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ENHANCED RESOLUTION IN STRUCTURED MEDIA*

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Abstract. The aim of this paper is to prove that we can achieve a resolution enhancement in detecting a target inclusion if it is surrounded by an appropriate structured medium. The physical notions of resolution and focal spot are revisited. Indeed, the resolution enhancement is estimated in terms of the material parameters of the structured medium.

Key words. imaging, resolution, superresolution, defect, periodic background structure

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1. Introduction.

In this paper, we consider the problem of imaging a target inclusion in a structured medium. We rigorously prove that the structured medium can improve the resolution in imaging the target from far-field measurements. Indeed, we precisely quantify the resolution enhancement in terms of the material parameters and the geometry of the structured medium.

Our mathematical analysis connects to a series of experiments recently performed by Lerosey et al. [25]. These experiments show the response to electromagnetic excitations with wavelength λ of a row of stick antennas placed on a metallic plane. The spacing between the antennas is $\lambda/30$. The group of antennas is surrounded by a network of thin conducting rods, which occupies a bounded region in space. The spacing between these thin scatterers is of the order of $\lambda/100$.

As one of the antennas radiates at frequency ω, the signal is recorded by an array of transducers, far from the region occupied by the antennas and the scatterers (at a distance of about 10λ). The signal is then time reversed and re-emitted by the transducers.

Were the scatterers not present, one would observe a refocalization of the time-reversed signal with a focal spot of size roughly $\lambda/2$, and thus one would not be able to distinguish which antenna was the original point of emission. In the presence of the scatterers, however, the results reported in [25] show that the time-reversed signal sharply refocuses on the originating antenna; i.e., one is able to distinguish between objects which are only $\lambda/30$ apart. In the authors' words, superresolution beyond the diffraction limit takes place.

Instead of trying to identify sources as in [25], we consider the problem of imaging a small target inclusion. In section 2, we define the resolution in mathematical terms by analyzing the response operator, which describes how the target affects the far-field.

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It is worth emphasizing that the notion of resolution is independent of the imaging functional or reconstruction procedure used to detect the target. The analysis of the response operator shows that, at leading order in terms of the target size, it is a projection on a particular eigenspace. Reconstruction methods such as time-reversal, back-propagation, and multiple signal classification (MUSIC) are known to be particularly efficient in that case [20, 16, 33, 17, 28, 3]. We only briefly mention this question in this paper, as we focus on the notion of resolution only.

To formulate the problem mathematically, we consider the Helmholtz equation in a bounded domain O in $\mathbb{R}^2$ as a model for the propagation of electromagnetic waves in a three-dimensional medium which is translation invariant in one direction (say the x_3 direction). In the transverse magnetic (TM) polarization, the Helmholtz equation describes the x_3 component of the magnetic field and has the form

$$\nabla \cdot (a(x)\nabla u(x)) + \omega^2 \mu(x) u(x) = 0 \qquad \text{in } O,$$

where $\omega = \frac{2\pi}{\lambda}$ denotes the frequency and the coefficients of the PDE

$$a(x) = \frac{1}{\varepsilon(x) + i\sigma(x)/\omega}$$

are related to the local material parameters $\varepsilon(x)$, $\sigma(x)$, and $\mu(x)$, which denote the electric permittivity, the conductivity, and the magnetic permittivity, respectively. Throughout the paper, we assume that the background medium (the reference medium) has constant dielectric parameters $\varepsilon_0 > 0$, $\sigma_0 = 0$, $\mu_0 > 0$.

Our target is a smooth dielectric inclusion D_δ, such that $|D_\delta| \to 0$ as $\delta \to 0$, centered at 0, with material parameters $\varepsilon_D > 0$, $\sigma_D = 0$, $\mu_D > 0$.

We fix a bounded region Ω, with $\overline{B(0,1)} \subset \Omega \subset O$, that contains the scatterers as well as the dielectric inclusion, $D_\delta \Subset \Omega \subset O$.

To define the scatterers, let B be a fixed open connected domain, with smooth boundary (say at least $\mathcal{C}^{1,\alpha}$, $\alpha > 0$) and diameter smaller than 1. Let $x_{\delta,j} = \delta j$, with $j \in \mathbb{Z}^2$, and let $\mathcal{S}_\delta$ denote the set of indices j such that $\delta B + x_{\delta,j} \subset \Omega$. For $j \in \mathcal{S}_\delta$, we set

$$B_{\delta,j} = \delta B + x_{\delta,j} = \{x \in \Omega \text{ such that } (x - x_j)\delta^{-1} \in B\};$$

see Figure 1.1. For simplicity, we assume that B is unchanged by a rotation of $\pi/2$. We assume that the scatterers are conducting inclusions, i.e., that their dielectric constants are

$$\mu_s > 0, \quad \varepsilon_s > 0, \quad \sigma_s \geq 0.$$

We can view D_δ as a defect in a periodic network of conducting inclusions within the region Ω, which is of size 1. We compare this medium to an ideal, defect-free medium. The ideal medium's material parameters are

$$(1.1) \qquad (a_\delta(x), \mu_\delta(x)) = \begin{cases} \left(\dfrac{1}{\varepsilon_s + i\sigma_s/\omega}, \mu_s\right) & \text{in } \displaystyle\bigcup_{j \in \mathcal{S}_\delta} B_{\delta,j}, \\ (\varepsilon_0^{-1}, \mu_0) & \text{otherwise.} \end{cases}$$

As for the true (defective) medium, its material parameters are

$$(1.2) \qquad (a_{\delta,d}(x), \mu_{\delta,d}(x)) = \begin{cases} (\varepsilon_D^{-1}, \mu_D) & \text{in } D_\delta, \\ \left(\dfrac{1}{\varepsilon_s + i\sigma_s/\omega}, \mu_s\right) & \text{in } \displaystyle\bigcup_{j \in \mathcal{S}_\delta} B_{\delta,j} \setminus \overline{D_\delta}, \\ (\varepsilon_0^{-1}, \mu_0) & \text{otherwise.} \end{cases}$$

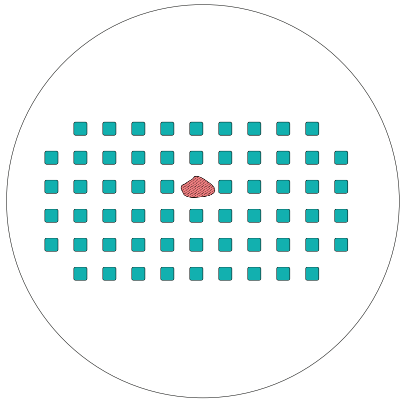

FIG. 1.1. *Sketch of the scatterers on a periodic grid together with the target inclusion.*

Assuming that a magnetic field ϕ is imposed on ∂O, the magnetic field $u_{\delta,d}$ solves

$$\nabla \cdot (a_{\delta,d}(x)\nabla u_{\delta,d}(x)) + \omega^2 \mu_{\delta,d}(x) u_{\delta,d}(x) = 0 \qquad \text{in } O, \tag{1.3}$$

$$u_{\delta,d} = \phi \qquad \text{on } \partial O. \tag{1.4}$$

We show in section 5 that the solutions u_δ to the Helmholtz equation in the defect-free medium are uniformly bounded and converge as δ tends to zero to the solution u_* of the homogenized equation

$$\nabla \cdot (A_*(x)\nabla u_*(x)) + \omega^2 \mu_*(x) u_*(x) = 0 \qquad \text{in } O, \tag{1.5}$$

$$u_* = \phi \qquad \text{on } \partial O. \tag{1.6}$$

The homogenized coefficients A_* and μ_* are given by (3.5) and (3.4). We note that because of the symmetry property of B (invariance under $\pi/2$-rotation), the homogenized tensor A_* is isotropic.

We proceed to derive an asymptotic formula for the difference response operator (the difference of Dirichlet-to-Neumann operators on ∂O):

$$r : H^{1/2}(\partial O) \to H^{-1/2}(\partial O),$$
$$\phi \mapsto \varepsilon_0^{-1} \frac{\partial}{\partial n} (u_{\delta,d} - u_\delta)\bigg|_{\partial O}.$$

As this operator can be recovered by polarization, we limit ourselves to the study of the quadratic form

$$R : H^{1/2}(\partial O) \to \mathbb{R},$$
$$\phi \mapsto \int_{\partial O} \varepsilon_0^{-1} \frac{\partial}{\partial n} (u_{\delta,d} - u_\delta)(s)\phi(s)ds.$$

Formulae for the trace of the difference $u_{\delta,d} - u_\delta$ have been used in the context of imaging by various authors [6, 15, 34, 14] (see in particular the books [4, 5]) as a means to detect inclusions of small volume from boundary measurements, in a known background reference medium. In this asymptotic, each inclusion contributes a term which is proportional to the gradient of the Green function of the reference medium (see section 2). In our situation, the size of the target inclusion is comparable to the scale of oscillations of the background medium, as in [10], where a three-dimensional conduction equation is studied. In this case, the asymptotics involve the Green function of the *homogenized* medium.

We show that, asymptotically, the response operator is given by

$$R(\phi) = \int_{D_\delta} M_* \nabla u_* \cdot \nabla u_* dx + \omega^2 \int_{D_\delta} \mu_* {u_*}^2 dx + o\left(|D_\delta|\right), \tag{1.7}$$

where M_* and μ_* are constant polarization terms that depend on the contrast in material constants, on the geometry of the inclusion, and on the geometry of the arrangement of scatterers, and where $o\left(|D_\delta|\right) / |D_\delta| \to 0$ uniformly for $\|\phi\|_{H^{1/2}(\partial O)} \leq 1$. The effective magnetic field u_* that appears in this formula is the solution to (1.5)–(1.6). Thus, the response from the medium perturbed by the scatterers is that of an effective medium. This is the main result of this paper. Its proof is given in section 3.

