SMALL VOLUME ASYMPTOTICS FOR ANISOTROPIC ELASTIC INCLUSIONS

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ABSTRACT. We derive asymptotic expansions for the displacement at the boundary of a smooth, elastic body in the presence of small inhomogeneities. Both the body and the inclusions are allowed to be anisotropic. This work extends prior work of CapdeBoscq and Vogelius (*Math. Modelling Num. Anal.* 37, 2003) for the conductivity case. In particular, we obtain an asymptotic expansion of the difference between the displacements at the boundary with and without inclusions, under Neumann boundary conditions, to first order in the measure of the inclusions. We impose no geometric conditions on the inclusions, which need only be measurable sets. The first-order correction contains a moment or polarization tensor $\mathbb{M}$ that encodes the effect of the inclusions. In the case of thin, strip-like, planar inhomogeneities we obtain a formula for $\mathbb{M}$ only in terms of the elasticity tensors, which we assume strongly convex, their inverses, and a frame on the curve that supports the inclusion. We prove uniqueness of $\mathbb{M}$ in this setting and recover the formula previously obtained by Beretta and Francini (*SIAM J. Math. Anal.*, 38, 2006).

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a smooth bounded domain representing the region occupied by an elastic body. Let $\mathbb{C}_0 = \mathbb{C}_0(x)$ be a smooth background elasticity tensor in Ω. Let $\omega_\epsilon \subset \Omega$ be a set of measurable small inhomogeneities and let $\mathbb{C}_1 = \mathbb{C}_1(x)$ be the smooth elasticity tensor inside the inhomogeneities. Let $\psi \in H^{-1/2}(\partial\Omega)$ represents a traction on $\partial\Omega$ and U the corresponding background displacement field which satisfies the system of linearized elasticity:

$$\begin{cases} \operatorname{div}(\mathbb{C}_0\widehat{\nabla}U) &= 0 \text{ in } \Omega \\ (\mathbb{C}_0\widehat{\nabla}U)\nu &= \psi \text{ on } \partial\Omega, \end{cases}$$

Let

$$\mathbb{C}_\epsilon = \mathbb{C}_0\chi_{\Omega\setminus\omega_\epsilon} + \mathbb{C}_1\chi_{\omega_\epsilon},$$

and consider the perturbed displacement field solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon) &= 0 \text{ in } \Omega \\ (\mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon)\nu &= \psi \text{ on } \partial\Omega. \end{cases}$$

One goal of this paper is to obtain an asymptotic formula for $u_\epsilon - U$ on the boundary of Ω as the measure of ω_ϵ approaches zero. The formula we derive generalizes those already available in case of homogeneous isotropic bodies with diametrically

small (see [2]) or for thin (see [3]) inhomogeneities. To derive the asymptotic expansion we follow the approach introduced by Capdeboscq and Vogelius in [5] for the conductivity equation and we establish a formula in the case of arbitrary elastic tensors $\mathbb{C}_0$ and $\mathbb{C}_1$. More precisely we show that for $y \in \partial\Omega$

$$(1.1) \qquad (u_{\epsilon_n} - U)(y) = |\omega_{\epsilon_n}| \int_\Omega \mathbb{M}(x)\widehat{\nabla}U(x) : \widehat{\nabla}N(x,y)d\mu_x + o(|\omega_{\epsilon_n}|).$$

along a sequence of configurations $\{\omega_{\epsilon_n}\}$ whose measure tends to 0. Here, N is the Neumann function corresponding to the operator $div(\mathbb{C}_0\widehat{\nabla}\cdot)$, μ is a Radon measure, the elastic moment tensor $\mathbb{M} \in L^2(\Omega, d\mu)$ and $\widehat{\nabla}U$ represents the symmetric deformation tensor.

For particular geometries like diametrically small or thin inhomogeneities the asymptotic expansion holds for $(u_\epsilon - U)(y)$ as $\epsilon \to 0$ and one can characterize the measure μ and the tensor $\mathbb{M}$. In particular, if $\omega_\epsilon = z + \epsilon B$, where the center $z \in \Omega$ and B is a bounded domain, then μ is a Dirac function concentrated at z. If further both $\mathbb{C}_0$ and $\mathbb{C}_1$ are homogeneous and isotropic, the tensor $\mathbb{M}$ can be explicitly computed and carries information about the geometry of B and about the elastic parameters of $\mathbb{C}_0$ and $\mathbb{C}_1$ ([2]). If $\omega_\epsilon = \{x \in \Omega, \quad dist(x, \sigma_0) < \epsilon\}$, where σ_0 is a simple smooth open curve in the plane, μ reduces to a Dirac measure supported on σ_0. If again the phases are isotropic, $\mathbb{M}$ can be explicitly determined by the transmission conditions for u_ϵ [3].

In the second part of the paper we analyze the case of thin inhomogeneities in a planar domain in the case of arbitrary elasticity tensors.

In the case of isotropic homogeneous tensors, the idea, used in [3] to derive the asymptotic expansion for $u_\epsilon - U$, is to apply fine regularity results for solutions of elliptic systems with discontinuous coefficients by Y.Y. Li and L. Nirenberg [10] and to use the transmission conditions to derive the tensor $\mathbb{M}$ which satisfies

$$(1.2) \qquad\qquad (\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}u_\epsilon^i(x) = \mathbb{M}(x)\widehat{\nabla}u_\epsilon^e(x),$$

whereas $\widehat{\nabla}u_\epsilon^i$ and $\widehat{\nabla}u_\epsilon^e$ denote the values of the deformation tensor inside and outside the inclusion at a point x on its boundary. Note that the deformation tensors are related by transmission conditions across $\partial\omega_\epsilon$.

One of the difficulties we encountered in deriving the expansion in the anisotropic case is the the direct derivation of (1.2) from the transmission conditions. To construct $\mathbb{M}$, we follow the work of Francfort and Murat [8] on the calculation of the effective properties of laminated 2-phase elastic composites. Indeed, one can view polarization tensors as limits of effective tensors as the volume fraction of one of the phases tends to 0.

The paper is organized as follows. In section 2, we state a general representation formula of the form (1.1) for anisotropic elastic inhomogeneities embedded in an anisotropic background medium. The asymptotic expansion is proved in section 3. Properties of the elastic moment tensor $\mathbb{M}$ are established in section 4. In section 5, ω_ϵ is assumed to be a thin strip-like planar inclusion. Firstly, relying on the uniform Hölder regularity of u_ϵ and on Meyer's theorem, we give a direct derivation of the asymptotic expansion similar to that in [3], under the assumption that there exists a tensor $\mathbb{M}$, independent of ϵ, that satisfies (1.2). Secondly, we prove existence of such M, invoking the result of Francfort and Murat mentioned above [8] . Thirdly, we show that the asymptotic expansion of theorem 2.1 coincides with that obtained in theorem 5.1. Finally, in the appendix, we recall classical regularity results for the

system of elasticity, and we prove how Caccioppoli inequality and Meyer's theorem also hold for the system of elasticity.

2. Notations, assumptions and main result

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$ be a bounded, smooth domain. For $x \in \partial\Omega$, let us denote by $\nu(x)$ the normal direction to $\partial\Omega$ at point x. We use the following notation:

Notation. *Let $\mathbb{C}$ be a 4-th order tensor, let A and B be $d \times d$ matrices, and let u, v denote vectors in $\mathbb{R}^d$. We set:*

$$u \cdot v = \sum_{j=1}^d u_j v_j \quad Av = \sum_{j=1}^d A_{ij} v_j \quad \mathbb{C}A = \sum_{k,l=1}^d \mathbb{C}_{ijkl} A_{kl}$$

$$(\mathbb{C}A)\, v = \sum_{j,k,l=1}^d \mathbb{C}_{ijkl} A_{kl} v_j \quad \mathbb{C}A : B = \sum_{i,j,k,l=1}^d \mathbb{C}_{ijkl} A_{kl} B_{ij},$$

$$|A| = \left(\sum_{ij} A_{ij}^2\right)^{1/2}.$$

Moreover, we denote by $\widehat{A} = (A + A^T)/2$ the symmetrization of the matrix A. In particular, given a vector valued function u defined in Ω, we denote by $\widehat{\nabla} u$ the strain $\widehat{\nabla} u = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$.

Let $\mathbb{C}_0 \in C^{1,\alpha}(\Omega)$, for some $\alpha \in (0,1)$, be a fourth order elasticity tensor that satisfies the full symmetry properties:

$$(2.1) \qquad (\mathbb{C}_0(x))_{ijkl} = (\mathbb{C}_0(x))_{klij} = (\mathbb{C}_0(x))_{jikl} \quad \forall\, 1 \leq i,j,k,l \leq d \text{ and } x \in \Omega,$$

and the strong convexity condition, i.e., there exists a constant $\lambda_0 > 0$ such that

$$(2.2) \qquad \mathbb{C}_0(x)A : A \geq \lambda_0 |A|^2 \text{ for every } d \times d \text{ symmetric matrix } A \text{ and } x \in \Omega.$$

Let $\psi \in H^{-1/2}(\partial\Omega)$ satisfying the compatibility condition

$$(2.3) \qquad \int_{\partial\Omega} \psi \cdot R = 0,$$

for every infinitesimal rigid motion R, that is $R(x) = Wx + c$ for some skew-symmetric matrix W and $c \in \mathbb{R}^d$.

The background displacement field $U \in \tilde{H}(\Omega)$ is defined as the solution to

$$(2.4) \qquad \begin{cases} \operatorname{div}(\mathbb{C}_0 \widehat{\nabla} U) &=& 0 \text{ in } \Omega \\ (\mathbb{C}_0 \widehat{\nabla} U)\nu &=& \psi \text{ on } \partial\Omega, \end{cases}$$

where $\tilde{H}(\Omega)$ is the space of vector valued functions given by

$$\tilde{H}(\Omega) = \left\{ u \in H^1(\Omega; \mathbb{R}^d) \text{ such that } \int_{\partial\Omega} u\, d\sigma = 0, \quad \int_\Omega \left(\nabla u - (\nabla u)^T\right) dx = 0 \right\}$$

Let ω_ϵ denote the a subset of Ω, that contains one or several inhomogeneities. We assume that ω_ϵ is measurable and separated from the boundary, that is $d(\omega_\epsilon, \partial\Omega) \geq d_0 > 0$. We also assume that the measure $|\omega_\epsilon| > 0$ tends to 0 as $\epsilon \to 0$.

