Consistency result for a non monotone scheme for anisotropic mean curvature flow

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Abstract

In this paper, we propose a new scheme for anisotropic motion by mean curvature in $\mathbb{R}^d$. The scheme consists of a phase-field approximation of the motion, where the nonlinear diffusive terms in the corresponding anisotropic Allen-Cahn equation are linearized in the Fourier space. In real space, this corresponds to the convolution with a specific kernel of the form

$$K_{\phi,t}(x) = \mathcal{F}^{-1}\left[e^{-4\pi^2 t\phi^o(\xi)}\right](x).$$

We analyse the resulting scheme, following the work of Ishii-Pires-Souganidis on the convergence of the Bence-Merriman-Osher algorithm for isotropic motion by mean curvature. The main difficulty here, is that the kernel $K_{\phi,t}$ is not positive and that its moments of order 2 are not in $L^1(\mathbb{R}^d)$. Still, we can show that in one sense the scheme is consistent with the anisotropic mean curvature flow.

1 Introduction and motivation

In the last decades, a lot of attention has been devoted to the motion of interfaces, and particularly to motion by mean curvature. Applications concern image processing (denoising, segmentation), material sciences (motion of grain boundaries in alloys, crystal growth), biology (modelling of vesicles and blood cells). This paper is interested in numerical schemes for the anisotropic mean curvature flow, that is, the "gradient flow" of an anisotropic perimeter

$$P_\phi(\Omega) = \int_{\partial\Omega} \phi^\circ(n(x))\, d\sigma \tag{1}$$

where $n(x)$ is the outer normal to ∂E at x and ϕ° is a convex, one-homogeneous surface tension (the isotropic case corresponds to $\phi^\circ(n(x)) = |n(x)| = 1$).

There is an important literature on numerical methods for the isotropic and anisotropic curvature flows. These can be roughly classified into three categories: Parametric methods [23, 24, 7, 8], Level set formulations [38, 36, 37, 28, 19] or Phase field approaches [35, 17, 10, 39]. See for instance [25] for a complete review and comparison beetween these three differents strategies.

In this work, we will consider a new scheme, proposed in [14], based on a phase field representation. It relies on the introduction of a specific anisotropic Laplacian (pseudo-differential) operator, which can be used both in a standard phase-field approximation (an anisotropic Allen-Cahn equation), or in a convolution/thresholding scheme [11, 33] which can be thought as a limiting case of the Allen-Cahn equation. The basic idea in [11] is to alternate the diffusion (with the heat equation) and the sharpening (by thresholding) of the characteristic functions of a set; [33] study a more general variant where the diffusion is replaced with the convolution by quite general kernels.

In the phase-field approach, anisotropic flows can be tackled either by a modified version of the Bence-Merriman-Osher algorithm [11], where the heat equation is replaced with a nonlinear variant built upon the anisotropy ϕ° [15], or by replacing the heat equation with the convolution with a nonnegative, nonsymmetric kernel f as in [33]. However, in the latter case, the inverse problem of finding an appropriate kernel f, given the anisotropy ϕ°, is solved only in 2D [42]. Some progress was done recently in relating the convolution kernel with ϕ° in [22], but the inverse problem is still considered untractable in higher dimension. In fact, it is not even clear that any anisotropy, even smooth, can be obtained in the framework of [33].

The aim of this work is to study a simple construction, proposed in [14] of a kernel f for all kind of anisotropy ϕ° in all dimension. This kernel can be seen as the fundamental solution of the heat equation with a particular pseudo-differential operator which can be seen as an anisotropic Laplacian. The most interesting feature of this approach is that the diffusion can be solved efficiently using the Fourier Transform, as proposed in [16] for the isotropic Allen-Cahn equation (and as can also be done for [42, 33]. A few numerical experiments with this approach have been already shown in [14] with smooth, but also cristalline or even non-convex anisotropies. Although the approach seems (numerically) to perform well in all these cases, a full justification is still missing. The essential contribution of this paper is to extend the consistency proof (see theorem (2)) of Ishii-Pires-Souganidis in the case of our specific kernel, for smooth, uniformly elliptic anisotropies. The main issue is that in this case, the kernel does not satisfy the assumptions which are needed in [33]. In particular, it is not even nonnegative, so that our scheme is non-monotone and a complete proof of convergence is still missing.

In the next section, we introduce our notation and a precise framework. We give a short introduction to level set formulations, phase field approximations and the Bence-Merriman-Osher algorithm in the case of the isotropic and anisotropic mean curvature flow. Next, we introduce our anisotropic heat kernel and establish some of its properties. Our main consistency result (Theorem 2) is given in Section 4. We show the consistency of an anisotropic Bence-Merriman-Osher scheme built upon that kernel. The last section shows numerical evidence of the convergence of a slightly modified scheme, which corresponds to a splitting of the anisotropic Allen-Cahn equation (hence the thresholding is replaced with a reaction term which only enhances the slope of the diffuse interface). Computationally, the scheme proves very efficient and very fast, even when the anisotropy is not smooth.

In comparison with other existing methods, our approach can be easily implemented: Contrarily to methods based on a parametric representations, it does not require special care in handling topological changes or in the case of 3D computations. Besides, it avoids direct discretization of the non linear anisotropic Allen Cahn equation (e.g. by finite elements) [10, 39]. Indeed, in our approach the non linear diffusion operator is replaced by an approximate linear operator, whose resolution can be easily performed by Fourier transform.

2 Preliminaries

2.1 Motion by isotropic mean curvature

The simplest case of motion by isotropic mean curvature concerns the evolution of a set $\Omega_t \subset \mathbb{R}^d$ with a boundary Γ_t of codimension 1, whose normal velocity V_n is proportional to its mean curvature κ

$$V_n(x) = \kappa(x), \quad \text{a.e. } x \in \Gamma_t, \tag{2}$$

with the convention that κ is negative if Ω_t is a convex set. It at $t = 0$ the initial set Ω_0 is smooth, then the evolution is well-defined until some time $T > 0$ when singularities may develop [2].

Viscosity solutions provide a more general framework, that defines evolution past singularities, or evolution from non-smooth initial sets. If g is a level set function of Ω_0, i.e.,

$$\Omega_0 = \left\{x \in \mathbb{R}^d \ ; \ g(x) \leq 0\right\}, \quad \Gamma_0 = \left\{x \in \mathbb{R}^d \ ; \ g(x) = 0\right\},$$

and if u denotes the viscosity solution to the geometric evolution equation

$$\begin{cases} u_t = \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)|\nabla u| \\ u(0,x) = g(x), \end{cases}$$

then the generalized mean curvature flow Ω_t starting from Ω_0 is defined by the 0-level set of u [27, 38, 20, 28]

$$\Omega_t = \left\{x \in \mathbb{R}^d \ ; \ u(t,x) \leq 0\right\}, \quad \partial\Gamma_t = \left\{x \in \mathbb{R}^d \ ; \ u(t,x) = 0\right\}.$$

Alternatively, one can define the motion by mean curvature as the limit of diffuse interface approximations obtained by solving the Allen-Cahn equation

$$\frac{\partial u}{\partial t} = \Delta u - \frac{1}{\epsilon^2}(W'(u)), \tag{3}$$

where ϵ is a small parameter (that determines the width of the diffuse interface) and where $W(s) = \frac{s^2(1-s)^2}{2}$ is a double well potential. This equation can be viewed as a gradient flow for the energy

$$J_\epsilon(u) = \int_{\mathbb{R}^d} \left(\frac{\epsilon}{2}|\nabla u|^2 + \frac{1}{\epsilon}W(u)\right) dx.$$

Modica and Mortola [35, 34] have shown that J_ϵ approximates (in the sense of Γ- convergence) the surface energy $c_W\, J$ where

$$J(\Omega) = \int_{\partial\Omega} 1\, d\sigma \quad \text{and} \quad c_W = \int_0^1 \sqrt{2W(s)}\, ds.$$

Existence, uniqueness, and a comparison principle have been established for (3) (see for example chapters 14 and 15 in [2] and the references therein).

Let u_ϵ solve (3) with the initial condition

$$u_\epsilon(x,0) = q\Big(\frac{d(x,\Omega_0)}{\epsilon}\Big),$$

where $d(x,\Omega)$ denotes the signed distance of a point x to the set Ω and where the profile q is defined by

$$\begin{aligned} q \quad = \quad \arg\min \Bigg\{ \int_{\mathbb{R}} \Big(\frac{1}{2}\gamma'^2 + W(\gamma)\Big); \gamma \in H^1_{loc}(\mathbb{R}),\ \gamma(-\infty) = +1, \\ \gamma(+\infty) = -1,\ \gamma(0) = \frac{1}{2} \Bigg\}. \end{aligned}$$

The set

$$\Omega_{t,\epsilon} = \left\{ x \in \mathbb{R}^d \ ;\ u_\epsilon(x,t) \geq \frac{1}{2} \right\},$$

approximates $\Omega(t)$ at the rate of convergence $O(\epsilon^2 |\log \epsilon|^2)$ in the case of smooth motion by mean curvature [17, 9].

In the case of generalized motion by mean curvature, convergence has been shown [3, 27] provided that the interior of the set Γ_t remains empty (i.e. no fatting occurs).

About numerical point of view, convergence has been established for a finite element method in [40] and for a finite difference method in [18]. A splitting spectral Fourier method is also been addressed in [16, 14].

The Bence-Merriman-Osher algorithm [11] is yet another approximation to motion by mean curvature. Given a closed set $E \subset \mathbb{R}^d$, and denoting χ_E its characteristic function, one defines

$$T_h E = \left\{ x \in \mathbb{R}^d \ ;\ u(x,h) \geq \frac{1}{2} \right\},$$

where u solves the heat equation

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \quad t > 0 \quad x \in \mathbb{R}^d \\ u(x,0) = \chi_E(x). \end{cases}$$

Setting $E_h(t) = T^{[t/h]}E$, where $[t/h]$ is the integer part of t/h, Evans [26], and Barles and Georgelin [4] have shown that $E_h(t)$ converges to E_t, the evolution by mean curvature from E. Remark also that this algorithm can be seen as a splitting algorithm for the Allen-Cahn equation in the limite case $\epsilon \to 0$. See also [42] where an efficient numerical resolution is presented.

2.2 Motion by anisotropic mean curvature

We use the framework of the Finsler geometry as described in [10]. Let $\phi : \mathbb{R}^d \to [0, +\infty[$ denote a strictly convex function in $C^2(\mathbb{R}^d \setminus \{0\}))$, which is 1-homogeneous and bounded, i.e.,

$$\begin{cases} \phi(t\xi) = |t|\phi(\xi) & \xi \in \mathbb{R}^d,\ t \in \mathbb{R}, \\ \lambda|\xi| \leq \phi(\xi) \leq \Lambda|\xi| & \xi \in \mathbb{R}^d, \end{cases}$$

for two positive constants $0 < \lambda \leq \Lambda < +\infty$. We assume that its dual function $\phi^o : \mathbb{R}^N \to [0, +\infty[$, defined by

$$\phi^o(\xi^*) = \sup \{\xi^*.\xi \ ;\ \phi(\xi) \leq 1\}$$

is also in $C^2(\mathbb{R}^N \setminus \{0\}))$. Given a smooth set E and a smooth function $u : \mathbb{R}^d \to \mathbb{R}$ such that $\partial E = \left\{x \in \mathbb{R}^d \ ;\ u(x) = 0\right\}$, we define

- the Cahn-Hoffman vector field $n_\phi = \phi^o_\xi(\nabla u)$.
- the ϕ-curvature $\kappa_\phi = \mathrm{div}(n_\phi)$.

We say that $E(t)$ is the evolution from E by ϕ-curvature, if at each time t, the normal velocity V_n is given by

$$V_n = -\kappa_\phi n_\phi.$$

As in the case of isotropic flows, one can define motion by ϕ-curvature using a level set formulation, i.e., following the level lines of the solution to the anisotropic evolution equation

$$u_t = \phi^o(\nabla u)\ \phi^o_{\xi\xi}(\nabla u) : \nabla^2 u. \tag{4}$$

Existence, uniqueness and a comparison principle have been established in [21, 19, 6, 5].

The anisotropic surface energy

$$J(\Omega) = \int_{\partial\Omega} \phi^o(n)\, d\sigma.$$

can be approximated by the Ginzburg-Landau-like energy

$$J_{\epsilon,\phi}(u) = \int_{\mathbb{R}^d} \left(\frac{\epsilon}{2} \phi^o(\nabla u)^2 + \frac{1}{\epsilon} W(u) \right) dx,$$

and its gradient flow leads to the anisotropic Allen–Cahn equation [1]

$$\frac{\partial u}{\partial t} = \Delta_\phi u - \frac{1}{\epsilon^2} W'(u). \tag{5}$$

The operator $\Delta_\phi := \operatorname{div}\left(\phi^o_\xi(\nabla u)\phi^o(\nabla u)\right)$ is called the anisotropic Laplacian.

This equation can be numerically solved by a semi finite elements method (see [39] for instance) but the complexity of this algorithm is much greater than in the isotropic case because it needs a resolution of a new linear system at each iteration in time.

The Bence-Merriman-Osher algorithm has also been extended to anisotropic motion by mean curvature. One generalization was proposed by Chambolle and Novaga [15] as follows: Given a closed set E, let $T_h(E) = \left\{ x \in \mathbb{R}^d \ ; \ u(x,h) \geq \frac{1}{2} \right\}$, where $u(x,t)$ is the solution to

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta_\phi u(x,t), \quad t > 0 \quad x \in \mathbb{R}^d \\ u(x,0) = \chi_E(x). \end{cases} \tag{6}$$

Define then $E_h(t) = T_h^{[t/h]} E$. The convergence of $E_h(t)$ to the generalized anisotropic mean curvature flow from E is established in [15]. The result holds for very general anisotropic surface tensions and even in the cristalline case. However, as for the phase field formulation, because of the strongly nonlinear character of Δ_ϕ, the numerical resolution of (6) is much harder and not also efficient than in the isotropic case.