This result should be compared to the usual polarization result in a homogeneous medium [4, 34], where, asymptotically, the response takes the form

$$R(\phi) = \int_{D_\delta} M \nabla u_0 \cdot \nabla u_0 dx + \omega^2 \int_{D_\delta} \mu\, u_0^2 dx + o\left(|D_\delta|\right).$$

Here, u_0 is the background field. As shown in section 2, the resolution in this case is proportional to $k^{-1} = (\varepsilon_0 \mu_0)^{-1/2}$. In the case of a real medium containing periodically distributed small scatterers, the resolution is proportional to $k_*^{-1} = (\varepsilon_* \mu_*)^{-1/2}$ if the effective medium is isotropic. A proper choice for dielectric properties of the scatterers may thus guarantee that $k_*^{-1} \ll k^{-1}$ and the resolution is enhanced. The magnitude of this enhancement depends on the dielectric properties of the scatterers and on their distribution. How to maximize the resolution with appropriate choice of these parameters is the subject of a forthcoming work. Our result also applies to conducting scatterers: in that case, there is a trade-off between the resolution enhancement and the losses due to the energy dissipated in the medium. The size of the area where periodic scatterers are placed is thus also relevant. In a forthcoming work we also investigate how this size can be taken into account in the scatterers' parameter choices and we perform numerical experiments.

The paper is structured as follows. In section 2, we propose a definition for the notion of resolution when trying to detect an inclusion from far-field measurements. In the simple case of a real homogeneous medium, we highlight the dependence of the focal spot on the material properties of the background medium.

In section 3, we state our main result, namely, formula (1.7), for the leading order term in the response operator, when the target inclusion is surrounded by a set of periodically distributed, possibly complex, scatterers.

This formula is established on the basis of several intermediate results, which are proved in the following sections: section 5 addresses the question of uniform well-posedness with respect to the period δ of the scatterers. In section 6, we establish precise error estimates for the homogenization of (1.3). These results rely on fine regularity estimates in media with piecewise Hölder coefficients [26]. The next section establishes error estimates between the perturbed and unperturbed fields $u_{\delta,d}$ and u_δ. In section 8 we derive an asymptotic formula for $u_{\delta,d} - u_\delta$. Finally, symmetry and positivity properties of the polarization tensor M_* are established in section 9.

2. Resolution in a real homogeneous background medium.

This section is to introduce the notion of resolution. We consider the two-dimensional case. Similar results hold in three dimensions. In the context of anomaly detection, that is, a passive medium case, we consider the response operator on the boundary of a ball $B_1 = B(0,1)$, that is, the unit circle S_1. Assuming that the background is homogeneous and that the inclusion is a small ball or constant permeability and permittivity, $B_\delta = B(0,\delta)$, we remind the reader of the result of the explicit computation of the response operator.

We consider two functions $\phi, \psi \in L^2(S_1)$. The response operator is given by

$$R(\phi,\psi) = \int_{B_\delta} M_B \nabla u_\phi \cdot \nabla u_\psi dy + \omega^2 \int_{B_\delta} \frac{\mu_1 - \mu_0}{\mu_0} u_\phi u_\psi dy + o(r^2). \tag{2.1}$$

Here M_B is a scalar constant. The function u_ϕ (resp., u_ψ) satisfies

$$\begin{cases} \nabla \cdot (\epsilon_0^{-1} \nabla u_\phi) + \omega^2 \mu_0 u_\phi = 0 & \text{in } B_1, \\ u_\phi = \phi \text{ (resp., } \psi) & \text{on } S_1. \end{cases} \tag{2.2}$$

Writing ϕ and ψ in an appropriate basis of $L^2(S_1)$, we can compute u_ϕ and u_ψ using separation of variables.

Using polar coordinates, we decompose ϕ and ψ in the basis of trigonometric functions:

$$\phi = a_0 + \sum_{n\geq 1} a_n \frac{1}{\sqrt{\pi}} \cos(n\theta) + b_n \frac{1}{\sqrt{\pi}} \sin(n\theta),$$
$$\psi = \alpha_0 + \sum_{n\geq 1} \alpha_n \frac{1}{\sqrt{\pi}} \cos(n\theta) + \beta_n \frac{1}{\sqrt{\pi}} \sin(n\theta),$$

with $\sum_{n=0}^{\infty}(a_n^2 + b_n^2)$ and $\sum_{n=0}^{\infty}(\alpha_n^2 + \beta_n^2) < \infty$. Set $b_0 = \beta_0 = 0$. An elementary computation shows that

$$u_\phi = \sum_{n\geq 0} \frac{J_n(kr)}{J_n(k)} \left(a_n \frac{1}{\sqrt{\pi}} \cos(n\theta) + b_n \frac{1}{\sqrt{\pi}} \sin(n\theta) \right),$$
$$u_\psi = \sum_{n\geq 0} \frac{J_n(kr)}{J_n(k)} \left(\alpha_n \frac{1}{\sqrt{\pi}} \cos(n\theta) + \beta_n \frac{1}{\sqrt{\pi}} \sin(n\theta) \right),$$

where J_n is the nth Bessel function of the first kind and $k = \omega\sqrt{\epsilon_0 \mu_0}$.

Replacing these formulae in the response operator, we obtain that

$$R(\phi,\psi) = \sum_{n\geq 0} (a_n \alpha_n + b_n \beta_n) R_n + o(\delta^2),$$

$$R_n = \frac{1}{|J_n(k)|^2} \left(\int_0^{\delta k} M_B \left((J_n'(r))^2 + \frac{n^2}{r^2} (J_n(r))^2 \right) r dr + \frac{1}{\mu_0 \epsilon_0} \int_0^{\delta k} \frac{\mu_1 - \mu_0}{\mu_0} (J_n(r))^2 r dr \right).$$

It is well known (see, for instance, [1, Formula 9.5.10]) that, for all $0 \leq r \leq 1$ and all $m > 0$,

$$J_m(r) \leq C r^m \quad \text{and} \quad |J_m'(r)| \leq C r^{m-1}, \tag{2.3}$$

where the constant C is independent of r and m. Consequently, for all $n \geq 2$,

$$|R_n| \leq o\left(\delta^2\right).$$

Considering the actual formulae for $n = 0$ and $n = 1$, we find that

$$R_1 = C_1 M_B \delta^2 + o\left(\delta^2\right) \text{ and } R_0 = C_0 \frac{\mu_1 - \mu_0}{\mu_0}\delta^2 + o\left(\delta^2\right),$$

where $C_{0(1)}$ are constants independent of δ and are given by

$$C_1 = \frac{1}{4}\frac{k^2}{|J_1(k)|^2}, \quad C_0 = \frac{1}{2}\frac{\omega^2}{|J_0(k)|^2}.$$

We have recovered the classical result that, at first order in the size of the inclusion, the response has a finite-dimensional range. We can identify a permittivity response, corresponding to the terms where M_B appears, and a permeability response, corresponding to the terms where $\mu_0 - \mu_1$ appears.

Note that the response operator R as defined in (2.1) is a function of ϕ and ψ by means of u_ϕ and u_ψ, solutions to (2.2). Thus, with a small abuse of notation, we now write

$$R(u_\phi, u_\psi) = R(\phi, \psi).$$

We have shown that $R(u_\phi, \cdot)$ is given at first order by

$$R(u_\phi, \cdot) = \delta^2 c_0 \langle J_0(kr), \cdot\rangle.$$

Here, the constant c_0 is a function of ϕ independent of δ, and $\langle\cdot,\cdot\rangle$ denotes the scalar product on $L^2(B_1)$.

Similarly, for the permittivity response, we have shown that $R(u_\phi, \cdot)$ is given at first order by

$$R(u_\phi, \cdot) = \delta^2 c_1 \langle J_1(kr)e^{i\theta}, \cdot\rangle,$$

where the constant c_1 is a function of ϕ and independent of δ.

The Bessel function J_0 (resp., J_1) has one (resp., two) large central lobe and decays rapidly afterwards. We may therefore define the permittivity (resp., permeability) resolution as the size of the inverse image of the first lobe of J_0 (resp., J_1). The above expressions show that both permittivity and permeability resolutions are proportional to the wavelength k^{-1}. This is the Rayleigh resolution limit (see [3, sect. 8.3.2] and [25]).

In particular, we note that the resolution depends only on the dielectric properties of the medium surrounding the target (i.e., on k). The parameters of the inclusion δ, μ_1, and ε_1 modify its amplitude by a factor but leave its support unchanged.

This is the key observation. If the background solution u_0 were replaced by the effective solution u_* corresponding to ε_* and μ_*, this would modify the size of the first lobe, that is, the resolution, by changing $k = \omega\sqrt{\varepsilon_0\mu_0}$ to $k_* = \omega\sqrt{\varepsilon_*\mu_*}$. Increasing the effective permeability or the effective permittivity around the target, as in the experiment reported in [25], therefore increases the resolution. We refer the reader to [23, 24, 13] for additional references in connection to this question.

Note that the resolution defined above does not depend on the method used to solve the inverse problem itself. Alternatively, one can observe that the response operator is a projection on the lower eigenmodes of the background problem. Reconstruction methods such as time-reversal, back-propagation, and MUSIC [20, 16, 33, 17, 28, 3] take advantage of that fact very efficiently. This is not, however, the purpose of this work, which focuses on resolution enhancement.

3. Main result. Throughout the text we use the following notation: Ω denotes a bounded domain strictly contained in a smooth bounded domain $O \subset \mathbb{R}^2$. We assume that the closed unit ball $\overline{B(0,1)}$ lies strictly within Ω. The norm of the Sobolev space $W^{m,p}(X)$ is denoted by $||\cdot||_{m,p,X}$ for a bounded domain X. Similarly, the norm of $W^{m,p}(\partial X)$ is denoted by $||\cdot||_{m,p,\partial X}$.

