finally we argue exactly as in the proof of Theorem 5.1 to get the asymptotics in $H^1_{\rm loc}(\Omega_e)$ for the far field, and in $H^1(B_1^+)$ for the near field. $\square$

Remark 5.5. Applying the Helmholtz operator $\Delta + k^2$ to $\kappa^{\pm}_{\ell,w}$ implies, according to Remark 5.3, that

$$\begin{cases} (\Delta + k^2)\kappa^{+}_{\ell,w} = c^{+}_{\ell} w\delta_0, \\ (\Delta + k^2)\kappa^{-}_{\ell,w} = c^{-}_{\ell} w^3 D_{x_1}\delta_0, \end{cases}$$

in $\mathcal{D}^{+}(\Omega_e)$, where $c^{\pm}_{\ell}$ are the constants defined by

$$c^{+}_{\ell} := -\frac{2u_e(0)}{\alpha'(k_\ell)},$$

and

$$c^{-}_{\ell} := -\frac{1}{2\alpha'(k_\ell)}\frac{\partial u_e}{\partial x_1}(0)\left(2d - \frac{\langle xe_2 + xe_2, \mathcal{S}^{-1}(e_1 - e_2)\rangle}{q_0^-}\right)^2.$$

These equations essentially say that at the frequency $k^{+}_{\ell,w}$, the spatial singularity sensed in far field is that of a Dirac mass, which is a macroscopic manifestation of a dipole placed on a metallic plane. On the other hand, when the other resonance $k^{-}_{\ell,w}$ is excited, the spatial singularity is the derivative in the x_1-direction of a Dirac mass; this is exactly the asymptotic response of a quadripole placed on the metallic plane.

Moreover, when the frequency k of the incident wave is close to the resonance $k^{\pm}_{\ell,w}$, the term $k - k^{\pm}_{\ell,w}$ scales like $-\mathrm{Im}(k^{\pm}_{\ell,w})$ since k is purely real. Then by Theorem 4.13 and Remark 4.14, the dipolar and quadripolar momentum are of order one, and they are respectively given by

$$\frac{c^{+}_{\ell} w}{k - k^{+}_{\ell,w}} = 2u_e(0) + o(1),$$

and

$$\frac{c^{-}_{\ell} w^3}{k - k^{-}_{\ell,w}} = \frac{4}{k_\ell^2}\frac{\partial u_e}{\partial x_1}(0) + o(1).$$

Remark 5.6. The asymptotic of the field close to the resonances shows that the field u concentrates on the top of the cavities as their width w shrinks. Indeed,using Theorems 4.13 and 5.4, we infer that when k is close to $k^{+}_{\ell,w}$, then

$$\begin{aligned} \frac{\kappa^{+}_{\ell,w}(wy_1 \pm wd, wy_2)}{k - k^{+}_{\ell,w}} &= \frac{2u_e(0)}{\pi}\ln w \\ &\quad + 2u_e(0)\left(\delta + \frac{\ln k_\ell}{\pi} + \frac{\ln|(y_1 \pm d, y_2)|}{\pi}\right) + o(1), \end{aligned}$$

while if k is close to $k^{-}_{\ell,w}$, then

$$\begin{aligned} &\frac{\kappa^{-}_{\ell,w}(wy_1 \pm wd, wy_2)}{k - k^{-}_{\ell,w}} \\ &\quad = \frac{4}{\pi w}\frac{\langle \Gamma^{\pm}(\cdot, y_1, y_2), \mathcal{S}^{-1}(e_1 - e_2)\rangle}{\left(2dq_0^- - \langle xe_2 + xe_2, \mathcal{S}^{-1}(e_1 - e_2)\rangle\right)}\frac{\partial u_e}{\partial x_1}(0) + o(1). \end{aligned}$$

In both cases, we can see that the field blows up as $w \to 0$. The concentration pattern scales like $\ln w$ close to the resonance $k^{+}_{\ell,w}$, and like $1/w$ close to the resonance $k^{-}_{\ell,w}$.

Remark 5.7. Thanks to the expressions of the Green functions G_i, it is possible to prove by similar arguments than those used in the proof of Theorem 5.4 and Remarks 5.5 and 5.6, that close to the resonances, the field u is asymptotically very large inside each cavity. Indeed, for every $y \in C_1 \cup C_2$, one has

$$u(y) = \sum_{\ell=0}^{\ell_0}\frac{\gamma^{+}_{\ell,w}(y)}{k - k^{+}_{\ell,w}} + \sum_{\ell=0}^{\ell_0}\frac{\gamma^{-}_{\ell,w}(y)}{k - k^{-}_{\ell,w}} + V^w(k, y),$$