Another generalization of the Bence-Merriman-Osher algorithm has been studied by Ishii, Pires and Souganidis [33]. The main idea is to represent the solution u of (6) as the convolution of χ_E with a geometric kernel. More precisely, Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function which satisfies the following conditions

(A_1) Positivity and symmetry :

$$f(x) \geq 0, \quad f(-x) = f(x), \quad \text{and} \int_{\mathbb{R}^d} f(x)dx = 1$$

(A_2) Boundedness of the moments :

$$\int_{\mathbb{R}^d} |x|^2 f(x)dx < +\infty,$$

$$0 < \int_{p^\perp} (1 + |x|^2) f(x) d\mathcal{H}^{d-1} < \infty, \qquad \text{for all } p \in \mathbb{S}^{d-1}.$$

(A_3) Smoothness :

$$p \to \int_{p^\perp} f(x) d\mathcal{H}^{d-1} \text{ and } p \to \int_{p^\perp} x_i x_j f(x) d\mathcal{H}^{d-1} \quad \text{are continous on } \mathbb{S}^{d-1}.$$

Here $p^\perp$ denotes the orthogonal complement of the vector p, i.e, $p^\perp = \Big\{x \in \mathbb{R}^d \ ; \ \langle x \, , \, p \rangle = 0\Big\}$.

Given $E \subset \mathbb{R}^d$, define $T_h E = \Big\{x \in \mathbb{R}^d \ ; \ u(x,h) \geq \frac{1}{2}\Big\}$, where

$$u(x,h) = \int_{\mathbb{R}^d} \tilde{K}_h(y)\, \chi_E(y - x)\, dy,$$

with the kernel

$$\tilde{K}_t(x) = \frac{1}{t^{d/2}} f(\sqrt{t}x), \quad x \in \mathbb{R}^d.$$

They showed [33] that $T_h^{[t/h]} E$ converges to the set $E(t)$ obtained from E as the generalized motion by anisotropic mean curvature via the geometric evolution equation

$$u_t = F(D^2 u, Du)$$

where

$$F(X,p) = \left(\int_{p^\perp} f(x) d\mathcal{H}^{d-1}(x)\right)^{-1} \left(-\frac{1}{2}\int_{p^\perp} \langle Xx \, , \, x\rangle\, f(x) d\mathcal{H}^{d-1}(x)\right).$$

This algorithm appears to be more efficient than the last one (with the non linear operator Δ_ϕ), but raises a natural question: Given an anisotropy ϕ^o, can one find a kernel f, so that the generalized front $\partial E(t)$ defined by the associated evolution equation evolves by ϕ-mean curvature ? This problem has been addressed by Ruuth and Merriman [43] in dimension 2. They propose a class of kernels and study the corresponding numerical schemes, which prove very efficient. However, their approach cannot be generalized to higher dimensions. In contrast, our algorithm is not specific to the dimension 2.

2.3 A new algorithm for motion by anisotropic mean curvature

In this work, our objective is to extend Ishii-Pires-Souganidis' analysis to study the following algorithm. Starting from a bounded closed set $E \subset \mathbb{R}^d$, we define an operator $T_h E$ by

$$T_h E = \left\{ x \in \mathbb{R}^d \ ; \ u(x,h) \geq \frac{1}{2} \right\}, \tag{7}$$

where u solves the following parabolic equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) = \tilde{\Delta}_\phi u(x,t), \quad t > 0 \quad x \in \mathbb{R}^d \\ u(x,0) = \chi_E(x). \end{cases} \tag{8}$$

Denoting by $\mathcal{F}(u)$ the Fourier transform of a function u,

$$\mathcal{F}(u)(\xi) \quad = \quad \int_{\mathbb{R}^d} u(x) e^{-2\pi i x \cdot \xi} \, dx,$$

the operator $\tilde{\Delta}_\phi$ is defined by

$$\tilde{\Delta}_\phi u = \mathcal{F}^{-1} \left(-4\pi^2 \phi^o(\xi)^2 \mathcal{F}(u)(\xi) \right).$$

and can be seen as a linearization of Δ_ϕ in the Fourier space. The solution u of (8) can be expressed as a convolution product of the characteristic function of E and of the anisitropic kernel

$$K_{\phi,t}(x) = \mathcal{F}^{-1} \left(e^{-4\pi^2 t \phi^o(\xi)^2} \right)(x).$$

However, this kernel (more precisely $K_{\phi,t=1}$) does not satisfy the hypotheses (A_1) and (A_2) above: $K_{\phi,1}$ is not positive and $x \to \int_{\mathbb{R}^d} |x|^2 K_\phi(x)$ is not in $L^1(\mathbb{R})$. But we will show that the associated Hamiltonian flow is

$$\begin{aligned} F(X,p) \quad &= \quad \left(\int_{p^\perp} K_\phi d\mathcal{H}^{d-1} \right)^{-1} \left(\frac{1}{2} \int_{p^\perp} < Xx, x > K_\phi(x) d\mathcal{H}^{d-1} \right) \\ &= \quad \phi^o(p) \phi^o_{\xi\xi}(p) : X, \end{aligned}$$

which establishes a link between K_ϕ and ϕ-anisotropic mean curvature flow.

3 The operator $\tilde{\Delta}_\phi$ and properties of the anisotropic kernel K_ϕ

Let $\phi = \phi(\xi)$ denote a strictly convex smooth Finsler metric and let ϕ^o denote its dual (see [10]). We assume that that ϕ^o is a 1-homogenous, symmetric function in $C^\infty(\mathbb{R}^d \setminus \{0\})$ that satisfies

$$\lambda|\xi| \leq \phi^o(\xi) \leq \Lambda|\xi|. \tag{9}$$

In particular, it follows that for any $\xi \in \mathbb{R}^d$ and $t \in \mathbb{R}$,

$$\begin{cases} \phi^o(t\xi) = |t|\phi^o(\xi) \\ \phi^o_\xi(t\xi) = \frac{t}{|t|}\phi^o_\xi(\xi) \\ \phi^o_\xi(\xi).\xi = \phi^o(\xi). \end{cases}$$

The associated anisotropic mean curvature is defined as the anisotropic Laplacian operator

$$\triangle_\phi u = \operatorname{div}\left(\phi^o(\nabla u)\phi^o_\xi(\nabla u)\right), \quad \forall u \in H^2(\Omega)$$

A direct computation shows that for any $\xi \in \mathbb{R}^d$,

$$\begin{cases} \triangle_\phi\left[\cos(2\pi\xi.x)\right] = -4\pi^2\phi^o(\xi)^2\cos(2\pi\xi.x) \\ \triangle_\phi\left[\sin(2\pi\xi.x)\right] = -4\pi^2\phi^o(\xi)^2\sin(2\pi\xi.x), \end{cases}$$

i.e., that plane waves are eigenfunctions of the anisotropic Laplacian (albeit non-linear). We define $\tilde{\triangle}_\phi : H^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ by

$$\tilde{\triangle}_\phi u = \mathcal{F}^{-1}\left[-4\pi^2\phi^o(\xi)^2\mathcal{F}[u](\xi)\right],$$

Given an initial condition $u_0 \in L^2(\mathbb{R}^d)$, we study the solution u of,

$$\begin{cases} u_t(t,x) = \tilde{\triangle}_\phi u(t,x), \\ u(0,x) = u_0 \end{cases}$$

The function u can also be expressed as the convolution product $u = K_{\phi,t} * u_0$, where the anisotropic heat kernel $K_{\phi,t}$ is defined by

$$K_{\phi,t} = \mathcal{F}^{-1}\left[e^{-4\pi^2 t\phi^o(\xi)^2}\right].$$

We also set $K_\phi = K_{\phi,1}$. In the rest of this section, we establish some properties of this operator.

Proposition 1 (Regularity of $\hat{K}_\phi$)**.**
The function $\hat{K}_\phi : \xi \to e^{-4\pi^2\phi^o(\xi)^2}$ is in $W^{d+1,1}(\mathbb{R}^d)$, and $D^{d+2}\hat{K}_\phi$ is a function.

Proof. First, we notice that

$$D\hat{K}_\phi(\xi) = -8\pi^2 \phi^o_\xi(\xi)\phi^o(\xi)e^{-4\pi^2\phi^o(\xi)^2},$$

and

$$\begin{aligned} D^2\hat{K}_\phi(\xi) &= 64\pi^4\phi^o(\xi)^2\left(\phi^o_\xi(\xi)\otimes\phi^o_\xi(\xi)\right)e^{-4\pi^2\phi^o(\xi)^2} \\ &\quad -8\pi^2\left(\phi^o(\xi)\phi^o_{\xi\xi}(\xi)+\phi^o_\xi(\xi)\otimes\phi^o_\xi(\xi)\right)e^{-4\pi^2\phi^o(\xi)^2}. \end{aligned}$$

We note that ϕ^o_ξ is discontinuous at $\xi = 0$. Nevertheless, we next prove that the $d-1^{th}$ derivative of $D^2\hat{K}_\phi$ belongs to $L^1(\mathbb{R}^d)^{d^2}$. Assume that $f = D^{n+2}\hat{K}_\phi$ is an integrable function on $\mathbb{R}^d$ for some integer $n < d$. The homogeneity of ϕ^o shows the existence of a constant C_n such that

$$|D^{n+2}\hat{K}_\phi| \le C_n\frac{1}{|\xi|^n}e^{-\lambda|\xi|^2}, \quad \text{for all } \xi\in\mathbb{R}^d\setminus\{0\}.$$

Since f is smooth away from $\xi = 0$, the distributional derivative of f is the sum of a function and of possibly a Dirac mass at $\xi = 0$:

$$Df = \{\nabla f\} + c\,\delta,$$

where c is a constant and ∇f denotes the pointwise derivative of f. Let $\varphi \in \mathcal{D}(\mathbb{R}^d)^{d^{n+2}}$ and let $\epsilon > 0$. Then

$$\begin{aligned} \langle Df\,,\,\varphi\rangle &= -\langle f\,,\,\mathrm{div}\varphi\rangle = -\int_{\mathbb{R}^d} f.\mathrm{div}\varphi dx \\ &= -\int_{\mathbb{R}^d\setminus B(0,\epsilon)} f.\mathrm{div}\varphi dx - \int_{B(0,\epsilon)} f.\mathrm{div}\varphi dx \\ &= \int_{\mathbb{R}^d\setminus B(0,\epsilon)} \nabla f.\varphi dx - \int_{\partial B(0,\epsilon)} f.(\varphi.\vec{n})d\sigma - \int_{B(0,\epsilon)} f.\mathrm{div}\varphi dx, \end{aligned}$$

Since we assumed that $f \in L^1(\mathbb{R}^d)^{d^{n+2}}$, the last integral above tends to 0, as $\epsilon \to 0$. Moreover as $n < d$, we have

$$\begin{aligned} \left|\int_{\partial B(0,\epsilon)} f\ \varphi.\vec{n}d\sigma\right| &\le \|\varphi\|_{L^\infty}\int_{\partial B(0,\epsilon)} C_n\frac{1}{|\xi|^n}e^{-\lambda|\xi|^2}d\sigma \\ &\le \|\varphi\|_{L^\infty}C_n\int_{\partial B(0,\epsilon)}\epsilon^{-n}d\sigma \le C_n\|\varphi\|_{L^\infty}\epsilon^{d-1-n}, \end{aligned}$$

so that

$$\lim_{\epsilon\to 0}\left|\int_{\partial B(0,\epsilon)} f\ \varphi.\vec{n}d\sigma\right| = 0.$$

It follows that $c = 0$, which concludes the proof. □

Proposition 2 (Decay properties of K_ϕ).
Let $s \in [0,1[$. There exists a constant $C_{\phi^o,s}$, which only depends on the anisotropy ϕ^o and on s, such that

$$|K_\phi(x)| \quad \leq \quad \frac{C_{\phi^o,s}}{1+|x|^{d+1+s}}, \qquad \forall x \in \mathbb{R}^d. \tag{10}$$

Remark 1. *The case $s = 0$ is easy: According to proposition 1, the function $\triangle^{\frac{d+1}{2}} \hat{K}_\phi(\xi)$ is in $L^1(\mathbb{R}^d)$. The continuity of the Fourier transform from L^1 to L^∞ shows that*

$$\|(1+|x|^{d+1})K_\phi\|_{L^\infty} \leq C\|\hat{K}_\phi(\xi) + \triangle^{\frac{d+1}{2}}\hat{K}_\phi(\xi)\|_{L^1(\mathbb{R}^d)},$$

and since $\hat{K}_\phi(\xi) = e^{-4\pi^2\phi^o(\xi)^2}$,

$$|K_\phi(x)| \leq \frac{C_{\phi^o,0}}{1+|x|^{d+1}}, \quad \forall x \in \mathbb{R}^d.$$

The proof uses properties of interpolation spaces [12]. Consider X, Y two Banach spaces, and for $u \in X + Y$ and $t \in \mathbb{R}^+$, let

$$k(t,u) = \inf_{u=u_0+u_1} \{\|u_0\|_X + t\|u_1\|_Y\}.$$

For $s \in [0,1]$ and $p \geq 1$, the interpolation space $[X,Y]_{s,p}$ beetween X and Y is defined by

$$[X,Y]_{s,p} = \left\{u \in X+Y \ ; \ t^{-s}K(t,u) \in L^p\left(\frac{1}{t}\right)\right\}.$$

In particular, given a strictly positive function $h : \mathbb{R}^d \to \mathbb{R}$, consider the weighted space L^∞_h defined by

$$L^\infty_h(\mathbb{R}^d) = \left\{u \in L^\infty(\mathbb{R}^d) \ ; \ \sup_{x\in\mathbb{R}^d}\{h(x)u(x)\} < \infty\right\}.$$

One can interpolate between L^∞ and L^∞_h according to the following lemma.