Let $\mathbb{C}_1$ denote the elasticity tensor inside ω_ϵ. We assume that $\mathbb{C}_1 \in C^1(\Omega)$ is fully symmetric and strongly convex, i.e.

$$(2.5) \qquad \mathbb{C}_1(x)A : A \geq \lambda_0 |A|^2, \quad \text{for every } d \times d \text{ symmetric matrix } A \text{ and } x \in \Omega.$$

Let $\mathbb{C}_\epsilon$ be the elasticity tensor in the presence of the inhomogeneity

$$(2.6) \qquad \mathbb{C}_\epsilon = \mathbb{C}_0 \chi_{\Omega \setminus \omega_\epsilon} + \mathbb{C}_1 \chi_{\omega_\epsilon},$$

and consider the corresponding displacement field $u_\epsilon \in \tilde{H}(\Omega)$ solution to

$$(2.7) \qquad \begin{cases} \operatorname{div}(\mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon) &= 0 \text{ in } \Omega \\ (\mathbb{C}_\epsilon \widehat{\nabla} u_\epsilon)\nu &= \psi \text{ on } \partial\Omega. \end{cases}$$

For existence and uniqueness of solutions to (2.4) and (2.7) in $\tilde{H}(\Omega)$ we refer to [13] for example.

Since $d(\omega_\epsilon, \partial\Omega) \geq d_0 > 0$, there exists a compact set K_0, independent of ϵ, such that

$$(2.8) \qquad \omega_\epsilon \subset K_0 \subset \Omega \text{ and } \operatorname{dist}(\omega_\epsilon, \Omega \setminus K_0) > d_0/2 > 0.$$

We also introduce the Neumann matrix for the operator $\operatorname{div}(\mathbb{C}_0 \widehat{\nabla} \cdot)$, i.e. the weak solution to

$$(2.9) \qquad \begin{cases} \operatorname{div}(\mathbb{C}_0 \widehat{\nabla} N(\cdot, y)) &= -\delta_y \mathbf{I}_d \text{ in } \Omega \\ (\mathbb{C}_0 \widehat{\nabla} N(\cdot, y)\nu &= -\frac{1}{|\partial\Omega|} \mathbf{I}_d \text{ on } \partial\Omega, \end{cases}$$

that satisfies the normalization conditions

$$(2.10) \qquad \int_{\partial\Omega} N(x, y)\, d\sigma_x = 0, \quad \int_\Omega (\nabla_x N(x, y) - \nabla_x N(x, y)^T)\, dx = 0,$$

where $\mathbf{I}_d$ is the d-dimensional identity matrix.

For the existence of such Neumann matrix and its behavior for x close to y we refer to [9] where existence and regularity of the Green's matrix for weakly elliptic systems is considered.

The following result generalizes the compactness result of [5] to the case of elastic inclusions:

Theorem 2.1. *Let ω_{ϵ_n} be a sequence of measurable subsets satisfying (2.8) such that, as $n \to \infty$, $|\omega_{\epsilon_n}| \to 0$ and*

$$(2.11) \qquad |\omega_{\epsilon_n}|^{-1} \chi_{\omega_{\epsilon_n}} dx \to d\mu \text{ in the weak}^* \text{ topology of } (C(\overline{\Omega}))',$$

for some regular positive Borel measure μ, such that $\int_\Omega d\mu = 1$.
Given $\psi \in H^{-1/2}(\partial\Omega)$ satisfying (2.3), let U and u_{ϵ_n} denote the solutions to (2.4) and (2.7) respectively. There exists a subsequence, not relabeled, and a fourth order tensor $\mathbb{M} \in L^2(\Omega, d\mu)$ such that, for $y \in \partial\Omega$,

$$(2.12) \qquad (u_{\epsilon_n} - U)(y) = |\omega_{\epsilon_n}| \int_\Omega \mathbb{M}(x) \widehat{\nabla} U(x) : \widehat{\nabla} N(x, y) d\mu_x + o(|\omega_{\epsilon_n}|).$$

We will prove this result in the next section.

3. Proof of theorem 2.1

3.1. **Preliminary estimates.** Let $F \in H^{-1}(\Omega)$ and $f \in H^{-1/2}(\Omega)$ satisfying the compatibility conditions

$$\int_\Omega F \, dx = \int_{\partial\Omega} f \, d\sigma_x \quad \text{and} \quad \int_\Omega F \cdot R \, dx = \int_{\partial\Omega} f \cdot R \, d\sigma_x,$$

for every infinitesimal rigid motion R.

Let V and v_ϵ in $\tilde{H}(\Omega)$ solve

$$(3.1) \qquad \begin{cases} \operatorname{div}(\mathbb{C}_0 \widehat{\nabla} V) & = & F \text{ in } \Omega \\ (\mathbb{C}_0 \widehat{\nabla} V)\nu & = & f \text{ on } \partial\Omega, \end{cases}$$

and

$$(3.2) \qquad \begin{cases} \operatorname{div}(\mathbb{C}_\epsilon \widehat{\nabla} v_\epsilon) & = & F \text{ in } \Omega \\ (\mathbb{C}_0 \widehat{\nabla} v_\epsilon)\nu & = & f \text{ on } \partial\Omega, \end{cases}$$

respectively.

Lemma 3.1. *Let $F \in C^\alpha(\overline{\Omega})$, with $0 < \alpha < 1$ and let $0 < \eta < 1/d$. There exists a constant $C > 0$, such that*

$$(3.3) \qquad \|v_\epsilon - V\|_{H^1(\Omega)} \le C|\omega_\epsilon|^{1/2} \left(\|F\|_{C^\alpha(\Omega)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} \right),$$

and

$$(3.4) \qquad \|v_\epsilon - V\|_{L^2(\Omega)} \le C|\omega_\epsilon|^{\frac{1}{2} + \frac{1}{d} - \eta} \left(\|F\|_{C^\alpha(\Omega)} + \|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} \right).$$

Proof. We adapt the arguments of ([5]) to the system of elasticity. Since V and v_ϵ solve (3.1) and (3.2) respectively, for every $w \in H^1(\Omega)$, we have

$$\int_\Omega \mathbb{C}_\epsilon \widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla} w \, dx = \int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1) \widehat{\nabla} V : \widehat{\nabla} w \, dx.$$

By choosing $w = v_\epsilon - V$ and applying Korn's inequality, we show that

$$\int_\Omega |\nabla(v_\epsilon - V)|^2 dx \le C \int_\Omega |\widehat{\nabla}(v_\epsilon - V)|^2 dx \le C|\omega_\epsilon|^{1/2} \|\nabla V\|_{L^\infty(\omega_\epsilon)} \|\nabla(v_\epsilon - V)\|_{L^2(\Omega)}.$$

It follows from interior regularity results for the elasticity system with regular coefficients (see, for example, Theorem 6.III, chapter 2 in [4]) that

$$(3.5) \qquad \begin{aligned} \|\nabla V\|_{L^\infty(\omega_\epsilon)} & \le C \left(\|V\|_{H^1(\Omega)} + \|F\|_{C^\alpha(K_0)} \right) \\ & \le C \left(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} + \|F\|_{C^\alpha(K_0)} \right), \end{aligned}$$

and, hence,

$$\|v_\epsilon - V\|_{H^1(\Omega)} \le C|\omega_\epsilon|^{1/2} \left(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} + \|F\|_{C^\alpha(K_0)} \right).$$

We also have, for any $w \in H^1(\Omega)$,

$$(3.6) \qquad \int_\Omega \mathbb{C}_0 \widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla} w \, dx = \int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1) \widehat{\nabla} v_\epsilon : \widehat{\nabla} w \, dx.$$

Let us select $w \in \tilde{H}(\Omega)$ as the solution to

$$\begin{cases} \operatorname{div}(\mathbb{C}_0 \widehat{\nabla} w) & = & V - v_\epsilon \text{ in } \Omega \\ (\mathbb{C}_0 \widehat{\nabla} w)\nu & = & \frac{1}{|\partial\Omega|} \int_\Omega (V - v_\epsilon) dx \text{ on } \partial\Omega, \end{cases}$$

By the smoothness assumption on $\mathbb{C}_0$ and by interior regularity estimates (see, for example, Theorem 2.I, chapter 2 in [4]) we have that

$$\|w\|_{H^2(K_0)} \leq C(\|v_\epsilon - V\|_{L^2(\Omega)} + \|w\|_{H^1(\Omega)}).$$

By Korn and Poincaré inequalities

$$\|w\|_{H^1(\Omega)} \leq C\|\nabla w\|_{L^2(\Omega)} \leq C\|\widehat{\nabla}w\|_{L^2(\Omega)} \leq \|V - v_\epsilon\|_{L^2(\Omega)}$$

and, hence,

$$(3.7) \qquad \|w\|_{H^2(K_0)} \leq C\|v_\epsilon - V\|_{L^2(\Omega)}$$

By Sobolev Embedding Theorem, we have that $\nabla w \in L^p(K_0)$ for every $1 < p < d^*$ where $d^* = \frac{2d}{d-2}$ for $d > 2$ and $d^* = +\infty$ for $d = 2$, and

$$(3.8) \qquad \left(\int_{K_0} |\nabla w|^p dx\right)^{\frac{1}{p}} \leq C_p\|w\|_{H^2(K_0)} \leq C\|v_\epsilon - V\|_{L^2(\Omega)}.$$

Let us choose $q \in (\frac{2d}{d+2}, 2)$ and p such that $\frac{1}{p} + \frac{1}{q} = 1$. Notice that, $p \in (1, d^*)$. By inserting w into (3.6) we obtain

$$
\begin{aligned}
\int_\Omega (v_\epsilon - V)^2 dx &= \int_\Omega \mathbb{C}_0 \widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla}w \, dx = \int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}v_\epsilon : \widehat{\nabla}w \, dx \\
(3.9) \qquad &\leq C\left(\int_{\omega_\epsilon} |\nabla v_\epsilon|^q\right)^{\frac{1}{q}}\left(\int_{\omega_\epsilon} |\nabla w|^p\right)^{\frac{1}{p}} \\
(3.10) \qquad &\leq C\left(\int_{\omega_\epsilon} |\nabla v_\epsilon|^q\right)^{\frac{1}{q}}\|v_\epsilon - V\|_{L^2(\Omega)}.
\end{aligned}
$$

Now, by Hölder inequality and (3.5) we get

$$
\begin{aligned}
\|\nabla v_\epsilon\|_{L^q(\omega_\epsilon)} &\leq \|\nabla(v_\epsilon - V)\|_{L^q(\omega_\epsilon)} + \|\nabla V\|_{L^q(\omega_\epsilon)} \\
(3.11) \qquad &\leq |\omega_\epsilon|^{\frac{1}{q} - \frac{1}{2}}\|\nabla(v_\epsilon - V)\|_{L^2(\omega_\epsilon)} + |\omega_\epsilon|^{\frac{1}{q}}\|\nabla V\|_{L^\infty(\omega_\epsilon)} \\
&\leq C|\omega_\epsilon|^{1/q}\left(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} + \|F\|_{L^\infty(\Omega)}\right)
\end{aligned}
$$

A combination of (3.9), (3.8) and (3.11) yields

$$\|v_\epsilon - V\|_{L^2(\Omega)} \leq C_q|\omega_\epsilon|^{1/q}\left(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} + \|F\|_{C^\alpha(\Omega)}\right)$$