We set $Y = [0,1]^2$. The space $H^1_{\#}(Y)$ is the subspace of functions in $H^1(Y)$ which are Y-periodic. The rescaled scatterer $B \Subset Y$ is a fixed open connected domain, with a smooth boundary: we assume that ∂B is at least $\mathcal{C}^{1,\alpha}$ for some $\alpha > 0$. For simplicity, we assume that B is symmetric with respect to a rotation by $\pi/2$.

We suppose that the inclusion D_δ is located well within Ω; that is, $D_\delta \subset B(0,1)$. For ease of notation, we will use here as before the convention that $B_r := B(0,r)$.

For $u \in H^1(O)$, we set

$$L_\delta u(x) = \nabla\cdot(a_\delta(x)\nabla u(x)) + \omega^2\mu_\delta(x)u(x),$$
$$L_{\delta,d}u(x) = \nabla\cdot(a_{\delta,d}(x)\nabla u(x)) + \omega^2\mu_{\delta,d}(x)u(x)$$

for the reference and the perturbed media, where a_δ and $a_{\delta,d}$ are given by (1.1) and (1.2), respectively. It is convenient to introduce the periodic functions a and μ given by

$$(a(y),\mu(y)) = \begin{cases} \left(\dfrac{1}{\varepsilon_s + i\sigma_s/\omega}, \mu_s\right) & \text{for } y \in B, \\ (\varepsilon_0^{-1}, \mu_0) & \text{for } y \in Y \setminus B, \end{cases} \tag{3.1}$$

so that

$$(a_\delta(x),\mu_\delta(x)) = \chi_\Omega(x)\left(a\left(\frac{x}{\delta}\right), \mu\left(\frac{x}{\delta}\right)\right) + (1-\chi_\Omega(x))\left(\varepsilon_0^{-1},\mu_0\right).$$

Given $\phi \in L(\partial O)$, u_δ and $u_{\delta,d}$ are the solutions in $H^1(O)$ to

$$L_\delta(u_\delta) = 0 \text{ in } O, \quad u_\delta = \phi \text{ on } \partial O, \tag{3.2}$$
$$L_{\delta,d}(u_{\delta,d}) = 0 \text{ in } O, \quad u_{\delta,d} = \phi \text{ on } \partial O. \tag{3.3}$$

The associated homogenized operator is L_*, given by

$$L_* u = \nabla\cdot(A_*(x)\nabla u(x)) + \omega^2\mu_*(x)u(x),$$

where μ_* is the effective permeability

$$\mu_*(x) = \begin{cases} \displaystyle\int_Y \mu(y) & \text{for } x \in \Omega, \\ \mu_0 & \text{for } x \in O \setminus \overline{\Omega}. \end{cases} \tag{3.4}$$

The matrix of homogenized coefficients A_* is defined by

$$(A_*)_{ij}(x) = \begin{cases} \displaystyle\int_Y a(y)(\delta_{ij} + \partial_j\chi_i(y))(y)dy & \text{for } x \in \Omega,\ 1 \le i,j \le 2, \\ \varepsilon_0^{-1}\delta_{ij} & \text{for } x \in O \setminus \overline{\Omega}, \end{cases} \tag{3.5}$$

in terms of the vector of corrector functions χ, the components of which solve the cell problems

$$\begin{cases} \nabla \cdot (a(y)\nabla(\chi_j(y) + y_j)) = 0 & \text{in } Y, \\ \chi_j \in H^1_{\#}(Y). \end{cases} \tag{3.6}$$

It is convenient to also introduce the corrector matrix $P := (\delta_{ij} + \frac{\partial \chi_i}{\partial x_j})_{1\le i,j\le 2}$.

Note that, because of the symmetry of B, it is easy to see that A_* is isotropic. The Bergman–Milton bounds (see, e.g., [30]) for composites with complex parameters ensure in this case that the homogenized problem is indeed elliptic.

Throughout the paper, we assume that

$$\begin{gathered} \omega^2 \text{ is not an eigenvalue of the eigenvalue problem} \\ -\nabla \cdot (A_* \nabla v) = \lambda \mu_* v \quad \text{in } O, \quad v \in H^1_0(O). \end{gathered} \tag{3.7}$$

Under this assumption, problems (3.2) and (3.3) are both well-posed for δ small enough. It is an easy consequence of the following proposition, which is proved in section 5.

Proposition 3.1. *Assume that* (3.7) *is satisfied. There exists* $\delta_0 > 0$ *such that, for all* $0 < \delta < \delta_0$ *and for all* F *in* $H^{-1}(O)$*, each of the problems*

$$L_{\delta,d}(v_{\delta,d}) = F \quad \textit{and} \quad L_\delta(v_\delta) = F \tag{3.8}$$

has a unique solution $v_{\delta,d} \in H^1_0(O)$ *and* $v_\delta \in H^1_0(O)$*, respectively. Furthermore,*

$$\|v_{\delta,d}\|_{1,2,O} + \|v_\delta\|_{1,2,O} \le C \|F\|_{-1,2,O}, \tag{3.9}$$

where the constant C *is independent of* D_δ *and* δ.

We can therefore introduce the difference response operator

$$\begin{aligned} r : H^{1/2}(\partial O) &\to H^{-1/2}(\partial O), \\ \phi &\to \varepsilon_0^{-1} \frac{\partial}{\partial n} (u_{\delta,d} - u_\delta)\Big|_{\partial O}. \end{aligned}$$

It is well known that r is uniquely determined by the quadratic form

$$\begin{aligned} R : H^{1/2}(\partial O) &\to \mathbb{R}, \\ \phi &\to \int_{\partial O} \varepsilon_0^{-1} \frac{\partial}{\partial n} (u_{\delta,d} - u_\delta)(s)\phi(s) ds. \end{aligned} \tag{3.10}$$

We can now state our main result.

Theorem 3.2. *Assume that the scatterers lie on a periodic uniform grid of size* δ*, where* δ *tends to zero with* $|D_\delta|$*. Then, the bilinear response form* R *given by* (3.10) *has the following asymptotic form:*

$$R(\phi) = \int_{D_\delta} M_* \nabla u_*(x) \cdot \nabla u_*(x) dx + \omega^2 \int_{D_\delta} \mu_* u_*^2(x) dx + o(|D_\delta|),$$

where u_* *denotes the solution to the homogenized problem*

$$L_* u_* = 0 \quad \textit{in } O, \quad \varepsilon_0^{-1} \frac{\partial u_*}{\partial n} = \phi \quad \textit{on } \partial O. \tag{3.11}$$

The constant matrix M_* *is symmetric and independent of* ϕ, ω, μ_1, *and* μ_0. *The constant* μ_* *is given by* $\mu_* := \frac{1}{|D_\delta|}\int_{D_\delta}(\mu_{\delta,d} - \mu_d)$.

REMARK 3.3. *Note that the result holds when the scale* δ *of the microstructure is comparable to the scale of the inclusion* D_δ. *It also holds when it is smaller, i.e.,* $\delta \ll |D_\delta|^{1/2}$. *This is to be expected, since when there is a separation of scales, the theory of homogenization is known to apply. It is worth mentioning that* δ *can also be larger, e.g.,* $\delta = |D_\delta|^\alpha$ *with* $0 < \alpha < 1/2$. *As highlighted in section 2, the exact formulae of* M_* *and* μ_* *are not of great interest as far as the enhancement of resolution is concerned. What really matters is the fact that they are (nonzero) constants. We also remark that if* M_* *is anisotropic, then the first eigenmodes of the response operator do not have radial symmetry and therefore the resolution depends on the direction of the background field.*

4. Proof of Theorem 3.2. The proof of Theorem 3.2 uses four main ingredients, which we list below and which are proved in sections 6–8.

• *The smoothness of* u_* *inside* Ω. Since A_* and μ_* are constant in Ω, u_* is smooth in B_1: for some constant $C > 0$ and for any $k \geq 0$, $||u_*||_{\mathcal{C}^k(B_1)} \leq C||\phi||_{1/2,2,\partial\Omega}$; see, e.g., [19, p. 314].

• *Pointwise uniform estimates on* $u_\delta - u_*$.

THEOREM 4.1. *Let* $\delta < \delta_0$, *where* δ_0 *is given by Proposition* 3.1. *The solutions* u_δ *and* u_* *to* (3.2) *and to* (3.11) *satisfy*

$$\left\{ \begin{array}{rl} ||u_\delta - u_*||_{0,\infty,D_\delta} \leq & o(1)\,||\phi||_{1/2,2,\partial O}, \\ ||\nabla u_\delta - P_\delta \nabla u_*||_{0,\infty,D_\delta} \leq & o(1)\,||\phi||_{1/2,2,\partial O}, \end{array} \right. \tag{4.1}$$

where the term $o(1)$ *tends to* 0 *as* $\delta \to 0$ *and where* $P_\delta := I + \nabla_y \chi(\cdot/\delta)$ *satisfies*

$$||P_\delta||_{0,\infty,D_\delta} \leq C$$

for a constant $C > 0$ *independent of* δ *and* ϕ.

The proof of this homogenization result is given in section 6. In particular, we note that it implies a uniform $W^{1,\infty}(D_\delta)$-bound for u_δ with respect to δ. This, in turn, allows the derivation of asymptotic estimates for the difference $u_{\delta,d} - u_\delta$, as proved in [14].

• *A convergence estimate for* $u_{\delta,d} - u_\delta$.

LEMMA 4.2. *Let* $\delta < \delta_0$, *where* δ_0 *is given by Proposition* 3.1. *Let* u_δ *and* $u_{\delta,d}$ *be solutions to* (3.2) *and* (3.3), *respectively. There exists a constant* $C > 0$, *independent of* δ, *such that*

$$||u_{\delta,d} - u_\delta||_{1,2,O} \leq C\, |D_\delta|^{1/2}\, ||u_\delta||_{1,\infty,D_\delta}. \tag{4.2}$$

Further, there exists $0 < \eta < 1/2$, *which depends only on the dielectric constants* $\varepsilon_0, \varepsilon_s, \varepsilon_\omega$, *on* k *and on* O, *such that*

$$||u_{\delta,d} - u_\delta||_{0,2,O} \leq C\, |D_\delta|^{1/2+\eta}\, ||u_\delta||_{1,\infty,D_\delta}. \tag{4.3}$$

This lemma is proved in section 7.