Lemma 1. *Let h be a strictly positive function $\mathbb{R}^d \to \mathbb{R}$, and let $s \in]0,1[$. Then*

$$[L^\infty(\mathbb{R}^d), L^\infty_h(\mathbb{R}^d)]_{s,\infty} = L^\infty_{h^s}(\mathbb{R}^d)$$

Proof. 1) Assume that $u \in L^\infty_{h^s}(\mathbb{R}^d)$. There exists a constant C such that for a.e. $x \in \mathbb{R}^d$,

$$|u(x)| \leq \frac{C}{h(x)^s}. \tag{11}$$

To estimate $k(t,u) = \inf_{u=u_0+u_1}\left\{\|u_0\|_{L^\infty} + t\|u_1\|_{L^\infty_h}\right\}$, we note that

- If $t \geq 1$, the choice $u_0 = u$ and $u_1 = 0$, shows that $K(t,u) \leq \|u\|_{L^\infty}$.
- If $t < 1$, we consider the set $A = \left\{x \in \mathbb{R}^d \ ; \ |u(x)|h(x) \leq t^{s-1}\right\}$, and we choose $u_0 = \chi_{A^c}\, u$ and $u_1 = \chi_A\, u$, so that $\|u_1\|_{L^\infty_h} \leq t^{s-1}$. Moreover, we remark that for all $x \in A^c$, $|u(x)|h(x) \geq t^{s-1}$ so that, in view of (11)

$$|u_0(x)| \ \leq \ Ch(x)^{-s} \ \leq \ C|u_0(x)|^s t^{s(1-s)},$$

and thus $k(t,u) \leq (C+1)t^s$.

In summary, these estimates show that

$$K(t,u) \leq \begin{cases} \|u\|_{L^\infty} & \text{if } t \geq 1 \\ (C+1)t^s & \text{if } t < 1, \end{cases}$$

which proves that $u \in [L^\infty, L^\infty_h]_{s,\infty}$.
2) Conversely, we consider $u \in [L^\infty, L^\infty_h]_{s,\infty}$. For all $t > 0$, there exists a decomposition $u = u_{0,t} + u_{1,t}$ such that

$$|u_{0,t}|_{L^\infty} + t|u_{1,t}|_{L^\infty_h} \leq Ct^s.$$

It follows that for all $t > 0$, we have

$$h(x)^s|u(x)| \ \leq \ |h(x)^s\,|u_{0,t}(x) + u_{1,t}(x)| \ \leq \ C\left(h(x)^s t^s + h(x)^{s-1}t^{s-1}\right).$$

Choosing $t = h(x)^{-1}$ in the above inequality shows that for all $x \in \mathbb{R}^d$, $h(x)^s\,|u(x)| \leq 2C$, which concludes the proof. □

We use the following properties of interpolation spaces:

(P_1) if T is continous from $X \to \tilde{X}$ and from $Y \to \tilde{Y}$, then T is continous from $[X,Y]_{s,p}$ to $[\tilde{X},\tilde{Y}]_{s,p}$.

(P_2) if $p < p'$, then $[X,Y]_{s,p} \subset [X,Y]_{s,p'}$ for any $0 < s < 1$ and $p \geq 1$.

(P_3) $[L^\infty(\mathbb{R}^d), L^\infty_{(1+|x|)}(\mathbb{R}^d)]_{s,\infty} = L^\infty_{(1+|x|)^s}(\mathbb{R}^d)$ for any $0 < s < 1$.

In the following, we consider the case where T is the Fourier transform, $X = L^1(\mathbb{R}^d)$, $Y = L^\infty(\mathbb{R}^d)$, $\tilde{X} = W^{1,1}(\mathbb{R}^d)$ and $\tilde{Y} = L^\infty_{(1+|x|)}(\mathbb{R}^d)$.

Proof of Proposition 2. We claim that it suffices to show that for any $0 < s < 1$

$$u(\xi) \quad := \quad \triangle^{\frac{d+1}{2}} \hat{K}_\phi(\xi) \ \in [X,Y]_{s,1}. \tag{12}$$

Indeed, the inclusion $[X,Y]_{s,1} \subset [X,Y]_{s,\infty}$ implies then that $u \in [X,Y]_{s,\infty}$, so that in view of (P_1) and (P_3) we obtain

$$\hat{u} \in [\tilde{X},\tilde{Y}]_{s,\infty} = [L^\infty(\mathbb{R}^d), L^\infty_{(1+|x|)}(\mathbb{R}^d)]_{s,\infty} = L^\infty_{(1+|x|)^s}(\mathbb{R}^d),$$

and consequently

$$|(1+|x|^s)\hat{u}(x)| = |(1+|x|^{d+1})K_\phi(x)(1+|x|)^s| \le C_{\phi^o,s}, \quad \text{ for all } x \in \mathbb{R}^d.$$

It follows that for some constant $C_{\phi^o,s}$

$$|K_\phi(x)| \le \frac{C_{\phi^o,s}}{1+|x|^{d+1+s}}, \quad \text{for all } x \in \mathbb{R}^d.$$

We now prove (12). The homogeneity of ϕ^o shows that for some $c_1 > 0$ and $c_2 > 0$, and for $\xi \in \mathbb{R}^d \setminus \{0\}$,

$$|u(\xi)| \le \frac{c_1}{|\xi|^{d-1}} e^{-\lambda|\xi|^2} \qquad \text{and} \qquad |\nabla u(\xi)| \le \frac{c_2}{|\xi|^d} e^{-\lambda|\xi|^2},$$

which shows that $u \in X = L^1(\mathbb{R}^d)$. However, u may not belong to $Y = L^\infty(R^d)$. We now estimate $k(u,t)$, for $t \in \mathbb{R}^+$. If $t \ge 1$, we set $u_0 = u$, $u_1 = 0$, so that

$$k(t,u) \le \|u\|_X, \quad \forall t \ge 1. \tag{13}$$

If $t < 1$, consider the function $\rho_t(\xi)$ defined by

$$\rho_t(\xi) = \begin{cases} 0 & \text{if} \quad |x| \le t \\ 1 & \text{if} \quad |x| > 2t \\ \sin\left(\frac{\pi}{2}\frac{|\xi|-t}{t}\right) & \text{otherwise.} \end{cases}$$

We choose $u_0 = (1-\rho_t)u$ and $u_1 = \rho_t u$ and check that

$$|u_0|_{L^1(\mathbb{R}^d)} \quad \le \quad \int_{B(0,2t)} |u(\xi)| d\xi \le \int_{B(0,2t)} \frac{C}{|\xi|^{d-1}} d\xi \le 2C|\mathbf{S}^d|t.$$

Moreover,

$$\begin{aligned} \|\nabla u_1\|_{L^1(\mathbb{R}^d)} &\le \|\nabla\rho_t u + \rho_t \nabla u\|_{L^1(\mathbb{R}^d)} \\ &\le \int_{R^d\setminus B(0,t)} |\nabla\rho_t|u(\xi) d\xi + \int_{R^d\setminus B(0,t)} |\nabla u(\xi)| d\xi \\ &\le \frac{\pi}{2t}\int_{B(0,2t)\setminus B(0,t)} \frac{C}{|\xi|^{d-1}} e^{-\lambda|\xi|^2} d\xi + \int_{R^d\setminus B(0,t)} \frac{C}{|\xi|^d} e^{-\lambda|\xi|^2} d\xi. \end{aligned}$$

First, we have

$$\frac{\pi}{2t}\int_{B(0,2t)\setminus B(0,t)} \frac{C}{|\xi|^{d-1}} e^{-\lambda|\xi|^2} d\xi \quad \leq \quad \frac{C\pi}{2t}|\mathbf{S}^d| \int_t^{2t} dr \leq \frac{|\mathbf{S}^d| C\pi}{2}.$$

Second,

$$\begin{aligned}
\int_{R^d\setminus B(0,t)} \frac{C}{|\xi|^d} e^{-\lambda|\xi|^2} d\xi \quad &\leq \quad \int_{B(0,1)\setminus B(0,t)} \frac{C}{|\xi|^d} e^{-\lambda|\xi|^2} d\xi + \int_{\mathbb{R}^d\setminus B(0,1)} \frac{C}{|\xi|^d} e^{-\lambda|\xi|^2} d\xi \\
&\leq \quad C|\mathbf{S}^d| \int_t^1 \frac{1}{r} dr + C|\mathbf{S}^d| \int_1^\infty e^{-\lambda r^2} dr \\
&\leq \quad C|\mathbf{S}^d| \left(|\ln(t)| + \frac{1}{\sqrt{\lambda}} \frac{\sqrt{\pi}}{2} \right),
\end{aligned}$$

so that

$$\|u_1\|_Y \quad \leq \quad C\left[|\mathbf{S}^d| \left(\frac{\pi}{2} + \frac{1}{\sqrt{\lambda}} \frac{\sqrt{\pi}}{2} + |\ln(t)| \right) \right].$$

Consequently, this decomposition of u shows that

$$k(u,t) \leq C(1 + |\ln(t)|)t, \quad \forall t < 1, \tag{14}$$

for some constant $C > 0$. In summary,

$$k(u,t) \leq \begin{cases} \|u\|_X & \text{if} \quad t \geq 1 \\ C(1 + |\ln(t)|)t & \text{if} \quad t < 1, \end{cases}$$

and therefore we obtain

$$\begin{aligned}
\|t^{-s}\, k(t,u)\|^1_{L^1(1/t)} \quad &= \quad \int_{\mathbb{R}^+} |k(t,u)t^{-s}| \frac{1}{t} dt \\
&\leq \quad \int_0^1 \frac{(C_0 + C_1|\ln(t)|)}{t^s} dt + \int_1^\infty \frac{\|u\|^1_X}{t^{1+s}} dt < +\infty,
\end{aligned}$$

which proves that $u \in [X,Y]_{s,1}$ as claimed. $\square$

Corollary 1. *For any $s \in [0,1[$ and $p \in \mathbf{S}^d$,*

$$|x|^{1+s} K_\phi \in L^1(\mathbb{R}^d), \qquad (K_\phi)_{|p^\perp} \in L^1(\mathbb{R}^{d-1}), \qquad (x \otimes x K_\phi)_{|p^\perp} \in L^1(\mathbb{R}^{d-1}).$$

Proposition 3 (Decay of averages of K_ϕ on spheres)**.**
The integral

$$I(R) = \int_{\partial B(0,R)} K_\phi d\mathcal{H}^{d-1},$$

is stricly positive, and decays rapidly as

$$\frac{R^{d-1}|\mathbb{S}^{d-1}|}{(4\pi)^{d/2}\Lambda^d} e^{-\frac{R^2}{4\Lambda^2}} \leq I(R) \leq \frac{R^{d-1}|\mathbb{S}^{d-1}|}{(4\pi)^{d/2}\lambda^d} e^{-\frac{R^2}{4\lambda^2}}$$

where λ and Λ are bounds for ϕ^o as in (9).

Proof. Since the measure $\mu := \delta_{\partial B(0,R)}$ has finite mass, its Fourier transform is the continuous and bounded function

$$\hat{\mu}(\xi) \;\; = \;\; \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} d\mu \;\; = \;\; \int_{\partial B(0,R)} e^{-2\pi i x\cdot\xi}.$$

As μ is radially symmetric, $\hat{\mu}$ can be expressed in the form

$$\hat{\mu}(\xi) \;\; = \;\; R^{d-1} J(R|\xi|),$$

where J is a function $\mathbb{R}^+ \to \mathbb{R}$. It follows that

$$\begin{aligned} I(R) \;\; &= \;\; \left\langle \delta_{\partial B(0,R)} \, , \, K_\phi \right\rangle = \left\langle R^{d-1} J(R|\xi|) \, , \, e^{-4\pi^2 \phi^o(\xi)^2} \right\rangle \\ &= \;\; R^{d-1} \int_{\mathbb{S}^{d-1}} \int_0^{+\infty} r^{d-1} J(Rr) e^{-4\pi^2\phi^o(\theta)^2 r^2} dr d\mathcal{H}^{d-1}. \end{aligned} \tag{15}$$

We use the particular case when $\phi^o(\xi)$ is isotropic, i.e., $\phi^o(\xi) = |\xi|$ to estimate the previous integral. In this case, $K_\phi = \frac{1}{(4\pi)^{d/2}} e^{-\frac{x^2}{4}}$ is the heat kernel, and by a direct calculation we see that the corresponding integral is $I(R) =< \delta_{B(0,R)}, K_\phi >= \frac{R^{d-1}|\mathbb{S}^{d-1}|}{(4\pi)^{d/2}} e^{-\frac{R^2}{4}}$. Comparing this expression to (15) and using the radial symmetry of K_ϕ shows that

$$\int_0^{+\infty} r^{d-1} J(Rr) e^{-4\pi^2 r^2} dr \;\; = \;\; \frac{1}{(4\pi)^{d/2}} e^{-\frac{R^2}{4}},$$

or, after a change of variable, that

$$\int_0^{+\infty} r J(Rr) e^{-4\pi^2\phi^o(\theta)^2 r^2} dr = \frac{1}{(4\pi)^{d/2}\phi^o(\theta)^d} e^{-\frac{R^2}{4\phi^o(\theta)^2}}. \tag{16}$$

Returning to a general kernel K_ϕ, we deduce from (15) and (16) that

$$I(R) \;\; = \;\; \frac{R^{d-1}}{(4\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \frac{1}{\phi^o(\theta)^d} e^{-\frac{R^2}{4\phi^o(\theta)^2}} d\mathcal{H}^{d-1},$$

which in view of (9) concludes the proof. □

Proposition 4 (Positivity on hyperplanes)**.**
For all $p \in \mathbb{S}^{d-1}$, the integral $\int_{p^\perp} K_\phi d\mathcal{H}^{d-1}$ is well defined, and satisfies

$$\int_{p^\perp} K_\phi d\mathcal{H}^{d-1} = \frac{1}{2\sqrt{\pi}\phi^o(p)}.$$

In particular, we have

$$\frac{1}{2\sqrt{\pi}\Lambda} \leq \int_{p^\perp} K_\phi d\mathcal{H}^{d-1} \leq \frac{1}{2\sqrt{\pi}\lambda}.$$

Proof. Let $p \in \mathbb{S}^{d-1}$. We already know from Corollary 1 that $\int_{p^\perp} K_\phi d\mathcal{H}^{d-1}$ is well defined. Consider for $\mu > 0$, the approximating functions f_μ, defined by

$$\begin{cases} f_\mu(x) = K_\phi(x) e^{-\pi|x|^2/\mu^2}, \\ \hat{f}_\mu(\xi) = e^{-4\pi^2\phi^o(\xi)^2} * \frac{1}{\mu^2} e^{-\pi\mu^2|\xi|^2}. \end{cases}$$