Note that since for $q \searrow \frac{2d}{d+2}$, we have $\frac{1}{q} \nearrow \frac{1}{d} + \frac{1}{2}$. It follows that given any $0 < \eta < 1/d$ there exists a constant C such that

$$\|v_\epsilon - V\|_{L^2(\Omega)} \leq C|\omega_\epsilon|^{\frac{1}{d} + \frac{1}{2} - \eta}\left(\|F\|_{H^{-1}(\Omega)} + \|f\|_{H^{-1/2}(\partial\Omega)} + \|F\|_{C^\alpha(\Omega)}\right)$$

$\square$

3.2. Definition of the polarization tensor. Let

$$(3.12) \qquad v^{ij} = \frac{1}{2}\left(\mathbf{e}_i x_j + \mathbf{e}_j x_i\right) - c_{ij}$$

where $\mathbf{e}_i$ is the i-th coordinate direction and

$$c_{ij} = \frac{1}{2|\partial\Omega|}\int_{\partial\Omega}\left(\mathbf{e}_i x_j + \mathbf{e}_j x_i\right) d\sigma$$

and consider $v_\epsilon^{ij} \in \tilde{H}(\Omega)$ solution to the problem

$$(3.13) \qquad \begin{cases} \operatorname{div}(\mathbb{C}_\epsilon \widehat{\nabla} v_\epsilon^{ij}) &= \operatorname{div}(\mathbb{C}_0 \widehat{\nabla} v^{ij}) \text{ in } \Omega \\ (\mathbb{C}_\epsilon \widehat{\nabla} v_\epsilon^{ij})\nu &= (\mathbb{C}_0 \widehat{\nabla} v^{ij})\nu \text{ on } \partial\Omega, \end{cases}$$

Observe now that

$$\left\| \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} v_\epsilon^{ij} \right\|_{L^1(\Omega)} \leq \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \left| (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} (v_\epsilon^{ij} - v^{ij}) \right| dx$$
$$+ \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \left| (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} v^{ij} \right| dx.$$

By Lemma 3.1 and recalling that $\mathbb{C}_0$ and $\mathbb{C}_1$ are bounded and that

$$(3.14) \qquad \nabla v^{ij} = \widehat{\nabla} v^{ij} = \frac{1}{2}(\mathbf{e}_i \otimes \mathbf{e}_j + \mathbf{e}_j \otimes \mathbf{e}_i),$$

we have

$$\left\| \frac{1}{|\omega_\epsilon|} \chi_{\omega_\epsilon}(\mathbb{C}_1 - \mathbb{C}_0) \nabla v_\epsilon^{ij} \right\|_{L^1(\Omega)} \leq C.$$

Hence, possibly extracting a subsequence, we may assume that

$$(3.15) \qquad |\omega_{\epsilon_n}| \chi_{\omega_{\epsilon_n}} (\mathbb{C}_1 - \mathbb{C}_0) \nabla v_{\epsilon_n}^{ij} \to d\mathbb{M}_{ijlm},$$

in the weak* topology of $C^0(\overline{\Omega})$, where $d\mathbb{M}_{ijlm}$ is a regular Borel Measures with support in K_0. Let $\Phi \in C^0(\overline{\Omega})$. By definition of $d\mathbb{M}_{ijlm}$, we see that

$$\left| \int_\Omega \Phi d\mathbb{M}_{ijlm} \right| = \left| \lim_{n\to\infty} \frac{1}{|\omega_{\epsilon_n}|} \int_\Omega \chi_{\omega_{\epsilon_n}} (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} v_{\epsilon_n}^{ij} \Phi \, dx \right|$$
$$\leq \underline{\lim}_{n\to\infty} \frac{1}{|\omega_{\epsilon_n}|} \int_\Omega \chi_{\omega_{\epsilon_n}} \left| (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} (v_{\epsilon_n}^{ij} - v^{ij}) \right| |\Phi| \, dx$$
$$+ \lim_{n\to\infty} \frac{1}{|\omega_{\epsilon_n}|} \int_\Omega \chi_{\omega_{\epsilon_n}} \left| (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} v^{ij} \right| |\Phi| \, dx$$
$$\leq \underline{\lim}_{n\to\infty} \frac{C}{|\omega_{\epsilon_n}|^{1/2}} \left(\int_\Omega |\widehat{\nabla} (v_{\epsilon_n}^{ij} - v^{ij})|^2 \, dx \right)^{1/2} \left(\int_\Omega \frac{1}{|\omega_{\epsilon_n}|} \chi_{\omega_{\epsilon_n}} |\Phi|^2 dx \right)^{1/2}$$
$$+ C \left(\int_\Omega |\Phi|^2 d\mu \right)^{1/2} \leq C \left(\int_\Omega |\Phi|^2 d\mu \right)^{1/2}.$$

Hence

$$\Phi \to \int_\Omega \Phi dM_{ijlm}$$

is a bounded functional on $L^2(\Omega, d\mu)$, and

$$\int_\Omega \Phi dM_{ijlm} = \int_\Omega \Phi \mathbb{M}_{ijlm} d\mu$$

for some function $\mathbb{M}_{ijlm} \in L^2(\Omega, d\mu)$. The tensor $\mathbb{M}$ actually relates to the weak limit of u_ϵ, as the next lemma expresses:

Lemma 3.2. *Let U and u_ϵ denote the solutions to (2.4) and (2.7) for $\psi \in H^{-1/2}(\partial\Omega)$ satisfying the compatibility conditions (2.3). Let ω_{ϵ_n} such that $|\omega_{\epsilon_n}| \to 0$ be a sequence for which (2.8), (2.11) and (3.15) hold.*

Then, $\frac{1}{|\omega_{\epsilon_n}|}\chi_{\omega_{\epsilon_n}}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}u_{\epsilon_n}dx$ is convergent in the weak* topology of $(C^0(\overline{\Omega}))'$ with

$$\lim_{n\to\infty}\frac{1}{|\omega_{\epsilon_n}|}\chi_{\omega_{\epsilon_n}}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}u_{\epsilon_n}dx=\mathbb{M}\widehat{\nabla}Ud\mu$$

Proof. It suffices to prove that we may extract a subsequence of $\{\omega_{\epsilon_n}\}$ such that $\frac{1}{|\omega_{\epsilon_{n_j}}|}\chi_{\omega_{\epsilon_{n_j}}}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}u_{\epsilon_{n_j}}dx$ converges to $\mathbb{M}\widehat{\nabla}Ud\mu$. The fact that the limit is independent of the particular subsequence guarantees that the entire sequence is convergent.

Proceeding as for v_ϵ^{ij}, we see that

$$
\begin{aligned}
\left\|\frac{1}{|\omega_{\epsilon_n}|}\chi_{\omega_{\epsilon_n}}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}u_{\epsilon_n}\right\|_{L^1(\Omega)} &\leq \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}}\left|(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}(u_{\epsilon_n}-U)\right|dx\\
&+ \frac{1}{|\omega_{\epsilon_n}|}\int_{\omega_{\epsilon_n}}\left|(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}U\right|dx\\
&\leq C\|\psi\|_{H^{-1/2}(\partial\Omega)},
\end{aligned}
$$

hence, possibly extracting a subsequence, that we do not relabel, we may assume that, for some matrix-valued measure η,

$$\frac{1}{|\omega_{\epsilon_n}|}\chi_{\omega_{\epsilon_n}}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}u_{\epsilon_n}dx\to d\eta$$

in the weak* topology of $(C^0(\overline{\Omega}))'$.

We must now show that, for any scalar function Φ,

$$\text{(3.16)} \qquad \int_\Omega\Phi d\eta=\int_\Omega\Phi\mathbb{M}\widehat{\nabla}Ud\mu.$$

In order to do this, it is enough to prove that

$$\text{(3.17)} \qquad \int_{\omega_\epsilon}(\mathbb{C}_0-\mathbb{C}_1)\widehat{\nabla}U:\widehat{\nabla}v_\epsilon^{ij}\Phi\,dx=\int_{\omega_\epsilon}(\mathbb{C}_0-\mathbb{C}_1)\widehat{\nabla}u_\epsilon:\widehat{\nabla}v^{ij}\Phi\,dx+o(|\omega_\epsilon|),$$

because then, by passing to the limit along subsequences of ω_{ϵ_n} in (3.17), we get (3.16).

Let us notice that, since

$$
\left\{
\begin{array}{rcll}
\text{div}\left(\mathbb{C}_0\widehat{\nabla}U\right) &=& \text{div}\left(\mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon\right) & \text{in}\quad\Omega\\
(\mathbb{C}_0\widehat{\nabla}U)\nu &=& (\mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon)\nu & \text{on}\quad\partial\Omega,
\end{array}
\right.
$$

for every vector valued test function Ψ we have that

$$\text{(3.18)} \qquad \int_\Omega\mathbb{C}_0\widehat{\nabla}U:\widehat{\nabla}\Psi\,dx=\int_\Omega\mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon:\widehat{\nabla}\Psi\,dx.$$

For the same reason

$$\text{(3.19)} \qquad \int_\Omega\mathbb{C}_0\widehat{\nabla}v^{ij}:\widehat{\nabla}\Psi\,dx=\int_\Omega\mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon^{ij}:\widehat{\nabla}\Psi\,dx.$$

We can calculate

$$\int_{\Omega}(\mathbb{C}_0 - \mathbb{C}_\epsilon)\widehat{\nabla}U : \widehat{\nabla}v_\epsilon^{ij}\Phi\, dx - \int_{\Omega}(\mathbb{C}_0 - \mathbb{C}_\epsilon)\widehat{\nabla}u_\epsilon : \widehat{\nabla}v^{ij}\Phi\, dx$$

$$= \int_{\Omega}\Big(\mathbb{C}_0\widehat{\nabla}U : \widehat{\nabla}(v_\epsilon^{ij}\Phi) - \mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon^{ij} : \widehat{\nabla}(U\Phi)\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_0\widehat{\nabla}v^{ij} : \widehat{\nabla}(u_\epsilon\Phi) - \mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon : \widehat{\nabla}(v^{ij}\Phi)\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_0\widehat{\nabla}U : (v_\epsilon^{ij} \otimes \nabla\Phi) - \mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon^{ij} : (U \otimes \nabla\Phi)\Big)\, dx$$

$$+ \int_{\Omega}\Big(\mathbb{C}_0\widehat{\nabla}v^{ij} : (u_\epsilon \otimes \nabla\Phi) - \mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon : (v^{ij} \otimes \nabla\Phi)\Big)\, dx$$