• *An asymptotic representation formula for* $\nabla(u_{\delta,d} - u_\delta)$, proved in section 8.

THEOREM 4.3. *Let* u_δ *and* $u_{\delta,d}$ *be solutions to* (3.2) *and* (3.3), *respectively. Then up to a subsequence, there exists a* 2×2 *matrix* P_M, *independent of* ϕ, *such that for any* $w \in \mathcal{C}^2(B_1)$

$$\int_{D_\delta} (a_{\delta,d} - a_\delta)\, P_\delta^T \nabla(u_{\delta,d} - u_\delta) \cdot \nabla w dx = \int_{D_\delta} P_M \nabla u_* \cdot \nabla w dx + o(|D_\delta|)\, \|w\|_{\mathcal{C}^2(D_\delta)},$$

where $o(|D_\delta|)/|D_\delta|$ *converges to zero uniformly for* $\|\phi\|_{1/2,2,\partial O} \leq 1$.

Proof of Theorem 3.2. A straightforward integration by parts shows that

$$R(\phi) = \int_{D_\delta} (a_{\delta,d} - a_\delta) \nabla u_{\delta,d} \cdot \nabla u_\delta dx - \omega^2 \int_{D_\delta} (\mu_{\delta,d} - \mu_\delta) u_{\delta,d} u_\delta dx, \tag{4.4}$$

which we rewrite in the form

$$\begin{aligned} R(\phi) &= \int_{D_\delta} (a_{\delta,d} - a_\delta) (P_\delta \nabla u_*) \cdot (P_\delta \nabla u_*) dx - \omega^2 \int_{D_\delta} (\mu_{\delta,d} - \mu_\delta) u_*^2 dx \\ &\quad + \int_{D_\delta} (a_{\delta,d} - a_\delta) \left(\varepsilon_{\delta,d}^{-1} - \varepsilon_d\right) \nabla (u_{\delta,d} - u_\delta) \cdot (P_\delta \nabla u_*) dx + r_\delta, \\ \text{with } r_\delta &= \int_{D_\delta} (a_{\delta,d} - a_\delta) [\nabla u_\delta \cdot \nabla u_\delta - (P_\delta \nabla u_*) \cdot (P_\delta \nabla u_*)] dx \\ &\quad - \omega^2 \int_{D_\delta} (\mu_{\delta,d} - \mu_\delta) \left((u_{\delta,d} - u_\delta) u_\delta + u_\delta^2 - u_*^2\right) dx \\ &\quad + \int_{D_\delta} (a_{\delta,d} - a_\delta) \nabla (u_{\delta,d} - u_\delta) \cdot (P_\delta \nabla u_* - \nabla u_\delta) dx. \end{aligned}$$

Using the uniform $W^{1,\infty}$-estimates (4.1) for the differences of u_* and u_δ and using (4.2) and (4.3) for the differences of $u_{\delta,d}$ and u_δ shows that

$$|r_\delta| \leq C \left(o(1)|D_\delta| + \omega^2 \left(|D_\delta|^{1/2+\eta} |D_\delta|^{1/2} + o(1)|D_\delta|\right) + o(1)|D_\delta|\right) ||\phi||_{1/2,2,\partial O}^2, \tag{4.5}$$

so that $|r_\delta| = o(|D_\delta|)$, uniformly for $||\phi||_{1/2,2,\partial O} \leq 1$. Thus, we obtain

$$\begin{aligned} R(\phi) &= \int_{D_\delta} (a_{\delta,d} - a_\delta) P_\delta^T P_\delta \nabla u_* \cdot \nabla u_* dx - \omega^2 \int_{D_\delta} (\mu_{\delta,d} - \mu_\delta) u_*^2 dx \\ &\quad + \int_{D_\delta} (a_{\delta,d} - a_\delta) P_\delta^T \nabla (u_{\delta,d} - u_\delta) \cdot \nabla u_* dx + o(|D_\delta|). \end{aligned}$$

Replacing the first two terms by their averages over D_δ and using Theorem 4.3 for the last term yields

$$R(\phi) = \int_{D_\delta} M_{D_\delta} \nabla u_* \cdot \nabla u_* dx - \omega^2 \int_{D_\delta} \mu_* u_*^2 dx + o(|D_\delta|),$$

with

$$M_D = |D_\delta|^{-1} \int_{D_\delta} (a_{\delta,d} - a_\delta) P_\delta^T P_\delta dx + P_M \tag{4.6}$$

and

$$\mu_* = |D_\delta|^{-1} \int_{D_\delta} (\mu_{\delta,d} - \mu_d) dx.$$

Replacing M_D by its limit, we arrive at

$$M_* = \lim_{|D_\delta| \to 0} |D_\delta|^{-1} \int_{D_\delta} (a_{\delta,d} - a_\delta) P_\delta^T P_\delta dx + P_M,$$

which concludes the proof of the theorem. For the properties of M_*, we refer the reader to section 9. $\square$

5. Well-posedness. Proof of Proposition 3.1. Let us start by checking that Proposition 3.1 does show that problems (3.2) and (3.3) are indeed well-posed. Let ψ be the canonical lift of the boundary condition, i.e., a function such that

$$\Delta\psi = 0 \text{ in } O, \quad \psi = \phi \text{ on } \partial O,$$

which satisfies by definition

$$\|\psi\|_{1,2,O} = \|\phi\|_{1/2,2,O}\,.$$

It follows that $v_\delta := u_\delta - \psi \in H_0^1(O)$ and $v_{\delta,d} := u_{\delta,d} - \psi \in H_0^1(O)$ solve

$$L_\delta(v_\delta) = F_\delta \quad \text{and} \quad L_{\delta,d}(v_{\delta,d}) = F_{\delta,d}$$

with

$$F_\delta = -L_\delta(\psi) \in H^{-1}(O) \quad \text{and} \quad F_{\delta,d} = -L_{\delta,d}(\psi) \in H^{-1}(O).$$

Therefore, Proposition 3.1 ensures well-posedness for $0 < \delta < \delta_0$. Note that this result is well known; see, e.g., [22] for a fixed δ provided that ω^2 is not in the spectrum of L_δ or $L_{\delta,d}$.

We recall that a family $(T_n)_{n\geq 1}$ of bounded linear operators from a Banach space B into itself is called collectively compact if the set $\{T_n(u),\ n \geq 1,\ ||u||_B \leq 1\}$ is precompact. Collectively compact sequences of operators satisfy the following property [7].

PROPOSITION 5.1. *Let $(T_n)_{n\geq 1}$ be a collectively compact family of bounded linear operators from a Banach space B into itself, which converge pointwise to an operator $T \in \mathcal{L}(B)$, and let $\lambda \in \mathbf{C}$. Then the following statements are equivalent:*

1. *$(\lambda - T)$ is an isomorphism.*
2. *There exists N such that, for $n \geq N$, the operator $(\lambda - T_n)$ is an isomorphism, and the operators $(\lambda - T_n)$ are uniformly norm bounded for $n \geq N$.*

Let $s = \omega^2 \max(\mu_\delta) + 1$. We define $\Lambda_\delta : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ by

$$\Lambda_\delta v = -L_\delta v + sv.$$

Since

$$\begin{aligned} Re\langle \Lambda_\delta v, v\rangle &= \int_O Re(a_\delta)|\nabla v|^2 + \int_O (s - \omega^2\mu)|v|^2 \\ &\geq \alpha \int_O |\nabla v|^2 + |v|^2, \end{aligned}$$

the operator Λ_δ is coercive, and the Lax–Milgram theorem shows that Λ_δ is invertible and that there exists a constant $C > 0$, independent of δ such that

$$\left\{ \begin{array}{ll} ||\Lambda_\delta||_{\mathcal{L}(H_0^1,H^{-1})} & \leq C, \\ ||\Lambda_\delta^{-1}||_{\mathcal{L}(H^{-1},H_0^1)} & \leq C. \end{array} \right. \tag{5.1}$$

Denoting by I the injection from $H_0^1(O)$ into $L^2(O)$, we see that a solution v_δ to (5.1) also satisfies

$$(I - sT_\delta)v_\delta \;=\; \Lambda_\delta^{-1}F, \tag{5.2}$$

where we view $T_\delta := s\Lambda_\delta^{-1} I$ as an operator from $L^2(O)$ into itself.

LEMMA 5.2. *Let δ_n be a sequence such that $\delta_n \to 0$. The sequence of operators $(T_{\delta_n})_{n>0}$ is collectively compact and converges pointwise to the operator $T = s\Lambda_*^{-1} I : L^2(O) \longrightarrow L^2(O)$, where $\Lambda_* : H_0^1(O) \longrightarrow H^{-1}(O)$ is the homogenized operator*

$$\Lambda_* = -L_* + sI.$$

Proof. To show the pointwise convergence, let $F \in L^2(O)$ and let $v_{\delta_n} \in H_0^1(O)$ denote the solution to

$$-\nabla \cdot (a_{\delta_n} \nabla v_{\delta_n}) + (s - \mu\omega^2) v_{\delta_n} = F.$$

The compactness theorem of homogenization [32] shows that v_{δ_n} converges weakly in $H_0^1(O)$ to $v_* \in H_0^1(O)$, the solution to the homogenized equation

$$-\nabla \cdot (A_* \nabla u_*) + (s - \omega\mu_*^2) v_* = F.$$

In other words, $\Lambda_{\delta_n}^{-1} F \rightharpoonup \Lambda_*^{-1} F$ in $H_0^1(O)$, so that $T_{\delta_n} F \longrightarrow TF$ strongly in $L^2(O)$.