The function f_μ belongs to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. Moreover, $\hat{f}_\mu \to \hat{K_\phi}$ in $W^{d-1,1}(\mathbb{R}^d)$, and the trace trace theorem [32] shows that one also has

$$\lim_{\mu\to\infty} \int_{\mathbb{R}} \hat{f}_\mu(sp)ds \quad = \quad \int_{\mathbb{R}} \hat{K_\phi}(sp)ds. \tag{17}$$

On the other hand, it follows from the Lebesgue dominated convergence theorem and from (10) that

$$\lim_{\mu\to\infty} \int_{p^\perp} f_\mu d\mathcal{H}^{d-1} \quad = \quad \int_{p^\perp} K_\phi d\mathcal{H}^{d-1}. \tag{18}$$

As $f_\mu \in \mathcal{S}(\mathbb{R}^d)$, we infer that

$$\int_{p^\perp} f_\mu d\mathcal{H}^{d-1} = \left\langle \delta_{p^\perp} \, , \, f_\mu \right\rangle = \langle \delta_p \, , \, \mathcal{F}[f_\mu] \rangle = \int_{\mathbb{R}} \hat{f}_\mu(sp)ds.$$

so that (17) and (18) yield

$$\begin{aligned} \int_{p^\perp} K_\phi d\mathcal{H}^{d-1} \quad &= \quad \int_{\mathbb{R}} \hat{K_\phi}(sp)\, ds = \int_{\mathbb{R}} e^{-4\pi^2 s^2 \phi^o(p)^2}\, ds \\ &= \quad \int_{\mathbb{R}} e^{-\pi\left(2\sqrt{\pi}\phi^o(p)s\right)^2}\, ds = \frac{1}{2\sqrt{\pi}\phi^o(p)}, \end{aligned}$$

which concludes the proof. ∎

Proposition 5 (Moments of order 2)**.**
Let $p \in \mathbb{S}^{d-1}$. Then $\frac{1}{2}\int_{p^\perp} x \otimes x K_\phi d\mathcal{H}^{d-1}$ is well defined and satisfies

$$\frac{1}{2}\int_{p^\perp} x \otimes x K_\phi d\mathcal{H}^{d-1} = \phi^o_{\xi\xi}(p)\frac{1}{2\sqrt{\pi}}.$$

Proof. Corollary 1 states that the integral $\int_{p^\perp} |x|^2 K_\phi d\mathcal{H}^{d-1}$ is well defined. Recalling the sequence f_μ used in the previous proposition, we observe that $D^2\hat{f}_\mu \to D^2\hat{K}_\phi$ in $W^{d-1,1}(\mathbb{R}^d)$, so that the trace theorem implies

$$\lim_{\mu\to\infty}\int_{\mathbb{R}} D^2\hat{f}_\mu(sp)ds \quad = \quad \int_{\mathbb{R}} D^2\hat{K}_\phi(sp)ds. \tag{19}$$

From proposition 2 and the Lebesgue dominated convergence, we obtain

$$\lim_{\mu\to\infty}\int_{p^\perp} x\otimes x \; f_\mu(x) \; d\mathcal{H}^{d-1} \quad \to \quad \int_{p^\perp} x\otimes x \; K_\phi(x) \; d\mathcal{H}^{d-1}. \tag{20}$$

Moreover, we have

$$\begin{aligned}\int_{p^\perp} x\otimes x f_\mu(x) d\mathcal{H}^{d-1} &= \left\langle \delta_{p^\perp}\,,\, x\otimes x f_\mu \right\rangle = -\frac{1}{4\pi^2}\left\langle \delta_p\,,\, D^2\hat{f}_\mu \right\rangle \\ &= -\frac{1}{4\pi^2}\int_{\mathbb{R}} D^2\hat{f}_\mu(sp)ds,\end{aligned}$$

so that in view of (19)

$$\int_{p^\perp} x\otimes x K_\phi(x) d\mathcal{H}^{d-1} = -\frac{1}{4\pi^2}\int_{\mathbb{R}} D^2\hat{K}_\phi(sp)ds.$$

We next estimate the above right-hand side by a direct calculation:

$$\begin{aligned}-\frac{1}{4\pi^2}\int_{\mathbb{R}} D^2\hat{K}_\phi(sp)\,ds &= \left[2\phi^o(p)\phi^o_{\xi\xi}(p) + 2\phi^o_\xi(p)\otimes\phi^o_\xi(p)\right]\int_{\mathbb{R}} e^{-4\pi^2 s^2\phi^o(p)^2}\,ds \\ &\quad - \left[2\phi^o_\xi(p)\otimes\phi^o_\xi(p)\right]\int_{\mathbb{R}} 8\pi^2 s^2\phi^o(p)^2 e^{-4\pi^2 s^2\phi^o(p)^2}\,ds.\end{aligned}$$

Further, we see by integration by parts that

$$\begin{aligned}\int_{\mathbb{R}} 8\pi^2 s^2\phi^o(p)^2 e^{-4\pi^2 s^2\phi^o(p)^2}\,ds &= \int_{\mathbb{R}}\left\{4\pi^2 2s\phi^o(p)^2 e^{-4\pi^2 s^2\phi^o(p)^2}\right\}\{s\}\,ds \\ &= \int_{\mathbb{R}} e^{-4\pi^2 s^2\phi^o(p)^2}ds \;=\; \frac{1}{2\sqrt{\pi}\phi^o(p)},\end{aligned}$$

and we conclude that

$$\frac{1}{2}\int_{p^\perp} x\otimes x K_\phi(x) d\mathcal{H}^{d-1} = \phi^o_{\xi\xi}(p)\frac{1}{2\sqrt{\pi}}.$$

□

Corollary 2 (The operator $F(X,p)$)**.**
Given $X\in\mathbb{R}^{d\times d}$ and $p\in\mathbb{S}^{d-1}$, let

$$F(X,p) \;=\; \left(\int_{p^\perp} K_\phi(x) d\mathcal{H}^{d-1}\right)^{-1}\left(\frac{1}{2}\int_{p^\perp} < Xx, x > K_\phi(x) d\mathcal{H}^{d-1}\right). \quad (21)$$

This operator is elliptic and satisfies

$$F(X,p) \;=\; \phi^o(p)\phi^o_{\xi\xi}(p) : X. \quad (22)$$

Proof. Equation (22) is a direct consequence of propositions 4 and 5, while the ellipticity of F follows from the convexity of ϕ^o. □

Remark 2. *In the next section, we introduce an algorithm for motion by anisotropic mean curvature, and show its consistency with an evolution equation of the form $u_t = -F(D^2u, \frac{\nabla u}{|\nabla u|})$ where F is defined by (21). The expression (22) shows that this operator is precisely the one corresponding to motion by anisotropic mean curvature (see [10]).*

Proposition 6 (Positivity of order moment s)**.** *Let V be a subspace of $\mathbb{R}^d$ of dimension $1\le m\le d$, and let $0<s<2$. Then*

$$\int_V |x|^s K_\phi d\mathcal{H}^m > 0.$$

Proof. We first consider the case $m=d$ and $V=\mathbb{R}^d$. we consider the finite part $Pf\left(\frac{1}{|x|^{d+s}}\right)$ as a temperate distribution, defined for $\varphi\in\mathcal{S}(\mathbb{R}^d)$ by

$$\left\langle Pf\left(\frac{1}{|x|^{d+s}}\right), \varphi\right\rangle = \lim_{\epsilon\to 0}\left\{\int_{\mathbb{R}^d\setminus B(0,\epsilon)} \frac{\varphi(x)-\varphi(0)}{|x|^{d+s}} dx\right\}.$$

This function happens to be the Fourier transform of the distribution $|x|^s$. More precisely,

$$\mathcal{F}\left[|x|^s\right] = C_{s,d} Pf\left(\frac{1}{|2\pi\xi|^{d+s}}\right), \qquad \text{with} \quad C_{s,d} = 2^{s+d}\pi^{d/2}\frac{\Gamma((s+d)/2)}{\Gamma(-s/2)}, \quad (23)$$

(see for instance [31], Γ denotes the Gamma function). We can thus write

$$\begin{aligned}\int_{\mathbb{R}^d} |x|^s K_\phi \, dx &= \langle |x|^s \, , \, K_\phi \rangle = \left\langle C_{s,d} Pf\left(\frac{1}{|2\pi\xi|^{d+s}}\right) \, , \, e^{-4\pi^2\phi^o(\xi)^2} \right\rangle && (24)\\ &= C_{s,d} \lim_{\epsilon\to 0} \int_{\mathbb{R}^d \setminus B(0,\epsilon)} \frac{e^{-4\pi^2\phi^o(\xi)^2} - 1}{|2\pi\xi|^{d+s}} \, > \, 0, && (25)\end{aligned}$$

a stricly positive quantity, in view of the sign of $C_{s,d}$.

Suppose now that $m < d$ and consider the subspace $V = \mathrm{Vect}\{e_1, \ldots, e_m\}$. We write $x = (x', x'')$, $\xi = (\xi', \xi^{prime\prime})$, with $x', \xi' \in V$. A straightforward computation shows that

$$\begin{aligned}\int_V |x'|^s K_\phi \, d\mathcal{H}^m &= \langle |x'|^s \, , \, K_\phi(x', 0) \rangle_{\mathcal{D}'(\mathbb{R}^m), \mathcal{D}(\mathbb{R}^m)}\\ &= \left\langle \mathcal{H}^{d-m}_{\llcorner\{\xi''=0\}} \otimes |x'|^s \, , \, K_\phi(x', x'') \right\rangle_{\mathcal{D}'(\mathbb{R}^d), \mathcal{D}(\mathbb{R}^d)}\\ &= \left\langle C_{s,m} Pf\left(\frac{1}{|2\pi\xi'|^{m+s}}\right) \, , \, h(\xi') \right\rangle_{\mathcal{D}'(\mathbb{R}^m), \mathcal{D}(\mathbb{R}^m)},\end{aligned}$$

where the function $h : \mathbb{R}^m \to \mathbb{R}$ is defined by

$$h(\xi') = \int_{\mathbb{R}^{d-m}} e^{-4\pi^2\phi^o((\xi', \xi''))^2} d\xi''.$$

The next lemma states that h is C^1 and maximal at $\xi' = 0$, which in view of (23) and of the sign of $C_{s,m}$ concludes the proof. □

Lemma 2. *The function $h : \mathbb{R}^m \to \mathbb{R}$, defined by*

$$h(\xi') = \int_{\mathbb{R}^{d-m}} e^{-4\pi^2\phi^o((\xi', \xi'')^2} d\xi''$$

is C^1, with fast decay as $|\xi'| \to \infty$, and is maximal at $\xi' = 0$.

Proof. recalling (9), we first remark that

$$e^{-4\pi^2\phi^o(\xi', \xi_{\prime\prime})^2} \leq e^{-4\pi^2\lambda^2|\xi|^2} \leq e^{-4\pi^2\lambda^2|\xi'|^2},$$

so that the functions $\xi' \to e^{-4\pi^2\phi^o(\xi', \xi_{\prime\prime})^2}$ and their derivatives are uniformly bounded in $L^1(\mathbb{R}^{d-m})$. The C^1 regularity of h is thus a consequence of the Lebesgue theorem. The above estimate also shows that

$$|h(\xi')| \leq \int_{\mathbb{R}^{d-m}} e^{-4\pi^2\lambda^2(\xi_1^2+\xi_2^2+\ldots+\xi_d^2)} d\xi_{m+1} \ldots d\xi_d \leq \frac{1}{2\lambda^m\sqrt{\pi}^m} e^{-4\pi^2\lambda^2\xi'^2}.$$

To determine the maximal value of h, we consider the sets $A_{\xi',t}$, defined for all $\xi' \in \mathbb{R}^m$ and $t \in]0,1[$ by

$$A_{\xi',t} \quad = \quad \left\{ \xi'' \in \mathbb{R}^{d-m} \ ; \ e^{-4\pi^2 \phi^o((\xi',\xi''))^2} \geq t \right\}$$

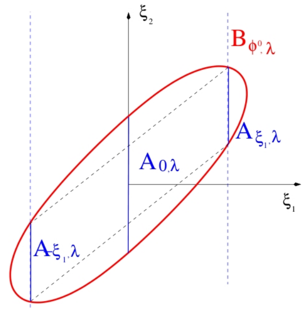

Figure 1:

Fix $\xi'_0 \in \mathbb{R}^m$. The set $A_{\xi'_0,t}$ can be defined as the intersection of the hyperplane $\left\{ \xi \in \mathbb{R}^d \ ; \ \xi' = \xi'_0 \right\}$ with the Frank shape

$$B_{\phi^o,t} = \left\{ \xi \in \mathbb{R}^d \ ; \ \phi^o(\xi) \leq \frac{1}{2\pi} \sqrt{-ln(t)} \right\}.$$

The set $B_{\phi^o,t}$ is convex since ϕ^o is convex. Moreover, from the symmetry of ϕ^o, $(\phi^o(\xi) = \phi^o(-\xi))$, we have

$$|A_{\xi'_0,t}| = |A_{-\xi'_0,t}|.$$

Next, let

$$\begin{aligned} \tilde{A}_{\xi'_0,t} \quad &= \quad \frac{1}{2} \left(A_{\xi'_0,t} + A_{\xi'_0,t} \right) \\ &= \quad \left\{ \xi'' \in \mathbb{R}^{d-m} \ ; \ \exists (\xi''_1, \xi''_2) \in A_{\xi'_0,t} \times A_{-\xi'_0,t}, \quad \xi'' = \frac{1}{2} \left(\xi''_1 + \xi''_2 \right) \right\}. \end{aligned}$$