By (3.18) and (3.19) and recalling that $\mathbb{C}_\epsilon = \mathbb{C}_0$ in $\Omega \setminus \omega_\epsilon$, we can write

$$\int_{\omega_\epsilon}(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}U : \widehat{\nabla}v_\epsilon^{ij}\Phi\, dx - \int_{\omega_\epsilon}(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}u_\epsilon : \widehat{\nabla}v^{ij}\Phi\, dx$$

$$= \int_{\Omega}\Big(\mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon : \widehat{\nabla}(v_\epsilon^{ij}\Phi) - \mathbb{C}_0\widehat{\nabla}v^{ij} : \widehat{\nabla}(U\Phi)\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon^{ij} : \widehat{\nabla}(u_\epsilon\Phi) - \mathbb{C}_0\widehat{\nabla}U : \widehat{\nabla}(v^{ij}\Phi)\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_0\widehat{\nabla}U : (v_\epsilon^{ij} \otimes \nabla\Phi) - \mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon^{ij} : (U \otimes \nabla\Phi)\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_0\widehat{\nabla}v^{ij} : (u_\epsilon \otimes \nabla\Phi) - \mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon : (v^{ij} \otimes \nabla\Phi)\Big)\, dx$$

$$= \int_{\Omega}\Big(\mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon : \widehat{\nabla}v_\epsilon^{ij}\Phi - \mathbb{C}_0\widehat{\nabla}v^{ij} : \widehat{\nabla}U\Phi\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon^{ij} : \widehat{\nabla}u_\epsilon\Phi - \mathbb{C}_0\widehat{\nabla}U : \widehat{\nabla}v^{ij}\Phi\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon : (v_\epsilon^{ij} \otimes \nabla\Phi) - \mathbb{C}_0\widehat{\nabla}v^{ij} : (U \otimes \nabla\Phi)\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon^{ij} : (u_\epsilon \otimes \nabla\Phi) - \mathbb{C}_0\widehat{\nabla}U : (v^{ij} \otimes \nabla\Phi)\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_0\widehat{\nabla}U : (v_\epsilon^{ij} \otimes \nabla\Phi) - \mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon^{ij} : (U \otimes \nabla\Phi)\Big)\, dx$$

$$- \int_{\Omega}\Big(\mathbb{C}_0\widehat{\nabla}v^{ij} : (u_\epsilon \otimes \nabla\Phi) - \mathbb{C}_\epsilon\widehat{\nabla}u_\epsilon : (v^{ij} \otimes \nabla\Phi)\Big)\, dx$$

By symmetry of the elasticity tensors, the first two lines of last equality give zero. By rearranging the various integral in a suitable way we get,

$$\int_{\omega_\epsilon} (\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}U : \widehat{\nabla}v_\epsilon^{ij}\Phi\, dx - \int_{\omega_\epsilon}(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}u_\epsilon : \widehat{\nabla}v^{ij}\Phi\, dx$$

$$= \int_\Omega \mathbb{C}_\epsilon\left(\widehat{\nabla}u_\epsilon - \widehat{\nabla}U\right) : ((v_\epsilon^{ij} - v^{ij}) \otimes \nabla\Phi)dx$$

$$- \int_\Omega \mathbb{C}_\epsilon\left(\widehat{\nabla}v^{ij} - \widehat{\nabla}v_\epsilon^{ij}\right) : ((U - u_\epsilon) \otimes \nabla\Phi)dx$$

$$+ \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}U : ((v_\epsilon^{ij} - v^{ij}) \otimes \nabla\Phi)dx$$

$$- \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}v^{ij} : ((U - u_\epsilon) \otimes \nabla\Phi)dx$$

By Lemma 3.1 and by regularity of functions U and v^{ij} in K_0, we get (3.17). $\qquad\square$

3.3. **End of the proof of Theorem 2.1.** Let ω_{ϵ_n} as above. By the definition of the Neumann matrix it is easy to see that

$$(3.20) \qquad (u_{\epsilon_n} - U)(y) = \int_{\omega_{\epsilon_n}} (\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}u_{\epsilon_n} : \widehat{\nabla}N(\cdot, y)dx.$$

(See [3] for details).
Let $K_0 \subset \Omega$ the compact set introduced in (2.8). Given $y \in \partial\Omega$ it is possible to find $\Psi_y \in C^0(\overline{\Omega})$ a matrix valued function such that $\Psi_y(x) = \nabla_x N(x, y)$ for $x \in K_0$. Using the previous lemma we get

$$\begin{aligned}(u_{\epsilon_n} - U)(y) &= |\omega_{\epsilon_n}| \int_\Omega \frac{1}{|\omega_{\epsilon_n}|}\chi_{\omega_{\epsilon_n}}(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}u_{\epsilon_n} : \Psi_y\, dx \\ &= |\omega_{\epsilon_n}| \int_\Omega \mathbb{M}\widehat{\nabla}U : \Psi_y d\mu + o(|\omega_\epsilon|) \\ &= |\omega_{\epsilon_n}| \int_\Omega \mathbb{M}\widehat{\nabla}U : \widehat{\nabla}N(\cdot, y)\, d\mu + o(|\omega_\epsilon|).\end{aligned}$$

4. PROPERTIES OF THE ELASTIC POLARIZATION TENSOR

In this section we prove few basic properties of the polarization tensor $\mathbb{M}$:

Proposition 4.1. *The polarization tensor $\mathbb{M}$ has the same symmetry properties of the elasticity tensors $\mathbb{C}_0$ and $\mathbb{C}_1$, that is*

$$\mathbb{M}_{ijkl} = \mathbb{M}_{klij} = \mathbb{M}_{jikl}, \qquad \mu\text{-a.e.}$$

for any choice of indices i, j, k, l between 1 and d.
Moreover, for any symmetric matrix E,

$$(4.1) \qquad \mathbb{C}_0\mathbb{C}_1^{-1}(\mathbb{C}_1 - \mathbb{C}_0)E : E \le \mathbb{M}E : E \le (\mathbb{C}_1 - \mathbb{C}_0)E : E \qquad \mu\text{-a.e.}.$$

Proof. Firstly, we show that $\mathbb{M}$ enjoys the same symmetry as the elastic tensors $\mathbb{C}_1$ and $\mathbb{C}_0$. To this end, we recall the following equality, which was obtained in the

proof of lemma 3.2.

$$\frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}U:\widehat{\nabla}v_\epsilon^{ij}\Phi dx = \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}u_\epsilon:\widehat{\nabla}v^{ij}\Phi dx + o(1).$$

Substituting U and u_ϵ for v^{hk} and v_ϵ^{hk} respectively, we see that

$$\frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}v^{hk}:\widehat{\nabla}v_\epsilon^{ij}\Phi dx = \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}v_\epsilon^{hk}:\widehat{\nabla}v^{ij}\Phi dx + o(1).$$

Recalling (3.14) and by the symmetry of $\mathbb{C}_0$ and $\mathbb{C}_1$, we get on one hand that

$$\begin{aligned}\frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}v^{hk}\widehat{\nabla}v_\epsilon^{ij}\Phi dx &= \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}\sum_{lmpq}(\mathbb{C}_1-\mathbb{C}_0)_{lmpq}\delta_{hp}\delta_{kq}\frac{\partial(v_\epsilon^{ij})_m}{\partial x_l}\Phi dx\\ &= \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}\sum_{lm}(\mathbb{C}_1-\mathbb{C}_0)_{lmhk}\frac{\partial(v_\epsilon^{ij})_m}{\partial x_l}\Phi dx\\ &\to \int_\Omega\mathbb{M}_{ijhk}\Phi d\mu.\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\mathbb{C}_1-\mathbb{C}_0)\widehat{\nabla}v_\epsilon^{hk}\widehat{\nabla}v^{ij}\Phi dx &= \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}\sum_{lmpq}(\mathbb{C}_1-\mathbb{C}_0)_{lmpq}\frac{\partial(v_\epsilon^{hk})_q}{\partial x_p}\delta_{il}\delta_{jm}\Phi dx\\ &= \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}\sum_{pq}(\mathbb{C}_1-\mathbb{C}_0)_{ijpq}\frac{\partial(v_\epsilon^{hk})_q}{\partial x_p}\Phi dx\\ &\to \int_\Omega\mathbb{M}_{hkij}\Phi d\mu.\end{aligned}$$

It follows that

(4.2) $$\mathbb{M}_{ijhk}=\mathbb{M}_{hkij},\qquad \mu-\text{a.e.}$$

To obtain the minor symmetry, we observe that

$$\sum_{lm}|\omega_{\epsilon_n}|\chi_{\omega_{\epsilon_n}}(\mathbb{C}_1-\mathbb{C}_0)_{lmpq}\frac{\partial(v_{\epsilon_n}^{ij})_q}{\partial x_p} = \sum_{lm}|\omega_{\epsilon_n}|\chi_{\omega_{\epsilon_n}}(\mathbb{C}_1-\mathbb{C}_0)_{mlpq}\frac{\partial(v_{\epsilon_n}^{ij})_q}{\partial x_p},$$

since $\mathbb{C}_1-\mathbb{C}_0$ is a totally symmetric 4th-order tensor. Then, from (3.15), it follows that

$$\mathbb{M}_{ijlm}=\mathbb{M}_{ijml},\qquad \mu-\text{a.e.}$$

All the minor symmetries now follow from (4.2).

For the proof of (4.1), we follow [5], where the case of the scalar conductivity equation was discussed.

We begin by fixing a constant, symmetric matrix $E=[E_{ij}]$. We set

(4.3) $$V=\sum_{ij}E_{ij}v^{ij},\qquad v_\epsilon=\sum_{ij}E_{ij}v_\epsilon^{ij},$$

where v^{ij} and v_ϵ^{ij} are given in (3.12) and (3.13), and observe that V solves

(4.4) $$\begin{cases}\text{div}(\mathbb{C}_0\widehat{\nabla}V)=\text{div}(\mathbb{C}_0E), & \text{in }\Omega,\\ (\mathbb{C}_0\widehat{\nabla}V)\nu=(\mathbb{C}_0E)\nu, & \text{on }\partial\Omega,\end{cases}$$

and, consequently, v^ϵ solves

$$(4.5) \qquad \begin{cases} \operatorname{div}(\mathbb{C}_\epsilon \widehat{\nabla} v_\epsilon) = \operatorname{div}(\mathbb{C}_0 E), & \text{in } \Omega, \\ (\mathbb{C}_\epsilon \widehat{\nabla} v_\epsilon)\nu = (\mathbb{C}_0 E)\nu, & \text{on } \partial\Omega. \end{cases}$$

We next recall that $\mathbb{M}$ is obtained as the weak limit (3.15). Therefore, given any function $\Phi \in C^0(\bar\Omega)$, we have

$$\int_\Omega \mathbb{M}_{ijkl} \Phi \, d\mu = \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \left[(\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} v_\epsilon^{ij} \right]_{kl} \Phi(x) \, dx \; + \; o(1),$$

where ϵ is an element of the sequence $\{\epsilon_n\}$, and square brackets indicate components. We choose $\Phi(x) = E_{ij} E_{kl} \phi \chi_{\bar\Omega}$, where ϕ is a positive smooth function, and sum over repeated indices:

$$\int_\Omega \mathbb{M} E : E \phi d\mu = \sum_{ijkl} \int_\Omega E_{ij} \, \mathbb{M}_{ijkl} \, E_{kl} \phi dx$$
$$= \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \sum_{ijkl} E_{ij} \left[(\mathbb{C}_1 - \mathbb{C}_0) \cdot \widehat{\nabla} v_\epsilon^{ij} \right]_{kl} E_{kl} \phi dx \; + \; o(1).$$