where $k \mapsto V^w(k,y)$ is a holomorphic function for all $y \in \mathcal{C}_1 \cup C_2$, and $V^w(k,\cdot) \in H^1(C_1 \cup C_2)$ for all $k \in D^+_{r_0}$. Moreover, the functions $\gamma^\pm_{\ell,w}$ admit the following expansion:

$$\gamma^+_{\ell,w}(y) := \frac{2u_e(0)}{h\alpha'(k_\ell)} \sum_{n=0}^{+\infty} \frac{(-1)^n \cos\left(\frac{n\pi}{h}(y_2+h)\right)}{k_\ell^2 - \left(\frac{n\pi}{h}\right)^2} + O(w\ln w) \quad \text{for every } y \in C_1 \cup C_2,$$

while

$$\gamma^-_{\ell,w}(y) := -\frac{w}{h\alpha'(k_\ell)} \left(\frac{\langle xe_1 + xe_2, \mathcal{S}^{-1}(e_1 - e_2)\rangle}{q_0^-} - 2d\right) \sum_{n=0}^{+\infty} \frac{(-1)^n \cos\left(\frac{n\pi}{h}(y_2+h)\right)}{k_\ell^2 - \left(\frac{n\pi}{h}\right)^2} \frac{\partial u_e}{\partial x_1}(0) + O(w^2)$$

if $y \in C_1$, and

$$\gamma^-_{\ell,w}(y) := \frac{w}{h\alpha'(k_\ell)} \left(\frac{\langle xe_1 + xe_2, \mathcal{S}^{-1}(e_1 - e_2)\rangle}{q_0^-} - 2d\right) \sum_{n=0}^{+\infty} \frac{(-1)^n \cos\left(\frac{n\pi}{h}(y_2+h)\right)}{k_\ell^2 - \left(\frac{n\pi}{h}\right)^2} \frac{\partial u_e}{\partial x_1}(0) + O(w^2)$$

if $y \in C_2$. Thus, using again Theorem 4.13, we infer that at a frequency k close to $k^+_{\ell,w}$, one has

$$\frac{\gamma^+_{\ell,w}(y)}{k - k^+_{\ell,w}} = \frac{a^+_\ell(y)}{w}\Big(1 + O(w\ln w)\Big) \quad \text{for every } y \in C_1 \cup C_2,$$

while at a frequency k close to $k^-_{\ell,w}$,

$$\frac{\gamma^-_{\ell,w}(y)}{k - k^-_{\ell,w}} = \begin{cases} \dfrac{a^-_\ell(y)}{w^2}\Big(1 + O(w)\Big) & \text{if} \quad y \in C_1, \\[2ex] -\dfrac{a^-_\ell(y)}{w^2}\Big(1 + O(w)\Big) & \text{if} \quad y \in C_2, \end{cases}$$

where $a^\pm_\ell(y)$ are functions defined by

$$a^+_\ell(y) := -\frac{2u_e(0)}{h} \sum_{n=0}^{+\infty} \frac{(-1)^n \cos\left(\frac{n\pi}{h}(y_2+h)\right)}{k_\ell^2 - \left(\frac{n\pi}{h}\right)^2},$$

and

$$a^-_\ell(y) := \frac{4}{k_\ell^2 h} \frac{\partial u_e}{\partial x_1}(0) \left(\frac{\langle xe_1 + xe_2, \mathcal{S}^{-1}(e_1 - e_2)\rangle}{q_0^-} - 2d\right)^{-1} \sum_{n=0}^{+\infty} \frac{(-1)^n \cos\left(\frac{n\pi}{h}(y_2+h)\right)}{k_\ell^2 - \left(\frac{n\pi}{h}\right)^2}.$$

The expressions highlight the symmetric or antisymmetric nature of the modes and the field enhancement inside the cavities. This witnesses of the symmetrical and anti-symmetrical modes occurring respectively at frequencies $k^+_{\ell,w}$ and $k^-_{\ell,w}$, as well as the field amplification inside the sub-wavelength cavities. Indeed, when the resonance $k^+_{\ell,w}$ is activated, the field u is oriented in same direction in both cavities: this is the symmetrical mode. In that case, the amplification factor scales like $1/w$ which is coherent with [11]. On the other hand, the field u is oriented in two opposite directions when the resonance $k^-_{\ell,w}$ is excited, and the amplification factor, much larger, scales like $1/w^2$: this is the anti-symmetrical mode. Finally, we remark that the field is independent of y_1 inside the cavities, and that it increases as y_2 tends to the top of the cavities. These modes are illustrated in a schematic way in Figure 2.

Appendix

Useful formulae. The following formulae can be found, *e.g.*, in [1].