We remark that the convexity of ϕ^o implies that $\tilde{A}_{\xi'_0,t} \subset A_{0,t}$. Indeed, let $\xi'' \in \tilde{A}_{\xi'_0,t}$,

$$\begin{aligned} \phi^o \left((0, \xi'') \right) \quad &= \quad \phi^o \left(\frac{1}{2} \left((\xi'_0, \xi''_1) + (-\xi'_0, \xi''_2) \right) \right) \\ &\leq \quad \frac{1}{2} \left(\phi^o \left((\xi'_0, \xi''_1) \right) + \phi^o \left((-\xi'_0, \xi''_2) \right) \right) \leq \frac{1}{2\pi} \sqrt{-ln(t)}, \end{aligned}$$

so that $e^{-4\pi^2\phi^o((0,\xi''))^2} \geq t$, i.e. $\xi'' \in A_{0,t}$. Invoking the Brunn-Minkowski inequality, we obtain

$$\begin{aligned} |\tilde{A}_{\xi_0',t}|^{1/(d-m)} &= \frac{1}{2}|A_{\xi_0',t} + A_{-\xi_0',t}|^{1/(d-m)} && (26) \\ &\geq \frac{1}{2}\left(|A_{\xi_0',t}|^{1/(d-m)} + |A_{-\xi_0',t}|^{1/(d-m)}\right) \geq |A_{\xi_0',t}|^{1/(d-m)}, && (27) \end{aligned}$$

and finally that,

$$|A_{0,t}| \geq |\tilde{A}_{\xi_0',t}| \geq |A_{\xi_0',t}|.$$

As this equality holds for any $\xi_0' \in \mathbb{R}^m$, it follows that h is maximal at $\xi' = 0$. $\square$

4 The Bence-Merriman-Osher-like algorithm

Barles and Souganidis [6] have studied the convergence of a general approximation scheme to viscosity solutions of nonlinear second-order parabolic PDE's of the type

$$u_t + F(D^2u, Du) = 0. \tag{28}$$

The main assumption on the function F is its ellipticity, i.e., F satisfies

$$\forall\, p \in \mathbb{R}^d \setminus \{0\}, \forall X, Y \in \mathbf{M}_s^{d\times d}, \quad X \leq Y \quad \Leftarrow \quad F(X,p) \leq F(Y,p). \tag{29}$$

Let $BUC(\mathbb{R}^d)$ denote the space of bounded uniformly continuous functions on $\mathbb{R}^d$. Thus, Barles and Souganidis study a family of operators $G_h : BUC(\mathbb{R}^d) \to BUC(\mathbb{R}^d)$ for $h > 0$, which satisfy, for all $u, v \in BUC(\mathbb{R}^d)$

- *Continuity*

$$\forall\, c \in \mathbb{R}, \quad G_h(u+c) = G_h u, \tag{30}$$

- *Monotonicity*

$$u \leq v \;\Leftarrow\; G_h u \leq G_h v + o(h), \tag{31}$$

  (see remark 2.1 in [6])

- *Consistency*

$$\forall\, \varphi \in \mathcal{C}^\infty(\mathbb{R}^d), \quad \begin{cases} \lim_{h\to 0} h^{-1}(G_h(\varphi) - \varphi)(x) & \leq -F_*(D^2\varphi(x), D\varphi(x)) \\ \lim_{h\to 0} h^{-1}(G_h(\varphi) - \varphi)(x) & \geq -F^*(D^2\varphi(x), D\varphi(x)) \end{cases} . \tag{32}$$

For all $T > 0$ and for all partitions $P = \{O = t_0 < ... < t_n = T\}$ of $[0, T]$, one can then define a sequence of fonctions $u_P : \mathbb{R}^d \times [0, T] \to \mathbb{R}$ by

$$u_P(., t) = \begin{cases} G_{t-t_i}(u_P(., t_i)) & \text{if} \quad t \in (t_i, t_{i+1}], \\ g & \text{if} \quad t = 0, \end{cases} \tag{33}$$

If additionally the following condition holds,

- *Stability*

$$\begin{cases} \text{there exists } \omega \in C([0, \infty], [0, \infty]), \text{ independent of } P \text{ and depending} \\ \text{ on } g \text{ only through the modulus of continuity of } g, \\ \text{such that } \omega(0) = 0 \text{ and for all } t \in [0, t], \\ \|u_P(., t) - g\|_{L^\infty} \leq \omega(t), \end{cases} \tag{34}$$

then the following theorem holds [6] :

Theorem 1. *Assume that $G_h : BUC(\mathbb{R}^d) \to BUC(\mathbb{R}^d)$ satisfies* (30)*,* (31)*,* (32)*, and* (34) *for all $T > 0$, $g \in BUC(\mathbb{R}^d)$ and all partitions P of $[0, T]$. Then, u_P defined in (33) converges uniformly in $\mathbb{R} \times [0, T]$ to the viscosity solution of* (28)*.*

This result was used by H. Ishii, G. Pires and P.E. Souganidis in [33] to study anisotropic mean curvature flow. These authors introduce a kernel f, which satisfies:

(H_1) $f(x) \geq 0, \quad f(-x) = f(x) \quad \text{for all } x \in \mathbb{R}^d, \quad \text{and} \quad \int_{\mathbb{R}^d} f(x)dx = 1$

(H_2) $\int_{p^\perp} (1 + |x|^2)|f(x)|d\mathcal{H}^{d-1} < \infty \quad \text{for all} \quad p \in \mathbb{S}^{d-1}$

(H_3) $\begin{cases} \text{the functions } p \to \int_{p^\perp} f(x)d\mathcal{H}^{d-1} \quad p \to \int_{p^\perp} x_i x_j f(x)d\mathcal{H}^{d-1}, \\ 1 \leq i, j \leq d, \quad \text{are continuous on } \mathbb{S}^{d-1} \end{cases}$

(H_4) $\int_{\mathbb{R}^d} |x|^2 |f(x)|dx < \infty$

(H_5) For all collections $\{R(\rho)\}_{0<\rho<1} \subset \mathbb{R}$ such that $R(\rho) \to \infty$ and $\rho R(\rho)^2 \to 0$ as $\rho \to 0$, and for all functions $g : \mathbb{R}^{d-1} \to \mathbb{R}$ of the form $g(\xi) = a + \langle A\xi , \xi \rangle$ where $a \in \mathbb{R}$ and where A is a symmetric matrix,

$$\lim_{\rho \to 0} \sup_{U \in \mathbf{O}(d)} \sup_{0<r<\rho} \left| \int_{B(0,R(\rho))} f_U(\xi, rg(\xi))g(\xi)d\xi - \int_{\mathbb{R}^{d-1}} f_U(\xi, 0)g(\xi)d\xi \right| = 0,$$

where $\mathbf{O}(n)$ denotes the group of $d \times d$ orthogonal matrices, and where $f_U : \mathbb{R}^d \to \mathbb{R}$ is defined for all $U \in \mathbf{O}(d)$ by $f_U(x) = f(U^* x)$.

Theorem 1 has been applied to schemes for anisotropic mean curvature motion (see theorem 3.3 in [33]) with G_h defined by

$$
\begin{aligned}
G_h\Psi(x) &= \sup\left\{\lambda \in \mathbb{R}\ ;\ S_h \mathbb{1}_{\Psi\geq\lambda}(x) \geq \theta_h\right\} && (35)\\
&= \inf\left\{\lambda \in \mathbb{R}\ ;\ S_h \mathbb{1}_{\Psi\geq\lambda}(x) < \theta_h\right\}, && (36)
\end{aligned}
$$

where

$$
S_h g(x) = h^{-d/2} f(./\sqrt{h}) * g(x) = h^{-d/2}\int_{\mathbb{R}^d} f(y/\sqrt{h}) g(x-y)\, dy, \quad \theta_h = \frac{1}{2} + c\sqrt{h},
$$

and where $F(X,p)$ is given by

$$
F(X,p) = -\left(\int_{p^\perp} f(x) d\mathcal{H}^{d-1}(x)\right)^{-1}\left(\frac{1}{2}\int_{p^\perp}\langle Xx\,,\, x\rangle\, f(x) d\mathcal{H}^{d-1}(x) + c|p|\right),
$$

(the last term in this integral models a forcing term).

In this section, we follow the proof in [33] to show a consistency result in our case when f is a non positive kernel and does not have moments of order two (ie. $x \to |x|^2 f(x) \notin L^1(\mathbb{R}^d)$). We introduce two operators G_h^+ and G_h^- defined by

$$
\begin{aligned}
G_h^+\Psi(x) &= \sup\left\{\lambda \in \mathbb{R}\ ;\ S_h \mathbb{1}_{\Psi\geq\lambda}(x) \geq \theta_h\right\}, && (37)\\
G_h^-\Psi(x) &= \inf\left\{\lambda \in \mathbb{R}\ ;\ S_h \mathbb{1}_{\Psi\geq\lambda}(x) < \theta_h\right\}, && (38)
\end{aligned}
$$

which are not necessarly equal as our kernel is not being nonnegative. To adapt these results to our context we modify the assumptions (H_1), (H_4) and (H_5) as follows

- (H_1') $\int_{p^\perp} f(x) d\mathcal{H}^{d-1} > 0$ for all $p \in \mathbb{S}^{d-1}$, $\quad f(-x) = f(x) \quad$ and $\int_{\mathbb{R}^d} f(x)dx = 1$,
- (H_4') $\int_{\mathbb{R}^d} |x|^{2-\mu}|f(x)|dx < \infty \quad$ for $0 < \mu < 2$,
- (H_5') Assume that $\mu \in]0,1/2]$. Then for all collections $\{R(\rho)\}_{0<\rho<1} \subset \mathbb{R}$ such that $R(\rho) \to \infty$ and $\rho R(\rho)^{2-\mu} \to 0$ as $\rho \to 0$, and for all functions $g : \mathbb{R}^{d-1} \to \mathbb{R}$ of the form $g(\xi) = a + \langle A\xi\,,\,\xi\rangle$ where $a \in \mathbb{R}$ and where A is a symmetric matrix,

$$
\lim_{\rho\to 0}\sup_{U\in O(d)}\sup_{0<r<\rho}\left|\int_{B(0,R(\rho))} f_U(\xi, rg(\xi))g(\xi)d\xi - \int_{\mathbb{R}^{d-1}} f_U(\xi,0)g(\xi)d\xi\right| = 0,
$$

$$
\lim_{\rho\to 0}\sup_{U\in O(d)}\sup_{0<r<\rho}\left|\int_{B(0,R(\rho))} |f_U(\xi, rg(\xi))|\, g(\xi)d\xi - \int_{\mathbb{R}^{d-1}} |f_U(\xi,0)|\, g(\xi)d\xi\right| = 0,
$$

In this last statement, $B(0,R(\rho))$ denotes the $(n-1)$-dimensional ball, centered at 0 and of radius $R(\rho)$.

4.1 K_ϕ satisfies (H_2, H_3) and (H_1', H_4', H_5')

We remark that $\hat{K}_\phi(\xi) = \hat{K}_\phi(-\xi)$ and $\mathcal{F}(K_\phi)(0) = 1$, so that

$$K_\phi(-x) = K_\phi(x) \quad \text{for all } x \in \mathbb{R}^d, \quad \text{and} \quad \int_{\mathbb{R}^d} K_\phi(x)dx = 1.$$

Moreover, proposition (4) shows that

$$\int_{p^\perp} K_\phi(x) d\mathcal{H}^{d-1} \geq \frac{1}{(4\pi)^{d/2}\Lambda^d} > 0 \quad \text{for all} \quad p \in \mathbb{S}^{d-1},$$

so that (H_1') is satisfied. Propositions (4) and (5) also imply that K_ϕ satisfies (H_2), i.e.,

$$\int_{p^\perp} (1+|x|^2)|K_\phi(x)| d\mathcal{H}^{d-1} \quad < \quad \infty \quad \text{for all} \quad p \in \mathbb{S}^{d-1}. \tag{39}$$

Concerning (H_3), we note that

$$\frac{1}{2}\int_{p^\perp} x \otimes x K_\phi(x) d\mathcal{H}^{d-1} \quad = \quad \frac{1}{2\sqrt{\pi}} \phi^o_{\xi\xi}(p),$$

and that

$$\int_{p^\perp} K_\phi d\mathcal{H}^{d-1} \quad = \quad \frac{1}{2\sqrt{\pi}\phi^o(p)}.$$

Since ϕ^o is smooth on $\mathbb{R}^d \setminus \{0\}$ and positive (in particular $\phi^o \geq \lambda$ on $\mathbf{S}^d$) we see that the functions

$$p \to \int_{p^\perp} K_\phi(x) d\mathcal{H}^{d-1} \qquad p \to \int_{p^\perp} x_i x_j K_\phi(x) d\mathcal{H}^{d-1}, \quad 1 \leq i,j \leq d,$$

are continuous on $\mathbf{S}^d$.

We next prove that if $0 < \mu < 2$, then

$$\int_{\mathbb{R}^d} |x|^{2-\mu}|f(x)|dx < \infty.$$

Indeed, proposition 2 with $s = 1 - \mu/2$ shows that

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^{2-\mu}|f(x)|\, dx \quad &\leq \quad \int_{\mathbb{R}^d} \frac{C_{\phi^o,s}|x|^{2-\mu}}{1+|x|^{d+1+(1-\mu/2)}} dx \;\leq\; \int_{\mathbb{R}^d} \frac{C}{1+|x|^{d+\mu/2}}\, dx \\ &\leq \quad C|\mathbf{S}^d| \int_0^\infty \frac{1}{(1+r^{1+\mu/2})} dr \;<\; \infty, \end{aligned}$$

for some generic constant C.