Recall that $\nabla v^{ij} = \widehat{(e_i \otimes e_j)}$, hence from (4.3), using that E is symmetric, we have

$$\widehat{\nabla} V = \sum_{ij} E_{ij} \, e_i \otimes e_j = E,$$

so that:

$$\begin{aligned}
\int_\Omega \mathbb{M} E : E \phi \, d\mu \;=\;& \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \sum_{klpq} (\mathbb{C}_1 - \mathbb{C}_0)_{klpq} \left(\sum_{ij} E_{ij} \partial_p [v_\epsilon^{ij}]_q \right) : [\widehat{\nabla} V]_{kl} \phi \, dx \\
& + o(1) \\
=\;& \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0) \, \widehat{\nabla} v_\epsilon : \widehat{\nabla} V \phi \, dx + \; o(1) \\
=\;& \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0) \, \widehat{\nabla} V : \widehat{\nabla} V \phi \, dx \\
(4.6) \qquad & + \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0) \, \widehat{\nabla} (v_\epsilon - V) : \widehat{\nabla} V \phi \, dx + \; o(1)
\end{aligned}$$

Let us now notice that

$$\int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0) \, \widehat{\nabla} (v_\epsilon - V) : \widehat{\nabla} V \phi \, dx = - \int_\Omega \mathbb{C}_\epsilon \widehat{\nabla} (v_\epsilon - V) : \widehat{\nabla} (v_\epsilon - V) \phi \, dx + R$$

where, by (4.4), (4.5) and by Lemma 3.1,

$$
\begin{aligned}
R \;:=\; & \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}(v_\epsilon - V) \;:\; \widehat{\nabla}V\phi\,dx + \int_\Omega \mathbb{C}_\epsilon\widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla}(v_\epsilon - V)\phi\,dx \\
=\; & \int_\Omega (\mathbb{C}_\epsilon - \mathbb{C}_0)\,\widehat{\nabla}(v_\epsilon - V) \;:\; \widehat{\nabla}V\phi\,dx + \int_\Omega \mathbb{C}_\epsilon\widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla}(v_\epsilon - V)\phi\,dx \\
=\; & \int_\Omega \mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon : \widehat{\nabla}(v_\epsilon - V)\phi\,dx - \int_\Omega \mathbb{C}_0\widehat{\nabla}V : \widehat{\nabla}(v_\epsilon - V)\phi\,dx \\
=\; & \int_\Omega \mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon : \widehat{\nabla}((v_\epsilon - V)\phi)\,dx - \int_\Omega \mathbb{C}_0\widehat{\nabla}V : \widehat{\nabla}((v_\epsilon - V)\phi)\,dx \\
& - \int_\Omega \mathbb{C}_\epsilon\widehat{\nabla}v_\epsilon : ((v_\epsilon - V)\otimes\nabla\phi\,dx + \int_\Omega \mathbb{C}_0\widehat{\nabla}V : ((v_\epsilon - V)\otimes\nabla\phi\,dx \\
=\; & -\int_\Omega \mathbb{C}_0\widehat{\nabla}(v_\epsilon - V) : ((v_\epsilon - V)\otimes\nabla\phi\,dx - \int_{\omega_\epsilon} \mathbb{C}_1\widehat{\nabla}v_\epsilon : ((v_\epsilon - V)\otimes\nabla\phi\,dx
\end{aligned}
$$

$$(4.7) =\; o(|\omega_\epsilon|)$$

By inserting the above relation in (4.6) we get

$$
\int_\Omega \mathbb{M}E : E\phi\,d\mu \;=\; \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}V \;:\; \widehat{\nabla}V\phi\,dx
$$

$$(4.8) \qquad\qquad - \int_\Omega \mathbb{C}_\epsilon\widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla}(v_\epsilon - V)\phi\,dx + o(1).$$

Since $\mathbb{C}_\epsilon$ is strongly convex, and $\phi > 0$,

$$
\int_\Omega \mathbb{C}_\epsilon\widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla}(v_\epsilon - V)\phi\,dx \geq 0
$$

and, hence,

$$
\int_\Omega \mathbb{M}E : E\phi\,d\mu \leq \frac{1}{|\omega_\epsilon|}\int_{\omega_\epsilon}(\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}V \;:\; \widehat{\nabla}V\phi\,dx + o(1)
$$

Since the left-hand side does not depend on ϵ we let $\epsilon \to 0$ and get

$$(4.9) \quad \int_\Omega \mathbb{M}E : E\phi\,d\mu \leq \int_\Omega(\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}V : \widehat{\nabla}V\phi\,d\mu = \int_\Omega(\mathbb{C}_1 - \mathbb{C}_0)\,E : E\phi\,d\mu.$$

Now, by (4.7) and by the fact that $\mathbb{C}_1$ is strongly convex, we have that

$$
\begin{aligned}
& \int_\Omega \mathbb{C}_\epsilon\widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla}(v_\epsilon - V)\phi\,dx \\
& = \int_{\omega_\epsilon}(\mathbb{C}_0 - \mathbb{C}_1)\,\widehat{\nabla}(v_\epsilon - V) \;:\; \widehat{\nabla}V\phi\,dx + o(|\omega_\epsilon|) \\
& = \int_{\omega_\epsilon}\mathbb{C}_1\,\widehat{\nabla}(v_\epsilon - V) : \mathbb{C}_1^{-1}(\mathbb{C}_0 - \mathbb{C}_1)\,\widehat{\nabla}V\phi\,dx + o(|\omega_\epsilon|) \\
& \leq \left(\int_{\omega_\epsilon}\mathbb{C}_1\widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla}(v_\epsilon - V)\phi\,dx\right)^{1/2} \cdot
\end{aligned}
$$

$$(4.10) \qquad \left(\int_{\omega_\epsilon}(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}V : \mathbb{C}_1^{-1}(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}V\phi\,dx\right)^{1/2} + o(|\omega_\epsilon|)$$

from which it follows that
(4.11)
$$\int_\Omega \mathbb{C}_\epsilon \widehat{\nabla}(v_\epsilon - V) : \widehat{\nabla}(v_\epsilon - V)\phi\, dx \le \int_{\omega_\epsilon} \mathbb{C}_1^{-1}(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}V : (\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}V\phi\, dx + o(|\omega_\epsilon|).$$

By inserting (4.11) into (4.8) we get

$$
\begin{aligned}
\int_\Omega \mathbb{M}E : E\phi\, d\mu \quad \ge \quad & \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\, \widehat{\nabla}V : \widehat{\nabla}V\phi\, dx \\
& -\frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \mathbb{C}_1^{-1}(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}V : (\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}V\phi\, dx + o(1) \\
= \quad & \frac{1}{|\omega_\epsilon|} \int_{\omega_\epsilon} \mathbb{C}_1^{-1}(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}V : \mathbb{C}_0\widehat{\nabla}V dx + o(1).
\end{aligned}
$$

By letting $\epsilon \to 0$ we get
(4.12)
$$\int_\Omega \mathbb{M}E : E\phi\, d\mu \ge \int_\Omega \mathbb{C}_0 \mathbb{C}_1^{-1}(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla}V : \widehat{\nabla}V\phi\, d\mu = \int_\Omega \mathbb{C}_0 \mathbb{C}_1^{-1}(\mathbb{C}_1 - \mathbb{C}_0)E : E\phi\, d\mu.$$

Notice that (4.9) and (4.12) hold for every positive ϕ, hence (4.1) follows. $\square$

5. The case of thin planar inclusions

In this section, we specialize to the case of thin inclusions in a planar domain $\Omega \subset \mathbb{R}^2$, modeled as an appropriate neighborhood ω_ϵ of a given simple curve $\sigma_0 \subset \Omega$, that is:

$$(5.1) \qquad\qquad \omega_\epsilon = \left\{ x \in \Omega \ : \ d(x, \sigma_0) < \epsilon \right\}.$$

We impose the following conditions on σ_0. We assume that σ_0 is of class C^3 and that there exists some $K > 0$ such that

$$(5.2) \qquad
\begin{aligned}
d(\sigma_0, \partial\Omega) &\ge& K^{-1} \\
\|\sigma_0\|_{C^3} &\le& K \\
K^{-1} \le \text{length}(\sigma_0) &\le& K.
\end{aligned}
$$

Moreover we assume that for every $x \in \sigma_0$ there exists two discs B_1 and B_2 of radius K^{-1}, such that

$$\overline{B}_1 \cap \overline{B}_2 = \overline{B}_1 \cap \sigma_0 = \overline{B}_2 \cap \sigma_0 = \{x\}.$$

The latter assumption guarantees that different parts of σ_0 do not get too close, so that ω_ϵ does not self-intersect for small ϵ. We refer to σ_0 as the support of ω_ϵ.

Let us fix an orthonormal system (n, τ) on σ_0 such that n is a unit normal vector field to the curve and τ is a unit tangent vector field. If σ_0 is a closed curve, then we take n to point in the outward direction of the domain it encloses.

We present a different derivation of the small volume asymptotic formula (2.12) for the displacement at the boundary in this case, which makes more explicit the measure and polarization tensor $\mathbb{M}$ that appear in (2.12).

The main result of this section is the following theorem, which is a counterpart to Theorem 2.1.

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain and let $\sigma_0 \subset\subset \Omega$ be a simple curve satisfying (5.2). Let u_ϵ and U be the solutions to (2.4) and (2.7) respectively. For every $x \in \sigma_0$, there exists a fourth order elastic tensor field $\tilde{\mathbb{M}}(x)$ such that, for $y \in \partial\Omega$*

$$(5.3) \qquad (u_\epsilon - U)(y) = 2\epsilon \int_{\sigma_0} \tilde{\mathbb{M}}(x)\widehat{\nabla}u_0(x) : \widehat{\nabla}N(x,y)\,d\sigma_0(x) + o(\epsilon).$$

The term $o(\epsilon)$ is bounded by $C\epsilon^{1+\theta}\|\psi\|_{H^{-1/2}(\partial\Omega)}$, for some $0 < \theta < 1$ and C depending only on θ, Ω, α_0, β_0 and K.