- $H_0^{(1)}(z) = \left\{1 + \frac{2i}{\pi}\left[\ln\left(\frac{z}{2}\right) + \gamma\right]\right\}\left(\sum_{n=0}^{+\infty} \frac{(-z^2/4)^n}{(n!)^2}\right) - \frac{2i}{\pi}\sum_{n=1}^{+\infty}\left(\sum_{m=1}^{n}\frac{1}{m}\right)\frac{(-z^2/4)^n}{(n!)^2}$, where γ is the Euler constant;
- $\sum_{n=0}^{+\infty} \frac{1}{r^2 - n^2} = \frac{1}{2r^2} + \frac{\pi}{2r}\cot(\pi r)$;

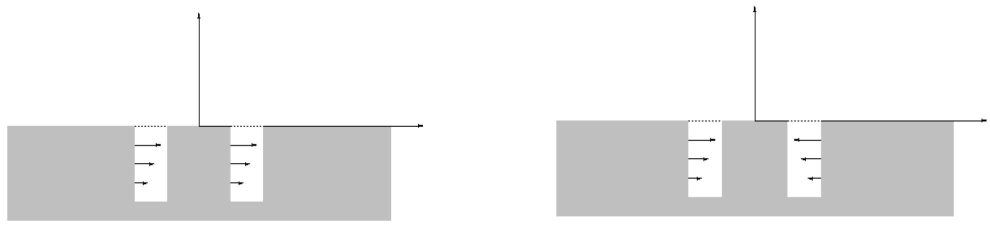


FIGURE 2. Symmetrical and anti-symmetrical modes

- $\displaystyle\sum_{n=1}^{+\infty} \frac{\cos(n\theta)}{n} = -\ln 2 - \ln\left|\sin\left(\frac{\theta}{2}\right)\right|$;
- $\displaystyle\sum_{n=1}^{+\infty} \frac{\cos(n\theta)}{n^2} = \frac{\theta^2}{4} - \frac{\pi|\theta|}{4} + \frac{\pi^2}{6}$.

Generalized Rouché theorem. In this section, we review the main results of [9]. Let $\mathcal{G}$ and $\mathcal{H}$ be two Banach spaces and let $\mathcal{L}(\mathcal{G},\mathcal{H})$ be the set of all bounded operators from $\mathcal{G}$ to $\mathcal{H}$. Let U be an open set in $\mathbb{C}$. Suppose that $\mathcal{A}(k)$ is an operator-valued function from U to $\mathcal{L}(\mathcal{G},\mathcal{H})$; k_0 is a *characteristic value* of $\mathcal{A}(k)$ if

- $\mathcal{A}(k)$ is holomorphic in some neighborhood of k_0, except possibly for k_0;
- there exists a holomorphic function $\phi(k)$, from a neighborhood of k_0 to $\mathcal{G}$, such that $\phi(k_0) \neq 0$, $\mathcal{A}(k)\phi(k)$ is holomorphic at k_0, and $\mathcal{A}(k_0)\phi(k_0) = 0$.

The function $\phi(k)$ in the above definition is called a *root function* of $\mathcal{A}(k)$ associated to k_0, and $\phi(k_0)$ is called an *eigenvector*. The closure of the space of eigenvectors corresponding to k_0 is denoted by $\operatorname{Ker}\mathcal{A}(k_0)$.

Let ϕ_0 be an eigenvector corresponding to k_0. The *rank* of ϕ_0 is the largest integer m such that there exists a complex neighborhood $V(k_0)$ of k_0, and two holomorphic functions $\phi : V(k_0) \to \mathcal{G}$ and $\psi : V(k_0) \to \mathcal{H}$ satisfying

$$\mathcal{A}(k)\phi(k) = (k-k_0)^m\psi(k), \quad \phi(k_0) = \phi_0 \quad \text{and} \quad \psi(k_0) \neq 0.$$

Suppose that $n = \dim\operatorname{Ker}\mathcal{A}(k_0) < +\infty$ and the ranks of all vectors in $\operatorname{Ker}\mathcal{A}(k_0)$ are finite. A system of eigenvectors ϕ_0^j $(j = 1,\dots,n)$ is called a *canonical system of eigenvectors* of $\mathcal{A}(k_0)$ if the rank of ϕ_0^j is the maximum of the ranks of all eigenvectors in some direct complement in $\operatorname{Ker}\mathcal{A}(k_0)$ of the linear space spanned by the vectors $\phi_0^1,\dots,\phi_0^{j-1}$. Then we define the *null multiplicity* of the characteristic value of k_0 to be the sum of the ranks of ϕ_0^j $(j = 1,\dots,n)$, which is denoted by $N(\mathcal{A}(k_0))$.

Suppose that $\mathcal{A}^{-1}(k)$ exists and is holomorphic in some neighborhood of k_0, except possibly at this point itself. Then the number

$$M(\mathcal{A}(k_0)) := N(\mathcal{A}(k_0)) - N(\mathcal{A}^{-1}(k_0))$$

is called the *multiplicity* of the characteristic value k_0.