It remains to prove (H_5'): Let $0 < \mu < 1/2$ and let $R : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that, as $\rho \to 0$, $R(\rho) \to \infty$ and $\rho R(\rho)^{2-\mu} \to 0$. Setting $f_U(x) = K_\phi(U^* x)$, we consider

$$\begin{aligned}
\int_{\mathbb{R}^{d-1}} \left| \Big(f_U(\tilde{x}, rg(\tilde{x})) \mathbb{1}_{B(0,R(\rho))}(\tilde{x}) - f_U(\tilde{x}, 0) \Big) g(\tilde{x}) \right| d\tilde{x} &\leq \int_{B(0,R(\rho))^c} |f_U(\tilde{x}, 0)| \, |g(\tilde{x})| d\tilde{x} \\
&+ \int_{B\left(0, R(\rho)^{\frac{2-\mu}{2}}\right)} |(f_U(\tilde{x}, rg(\tilde{x})) - f_U(\tilde{x}, 0)) \, g(\tilde{x})| \, d\tilde{x} \\
&+ \int_{B(0,R(\rho)) \setminus B\left(0, R(\rho)^{\frac{2-\mu}{2}}\right)} |(f_U(\tilde{x}, rg(\tilde{x})) - f_U(\tilde{x}, 0)) \, g(\tilde{x})| \, d\tilde{x}. \quad (40)
\end{aligned}$$

From the decay assumptions on K_ϕ (see proposition 2) we have

$$|f_U(\tilde{x}, rg(\tilde{x})) g(\tilde{x})| \leq \frac{C}{1 + |\tilde{x}|^{d-1+s}},$$

where C does not depend on U and r. Then, it holds that

$$\begin{aligned}
&\int_{B(0,R(\rho))^c} |f_U(\tilde{x}, 0) g(\tilde{x})| \, d\tilde{x} \\
\leq \; & C \int_{B(0,R(\rho))^c} \frac{1}{1 + |\tilde{x}|^{d-1+s}} d\tilde{x} \; \leq \; C \, |\mathbf{S}^{d-1}| \int_{R(\rho)}^{\infty} \frac{1}{1 + |r|^{1+s}} dr \\
\leq \; & C \, |\mathbf{S}^{d-1}| R(\rho)^{-s},
\end{aligned}$$

and

$$\begin{aligned}
&\int_{B(0,R(\rho)) \setminus B\left(0, R(\rho)^{\frac{2-\mu}{2}}\right)} | \, (f_U(\tilde{x}, rg(\tilde{x})) - f_U(\tilde{x}, 0)) \, g(\tilde{x})| \, d\tilde{x} \\
\leq \; & 2C \int_{B(0,R(\rho)) \setminus B\left(0, R(\rho)^{\frac{2-\mu}{2}}\right)} \frac{1}{1 + |\tilde{x}|^{d-1+s}} d\tilde{x} \; \leq \; 2C \, |\mathbf{S}^{d-1}| \int_{R(\rho)^{\frac{2-\mu}{2}}}^{R(\rho)} \frac{1}{1 + |r|^{1+s}} dr \\
\leq \; & C \, |\mathbf{S}^{d-1}| \left(R(\rho)^{\frac{-(2-\mu)s}{2}} - R(\rho)^{-s} \right),
\end{aligned}$$

and implies that the first and the third term in (40) converge to 0 uniformly with respect to U and r as $\rho \to 0$. Moreover, using smoothness property of K_ϕ ($C^\infty(\mathbb{R}^d)$), we have

$$|\nabla f_U(\tilde{x}, rg(\tilde{x}))| \, |g(\tilde{x})| \leq \frac{\tilde{C}}{1 + |\tilde{x}|^{d-1+s}} \in L^1(\mathbb{R}^{d-1}),$$

uniformly on U and r, and with $r < \rho$,

$$\int_{B\left(0,R(\rho)^{\frac{2-\mu}{2}}\right)} |(f_U(\tilde{x}, rg(\tilde{x})) - f_U(\tilde{x}, 0))\, g(\tilde{x})|\, d\tilde{x}$$

$$\leq rR(\rho)^{2-\mu} \int_{B\left(0,R(\rho)^{\frac{2-\mu}{2}}\right)} \left\{ \sup_{U\in O(d)} \sup_{0\leq s\leq rg(\tilde{x})} \{|\partial_{x_n} f_U(\tilde{x}, s) g(\tilde{x})|\} \right\} d\tilde{x}$$

$$\leq C\rho R(\rho)^{2-\mu},$$

for some generic constant C.

We conclude that

$$\lim_{\rho\to 0} \sup_{U\in O(d)} \sup_{0<r<\rho} \left| \int_{B(0,R(\rho))} f_U(\tilde{x}, rg(\tilde{x}))g(\tilde{x})\, d\tilde{x} - \int_{\mathbb{R}^{d-1}} f_U(\tilde{x}, 0)g(\tilde{x})\, d\tilde{x} \right| \quad = \quad 0.$$

The second statement in (H_5') is established similarly.

4.2 The consistency proof

Theorem 2. *Let $\varphi \in C^2(\mathbb{R}^d)$. For all $z \in \mathbb{R}^d$ and $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in B(z, \delta)$ and $h \in (0, \delta]$, if $\nabla\phi(x) \neq 0$ we have*

$$\begin{aligned} G_h^-\varphi(x) &\leq \varphi(x) + (-F(D^2\varphi(z), D\varphi(z)) + \epsilon)h \\ G_h^+\varphi(x) &\geq \varphi(x) + (-F(D^2\varphi, D\varphi(z)) - \epsilon)h. \end{aligned}$$

Proof. We closely follow the argument in [33].

1. We only prove the first inequality. The other one is obtained similarly.

2. Without loss of generality, we can assume that $z = 0$. Let us fix $a \in \mathbb{R}$, such that

$$a > -F(D^2\varphi(0), D\varphi(0)).$$

The inequality is proved if we can exhibit a $\delta > 0$ such that, for all $x \in B(0, \delta)$ and $h \in (0, \delta]$,

$$S_h \mathbb{1}_{\varphi\geq\varphi(x)+ah}(x) < \theta_h.$$

3. Fix $\delta_1 > 0$, such that $D\varphi \neq 0$ on $B(0, \delta_1)$ and choose a continuous family $\{U(x)\}_{x\in B(0,\delta_1)} \subset O(d)$, such that for all $x \in B(0, \delta_1)$,

$$U(x)\left(\frac{D\varphi(x)}{|D\varphi(x)|}\right) = e_d,$$

where e_d denotes the unit vector with components $(0, 0, ..., 0, 1) \in \mathbb{R}^d$. Note that if $x \in B(0, \delta_1)$, then

$$S_h \mathbb{1}_{\varphi \geq \varphi(x) + ah} = \int_{\mathbb{R}^d} f_{U(x)}(y) \mathbb{1}_{\varphi \geq \varphi(x) + ah}(x - \sqrt{h} U(x)^* y) dy.$$

4. Choosing δ smaller if necessary, (H_1') implies the inequality

$$a > -F(D^2\varphi, D\varphi) \quad \text{in} \quad B(0, \delta_1),$$

or in other words,

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^{d-1}} \left\langle P^* U(x) D^2\varphi(x) U(x)^* P\xi \,,\, \xi \right\rangle f_{U(x)}(\xi, 0) d\xi \quad &- \quad a \int_{\mathbb{R}^{d-1}} f_{U(x)}(\xi, 0) d\xi \\ &< \quad -c|D\varphi(x)|, \end{aligned} \tag{41}$$

where P denotes the $d \times (d-1)$ matrix with components $P_{ij} = \delta_{ij}$.

5. We next fix $\epsilon > 0$, and $\delta_2 \in (0, \delta_1[$, such that for all $x \in B(0, \delta_2)$,

$$\begin{aligned} &\frac{1}{2} \int_{\mathbb{R}^{d-1}} \left\langle P^* U(0)(D^2\varphi(0) + 3\epsilon^2 I) U(0)^* P\xi \,,\, \xi \right\rangle f_{U(x)}(\xi, 0) d\xi \\ &\qquad -(a - \epsilon^2) \int_{\mathbb{R}^{d-1}} f_{U(x)}(\xi, 0) d\xi < \; -(\xi + \epsilon)|D\varphi(0)|. \end{aligned} \tag{42}$$

6. The Taylor theorem yields a $\gamma > 0$ such that for all $h > 0$, $y \in \mathbb{R}^d$, and $x \in B(0, \delta_2)$, if $\sqrt{h}|y| \leq \gamma$, then

$$\begin{aligned} \varphi(x - \sqrt{h} U(x)^* y) \quad &\leq \quad \varphi(x) - \sqrt{h} \left\langle D\varphi(x) \,,\, U(x)^* y \right\rangle \\ &\qquad + \frac{h}{2} \left\langle U(x)(D^2\varphi(x) + \epsilon^2 I) U(x)^* y \,,\, y \right\rangle \\ &\leq \quad \varphi(x) - \sqrt{h} |D\varphi(x)| y_d + Chy_d^2 \\ &\qquad + \frac{h}{2} \left\langle P^* U(x)(D^2\varphi(x) + 2\epsilon^2 I) U(x)^* P y' \,,\, y' \right\rangle, \end{aligned}$$

and

$$\begin{aligned} \varphi(x - \sqrt{h} U(x)^* y) \quad &\geq \quad \varphi(x) - \sqrt{h} \left\langle D\varphi(x) \,,\, U(x)^* y \right\rangle \\ &\qquad + \frac{h}{2} \left\langle U(x)(D^2\varphi(x) - \epsilon^2 I) U(x)^* y \,,\, y \right\rangle \\ &\geq \quad \varphi(x) - \sqrt{h} |D\varphi(x)| y_d - Chy_d^2 \\ &\qquad + \frac{h}{2} \left\langle P^* U(x)(D^2\varphi(x) - 2\epsilon^2 I) U(x)^* P y' \,,\, y' \right\rangle, \end{aligned}$$

where we write $y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$, and where C is a positive constant.

7. Reducing γ and δ_2 if necessary, the previous inequalities imply that for $y \in B(0, \gamma/\sqrt{h})$ and $x \in B(0, \delta_2)$,

- if $\varphi(x-\sqrt{h}U(x)^*y) \geq \varphi(x)+ah$, then

$$\begin{aligned} y_d &\leq \frac{\sqrt{h}}{|D\varphi(x)|-C\sqrt{h}y_d}\left(-a+\frac{1}{2}\left\langle P^*U(x)(D^2\varphi(x)+2\epsilon^2 I)U(x)^*Py'\,,\,y'\right\rangle\right) \\ &\leq \frac{\sqrt{h}}{|D\varphi(0)|}\left(-a+\epsilon^2+\frac{1}{2}\left\langle P^*U(0)(D^2\varphi(0)+3\epsilon^2 I)U(0)^*Py'\,,\,y'\right\rangle\right) \end{aligned}$$

- if

$$y_d \leq \frac{\sqrt{h}}{|D\varphi(0)|}\left(-a-\epsilon^2+\frac{1}{2}\left\langle P^*U(0)(D^2\varphi(0)-3\epsilon^2 I)U(0)^*Py'\,,\,y'\right\rangle\right),$$

  then

$$\varphi(x-\sqrt{h}U(x)^*y) \geq \varphi(x)+ah.$$

We define

$$\begin{cases} a^\epsilon = (a-\epsilon^2)|D\varphi(0)|^{-1} \\ a_\epsilon = (a+\epsilon^2)|D\varphi(0)|^{-1} \\ A^\epsilon = |D\varphi(0)|^{-1}P^*U(0)\left(D^2\varphi(0)+3\epsilon^2 I\right)U(0)^*P \\ A_\epsilon = |D\varphi(0)|^{-1}P^*U(0)\left(D^2\varphi(0)-3\epsilon^2 I\right)U(0)^*P, \end{cases}$$

and for $y' \in \mathbb{R}^{d-1}$

$$g^\epsilon(y') = \left(-a^\epsilon+\frac{1}{2}\langle A^\epsilon y'\,,\,y'\rangle\right) \qquad g_\epsilon(y') = \left(-a_\epsilon+\frac{1}{2}\langle A_\epsilon y'\,,\,y'\rangle\right).$$

We also set

$$V_{h,x} = \left\{y \in R^d\ ;\ \varphi(x-\sqrt{h}U(x)^*y) \geq \varphi(x)+ah\right\},$$

and

$$\begin{cases} E^+_{\epsilon,h,x} = \left\{y \in \mathbb{R}^d\ ;\ y_d \leq \sqrt{h}g_\epsilon(y')\right\} \\ E^-_{\epsilon,h,x} = \left\{y \in \mathbb{R}^d\ ;\ y_d \leq \sqrt{h}g^\epsilon(y')\right\}. \end{cases}$$

We check that for all $x \in B(0,\delta_2)$,

$$\begin{cases} \left(V_{h,x}\cap B(0,\gamma/\sqrt{h})\right) \quad\subset\quad \left(E^+_{\epsilon,h,x}\cap B(0,\gamma/\sqrt{h})\right) \\ \left(E^-_{\epsilon,h,x}\cap B(0,\gamma/\sqrt{h})\right) \quad\subset\quad \left(V_{h,x}\cap B(0,\gamma/\sqrt{h})\right) \end{cases}$$

8. The assumption (H_4) yields the existence of a decreasing function $\omega \in C([0,\infty),[0,\infty))$ such that $\omega(R)\to 0$ as $R\to\infty$, and

$$\int_{B(0,R)^c} |f(y)||y|^{2-\mu}dy \leq \omega(R)^2, \text{ for all } R \geq 0.$$

For each $0 < t < 1$, we define the family of sets $R(t) \in (0,\infty)$ by

$$\omega(R(t)) \quad = \quad tR(t)^{2-\mu}, \tag{43}$$

which satify (H_5'). We then choose $\tau \in (0,1)$ such that

$$R(t) \leq \gamma/t, \text{ for all } t \in (0,\tau] \tag{44}$$

9. Let

$$\rho = \sqrt{h}, \quad T(\rho) = B_{n-1}(0,R(\rho)) \times \mathbb{R} \subset \mathbb{R}^d.$$

For all $h \in]0,\tau^2)$ and for all $x \in B(0,\delta_2)$, we estimate

$$\begin{aligned}
\int_{V_{h,x}} f_{U(x)}(y)dy &= \int_{\mathbb{R}^d} f_{U(x)}(y) \mathbb{1}_{\varphi \geq \varphi(x)+ah}(x - \sqrt{h}U^*(x)y)dy \\
&\leq \int_{V_{h,x}\cap B(0,R(\rho))} f_{U(x)}(y)dy + \int_{B(0,R(\rho))^c} |f_{U(x)}(y)|dy \\
&\leq \int_{E^+_{\epsilon,h,x}\cap B(0,R(\rho))} f_{U(x)}(x)dx + \int_{B(0,R(\rho))^c} |f_{U(x)}(y)|dy \\
&\qquad + \int_{\left(E^+_{\epsilon,h,x}\setminus E^-_{\epsilon,h,x}\right)\cap B(0,R(\rho))} |f_{U(x)}(y)|dy \\
&\leq \int_{E^+_{\epsilon,h,x}\cap T(\rho)} f_{U(x)}(y)dy + \int_{\left(E^+_{\epsilon,h,x}\setminus E^-_{\epsilon,h,x}\right)\cap T(\rho)} |f_{U(x)}(y)|dy \\
&\qquad +3\int_{B(0,R(\rho))^c} |f_{U(x)}(y)|dy
\end{aligned}$$