5.1. **Proof of Theorem 5.1.** The proof of the theorem closely follows the proof of the corresponding result in the isotropic case (see [3]). We only detail those steps, where the proof differs from that case. In the following we set $u_\epsilon^i = u_{\epsilon|\omega_\epsilon}$ and $u_\epsilon^e = u_{\epsilon|\Omega\setminus\omega_\epsilon}$. We simply use u_ϵ when no confusion can occur. Firstly, we write $(u_\epsilon - U)_{|\partial\Omega}$ in terms of an integral over ω_ϵ of the product of $\widehat{\nabla}u_\epsilon^i$ and $\widehat{\nabla}N$. Secondly, using some regularity estimates for solutions to the elastic system in a laminar domain due to Li and Nirenberg [10], we approximate this integral by an integral over a portion of $\partial\omega_\epsilon$, which we rewrite, in a third step, using the transmission conditions and a tensor $\tilde{\mathbb{M}}$ satisfying (5.7). The existence of $\tilde{\mathbb{M}}$ is proved later in Subsection 5.2. Finally, taking limits in the resulting expression as $\epsilon \to 0$ and using fine regularity estimates for u_ϵ proves the theorem.

First step.

We recall (see (3.20)) that, for $y \in \partial\Omega$

$$(5.4) \qquad (u_\epsilon - U)(y) = \int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}u_\epsilon^i : \widehat{\nabla}N(\cdot,y)\,dx.$$

Second step. Let β be a constant, $0 < \beta < 1$, and set

$$\omega_\epsilon' = \left\{ x + t\,n(x) \,:\, x \in \sigma_0,\, d(x,\partial\sigma_0) > \epsilon^\beta,\, t \in (-\epsilon, \epsilon) \right\}.$$

Notice that if σ_0 is a closed simple curve, then $\omega_\epsilon' = \omega_\epsilon$.

By Theorem 2.1, chapter 2 in [4] combined with Sobolev Embedding Theorem, we have that $\|\nabla U\|_{L^\infty(\omega_\epsilon)}$ and $\|\nabla N(\cdot,y)\|_{L^\infty(\omega_\epsilon)}$ (for $y \in \partial\Omega$) are bounded uniformly in ϵ. Using this fact together with the energy estimate (3.3), one can easily show as in [3] that

$$(5.5)$$
$$\int_{\omega_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}u_\epsilon^i : \widehat{\nabla}N(\cdot,y)\,dx = \int_{\omega_\epsilon'} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}u_\epsilon^i : \widehat{\nabla}N(\cdot,y)\,dx + O(\epsilon^{1+\beta/2}).$$

Let σ_ϵ' denote the curve

$$\sigma_\epsilon' = \left\{ x + \epsilon\,n(x) \,:\, x \in \sigma_0,\, d(x,\partial\sigma_0) > \epsilon^\beta \right\}.$$

A crucial ingredient, at this point, is a C^α regularity estimates for the gradient of solutions to laminated systems due to Li and Nirenberg (see [10]). Using this estimate and proceeding as in [3], we can approximate the values of ∇u_ϵ in ω_ϵ' by its values on σ_ϵ', so that

$$\int_{\omega_\epsilon'} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}u_\epsilon^i(x) : \widehat{\nabla}N(x,y)\,dx$$

$$(5.6) \qquad\qquad = 2\epsilon \int_{\sigma_\epsilon'} (\mathbb{C}_1 - \mathbb{C}_0)\,\widehat{\nabla}u_\epsilon^i : \widehat{\nabla}N(\cdot,y) + O(\epsilon^{1+\alpha-\beta(1+\alpha)}),$$

for $\beta < \alpha(1+\alpha)^{-1}$.

Third step.

From the results of the section 5.2, for every $x \in \sigma'_\epsilon$, there exists a fourth-order, symmetric tensor $\tilde{\mathbb{M}}(x)$, independent of ϵ, such that

$$(5.7) \qquad (\mathbb{C}_1(x) - \mathbb{C}_0(x))\, \widehat{\nabla} u^i_\epsilon(x) = \tilde{\mathbb{M}}(x)\widehat{\nabla} u^e_\epsilon(x)$$

Inserting (5.7) into (5.6), we get

$$(5.8)$$
$$\int_{\sigma'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0)\, \widehat{\nabla} u^i_\epsilon : \widehat{\nabla} N(\cdot, y)dx = 2\epsilon \int_{\sigma'_\epsilon} \tilde{\mathbb{M}}\widehat{\nabla} u^e_\epsilon : \widehat{\nabla} N(\cdot, y)dx + O(\epsilon^{1-\alpha-\beta(1+\alpha)}).$$

Fourth Step.

Now we show that

$$(5.9) \qquad \|\nabla u^e_\epsilon - \nabla U\|_{L^\infty(\sigma'_\epsilon)} \leq C\epsilon^\gamma \|\psi\|_{H^{-1/2}(\partial\Omega)}$$

for some positive γ.

Once estimate (5.9) is proved, then (5.8) holds with ∇U instead of ∇u^e_ϵ and (5.3) follows immediately by continuity.

In [3] the proof of estimate (5.9) strongly relies on the special features of a homogeneous and isotropic tensor $\mathbb{C}_0$. In the present case, we use of a Caccioppoli-type inequality proved in the Appendix.

Let $2\epsilon < d < d_0/2$ and $\Omega^\epsilon_d = \{x \in \Omega : d(x, \partial(\Omega\backslash\omega_\epsilon) > d\}$.

Since $u_\epsilon - u_0$ is solution to

$$\text{div}\left(\mathbb{C}_0 \widehat{\nabla}(u_\epsilon - u_0)\right) = 0 \quad \text{in} \quad \Omega \setminus \omega_\epsilon,$$

the regularity assumption on $\mathbb{C}_0$ implies that $u_\epsilon - u_0 \in H^2_{loc}(\Omega\backslash\omega_\epsilon)$ (see [4, Theorem 2.I chapter 2]).

Let $k \in \{1, 2\}$ and let $\phi^k_\epsilon := \partial_k(u_\epsilon - u_0)$. The function ϕ^k solves

$$\text{div}\left(\mathbb{C}_0 \widehat{\nabla}\phi^k_\epsilon\right) = F \quad \text{in} \quad \Omega \setminus \omega_\epsilon.$$

with

$$F = -\text{div}\left((\partial_k\mathbb{C}_0)\widehat{\nabla}(u_\epsilon - u_0)\right).$$

By Caccioppoli inequality (Theorem A.1 for $\bar{u} = 0$) and by (3.3), we see that

$$\begin{aligned}
\|\nabla\phi^k_\epsilon\|^2_{L^2(\Omega^\epsilon_{d/2})} &\leq \frac{C_1}{d^2}\|\phi^k_\epsilon\|^2_{L^2(\Omega^\epsilon_{d/4})} + \|F\|^2_{H^{-1}(\Omega^\epsilon_{d/4})} \\
&= C\left(\frac{1}{d^2}\|\phi^k_\epsilon\|^2_{L^2(\Omega^\epsilon_{d/4})} + \|\nabla(u_\epsilon - u_0)\|^2_{L^2(\Omega\backslash\omega_\epsilon)}\right) \\
&\leq C\|\psi\|^2_{H^{-1/2}}\left(d^{-2} + 1\right)\epsilon \leq C\|\psi\|^2_{H^{-1/2}}d^{-2}\epsilon.
\end{aligned}$$

and, by (3.3) again

$$\|\phi^k_\epsilon\|_{H^1((\Omega^\epsilon_{d/2}))} \leq C\|\psi\|_{H^{-1/2}}d^{-1}\sqrt{\epsilon}$$

Theorem B.1 applied to ϕ^k shows that

$$\|\nabla\phi^k\|_{L^{2+\eta}(\Omega^\epsilon_d)} \leq C\left(\|F\|_{H^{-1,2+\eta}(\Omega^\epsilon_{d/2})} + d^{\frac{2}{2+\eta}-1}\|\nabla\phi^k\|_{L^2(\Omega^\epsilon_{d/2})}\right).$$

Notice that, by applying Theorem B.1 to $u_\epsilon - u_0$ in $\Omega \setminus \omega_\epsilon$,

$$\|F\|_{H^{-1,2+\eta}(\Omega^\epsilon_{d/2})} = \|(\partial_k\mathbb{C}_0)\widehat{\nabla}(u_\epsilon - u_0)\|_{L^{2+\eta}(\Omega^\epsilon_{d/2})}$$

$$\leq C\|\widehat{\nabla}(u_\epsilon - u_0)\|_{L^{2+\eta}(\Omega^\epsilon_{d/2})} \leq \|\nabla(u_\epsilon - u_0)\|_{L^2(\Omega^\epsilon_{d/4})}$$

and, hence,

$$
\begin{aligned}
\|\nabla \phi^k\|_{L^{2+\eta}(\Omega_d^\epsilon)} &\leq C\left(\|\nabla(u_\epsilon - u_0)\|_{L^2(\Omega_{d/4}^\epsilon)} + d^{\frac{2}{2+\eta}-1}\|\nabla \phi^k\|_{L^2(\Omega_{d/2}^\epsilon)}\right) \\
&\leq C\left(d^{\frac{2}{2+\eta}-2}\right)\sqrt{\epsilon}.
\end{aligned}
$$

On the other hand, applying Theorem B.1 to $u_\epsilon - u_0$ shows that

$$
\|\phi_\epsilon^k\|_{L^{2+\eta}(\Omega_d^\epsilon)} \leq C d^{\frac{2}{2+\eta}-1}\sqrt{\epsilon}.
$$

By Sobolev Embedding Theorem, it follows that, for $k = 1, 2$,

(5.10) $\qquad \|\partial_k(u_\epsilon - u_0)\|_{L^\infty(\Omega_d^\epsilon)} = \|\phi_\epsilon^k\|_{L^\infty(\Omega_d^\epsilon)} \leq C\left(d^{\frac{2}{2+\eta}-2}\right)\sqrt{\epsilon}$

Now, let $y \in \partial\sigma_\epsilon'$ and let y_d denote the closest point to y in the set Ω_d^ϵ. From the gradient estimates for u_ϵ and u_0 (see [10] and [3, Prop.3.3]), we have

(5.11) $\qquad |\nabla u_\epsilon^e(y) - \nabla u_\epsilon^e(y_d)| \leq C d^\alpha,$

which yields, by (5.10),

$$
|\nabla(u_\epsilon^e - u_0)(y)| \leq C(d^\alpha + d^{-2+\frac{2}{2+\eta}}\epsilon^{1/2}).
$$

Choosing $d = \epsilon^{\frac{1}{2(2+\alpha-\frac{2}{2+\eta})}}$, we get

(5.12) $\qquad |\nabla(u_\epsilon^e - u_0)(y)| \leq C\epsilon^\gamma,$

where $\gamma = \frac{\alpha}{2\left(2+\alpha-\frac{2}{2+\eta}\right)}$, and hence

$$
\|\nabla(u_\epsilon^e - u_0)\|_{L^\infty(\sigma_\epsilon')} \leq C\epsilon^\gamma .
$$

We conclude exactly as in [3] by noticing that the tensor $\mathbb{M}$ is continuous in Ω (see next section). $\qquad\square$

Remark 5.2. *If the elasticity tensor $\mathbb{C}_0$ is smoother than $C^{1,\alpha}$ in Ω, by differentiating again the equation for $u_\epsilon - u_0$ in $\Omega \setminus \omega_\epsilon$ and using again Caccioppoli inequality we obtain a better exponent γ in (5.12). Moreover, if $\mathbb{C}_0 \in C^\infty$ this result can be extended to any dimension for small neighborhoods of regular hypersurfaces contained in Ω.*

5.2. **Construction of the polarization tensor.** To establish the asymptotic formula (5.3), one seeks to express the term $(\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}u_\epsilon^i(x)$ as a linear function of $\widehat{\nabla}u_\epsilon^e(x)$, for $x \in \sigma_\epsilon'$ (with the notation introduced in the second step of the previous proof). This linear dependence defines the polarization tensor $\tilde{\mathbb{M}}$ in this context, which therefore must satisfy:

(5.13) $\qquad (\mathbb{C}_0 - \mathbb{C}_1)\widehat{\nabla}u_\epsilon^i(x) = \tilde{\mathbb{M}}(x)\widehat{\nabla}u_\epsilon^e(x)$

Let us also recall that $n(x)$ and $\tau(x)$ denote the normal and tangential directions to σ_ϵ' at point x.