Suppose that the Laurent expansion of $\mathcal{A}(k)$ at k_0 is given by

$$\mathcal{A}(k) = \sum_{j\geq -s} (k-k_0)^j A_j.$$

If the operators A_j $(j = -s,\dots,-1)$ are finite dimensional, then $\mathcal{A}(k)$ is called *finitely meromorphic* at k_0. If the operator A_0 is a Fredholm one, then $\mathcal{A}(k)$ is said to be of *Fredholm type* at k_0.

If $\mathcal{A}(k)$ is holomorphic and invertible at k_0, then k_0 is called a *regular point* of $\mathcal{A}(k)$. A point k_0 is called a *normal point* of $\mathcal{A}(k)$ if $\mathcal{A}(k)$ is finitely meromorphic and of Fredholm type at k_0, and if there exists some neighborhood $V(k_0)$ of k_0 in which all the points except k_0 are regular points of $\mathcal{A}(k)$.

Lemma 5.8. *Every normal point k_0 of $\mathcal{A}(k)$ is a normal point of $\mathcal{A}^{-1}(k)$.*

An operator-valued function $\mathcal{A}(k)$ which is finitely meromorphic and of Fredholm type in $V(k_0)$ and continuous on $\partial V(k_0)$ is called *normal* with respect to $\partial V(k_0)$ provided it is invertible in $\overline{V(k_0)}$, except for a finite number of points of $V(k_0)$ which are normal points of $\mathcal{A}(k)$.

Suppose that $\mathcal{A}(k)$ is normal with respect to $\partial V(k_0)$ and let k_i $(i = 1, \ldots, \sigma)$ be its characteristic values and poles lying in $V(k_0)$, we set

$$\mathcal{M}(\mathcal{A}(k), \partial V(k_0)) := \sum_{i=1}^{\sigma} M(\mathcal{A}(k)).$$

The generalization of Rouché's Theorem is stated below:

Theorem 5.9. *Let $\mathcal{A}(k)$ be an operator-valued function which is normal with respect to $\partial V(k_0)$. If $\mathcal{S}(k)$ is an operator-valued function which is finitely meromorphic in $V(k_0)$, continuous at $\partial V(k_0)$ and satisfying*

$$\|\mathcal{A}^{-1}(k)\mathcal{S}(k)\|_{\mathcal{L}(\mathcal{G},\mathcal{G})} < 1 \qquad \textit{for } k \in \partial V(k_0),$$

then $\mathcal{A}(k) + \mathcal{S}(k)$ is normal with respect to $\partial V(k_0)$ as well, and

$$\mathcal{M}(\mathcal{A}(k), \partial V(k_0)) = \mathcal{M}(\mathcal{A}(k) + \mathcal{S}(k), \partial V(k_0)).$$

The generalization of Steinberg's Theorem is given by

Theorem 5.10. *Suppose that $\mathcal{A}(k)$ is an operator-valued function which is finitely meromorphic and of Fredholm type in $V(k_0)$. If $\mathcal{A}(k)$ is invertible at one point of $V(k_0)$, then $\mathcal{A}(k)$ has a bounded inverse for all $k \in V(k_0)$, except possibly for certain isolated points.*

We finally state a generalization of Rouché's Theorem which is also called generalized argument principle.

Theorem 5.11. *Suppose that the operator-valued function $\mathcal{A}(k)$ is normal with respect to $\partial V(k_0)$. Let $f(k)$ be a scalar function which is holomorphic in $V(k_0)$ and continuous in $\overline{V(k_0)}$. Then we have*

$$\frac{1}{2i\pi} \operatorname{tr} \int_{\partial V(k_0)} f(k)\mathcal{A}^{-1}(k)\frac{d\mathcal{A}(k)}{dk}\, dk = \sum_{j=1}^{\sigma} M(\mathcal{A}(k_j)f(k_j)),$$

where k_j $(j = 1, \ldots, \sigma)$ are all the poles or characteristic values of $\mathcal{A}(k)$ in $V(k_0)$.

Here tr denotes the trace of the operator which is the sum of all its nonzero eigenvalues. We mention the following property of the trace

$$\operatorname{tr} \int_{\partial V(k_0)} \mathcal{A}(k)\mathcal{B}(k)\, dk = \operatorname{tr} \int_{\partial V(k_0)} \mathcal{B}(k)\mathcal{A}(k)\, dk,$$

where $\mathcal{A}(k)$ and $\mathcal{B}(k)$ are operator-valued functions which are finitely meromorphic in $V(k_0)$, and $V(k_0)$ contains no pole of $\mathcal{A}(k)$ and $\mathcal{B}(k)$ other than k_0.

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