10. For the last integral above, we have

$$\int_{B(0,R(\rho))^c} |f_{U(x)}|(y)dy \quad \leq \quad \frac{1}{R(\rho)^{2-\mu}} \int_{B(0,R(\rho))^c} |y|^{2-\mu}|f_{U(x)}|(y)dy \ \leq \ \omega(R(\rho))\rho,$$

and moreover, since K_ϕ is symmetric,

$$\frac{1}{2} \quad = \quad \int_{y_d\leq 0} f_{U(x)}(y)dy \ \leq \ \int_{T(\rho)\cap\{y_d\leq 0\}} f_{U(x)}|(y)dy + \omega(R(\rho))\rho.$$

We note that

$$\begin{aligned}
\int_{T(\rho)\cap E^+_{\epsilon,h,x}} f_{U(x)}(y)dy &= \int_{T(\rho)\cap\{y_d\leq\rho g^\epsilon(y')\}} f_{U(x)}(y)\,dy \\
&= \int_{T(\rho)\cap\{y_d\leq 0\}} f_{U(x)}(y)dy + \int_{B_{n-1}(0,R(\rho))} d\xi \int_0^{\rho g^\epsilon(y')} f_{U(x)}(\xi,r)\,dr \\
&= \int_{T(\rho)\cap\{y_d\leq 0\}} f_{U(x)}(y)\,dy + \int_0^\rho dr \int_{B_{n-1}(0,R(\rho))} f_{U(x)}(\xi,rg(\xi))g^\epsilon(\xi)\,d\xi.
\end{aligned}$$

It follows from (H_5') that as $\rho \to 0$,

$$\frac{1}{\rho}\left\{\int_{T(\rho)\cap E^+_{\epsilon,h,x}} f_{U(x)}(y)dy - \int_{T(\rho)\cap\{y_d\le 0\}} f_{U(x)}(y)dy\right\} \;\to\; \int_{\mathbb{R}^{d-1}} f_{U(x)}(\xi,0)g^\epsilon(\xi)d\xi,$$

uniformly with respect to x. Possibly reducing τ we may assume that for $x \in B(0,\delta_2)$,

$$\frac{1}{\rho}\left\{\int_{T(\rho)\cap E^+_{\epsilon,h,x}} f_{U(x)}(y)dy - \int_{T(\rho)\cap\{y_d\le 0\}} f_{U(x)}(y)dy\right\} \;\le\; \int_{\mathbb{R}^{d-1}} f_{U(x)}(\xi,0)g^\epsilon(\xi)d\xi + \epsilon^2.$$

Using same argument, we also conclude that

$$\begin{aligned}\int_{T(\rho)\cap\left(E^+_{\epsilon,h,x}\setminus E^-_{\epsilon,h,x}\right)} |f_{U(x)}(y)|dy &= \left\{\int_{T(\rho)\cap\{0\le y_d\le \rho g^\epsilon(y')\}} |f_{U(x)}(y)|dy\right\} \\ &\quad - \left\{\int_{T(\rho)\cap\{0\le y_d\le \rho g_\epsilon(y')\}} |f_{U(x)}(y)|dy\right\} \\ &\le \rho\int_{\mathbb{R}^{d-1}} |f_{U(x)}|(\xi,0)(g^\epsilon(\xi)-g_\epsilon(\xi))d\xi + \rho\epsilon^2 \\ &\le \rho\epsilon^2\left(1+\int_{\mathbb{R}^{d-1}}\left(2+3|\xi|^2\right)|f_{U(x)}|(\xi,0)d\xi\right) \\ &\le C_0\rho\epsilon^2,\end{aligned}$$

where

$$C_0 = \sup_{x\in B(0,\delta_2)}\left\{1+\int_{\mathbb{R}^{d-1}}\left(2+3|\xi|^2\right)|f_{U(x)}|(\xi,0)d\xi\right\}.$$

11. Finally, noting that from (41),

$$\int_{\mathbb{R}^{d-1}} f_{U(x)}(\xi,0)g^\epsilon(\xi)d\xi \le -c-\epsilon,$$

we get

$$\begin{aligned}\int_{\mathbb{R}^d} f(x)\mathbb{1}_{\varphi\ge\varphi(x)+ah}(x-\sqrt{h}z)dz &\le \frac{1}{2}+\int_{\mathbb{R}^{d-1}} f_{U(x)}(\xi,0)g(\xi)d\xi \\ &\qquad +\rho\left(\epsilon^2+4\omega(R(\rho))+C_0\epsilon^2\right) \\ &\le \frac{1}{2}+\rho\left(-c-\epsilon+\epsilon^2+4\omega(R(\rho))+C_0\epsilon^2\right) \\ &< \theta_h,\end{aligned}$$

for ϵ sufficiently small. $\square$

Even if the function ϕ is regular, $G_h^+\varphi$ and $G_h^-\varphi$ need not be equal and continuous. However, it is easy to check that if $\varphi = 1\!\!1_\Omega$ is a characteristic function then $G_h^+ 1\!\!1_\Omega = G_h^- 1\!\!1_\Omega$. The next proposition shows that if φ is smooth, $G_h^-\varphi(x) = G_h^+(x)\varphi + o(h)$, so that one could conceivably build a Bence Merriman Osher type scheme using either G_h^+ or G_h^-.

Proposition 7. *Let $\varphi \in C^2(\mathbb{R}^d)$. Let $x \in \mathbb{R}^d$ such as $\nabla\varphi(x) \neq 0$, then*

$$G_h^-\varphi(x) = G_h^+\varphi(x) + o(h).$$

proof. Let $x \in \mathbb{R}^d$ such as $\nabla\varphi(x) \neq 0$ and for all $h > 0$ let

$$\epsilon(h) = G_h^+\varphi(x) - G_h^-\varphi(x).$$

Introduce also $g_h(\lambda) : \mathbb{R} \to \mathbb{R}$ defined by

$$g_h(\lambda) = S_h\chi_{\varphi\geq\lambda}(x) = \int_{\mathbb{R}^d} K_{\phi,h}(y)\chi_{\varphi\geq\lambda}(x-y)dy.$$

This function may not be continuous. We claim that its jumps are bounded by $o(\sqrt{h})$. Indeed, for all $\lambda \in \mathbb{R}$, one can express $g_h(\lambda)$ as

$$\begin{aligned} g_h(\lambda) &= \int_{B(0,\sigma)} K_{\phi,h}(y)\chi_{\{\varphi\geq\lambda\}}(x-y)dy + \int_{\mathbb{R}^d\setminus B(0,\sigma)} K_{\phi,h}(y)\chi_{\{\varphi\geq\lambda\}}(x-y)dy \\ &= \tilde{g}_h(\lambda) + R_h(\lambda), \end{aligned}$$

where σ is chosen sufficiently small so that $|\nabla\varphi(y)| > 0$ for all $y \in B(x,\sigma)$. Let $0 < \mu < 1$, let

$$\omega(R) = \int_{B(0,R)^c} |y|^{2-\mu}|K_\phi(y)|dy,$$

and let $R(t)$ be defined by the equality $\omega(R(t)) = tR(t)^{2-\mu}$. Note that (H_4') implies that $\sqrt{h}R(h)^{2-\mu} \to 0$ as $h \to 0$, so that $\sqrt{h}R(h)^{1-\nu/2} < \sigma$ for h sufficiently small, it follows that

$$|R_h(\lambda)| \;\leq\; \int_{\mathbb{R}^d\setminus B(0,\sqrt{h}R(h)^{1-\mu/2})} |K_{\phi,h}(y)|dy.$$

Moreover, changing variables, we see that

$$\begin{aligned} \int_{\mathbb{R}^d\setminus B(0,\sqrt{h}R(h)^{1-\mu/2})} |K_{\phi,h}(y)|dy &\leq \int_{\mathbb{R}^d\setminus B(0,R(h)^{1-\mu/2})} |K_\phi(y)|dy \\ &\leq \frac{1}{R(h)^{(2-\mu)^2/2}} \int_{\mathbb{R}^d\setminus B(0,R(h)^{1-\mu/2})} |y|^{2-\mu}\,|K_\phi(y)|dy \\ &\leq \frac{\omega(R(h))}{R(h)^{(2-\mu)^2/2}}. \end{aligned}$$

Since $0 < (2-\mu)/2 < 1$, it follows that

$$|R_h(\lambda)| \;\leq\; \left(\frac{\omega(R(h))}{R(h)^{2-\mu}}\right)^{1-\mu/2} \;=\; h^{1-\mu/2} \;=\; o(\sqrt{h}).$$

Further, the fact that $|\nabla\varphi(y)| > 0$ on $B(x,\sigma)$ show that $\tilde{g}_h$ is continuous in λ, which proves the claim.

Recall that

$$\begin{cases} G_h^-\varphi(x) = \inf\{s \in \mathbb{R} \;;\; S_h\chi_{\varphi\geq s}(x) < \theta_h\} \\ G_h^+\varphi(x) = \sup\{s \in \mathbb{R} \;;\; S_h\chi_{\varphi\geq s}(x) \geq \theta_h\}, \end{cases}$$

it follows from the claim above that

$$S_h\chi_{\varphi\geq G_h^-\varphi(x)}(x) = \theta_h + o(\sqrt{h}), \text{ and } \quad S_h\chi_{\varphi\geq G_h^+\varphi(x)}(x) = \theta_h + o(\sqrt{h}),$$

and consequently

$$\int_{\mathbb{R}^d} K_{\phi,h}(y)\chi_{G_h^-\varphi(x)\leq\varphi\leq G_h^-\varphi(x)+\epsilon(h)}(x-y)dy = o(\sqrt{h}).$$

One can use the same argument as in the consistency proof, (in particular see point 7) to show that asymptotically, the above integral behaves like

$$\int_{\mathbb{R}^d} K_{\phi,h}(y)\chi_{G_h^-\varphi(x)\leq\varphi\leq G_h^-\varphi(x)+\epsilon(h)}(x-y)dy = \frac{\epsilon(h)}{|\nabla\varphi(x)|\sqrt{h}}\int_{p^\perp} K_\phi(x)d\mathcal{H}^{d-1}(x) + o(\sqrt{h}),$$

where, $p = \frac{\nabla\phi(x)}{|\nabla\phi(x)|}$. In conclusion, as $\int_{p^\perp} K_\phi(x)d\mathcal{H}^{d-1}(x) > 0$, we deduce that

$$\epsilon(h) = \frac{|\nabla\varphi(x)|}{\int_{p^\perp} K_\phi(x)d\mathcal{H}^{d-1}(x)}o(h),$$

which proves the proposition. □

Remark 3. *Our consistency result sheds light on the relationship between the kernel K_ϕ and the evolution equation (4). Proving convergence of a Bence Merriman Osher type algorithm in our context seems to be very difficult (if true at all). The argument of [33] does not apply here. The main difficulty is that $G_h^\pm\varphi$ may not be continuous, even if φ is regular. Further, we can only show monotonicity of the operators $G_h^\pm$ up to $o(h)$ for smooth functions whose gradients do not vanish. The source of these difficulties is really the thresholding in the definition of $G_h^\pm$.*

5 Numerical simulations

In the previous section, we proved a consistency result for a Bence Merriman Osher-type algorithm. Here we numerically investigate the convergence properties of a related scheme, based on a phase-field discretization. Both schemes consist in a diffusion step followed by a correction step. In the case of the BMO scheme, the correction is a simple thresholding, while the correction is obtained via a reaction term for the phase field scheme. More precisely, in the second case, given a small parameter $\epsilon > 0$, we set

$$G_{h,\epsilon}\varphi(x) = T_{h,\epsilon}(K_{\phi,h} * \varphi),$$

where $T_{h,\epsilon}$ is defined as follows: Given $\lambda \in \mathbb{R}$, $T_{h,\epsilon}(\lambda) = \psi(\lambda)$ where ψ is the solution of the ODE

$$\begin{cases} \psi_t & = -\frac{1}{\epsilon^2} W'(\psi) \\ \psi(0) & = \lambda, \end{cases}$$

and W a double well potential with wells located at $\psi = 0$ and $\psi = 1$. Note that if $\varphi = \mathbb{1}_\Omega$ is a characteristic function, then

$$\lim_{\epsilon \to 0} G_{h,\epsilon} \mathbb{1}_\Omega = G_h^+ \mathbb{1}_\Omega = G_h^- \mathbb{1}_\Omega,$$

which shows a formal relationship in the correction step of the BMO and phase field schemes.

The advantage of the phase field scheme, is that it produces smoother interfaces, which avoids numerical errors due to aliasing. Moreover, we wanted to test our method for approximating anisotropic diffusion on computations of Wulff shapes, a problem where one has to impose a volume constraint. Such constraint is easier to handle with a phase field scheme, where one can explicitely compute the associated Lagrange multiplier. The next paragraph, describes the phase-field algorithm for the operator $\tilde{\Delta}_\phi$.

5.1 The $\tilde{\Delta}_\phi$-phase field model and its discretisation

As an approximation to the anisotropic Allen-Cahn equation (5), we consider the following phase-field model

$$\begin{cases} u_t = \tilde{\Delta}_\phi u - \frac{1}{\epsilon^2} W'(u) \\ u(x,0) = q\left(\frac{\mathrm{dist}(x,\partial E)}{\epsilon}\right) \end{cases} \tag{45}$$

We also report tests, where we estimate the L^1-error on anisotropic Wulff sets (the sets which minimize the anisotropic perimeter under a volume constraint). To impose volume conservation, we consider a conserved phase-field model, of the form

$$\begin{cases} u_t(x,t) = \tilde{\Delta}_\phi u(x,t) - \frac{1}{\epsilon^2} W'(u(x,t)) + \frac{1}{\epsilon}\lambda(t)\sqrt{2W(u(x,t))}, \\ u(x,0) = q\left(\frac{\mathrm{dist}(x,\Omega_0)}{\epsilon}\right). \end{cases} \tag{46}$$

The parameter

$$\lambda(t) = \frac{\int_{\mathbb{R}^d} W'(u(x,t))dx}{\epsilon \int_{\mathbb{R}^d} \sqrt{2W(u(x,t))}dx},$$

can be seen as a Langange multiplier, which preserves the mass of u. See [13] where schemes of this form have been studied for isotropic mean curvature with a volume constraint.