From the discussion in the previous subsection, it suffices to obtain $\tilde{\mathbb{M}}$ along σ_ϵ', away from the vertices of the curve (more precisely, outside a disk of radius ϵ^β centered at the vertices).

We introduce a coordinate system in ω_ϵ', adapted to the geometry of the problem, as follows. Let τ be a unit tangent vector field along the curve σ' that supports ω_ϵ,

for instance the velocity vector field using arclength, and let n be a vector field normal to σ_0 so that $\{\tau, n\}$ forms a frame field on σ_0.

We employ again the frame field (τ, n) along σ_0', and extend it to ω_ϵ' by setting $\tau(x')$ and $n(x')$ to be constant along the segment $\{x = x' + hn(x'),\ 0 \le h \le \epsilon\}$. By construction a global coordinate system in ω_ϵ' is given by $x = x' + hn(x')$, where x' is a point along the curve σ_0' and $0 \le h < \epsilon$. We can extend this coordinate system smoothly to part of the boundary, that is, for $h = \epsilon$.

Since the part of the boundary ω_ϵ above σ_0' is given by the graphs of two smooth functions above the same curve, by the regularity results recalled in the previous section (see [10]), the interior and exterior strain fields are regular and satisfy the following transmission conditions *pointwise* on σ_ϵ':

$$(5.14\text{a}) \qquad\qquad \nabla u_\epsilon^e(x)\tau(x) = \nabla u_\epsilon^i(x)\tau(x)$$

$$(5.14\text{b}) \qquad\qquad \mathbb{C}_0 \widehat{\nabla} u_\epsilon^e(x)n(x) = \mathbb{C}_1 \widehat{\nabla} u_\epsilon^i(x)n(x).$$

A direct determination of a 4th-order tensor $\tilde{\mathbb{M}}$ that satisfies (5.13) directly from the above relations turns out to be rather complicated. One is led to invert a linear system, the determinant of which is not easily seen to be non zero when $\mathbb{C}_0$ and $\mathbb{C}_1$ are anisotropic tensors, unless additional assumptions are made.

Instead, we follow a construction due to G. Francfort and F. Murat [8]. In their work, effective elastic properties of laminate composites are calculated in terms of the geometric parameters of the phase layers (see also [12], P. 168). This construction precisely relies on the transmission conditions (5.14). We fix a point $x' \in \sigma_0'$, and denote the values of the interior and exterior strains respectively by

$$e^i = \widehat{\nabla} u_\epsilon^i(x) \quad \text{and} \quad e^e = \widehat{\nabla} u_\epsilon^e(x).$$

Proposition 5.3. *There exists a 4-th order tensor $\tilde{\mathbb{M}}$ such that*

$$(5.15) \qquad\qquad (\mathbb{C}_0 - \mathbb{C}_1)e^i = \tilde{\mathbb{M}}e^e,$$

where $\tilde{\mathbb{M}}$ depends on $\mathbb{C}_0$, $\mathbb{C}_1$, n, and τ, but not on e^e and e^i.

Proof. First of all let us notice that, by the first transmission condition (5.14a) we can write

$$(5.16) \qquad\qquad e^e \; = \; e^i + \delta \otimes n + n \otimes \delta, \qquad \text{for some } \delta \in \mathbb{R}^2$$

Let q^{-1} denote the (symmetric) linear operator associated to the quadratic form on $\mathbb{R}^2$:

$$(q^{-1}\zeta)\xi \;\; = \;\; (\mathbb{C}_1(\zeta \otimes n)) : (\xi \otimes n), \qquad \zeta, \xi \in \mathbb{R}^2.$$

Note that, since $\mathbb{C}_1$ is positive definite, q^{-1} is invertible since, by (2.5),

$$\forall \zeta \in \mathbb{R}^2, \quad (q^{-1}\zeta)\zeta \;\; = \;\; (\mathbb{C}_1(\zeta \otimes n) : (\zeta \otimes n) \;\; \ge \;\; \lambda_0 |\zeta|^2,$$

and thus its inverse q is well defined. Conditions (5.14b) and (5.16) imply that

$$(5.17) \qquad\qquad ((\mathbb{C}_0 - \mathbb{C}_1)e^e)n) \;\; = \;\; \mathbb{C}_1(e^i - e^e)n$$

$$(5.18) \qquad\qquad\qquad\qquad = \;\; (\mathbb{C}_1(\delta \otimes n + n \otimes \delta))\, n$$

On the other hand, for any $\xi \in \mathbb{R}^2$, we have

$$\begin{aligned} 2(q^{-1}\delta)\xi \;\; &= \;\; 2(\mathbb{C}_1(\delta \otimes n)) : (\xi \otimes n) \\ &= \;\; (\mathbb{C}_1(\delta \otimes n + n \otimes \delta)) : (\xi \otimes n) \\ &= \;\; \mathbb{C}_1(e^i - e^e) : (\xi \otimes n). \end{aligned}$$

This can equivalently be written as

$$2(q^{-1}\delta)\xi = \left(\mathbb{C}_1(e^i - e^e)n\right) \cdot \xi, \quad \forall \xi \in \mathbb{R}^2$$

that is, by (5.17)

$$2(q^{-1}\delta) = \mathbb{C}_1(e^i - e^e)n = ((\mathbb{C}_0 - \mathbb{C}_1)\,e^e)n.$$

From the above equation we deduce that

$$\delta = \frac{1}{2}q\left((\mathbb{C}_0 - \mathbb{C}_1)e^e n\right)$$

hence, by (5.16)

$$e^i = e^e + \frac{1}{2}q\left((\mathbb{C}_0 - \mathbb{C}_1)e^e n\right) \otimes n + n \otimes \frac{1}{2}q\left((\mathbb{C}_0 - \mathbb{C}_1)e^e n\right).$$

We can now conclude that

$$(\mathbb{C}_0 - \mathbb{C}_1)e^i = (\mathbb{C}_0 - \mathbb{C}_1)e^e + (\mathbb{C}_0 - \mathbb{C}_1)\left(q\left((\mathbb{C}_0 - \mathbb{C}_1)e^e n\right) \otimes n\right),$$

where, in the last expression we used symmetry of tensor $\mathbb{C}_0 - \mathbb{C}_1$.
Hence, the forth order tensor $\tilde{\mathbb{M}}$ defined by

$$(5.19) \qquad \tilde{\mathbb{M}}h = (\mathbb{C}_0 - \mathbb{C}_1)h + (\mathbb{C}_0 - \mathbb{C}_1)\left(q\left((\mathbb{C}_0 - \mathbb{C}_1)hn\right) \otimes n\right),$$

satisfies (5.15). As can be seen by its expression, $\tilde{\mathbb{M}}$ does not depend on e^e and e^i but only on the elasticity tensors $\mathbb{C}_0$ and $\mathbb{C}_1$ and on the directions n and τ. Moreover, this tensor $\mathbb{M}$ can be defined for every point x in ω_ϵ' and it is continuous with respect to x.

Remark 5.4. *Assume that $\{\omega_\epsilon\}$ is a sequence of thin strip-like inclusions as in (5.1). Then, the expansions (2.12) and (5.3) coincide. In other words, the measure $\mathbb{M}(x)d\mu_x$ that appears in (2.12) is precisely $\tilde{\mathbb{M}}(x)\delta_{\sigma_0}(x)$, where $\tilde{\mathbb{M}}(x)$ is defined in the section 5.2.*
Indeed, given the form of the sets ω_ϵ, it is immediate to see that

$$\frac{1}{|\omega_\epsilon|}\chi_{\omega_\epsilon} \quad \rightharpoonup \quad \delta_{\sigma_0}.$$

Thus, recalling Lemma 3.2, it suffices to show that

$$\frac{1}{|\omega_\epsilon|}\chi_{\omega_\epsilon}(\mathbb{C}_1 - \mathbb{C}_0)\widehat{\nabla} u_{\epsilon_n} \quad \rightharpoonup \quad \tilde{\mathbb{M}}(x)\widehat{\nabla}U(x)\delta_{\sigma_0},$$

in the weak topology of $(C^0(\overline{\Omega}))'$. This again is a consequence of the uniform regularity estimates on u_ϵ. Indeed, let $\phi \in C^0(\overline{\Omega})$. Using the same notations and*

the same analysis as in section 5.1, we see that

$$\int_{\omega_\epsilon} \frac{1}{|\omega_\epsilon|} (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} u_\epsilon \phi \, dx \;=\; \int_{\omega'_\epsilon} \frac{1}{|\omega_\epsilon|} (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} u_\epsilon \phi \, dx$$

$$+ \int_{\omega_\epsilon \backslash \omega'_\epsilon} \frac{1}{|\omega_\epsilon|} (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} u_\epsilon \phi \, dx$$

$$= \; 2 \frac{1}{|\sigma'_\epsilon|} \int_{\sigma'_\epsilon} (\mathbb{C}_1 - \mathbb{C}_0) \widehat{\nabla} u_\epsilon^i \phi \, d\sigma_x + O(\epsilon^{\inf(\alpha - \beta(1+\alpha), \beta)})$$

$$= \; 2 \frac{1}{|\sigma'_\epsilon|} \int_{\sigma'_\epsilon} \tilde{\mathbb{M}}(x) \widehat{\nabla} u_\epsilon^e \phi \, d\sigma_x + O(\epsilon^{\inf(\alpha - \beta(1+\alpha), \beta)})$$

$$\to \; 2 \int_{\sigma_0} \tilde{\mathbb{M}}(x) \widehat{\nabla} U \phi \, d\mu,$$

which proves the claim.