Since Λ_* is coercive and since the injection $H_0^1(O) \longrightarrow L^2(O)$ is compact, one easily checks that T_{δ_n} and T are compact operators.

Finally, to verify collective compactness, consider a sequence $(T_{\delta_n} F_j)_{j\geq 1}$ with $||F_j||_{0,2,O} \leq 1$. Since the injection $L^2(O) \subset H^{-1}(O)$ is compact, we can assume that a subsequence (still denoted F_j) converges strongly in $H^{-1}(O)$ to some function F.

Assume first that $n_j \to \infty$. Applying again the compactness theorem of homogenization shows that

$$\Lambda_{\delta_{n_j}}^{-1} I F_j \longrightarrow \Lambda_*^{-1} F$$

weakly in $H_0^1(O)$. It follows that a subsequence of $(T_{\delta_{n_j}} F_j) = (s\Lambda_{\delta_{n_j}}^{-1} I F_j)$ converges (strongly) in $L^2(O)$.

If, on the other hand, n_j does not converge to ∞, then there must be a value n' of the indices that is repeated infinitely often. In this case, the compactness of the operator $\Lambda_{\delta_{n'}}^{-1} I$ shows that a subsequence of $(T_{\delta_{n_j}} F_j)_{j\ /\ n_j = n'}$ is convergent. ∎

LEMMA 5.3. *The operator $I - T$ is invertible.*

Proof. Since the matrix A in (3.5) is elliptic, $\Lambda_* : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is invertible. Furthermore, since

$$s^{-1}(I - T) = (s^{-1} + \Lambda_*^{-1} I) \ = \ s^{-1}\Lambda_*^{-1}(L_*),$$

the assumption (3.7) shows that $I - T$ is an isomorphism of $L^2(\Omega)$. ∎

Proof of Proposition 3.1. We infer from Lemmas 5.2 and 5.3 and Proposition 5.1 that there exist $\delta_0 > 0$ and $C > 0$ such that for $0 < \delta < \delta_0$ the operators $I - T_\delta$ are invertible and for any $f \in L^2(O)$

$$||(I - T_\delta)^{-1} f||_{0,2,O} \leq C\, ||f||_{0,2,O}. \tag{5.3}$$

It follows that, for $\delta < \delta_0$ and for $F \in H^{-1}(O)$, there exists a unique $w_\delta \in L^2(O)$ such that

$$(I - T_\delta w_\delta) = w_\delta + s\Lambda_\delta^{-1} I w_\delta = \Lambda_\delta^{-1} F.$$

From (5.3) and (5.1), we see that

$$||w_\delta||_{0,2,O} \leq C\, ||\Lambda_\delta^{-1} F||_{0,2,O} \ \leq \ C\, ||F||_{-1,2,O}. \tag{5.4}$$

Rewriting the above equation in the form

$$\Lambda_\delta w_\delta = F \ + \ s w_\delta$$

shows that w_δ in fact lies in $H^1_0(O)$ and the estimates in (5.1) and (5.4) yield

$$||w_\delta||_{1,2,O} \le C\,||F||_{-1,2,O}.$$

To complete the proof of Proposition 3.1, we can apply the same argument to (3.3). For $s > \max(\omega^2 \mu_{\delta,d}) + 1$, we can define

$$\Lambda_{\delta,d} u = -L_{\delta,d} u + s u.$$

The assumptions we made on the dielectric coefficients guarantee that $\Lambda_{\delta,d}$ is uniformly elliptic, and we can proceed as above. We need only check that $\Lambda_{\delta,d}$ also homogenizes to Λ_*. This follows from the fact that two sequences of second order elliptic operators have the same H-limits (i.e., limits in the sense of homogenization) provided that the difference of their coefficients tends to 0 in $L^1(\Omega)$; see [31, 18]. This is clearly the case here, since

$$\int_\Omega |a_{\delta,d} - a_\delta| = O(|D_\delta|). \qquad \square$$

6. Homogenization and estimates on $u_\delta - u_*$. The goal of this section is to prove Theorem 4.1. Our analysis is based on the pointwise regularity results for solutions to second order elliptic equations with *periodic* coefficients. Results of this nature were first obtained by Avellaneda and Lin [9] (see also [8]) in the case of smooth coefficients, precisely, under the assumption that the coefficients are $\mathcal{C}^{0,\beta}$ for some $\beta > 0$. These results were generalized by Li and Nirenberg with the help of a regularity theorem for elliptic equations and strongly elliptic systems with "piecewise Hölder coefficients" [26] (see also [27]).

We give a version of this regularity result adapted to our context. Recall that $B \subset Y$ has a $\mathcal{C}^{1,\alpha}$ boundary and that $\mathrm{dist}(B, \partial Y) > 0$. Let $0 < \lambda < \Lambda$, $0 < \beta$. Let $\mathcal{A}(\lambda, \Lambda, \beta, \alpha)$ denote the set of Y-periodic functions a which are $\mathcal{C}^{0,\beta}$ in $\overline{B}$ and in $\overline{Y \setminus B}$, and such that $0 < \lambda < Re(a(x)) < \Lambda$ in Y.

THEOREM 6.1 (see [26, Theorem 0.1]). *Assume that $\beta \le \frac{\alpha}{2(\alpha+1)}$ and that $a \in \mathcal{A}(\lambda, \Lambda, \beta, \alpha)$. Let $f \in L^\infty(Y)$, with $\int_Y f = 0$, and let h be a periodic function such that $h \in \mathcal{C}^{0,\beta}(\overline{B}) \cup \mathcal{C}^{0,\beta}(\overline{Y \setminus B})$. Let $u \in H^1_\#(Y)$ denote the solution to*

$$-\nabla \cdot (a(y) \nabla u(y)) = f + \nabla \cdot (h),$$

with $\int_Y u = 0$. Then

$$||u||_{\mathcal{C}^{1,\beta}(B)} + ||u||_{\mathcal{C}^{1,\beta}(Y \setminus B)} \le C\left(||f||_{0,\infty,Y} + ||h||_{\mathcal{C}^{1,\beta}(B)} + ||h||_{\mathcal{C}^{1,\beta}(Y\setminus B)}\right).$$

Li and Nirenberg realized that this result could be used to generalize Avellaneda and Lin's results to the case of systems with piecewise Hölder coefficients, and they obtained uniform $W^{1,\infty}$-estimates on u_δ. To prove error estimates, we need a slightly modified version of their result, which was proved in [10].

THEOREM 6.2 (see [10, Theorem 3.4]). *Assume that $\beta \le \frac{\alpha}{2(\alpha+1)}$ and that $a \in \mathcal{A}(\lambda, \Lambda, \beta, \alpha)$.*

Let b denote a Y-periodic function which is $\mathcal{C}^{0,\alpha}$ in $\overline{B}$ and in $\overline{Y \setminus B}$, and define $b_\delta(x) = b(x/\delta)$. Let $F \in L^\infty(B_1)$, $h \in \mathcal{C}^{0,\beta}(B_1)^2$, and let w_δ solve

$$-\nabla \cdot (a\,(x/\delta)\,\nabla w_\delta) = F + \delta \nabla \cdot (b_\delta h) \qquad \text{in } B_1.$$

Then

$$||w_\delta||_{\mathcal{C}^{0,\mu}(B_{1/2})} \;+\; ||\nabla w_\delta||_{0,\infty,B_{1/2}} \le C\,\left(||w_\delta||_{0,2,B_1} + ||F||_{0,\infty,B_1} + ||h||_{\mathcal{C}^{0,\mu}(B_1)}\right),$$

where the constant C is independent of δ, w, F, and h.

REMARK 6.3. *This result extends to the Helmholtz equation. Indeed, let v_δ denote a solution to $-L_\delta v_\delta = F + \delta\nabla \cdot (b_\delta h)$ in B_1, under the assumptions of Theorem* 6.2*. Since this equation rewrites*

$$-\nabla \cdot (a\,(x/\delta)\,\nabla v_\delta) = \tilde{F} + \delta \nabla \cdot (b_\delta h) \qquad \text{in } B_1,$$

with $\tilde{F} = F + \mu_\delta \omega^2 v_\delta \in L^2(B_1)$, the de Giorgi–Nash theorem [21, *Theorem* 8.24] *shows that $v_\delta \in L^\infty(B_{3/4})$ with*

$$||v_\delta||_{0,\infty,B_{3/4}} \le C\,\left(||F||_{0,2,B_1} + ||v_\delta||_{0,2,B_1} + \delta||h||_{\mathcal{C}^{0,\mu}(B_1)}\right),$$

where C depends only on the ellipticity constants of a_δ and on $||b_\delta||_\infty$ and thus is independent of δ. Consequently, $\tilde{F} \in L^\infty(B_{3/4})$ and one can apply Theorem 6.2 *(in $B_{3/4}$ instead) to obtain*

$$||u_\delta||_{\mathcal{C}^{0,\mu}(B_{1/2})} \;+\; ||\nabla w_\delta||_{0,\infty,B_{1/2}} \le C\,\left(||u_\delta||_{0,\infty,B_{3/4}} + ||F||_{0,\infty,B_{3/4}} + ||h||_{\mathcal{C}^{0,\mu}(B_1)}\right).$$

REMARK 6.4. *Theorem* 6.2 *is based on scaling invariance properties of solutions of an elliptic PDE with periodic coefficients. A careful examination of its proof (see* [10]*) shows that it also holds when the right-hand side has the form*

$$F + \delta\nabla \cdot (b_{1,\delta} h_{1,\delta} + b_{2,\delta} h_{2,\delta}),$$

where the functions b_i are Y-periodic and piecewise $\mathcal{C}^{0,\beta}$ and $h_i \in \mathcal{C}^{0,\beta}(B_1)$ as in the statement of Theorem 6.2*.*

We summarize these remarks in the following corrollary.