We now describe the numerical method we use for solving the PDE's (45) and (46). Several studies of classical numerical schemes for the Allen–Cahn equation have already been conducted in the past: see for instance, [24, 40, 16, 18, 41, 30, 29]. Here, the computational domain is the fixed box $Q = [-1/2, 1/2]^d \subset \mathbb{R}^d$, $d = 2, 3$. The initial datum is $u_0 = q(\frac{\mathrm{dist}(x,\partial\Omega}{\epsilon})$, where Ω_0 is a smooth bounded set strictly contained Q. We assume that during the evolution, the set $\Omega_{\epsilon,t} := \{u_\epsilon(x,t) \geq 1/2\}$ remains strictly inside Q, so that we may impose periodic boundary conditions on ∂Q.

Our strategy consists in representing u as a Fourier series in Q, and in using a splitting method. First, one applies the diffusion operator, which given the form of $\tilde{\Delta}_\phi$, merely amounts to a multiplication in the Fourier space. The interesting feature of our approach is that this step is fast and very accurate. Next, the reaction term is applied.

More precisely, $u_\epsilon(x,t_n)$ at time $t_n = t_0 + n\delta t$ is approximated by

$$u_\epsilon^P(x,t_n) \quad = \sum_{\max_{1\leq i\leq d}|p_i|\leq P} u_{\epsilon,p}(t_n) e^{2i\pi p\cdot x}.$$

In the diffusion step, we set

$$u_\epsilon^P(x,t_n+1/2) = \sum_{\max_{1\leq i\leq d}|p_i|\leq P} u_{\epsilon,p}(t_n) e^{-4\pi^2\delta t\,\phi^o(p)^2} e^{2i\pi p\cdot x}.$$

We then integrate the reaction terms

$$u_\epsilon^P(x,t_n+1) \quad = \quad u_\epsilon^P(x,t_n+1) - \delta t\epsilon^2 W'_{i,\epsilon}(u_\epsilon^P(x,t_n+1/2)).$$

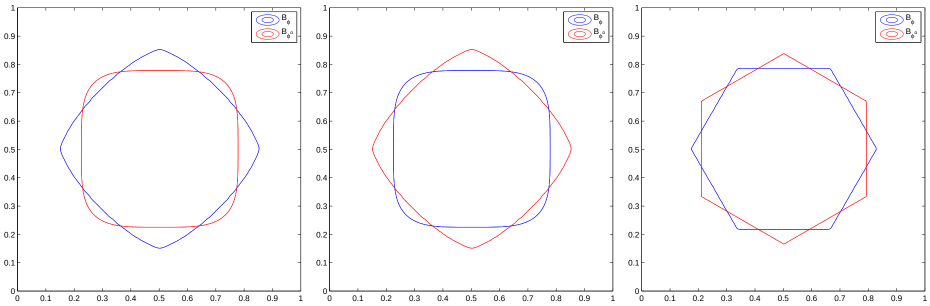

Figure 2: Wulff Set (blue) and Frank diagram (red) for the anisotropic densities (ϕ_1, ϕ_1^o), (ϕ_2, ϕ_2^o) and (ϕ_3, ϕ_3^o)

In practice, the first step is performed via a fast Fourier transform, with a computational cost $O(P^d \ln(P))$.

The corresponding numerical scheme turns out to be stable when solving (45), under the condition $\delta t \leq M\epsilon^2$, where $M = \left[\sup_{t\in[0,1]} \left\{ W''(t) \right\}\right]^{-1}$. Numerically, we observed that this condition is also sufficient for the conserved potential in (46). In the simulations, we used $W(s) = \frac{1}{2}s^2(1-s)^2$.

The isotropic version of our splitting scheme has been studied in [13]. It is shown there that this scheme converges with the same rate as phase-field approximations based on a spatial discretization by finite differences or by finite elements. Its advantages are greater precision, and unconditional stability.

5.2 Test of convergence in dimension 2

We consider following anisotropic densities

$$
\begin{aligned}
\phi_1^o(\xi) &= \|\xi\|_{\ell^4} = (|\xi_1|^4 + |\xi_2|^4)^{\frac{1}{4}} \\
\phi_2^o(\xi) &= \|\xi\|_{\ell^{\frac{4}{3}}} = \left(|\xi_1|^{\frac{4}{3}} + |\xi_2|^{\frac{4}{3}}\right)^{\frac{3}{4}} \\
\phi_3^o(\xi) &= \left(|\xi_1|^{1{,}001} + |\tfrac{1}{2}\xi_1 + \tfrac{\sqrt{3}}{2}\xi_2|^{1{,}001} + |\tfrac{1}{2}\xi_1 - \tfrac{\sqrt{3}}{2}\xi_2|^{1{,}001}\right)^{\frac{1}{1{,}001}}.
\end{aligned}
$$

See figure (5.2) for a representation of their Wulff sets B_{ϕ_i} and Frank diagrams $B_{\phi_i^o}$.

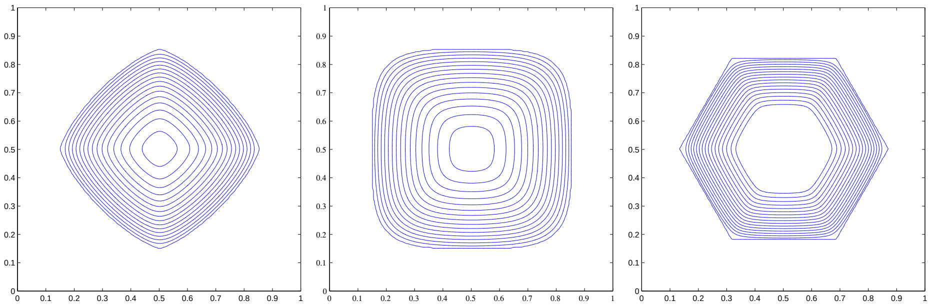

Figure 3: $\Omega(t)$ at different times for the anisotropic densities ϕ_1, ϕ_2, ϕ_3

1. *Evolution from a Wulff set.*
We consider the equation

$$\begin{cases} \partial_t u & = \tilde{\Delta}_\phi u - \frac{1}{\epsilon^2} W'(u) \\ u(0,x) & = q\left(dist(x, \Omega_0)/\epsilon\right), \end{cases}$$

where the initial set Ω_0 is a Wulff set of radius $R_0 = 0.25$

$$\Omega_0 = \left\{ x \in \mathbb{R}^2 \ ; \ \phi(x) \leq R_0 \right\}.$$

It is well known that the set $\Omega(t)$ obtained from Ω_0 through evolution by anisotropic mean curvature is a Wulff set with radius $R(t) = \sqrt{R_0^2 - 2t}$, which decreases to a point at the extinction time $t_{ext} = \frac{R_0^2}{2}$. In these simulations, the number of Fourier modes is $P = 2^8$, and the time step and phase-field parameter are chosen to be $\delta_t = 1/P^2$ and $\epsilon = 1/P$. On figure (5.2) the interface $\Omega(t)$ is plotted at different times. We observe a good agreement between the theoretical and computed curves, in spite of the smoothening of the corners of the latter.

2.*Convergence to the Wulff set*

This smoothening of corners actually depends on the thickness ϵ of the diffuse interface, as evidenced in the next series of tests, of evolution by anisotropic mean curvature under a volume constraint according to (46). The initial set Ω_0 is a circle centered at 0, of the same volume as $\Omega^* = \left\{ x \in \mathbb{R}^d \ ; \ \phi(x) < R_0 \right\}$. The evolution Ω_t from Ω_0 is expected to converge to the Wulff set Ω^*.

Figures 5.2-a,b represent the final sets Ω^*_ϵ obtained from the resolution of anisotropic Allen-Cahn equation, with respective anisotropic densities ϕ_1 and ϕ_2, and for dif-

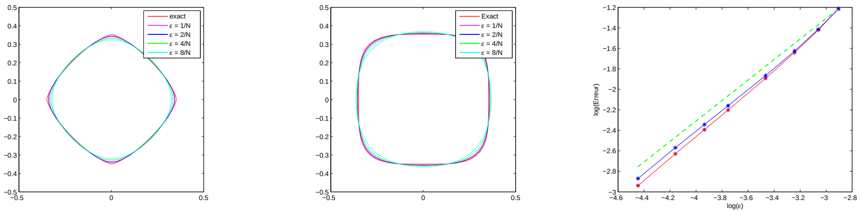

Figure 4: From left to right : $\Omega(t)$ at different times with anisotropy ϕ_1^o, $\Omega(t)$ at different times with anisotropy ϕ_2^o, error estimate $\epsilon \to \|\mathbb{1}_{B_\phi^\epsilon} - \mathbb{1}_{B_{\phi,R_0}}\|_{L_1(\mathbb{R}^d)}$ in logarithmic scale (ϕ_1^o in red and ϕ_2^o in blue)

ferents value of ϵ. We observe that the smaller ϵ, the better the approximation of the Wulff set. In figure 5.2-c, the L^1 error

$$\epsilon \to \|\mathbb{1}_{\Omega^*} - \mathbb{1}_{\Omega^*_\epsilon}\|_{L_1(\mathbb{R}^d)},$$

is plotted in a logarithmic scale. This graph indicates that this error is of order ϵ.

5.3 Some 3D simulations

As final illustrations, we consider the anisotropic densities

$$\begin{cases} \phi_4^o(\xi) & = \sqrt{\xi_1^2 + \xi_2^2} + |\xi_3| \\ \phi_5^o(\xi) & = |\xi_1| + |\xi_2| + |\xi_3| \end{cases}$$

The corresponding Wulff sets and Frank diagrams are plotted in figure (5).

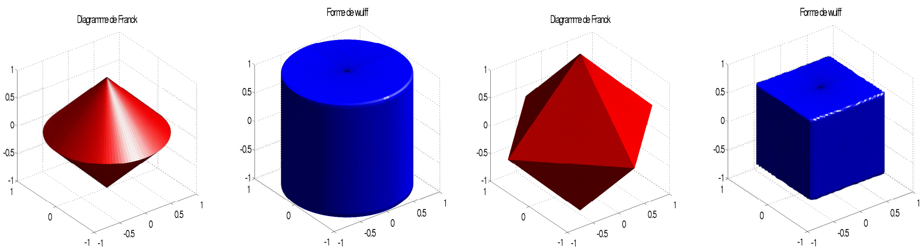

Figure 5: Frank diagram and Wulff set : $B_{\phi_4^o}$, B_{ϕ_4} , $B_{\phi_5^o}$, B_{ϕ_5}

We report in figure (6) (respectively in figure (7)) the evolution by ϕ_4^o (resp. ϕ_5^o) anisotropic mean curvature from an initial torus. The number of Fourier modes is $P = 2^7$, the time step and diffuse interface thickness are $\delta_t = 1/P^2$ and $\epsilon = 1/P$.

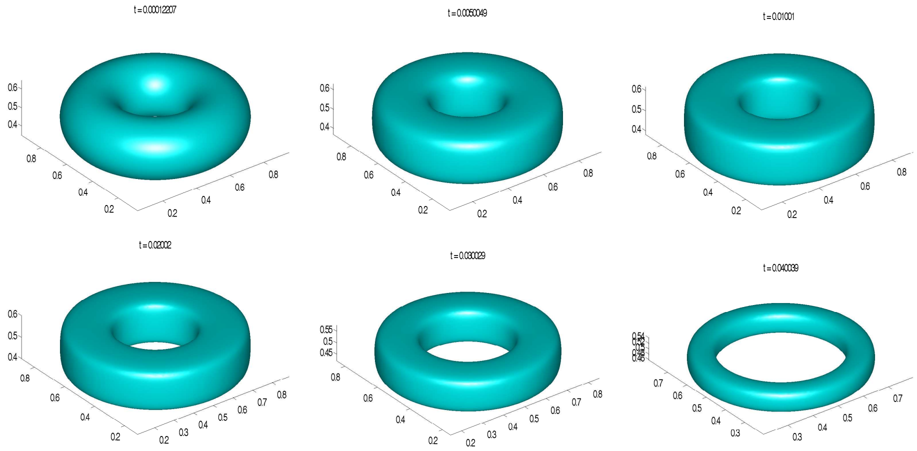

Figure 6: $\phi_4^o(\xi)$-evolution from an initial torus, at different times

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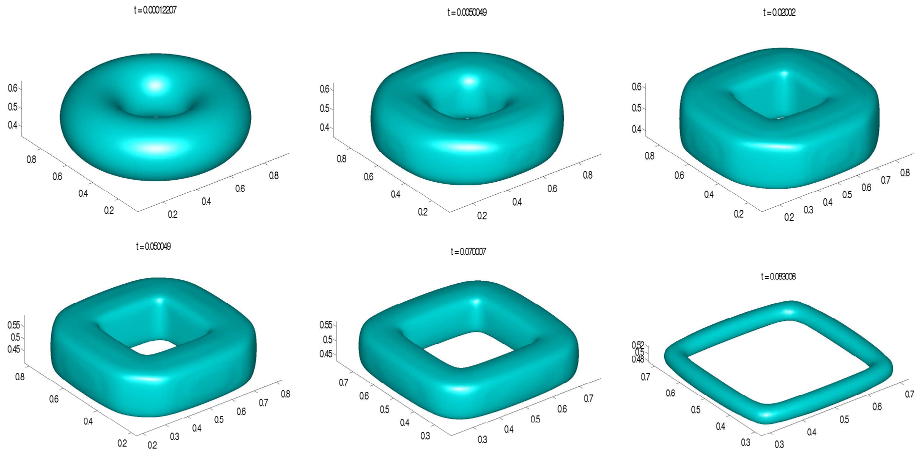

Figure 7: $\phi_5^o(\xi)$-evolution from an initial torus, at different times

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