Remark 5.5. *In the case of isotropic materials, an explicit formula for $\mathbb{M}$ can be obtained from (5.19). In fact, in this case $\mathbb{C}_0$ and $\mathbb{C}_1$ have the simple form:*

$$(5.20) \qquad \begin{aligned} \mathbb{C}_0 &= \lambda_0 \, I_2 \otimes I_2 + 2 \, \mu_0 \, I_4, \\ \mathbb{C}_1 &= \lambda_1 \, I_2 \otimes I_2 + 2 \, \mu_1 \, I_4 \end{aligned}$$

where λ_i, μ_i, $i = 0, 1$, are the Lamé parameters, I_2, I_4 are the identity elements on 2 and 4 symmetric tensors respectively. Hence, as linear maps over symmetric matrices, they are diagonal in any basis. These tensors are strongly convex if $\lambda_i > 0$ and $\lambda_i + \mu_i > 0$, $i = 0, 1$.

Then, it readily follows that the matrix q is given by (see [8, Equation 4.8]):

$$(5.21) \qquad q = \frac{1}{\mu_1} I_2 - \frac{\lambda_1 + \mu_1}{\mu_1 (\lambda_1 + 2\mu_1)} n \otimes n.$$

Combining (5.20) with (5.21) gives:

$$(q \, ((\mathbb{C}_0 - \mathbb{C}_1) h n) \otimes n) = \frac{\lambda_0 - \lambda_1}{\lambda_1 + 2\mu_1} \, Tr(h) \, n \otimes n$$

$$+ \frac{2 \, (\mu_0 - \mu_1)}{\mu_1} \left[(hn) \otimes n - \frac{\lambda_1 + \mu_1}{\lambda_1 + 2\mu_1} \, ((hn) \cdot n) \, n \otimes n \right]$$

Lengthy, but straightforward calculations yield:

$$(5.22) \qquad \mathbb{M}h = a \, Tr(h) \, I_2 + b \, h + c \, ((h\tau) \cdot \tau) \, \tau \otimes \tau + d \, ((hn) \cdot n) \, n \otimes n,$$

where

$$a = (\lambda_0 - \lambda_1) \frac{\lambda_0 + 2\mu_0}{\lambda_1 + 2\mu_1}, \qquad b = (\mu_0 - \mu_1) \frac{\mu_0}{\mu_1},$$

$$c = (\mu_0 - \mu_1) \frac{2\lambda_1(\mu_1 - \mu_0) + \mu_1(\lambda_1 - \lambda_0) + 2\mu_1(\mu_1 - \mu_0)}{\mu_1(\lambda_1 + 2\mu_1)},$$

$$d = 2 \, (\mu_0 - \mu_1) \frac{\mu_1 \lambda_0 - \lambda_1 \mu_0}{\mu_1(\lambda_1 + 2\mu_1)}.$$

As expected, this formula agrees with that obtained in [3, Theorem 2.1].

APPENDIX A. A CACCIOPPOLI TYPE INEQUALITY

For the sake of completness, we state and prove a Caccioppoli type inequality for solutions of strongly convex systems.

Theorem A.1. *Let B_ρ and $B_{2\rho}$ be two concentric balls contained in Ω and let $\bar{u}$ be any constant vector.*
Let $u \in H^1(\Omega)$ be solution to

$$\nabla \cdot \left(\mathbb{C} \widehat{\nabla} u \right) = f \quad in \quad \Omega,$$

where $\mathbb{C} \in L^\infty(\Omega)$ is a strongly convex tensor and $f \in H^{-1}(\Omega)$. Then

(A.1) $$\|\nabla u\|^2_{L^2(B_\rho)} \le \frac{C_1}{\rho^2} \|u - \bar{u}\|^2_{L^2(B_{2\rho})} + C_2 \|f\|^2_{H^{-1}(B_{2\rho})}$$

Remark A.2. *Note that $\bar{u} = (u)_{B_{2\rho}} := \frac{1}{|B_{2\rho}|} \int_{B_{2\rho}} u \, dx$ gives the minimum value for $\|u - \bar{u}\|^2_{L^2(B_{2\rho})}$.*

Proof of Theorem A.1
Let $\theta \in C_0^\infty(B_{2\rho})$ such that

$$0 \le \theta \le 1, \quad \theta = 1 \text{ in } B_\rho \text{ and } |\nabla \theta| \le \frac{C}{\rho}.$$

Let $v \in H^1(B_{2\rho}, \mathbb{R}^2)$ and observe that

$$\nabla(\theta^2 v) = \theta \nabla(\theta v) + \theta \nabla \theta \otimes v.$$

By assumption u is a weak solution of

$$\nabla \cdot \left(\mathbb{C} \widehat{\nabla} u \right) = f \quad in \quad B_{2\rho}$$

hence we have

$$a_0(u, \phi) = - < f, \phi > \quad \forall \phi \in H_0^1(B_{2\rho})$$

where

$$a_0(u, \phi) = \int_{B_{2\rho}} \mathbb{C} \widehat{\nabla} u : \widehat{\nabla} \phi.$$

Let $\psi = \theta(u - \bar{u})$ and let $\phi = \theta \psi$. We have that

$$a_0(\psi, \psi) = \int_{B_{2\rho}} \mathbb{C} \widehat{\nabla} \psi : \widehat{\nabla} \psi$$

$$= \int_{B_{2\rho}} [\mathbb{C}(\theta \widehat{\nabla}(u - \bar{u}) + (u - \bar{u}) \otimes \nabla \theta) : (\theta \widehat{\nabla}(u - \bar{u}) + (u - \bar{u}) \otimes \nabla \theta)]$$

After some straightforward calculations one gets

$$a_0(\psi, \psi) = a_0(u, \phi) + \int_{B_{2\rho}} \mathbb{C} \left(\nabla \theta \otimes (u - \bar{u}) \right) : \left(\nabla \theta \otimes (u - \bar{u}) \right) = < f, \phi > + I$$

We have

$$|I| \le C_3 \|\nabla \theta \otimes (u - \bar{u})\|^2_{L^2(B_{2\rho})} \le \frac{C_3}{\rho^2} \|u - \bar{u}\|^2_{L^2(B_{2\rho})}$$

and by Young inequality

$$| < f, \phi > | \le \|f\|_{H^{-1}(B_{2\rho})} \|\nabla \phi\|_{L^2(B_{2\rho})} \le \delta \|\nabla \phi\|^2_{L^2(B_{2\rho})} + c(\delta) \|f\|^2_{H^{-1}(B_{2\rho})}$$

So, we get

$$a_0(\psi, \psi) - \delta\|\nabla\phi\|^2_{L^2(B_{2\rho})} \leq \frac{C_3}{\rho^2}\|u - \bar{u}\|^2_{L^2(B_{2\rho})} + c(\delta)\|f\|^2_{H^{-1}(B_{2\rho})}$$

By the strong convexity of $\mathbb{C}$ and by Korn inequality applied to the test function ψ, we have

$$a_0(\psi, \psi) \geq \gamma\|\nabla\psi\|^2_{L^2(B_{2\rho})}$$

which gives

$$\frac{C_3}{\rho^2}\|u - \bar{u}\|^2_{L^2(B_{2\rho})} + c(\delta)\|f\|^2_{H^{-1}(B_{2\rho})} \quad \geq \quad \gamma\|\nabla\psi\|^2_{L^2(B_{2\rho})} - \delta\|\nabla\phi\|^2_{L^2(B_{2\rho})}$$
$$\geq \quad \gamma\|\nabla\psi\|^2_{L^2(B_{2\rho})} - \delta\|\nabla(\theta\psi)\|^2_{L^2(B_{2\rho})}$$

Hence, since $\theta \leq 1$ and $|\nabla\theta| \leq \frac{C}{\rho}$,

$$\frac{C_3}{\rho^2}\|u - \bar{u}\|^2_{L^2(B_{2\rho})} + c(\delta)\|f\|^2_{H^{-1}(B_{2\rho})}$$

$$\geq \gamma\|\nabla\psi\|^2_{L^2(B_{2\rho})} - 2\delta\|\nabla\psi\|^2_{L^2(B_{2\rho})} - \frac{2\delta}{\rho^2}\|\psi\|^2_{L^2(B_{2\rho})}$$

$$\geq \gamma\|\nabla\psi\|^2_{L^2(B_{2\rho})} - 2\delta\|\nabla\psi\|^2_{L^2(B_{2\rho})} - \frac{2\delta}{\rho^2}\|u - \bar{u}\|^2_{L^2(B_{2\rho})}$$

Choosing δ small enough, using the fact that $\theta \leq 1$, that $|\nabla\theta| \leq \frac{C}{\rho}$ and the definition of ψ we finally get

$$\|\nabla u\|^2_{L^2(B_\rho)} = \|\nabla\psi\|^2_{L^2(B_\rho)} \leq \|\nabla\psi\|^2_{L^2(B_{2\rho})} \leq \frac{C_1}{\rho^2}\|u - \bar{u}\|^2_{L^2(B_{2\rho})} + C_2\|f\|^2_{H^{-1}(B_{2\rho})}.$$

$$\square$$

APPENDIX B. A MEYER'S TYPE RESULT

We state a generalization of Meyer's theorem concerning the regularity of solutions to systems with bounded coefficients. For $\eta > 0$, define $H^{1,2+\eta}(\Omega)$ by

$$H^{1,2+\eta}(\Omega) := \left\{ u \in L^{2+\eta}(\Omega), \nabla u \in L^{2+\eta}(\Omega) \right\}$$

and let $H^{-1,2+\eta}(\Omega)$ be its dual. Introduce

$$H^{1,2+\eta}_{\mathrm{loc}}(\Omega) := \left\{ u \in H^{1,2+\eta}(K), \forall K \subset\subset \Omega \right\}.$$

Theorem B.1. *There exists $\eta > 0$ such that if $u \in H^1(\Omega)$ is solution to*

$$\nabla \cdot \left(\mathbb{C}\widehat{\nabla} u \right) = f \quad in \quad \Omega,$$

where $\mathbb{C} \in L^\infty(\Omega)$ is a strongly convex tensor and $f \in H^{-1,2+\eta}(\Omega)$ then $u \in H^{1,2+\eta}_{loc}(\Omega)$ and given B_ρ and $B_{2\rho}$ concentric balls contained in Ω,

$$(B.1) \qquad \|\nabla u\|_{L^{2+\eta}(B_\rho)} \leq C(\|f\|_{H^{-1,2+\eta}(B_{2\rho})} + \rho^{\frac{2}{2+\eta}-1}\|\nabla u\|_{L^2(B_{2\rho})}).$$

Theorem B.1 has been proved by Campanato in [4] in the case of strongly elliptic systems. From the proof of this result in [4, Chapter II, section 10], it is clear that the result can be extended to more general systems provided the Caccioppoli type inequality (Theorem A.1) holds.

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