COROLLARY 6.5. *Under the hypothesis of Theorem* 6.2*, let w_δ solve*

$$-L_\delta w_\delta = F + \delta\nabla \cdot (b_\delta h) \qquad \text{in } B_1.$$

Then

$$||w_\delta||_{\mathcal{C}^{0,\mu}(B_{1/2})} \;+\; ||\nabla w_\delta||_{0,\infty,B_{1/2}} \le C\,\left(||w_\delta||_{0,2,B_1} + ||F||_{0,\infty,B_1} + ||h||_{\mathcal{C}^{0,\mu}(B_1)}\right),$$

where the constant C is independent of δ.

As a consequence of Corollary 6.5, one can obtain the following interior $W^{1,\infty}$-estimates on $u_\delta - u_*$ (see also [10, Theorem 3.6]).

THEOREM 6.6. *Let $F \in \mathcal{C}^\infty(\overline{B_1})$, and let v_δ and v_* satisfy, respectively,*

$$L_\delta v_\delta = F, \qquad L_* v_* = F \qquad \text{in } B_1.$$

Then

$$\left\{\begin{aligned} ||v_\delta - v_*||_{0,\infty,B_{1/2}} &\le C\,\left(||v_\delta - v_*||_{0,2,B_1} + \delta\right), \\ ||\nabla v_\delta - P_\delta \nabla v_*||_{0,\infty,B_{1/2}} &\le C\,\left(||v_\delta - v_*||_{0,2,B_1} + \delta\right), \end{aligned}\right. \tag{6.1}$$

where the constant C depends on F but is independent of δ.

Proof. We first note that since $F \in \mathcal{C}^\infty(\overline{B_1})$, the function v_* is smooth inside Ω, and thus

$$||D^k v_*||_{0,\infty,B_1} \leq C(F) \qquad \text{for } k \leq 3.$$

For $1 \leq i, j \leq 2$, let ϕ_{ij} denote the $H^1_\#(Y)$ solution to

$$\begin{cases} -\nabla \cdot (a_1(y)\nabla\phi_{ij}(y)) = B_{ij}(y) - \displaystyle\int_Y B_{ij}(y)\, dy & \text{in } Y, \\ \displaystyle\int_Y \phi(y)\, dy = 0, \end{cases} \tag{6.2}$$

with $B_{ij}(y) = a(y)(\delta_{ij} + \partial_i\chi_j) + \nabla \cdot (a(y)e_i \otimes e_j\chi)$. The vectors e_1, e_2 denote the canonical basis vectors in $\mathbb{R}^2$, and χ is given by (3.6). Further, let τ be the $H^1_\#(Y)$ solution to

$$\begin{cases} -\nabla \cdot (a_1(y)\nabla\tau(y)) = \mu_\delta(y) - \mu_* & \text{in } Y, \\ \displaystyle\int_Y \tau(y)\, dy = 0. \end{cases} \tag{6.3}$$

It follows from Theorem 6.1 that $\chi \in \mathcal{C}^{1,\beta}(\overline{B}) \cup \mathcal{C}^{1,\beta}(\overline{Y \setminus B})$ for some $0 < \beta \leq 1$, so that invoking Theorem 6.1 again, we see that $\phi_{ij} \in \mathcal{C}^{1,\beta}(\overline{D_\delta}) \cup \mathcal{C}^{1,\beta}(\overline{Y \setminus D_\delta})$. Similarly, $\tau \in \mathcal{C}^{1,\beta}(\overline{D_\delta}) \cup \mathcal{C}^{1,\beta}(\overline{Y \setminus D_\delta})$. Let

$$z_\delta = v_\delta - \left(v_* + \delta\chi_i\left(\frac{x}{\delta}\right)\frac{\partial v_*}{\partial x_j} + \delta^2\phi_{ij}\left(\frac{x}{\delta}\right)\frac{\partial^2 v_*}{\partial x_i \partial x_j} + \delta^2\tau\left(\frac{x}{\delta}\right) v_* \right). \tag{6.4}$$

Denoting $\chi_\delta = \chi(x/\delta)$, $\phi_\delta = (\phi_{ij}(x/\delta))$, and $\tau_\delta = \tau(x/\delta)$, we compute

$$\begin{aligned}
&-\nabla \cdot (a_\delta \nabla z_\delta) - \omega^2\mu_\delta z_\delta \\
&= \nabla \cdot (a_\delta(I - A_*)\nabla v_*) + \omega^2(\mu_\delta - \mu_*)v_* \\
&\quad + \nabla \cdot \left(a_\delta \nabla\left[\delta\chi_\delta\nabla v_* + \delta^2\phi_\delta : \nabla^2 v_* + \delta^2\tau_\delta v_*\right]\right) \\
&\quad + \omega^2\mu_\delta\left(\delta\chi_\delta\nabla v_* + \delta^2\phi_\delta : \nabla^2 v_* + \delta^2\tau_\delta v_*\right) \\
&= \frac{1}{\delta}\nabla_y \cdot [a(I + \nabla_y\chi) - A] \cdot \nabla v_* + \nabla_y \cdot (a\nabla_y\tau)v_* + \omega^2(\mu_\delta - \mu_*)v_* \\
&\quad + [a(I + \nabla_y\chi) - A + \nabla_y(a\chi) + \nabla_y(a\nabla_y\phi)] : \nabla^2 v_* \\
&\quad + \delta\left(a_\delta\chi_\delta\nabla^3 v_* + a_\delta\nabla_y\phi\nabla^3 v_* + a_\delta\nabla_y\tau \cdot \nabla v_*\right) \\
&\quad + \delta^2\nabla \cdot \left(a_\delta\phi_\delta\nabla^3 v_* + a_\delta\tau_\delta\nabla v_*\right) + \omega^2\mu_\delta\left(\delta\chi_\delta\nabla v_* + \delta^2\phi_\delta : \nabla^2 v_* + \delta^2\tau_\delta v_*\right).
\end{aligned}$$

Recalling (6.2) and (6.3), and recalling that $\int_Y B(y) = A$, the above relation reduces to

$$-\nabla \cdot (a_\delta\nabla z_\delta) - kz_\delta = \delta\left[F_\delta + \delta\nabla \cdot (b_{1,\delta}H_{1,\delta} + b_{2,\delta}H_{2,\delta})\right],$$

where

$$\begin{aligned}
F_\delta(x) &= a_\delta(x)\chi_\delta(x)\nabla^3 u_*(x) + a_\delta(x)(\nabla_y\phi)\left(\frac{x}{\delta}\right)\nabla^3 u_*(x) + a_\delta(x)(\nabla_y\tau)\left(\frac{x}{\delta}\right) \cdot \nabla u_*(x) \\
&\quad + \omega^2\mu_\delta\left(\chi_\delta(x) \cdot \nabla u_*(x) + \delta\phi_\delta(x) : \nabla^2 u_*(x) + \delta\tau_\delta(x)u_*(x)\right), \\
b_{1,\delta}(y) &= a(y)\phi(y), \qquad H_{1,\delta}(x) = \nabla^3 u_*(x), \\
b_{2,\delta}(y) &= a(y)\tau(y), \qquad H_{2,\delta}(x) = \nabla u_*(x).
\end{aligned}$$

Because of the uniform $W^{1,\infty}$-estimates on χ, ϕ, and τ, Corollary 6.5 applies and yields

$$\begin{aligned}&||z_\delta||_{0,\infty,B_{1/2}} + ||\nabla z_\delta||_{0,\infty,B_{1/2}}\\ &\leq C\,\left(\delta\left[||F_\delta||_{0,\infty,B_1} + ||H_{1,\delta}||_{\mathcal{C}^\beta(B_1)} + ||H_{2,\delta}||_{\mathcal{C}^\beta(B_1)}\right] + ||z_\delta||_{0,2,B_1}\right).\end{aligned}$$

Consequently, we arrive at

$$\begin{aligned}||v_\delta - v_*||_{1,\infty,B_{1/2}} \leq C\,\big(&||\delta\chi_\delta\cdot\nabla v_* + \delta^2\phi_\delta : \nabla^2 v_* + \delta^2\tau_\delta v_*||_{1,\infty,B_{1/2}}\\ &+ \delta\left[||F_\delta||_{0,\infty,B_1} + ||H_{1,\delta}||_{\mathcal{C}^\beta(B_1)} + ||H_{2,\delta}||_{\mathcal{C}^\beta(B_1)}\right] + ||z_\delta||_{0,2,B_1}\big),\end{aligned}$$

from which the estimates (6.1) follow. ∎

Obtaining interior $W^{1,\infty}$-estimates on $u_\delta - u_*$ then reduces to controlling the L^2-norm of $u_\delta - u_*$ on a larger set. When the data in (3.2) is sufficiently smooth one can actually estimate this L^2-norm in terms of δ. This is the aim of the following proposition, which conclude the proof of Theorem 4.1.

PROPOSITION 6.7. *Assume that $F \in L^2(O)$ and $f \in H^{1/2}(\partial O)$. Let u_δ and u_* satisfy* (3.2) *and* (3.11)*, respectively. Then*

$$||u_\delta - u_*||_{0,2,O} \leq o(1)\,(||F||_{0,2,O} + ||f||_{1/2,2,\partial O}), \tag{6.5}$$

where $o(1) \to 0$ as $\delta \to 0$.

Proof. By Proposition 3.1, (u_δ) is uniformly bounded in $H^1(O)$. One can extract a subsequence (u_{δ_n}) that converges to some u_∞, weakly in $H^1(O)$ and strongly in $L^2(O)$. We note that u_{δ_n} solves

$$-\nabla\cdot(a_{\delta_n}\nabla u_{\delta_n}) = F_{\delta_n} \quad \text{in } O,$$

where $F_{\delta_n} := F + \mu_{\delta_n} u_{\delta_n}$. It is easily checked that $\mu_{\delta_n} u_{\delta_n}$ converges weakly in $L^2(O)$, and thus strongly in $H^{-1}(O)$. Classical homogenization results [11] then imply that u_∞ is the solution to (3.11), i.e., that $u_\infty \equiv u_*$. Moreover, uniqueness of the limit shows that the whole sequence (u_δ) converges strongly to u_* in $L^2(O)$. ∎

7. Proof of Lemma 4.2.

Proof of estimate (4.2). Using (3.2) and (3.3), we see that $u_{\delta,d} - u_\delta \in H_0^1(O)$ and

$$L_{\delta,d}(u_{\delta,d} - u_\delta) = F \text{ in } H^{-1}(O),$$

with $F = \nabla\cdot((a_{\delta,d} - a_\delta)\nabla u_\delta) + \omega^2(\mu_{\delta,d} - \mu_\delta)u_\delta$. It is apparent that

$$\|F\|_{-1,2,O} \leq \left(\|a_{\delta,d} - a_\delta\|_{0,\infty,D_\delta} + \omega^2\|\mu_{\delta,d} - \mu_\delta\|_{0,\infty,D_\delta}\right)\|u_\delta\|_{1,\infty,D_\delta}|D_\delta|^{1/2},$$

which implies the first estimate (4.2), thanks to the uniform estimate (3.9) of Proposition 3.1. Additionally, we obtain

$$\|u_{\delta,d} - u_\delta\|_{0,2,O} \leq C\,\|u_d\|_{1,\infty,D_\delta}|D_\delta|^{1/2}. \tag{7.1}$$ ∎

For the refined L^2-estimate of the difference $u_{\delta,d} - u_\delta$, we rely on a theorem of Meyers [29, 11].

THEOREM 7.1. *Assume that O is an open set in $\mathbb{R}^d$ with a $\mathcal{C}^2$ boundary. Let $\alpha \in L^\infty(\Omega)^d$ be such that*

$$0 \leq \lambda|\xi|^2 \leq Re(\alpha(x)\xi\cdot\xi) \leq \Lambda|\xi|^2 \qquad a.e.\ x \in O\ \forall\,\xi \in \mathbb{R}^d,$$

for some constants λ, Λ. *Let* $F \in H^{-1}(O)$, *and let* $u \in H_0^1(O)$ *denote the unique solution to*

$$-\nabla \cdot (\alpha \nabla u) = F \quad \text{in } O.$$

Then there exists a number $p_M > 2$ *such that for any* $2 \le p \le p_M$, *there exists* $C > 0$ *which depends only on* $\Omega, \lambda, \Lambda, p, n$ *such that if* $F \in W^{-1,p}(O)$, *then* $u \in W_0^{1,p}(O)$ *and satisfies*

$$||u||_{1,p,O} \le C \, ||F||_{-1,p,O}. \tag{7.2}$$

REMARK 7.2. *Meyers's theorem is usually stated with* $p = p_M$ *only. The extension to* $2 \le p \le p_M$ *is easily obtained via interpolation* [12].

Proof of (4.3). Fix $\delta < \delta_0$ and consider the solution to the adjoint problem

$$\begin{cases} L_\delta w = \overline{(u_{\delta,d} - u_\delta)} & \text{in } O, \\ \quad w \in H_0^1(O). \end{cases} \tag{7.3}$$

The uniform estimates (3.9) shows that

$$||w||_{0,2,O} \le C \, ||u_{\delta,d} - u_\delta||_{0,2,O}. \tag{7.4}$$

We further note that the equation for w can be rewritten as

$$\nabla \cdot (a_\delta \nabla w) = \overline{(u_{\delta,d} - u_\delta)} + \omega^2 \mu_\delta w. \tag{7.5}$$

We choose $2 < p < p_M$ such that $q^* > 2$, where

$$q^* := \frac{2q}{2-q} \qquad \text{with } \frac{1}{p} + \frac{1}{q} \; = \; 1$$

(which is always possible as $q \to 2^-$ when $p \to 2^+$, and $q^* \to +\infty$ as $q \to 2^-$). By the Sobolev embedding theorem, the injection $W^{1,q}(O) \subset L^2(O)$ is continuous, so that $L^2(O) \subset W^{-1,p}(O)$. Thus, the right-hand side of (7.5) lies in $W^{-1,p}(O)$, and Meyers's theorem implies that $w \in W_0^{1,p}(O)$ and that

$$||w||_{1,p,O} \le C \; (||u_{\delta,d} - u_\delta||_{0,2,O} \; + \; ||w||_{0,2,O}) \tag{7.6}$$

for some constant $C > 0$ independent of δ.

Multiplying (7.3) by $u_{\delta,d} - u_\delta$ and using (3.2)–(3.3), we obtain

$$\begin{aligned} \int_O |u_{\delta,d} - u_\delta|^2 &= \int_O a_\delta \nabla (u_{\delta,d} - u_\delta) \cdot \nabla w + \mu_\delta (u_{\delta,d} - u_\delta) w \\ &= \int_{D_\delta} (a_{\delta,d} - a_\delta) \nabla u_{\delta,d} \cdot \nabla w. \end{aligned}$$

Thus, it follows from (7.6) that

$$\begin{aligned} \int_O |u_{\delta,d} - u_\delta|^2 &\le C \left(\int_{D_\delta} |\nabla u_\delta|^q \right)^{1/q} \left(\int_{D_\delta} |\nabla w|^p \right)^{1/p} \\ &\le C \left(\int_{D_\delta} |\nabla u_\delta|^q \right)^{1/q} ||u_{\delta,d} - u_\delta||_{0,2,O}. \end{aligned}$$

We note that in the above expression the exponent q satisfies $1 < q = (1-1/p)^{-1} < 2$, so that

$$\begin{aligned}||\nabla u_{\delta,d}||_{0,q,D_\delta} &\le ||\nabla(u_{\delta,d}-u_\delta)||_{0,q,D_\delta} \ + \ ||\nabla u_\delta||_{0,q,D_\delta} \\ &\le \left(\int_{D_\delta} 1\right)^{1/q-1/2} ||u_{\delta,d}-u_\delta||_{0,2,O} \ + \ \left(\int_{D_\delta} 1\right)^{1/q} ||u_\delta||_{1,\infty,D_\delta} \\ &\le C \ \|u\|_{1,\infty,D_\delta} \left[\left(\int_{\omega_\delta} 1\right)^{1/q-1/2} |\omega_\delta|^{1/2} + |\omega_\delta|^{1/q}\right],\end{aligned}$$

where we have used (4.2) and (7.1). Since $1/q > 1/2$, choosing $\eta = 1/q - 1/2$ proves (4.3). ◻

8. Asymptotics of $u_{\delta,d} - u_\delta$. This section is devoted to the proof of Theorem 4.3. We follow the method introduced in [14]. To simplify the notation, we introduce $U_{\delta,d} := u_{\delta,d} - u_\delta$, the difference of the solutions to (3.3) and (3.2). Given two functions $\psi_i \in H^{1/2}(\partial O)$, $i \in \{1,2\}$, we also consider $V^i_{\delta,d} := v^i_{\delta,d} - v^i_d$, where $v^i_{\delta,d}$ and v^i_d are the respective solutions to (3.3) and (3.2) with respective boundary data

$$v_{i,\delta,d} = v_{i,\delta} = \psi_i \quad \text{on } \partial O.$$

As $\delta \to 0$, v^i_δ converges to v^i_*, the solution to

$$\nabla \cdot \left(a_* \nabla v^i_*\right) = 0 \text{ in } O, \quad v^i_* = \psi^i \text{ on } \partial O.$$

We choose ψ^i such that $v^i_*\big|_\Omega = x^i$. Applying Theorem 4.1 and Lemma 4.2 yields

$$\left\|v^i_\delta\right\|_{1,\infty,B_1} \le C, \quad \left\|\nabla v^i_\delta - P_\delta e_i\right\|_{0,2,B_1} \le o(1), \tag{8.1}$$

$$\left\|V^i_{\delta,d}\right\|_{1,2,O} \le C\ |D_\delta|^{1/2} \text{ and } \left\|V^i_{\delta,d}\right\|_{0,2,O} \le C\,|D_\delta|^{1/2+\eta}, \tag{8.2}$$

where the constant $0 < C$ is independent of δ and $0 < \eta < 1/2$ is given by Lemma 4.2. Let us check that $||\nabla V^i_{\delta,d}||_{0,1,\Omega}$ is uniformly bounded. Applying the Cauchy–Schwarz inequality yields

$$\left|\frac{1}{|D_\delta|}\int_{D_\delta} \nabla V^i_{\delta,d}\right| \le \frac{1}{|D_\delta|}\, C\ |D_\delta|^{1/2}\ \ ||V^i_{\delta,d}||_{1,2,\Omega} \le C, \tag{8.3}$$

thanks to (8.2). Recalling that $P_\delta = O(1)$, we may therefore extract a subsequence $(D_{\delta_n})_{n\ge 1} \to 0$ such that

$$|D_{\delta_n}|^{-1}\, 1_{D_{\delta_n}}\, dx \longrightarrow \delta_0, \tag{8.4}$$

$$|D_{\delta_n}|^{-1}\, 1_{D_{\delta_n}}\, (a_\delta - a_{d,\delta_n}) \left(P^T_{\delta_n} \nabla V^j_{\delta_n,d}\right)_i dx \longrightarrow d\mathcal{M}_{ij}, \tag{8.5}$$

where δ_0 denotes the Dirac mass at $x = 0$ and where $\mathcal{M}_{i,j}$, $1 \le i,j \le 2$, is a Borel measure with support in $\overline{\Omega}'$. The above convergence results hold in the weak* topology of $(\mathcal{C}^0(\overline{\Omega}))'$. Following [14], we see that, for any $f \in \mathcal{C}^0(\overline{\Omega})$,

$$\begin{aligned}\left|\int_\Omega f\, d\mathcal{M}_{ij}\right| &= \left|\lim_{n\to\infty} |D_{\delta_n}|^{-1} \int_{D_{\delta_n}} (a_\delta - a_{d,\delta_n}) \left(P^T_{\delta_n} \nabla V^j_{\delta_n,d}\right)_i f dx\right| \\ &\le \liminf_{n\to\infty} \frac{||\,(a_\delta - a_{d,\delta_n})\, P_{\delta_n}||_{0,\infty,D_{\delta_n}}}{\sqrt{|D_{\delta_n}|}} ||\nabla V^j_{\delta_n,d}||_{0,2,\Omega'} \left(\frac{1}{|D_{\delta_n}|}\int_{D_{\delta_n}} |f|^2\right)^{1/2}.\end{aligned}$$

Since $|D_{\delta_n}|^{-1}\int_{D_{\delta_n}}|f|^2 \to |f(0)|^2$, we conclude, using (8.2), that

$$\left|\int_\Omega f\,d\mathcal{M}_{ij}\right| \le C\,|f(0)|\,. \tag{8.6}$$

It follows that $d\mathcal{M}_{ij}$ is absolutely continuous with respect to δ_0 and, thus, there exists a 2×2 matrix P_M such that

$$\int_\Omega \phi d\mathcal{M}_{ij} = (P_M)_{ij}\, f(0).$$

The following lemma concludes the proof of Theorem 4.3.

LEMMA 8.1. *For all $f \in \mathcal{C}^1_c(B_1)$,*

$$|D_{\delta_n}|^{-1}\int_{D_{\delta_n}} P_\delta^T \nabla U_{\delta_n,d} f\,dx = |D_{\delta_n}|\int_{D_{\delta_n}} P_M \nabla u_* f\,dx + o(1)\,\|f\|_{\mathcal{C}^1(B_1)}\,,$$

where $o(1)$ converges to zero as $|D_{\delta_n}|$ tends to zero uniformly for $||\phi||_{1/2,2,\partial O}\le 1$.

Proof. To simplify the exposition, we henceforth drop the index n. Also, in the rest of this section, the notation "err" denotes various error terms, which are explicitly estimated, any time they are used.

Given any $f\in\mathcal{C}^1_c(B_1)$, we write $N_f = \|f\|_{\mathcal{C}^1(B_1)}$. For $i=1,2$,

$$\int_O a_{\delta,d}\nabla U_{\delta,d}\cdot\nabla\left(V^i_{\delta,d} f\right)dx = \int_O a_{\delta,d}\nabla\left(U_{\delta,d} f\right)\cdot\nabla V^i_{\delta,d}dx + \mathrm{err},$$

where

$$\begin{aligned}|\mathrm{err}| &\le \|a_{\delta,d}\|_{0,\infty,\Omega}\left(\|\nabla U_{\delta,d}\|_{0,2,\Omega}\left\|V^i_{\delta,d}\right\|_{0,2,\Omega} + \|\nabla U_{\delta,d}\|_{0,2,\Omega}\left\|V^i_{\delta,d}\right\|_{0,2,\Omega}\right)N_f\\ &\le C\,|D_\delta|^{1+\eta}\,|\phi||_{1/2,2,\partial O}N_f,\end{aligned}$$

from the estimates given by Lemma 4.2 for $U_{\delta,d}$ and from (8.2) for $V_{\delta,d}$.

We note that $U_{\delta,d}\in H^1_0(O)$, $V^i_{\delta,d}\in H^1_0(O)$ and that they satisfy

$$\begin{aligned}L_{\delta,d}\left(U_{\delta,d}\right) &= \nabla\cdot\left(\left(a_\delta - a_{\delta,d}\right)\nabla u_d\right) + \omega^2\left(\mu_d-\mu_{\delta,d}\right)u_d,\\ \nabla\cdot\left(a_{\delta,d}\nabla V^i_{\delta,d}\right) &= \nabla\cdot\left(\left(a_\delta - a_{\delta,d}\right)\nabla v^i_d\right).\end{aligned} \tag{8.7}$$

Thus, on the one hand we have

$$\begin{aligned}\int_O a_{\delta,d}\nabla U_{\delta,d}\cdot\nabla\left(V^i_{\delta,d}f\right)dx &= \int_{D_\delta}\left(a_\delta - a_{\delta,d}\right)\nabla u_d\cdot\nabla\left(V^i_{\delta,d}f\right)dx\\ &\quad -\int_{D_\delta}\omega^2\left(\mu_d-\mu_{\delta,d}\right)u_d V^i_{\delta,d} f\,dx\\ &= \int_{D_\delta}\left(a_\delta - a_{\delta,d}\right)\nabla u_d\cdot\nabla V^i_{\delta,d} f\,dx + \mathrm{err},\end{aligned}$$

with

$$\begin{aligned}|\mathrm{err}| &\le C\left(\|u_d\|_{0,2,\Omega}\left\|V^i_{\delta,d}\right\|_{0,2,\Omega} + \|\nabla u_d\|_{0,2,\Omega}\left\|V^i_{\delta,d}\right\|_{0,2,\Omega}\right)N_f\\ &\le C\,|D_\delta|^{1+\eta}\,|\phi||_{1/2,2,\partial O}N_f,\end{aligned}$$

and on the other hand

$$\int_O a_{\delta,d} \nabla V^i_{\delta,d} \cdot \nabla \left(U_{\delta,d} f\right) dx = \int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) \nabla v^i_d \cdot \nabla \left(U_{\delta,d} f\right) dx$$
$$= \int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) \nabla v^i_d \cdot \nabla U_{\delta,d} f dx + \mathrm{err},$$

with

$$|\mathrm{err}| \leq C \left\|\nabla v^i_d\right\|_{0,2,\Omega} \left\|U_{\delta,d}\right\|_{0,2,\Omega} N_f$$
$$\leq C \left|D_\delta\right|^{1+\eta} |\phi||_{1/2,2,\partial O} N_f.$$

We therefore obtain

$$\int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) \nabla v^i_d \cdot \nabla U_{\delta,d} f dx = \int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) \nabla u_d \cdot \nabla V^i_{\delta,d} f dx + \mathrm{err},$$

with

$$|\mathrm{err}| \leq C \left|D_\delta\right|^{1+\eta} |\phi||_{1/2,2,\partial O} \|f\|_{\mathcal{C}^1(\Omega')} .$$

Let us now remark that (8.1) indicates that ∇v^i_d can be approximated by $P_\delta e_i$, whereas (4.1) indicates that ∇u_d can be approximated by $P_\delta \nabla u_*$. Namely, these estimates show that

$$\int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) \nabla v^i_d \cdot \nabla U_{\delta,d} f dx = \int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) P_\delta e_i \cdot \nabla U_{\delta,d} f dx + \mathrm{err},$$

with

$$|\mathrm{err}| \leq C \left\|\nabla v^i_\delta - P_\delta e_i\right\|_{0,\infty,D_\delta} |D|^{1/2} \left\|\nabla U_{\delta,d}\right\|_{0,2,\Omega} N_f,$$
$$\leq C\, o(1)\ \left|D_\delta\right| \|\phi\|_{1/2,2,\partial O} N_f,$$

and symmetrically that

$$\int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) \nabla u_d \cdot \nabla V^i_{\delta,d} f dx = \int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) P_\delta \nabla u_* \cdot \nabla V^i_{\delta,d} f dx + \mathrm{err},$$

with

$$|\mathrm{err}| \leq C \left\|\nabla u_\delta - P_\delta u_*\right\|_{0,\infty,D_\delta} \left|D_\delta\right|^{1/2} \left\|\nabla V^i_{\delta,d}\right\|_{0,2,\Omega} N_f,$$
$$\leq C\, o(1)\ \left|D_\delta\right| \|\phi\|_{1/2,2,\partial O} N_f.$$

It follows that

(8.8)
$$\left|D_\delta\right|^{-1} \int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) P_\delta e_i \cdot \nabla U_{\delta,d} f dx = \left|D_\delta\right|^{-1} \int_{D_\delta} \left(a_\delta - a_{\delta,d}\right) P_\delta \nabla u_* \cdot \nabla V^i_{\delta,d} f dx + \mathrm{err},$$

with

$$|\mathrm{err}| \leq C \left(o(1)\ + \left|D_\delta\right|^\eta\right) \|\phi\|_{1/2,2,\partial O} N_f.$$

Since u_* is smooth inside B_1, the right-hand side can be reformulated as

$$\begin{aligned} |D_\delta|^{-1}\int_{D_\delta}(a_\delta-a_{\delta,d})\,P_\delta\nabla u_*\cdot\nabla V^i_{\delta,d}f\,dx &= |D_\delta|^{-1}\int_{D_\delta}(a_\delta-a_{\delta,d})\,P_\delta^T\nabla V^i_{\delta,d}\cdot\nabla u_* f\,dx \\ &= P_M\nabla u_*(0)f(0)+\mathrm{err} \\ &= |D_\delta|^{-1}\int_{D_\delta}(P_M\nabla u_*)\,f\,dx+\mathrm{err}, \end{aligned}$$

with

$$\mathrm{err}=o(1)\,N_f\,\|u_*\|_{\mathcal{C}^1(B_1)},$$

where $\lim_{|D_\delta|\to 0}o(1)=0$. Invoking the regularity of u_*, the term $\|u_*\|_{\mathcal{C}^1(B_1)}$ can be controlled by $C\,\|\phi\|_{1/2,2,\partial O}$, which completes the proof. $\square$

9. Properties of the polarization tensor. In this section, we focus on some properties of the polarization tensor M_*. We first clarify the definition of M_*. Note that in the absence of a background periodic structure, the polarization tensor M_* corresponds to the generalization of the polarization tensor arising in electrical impedance tomography to the case of a complex conductivity. The properties of that tensor are of independent interest and will be the subject of a future work. Here, we mostly focus on the question of the interaction with the periodic background in two dimensions.

PROPOSITION 9.1. *The polarization tensor M_* is a constant symmetric matrix, given by*

(9.1)
$$M_*\xi\cdot\xi=\lim_{|D_\delta|\to 0}\left(|D_\delta|^{-1}\int_{D_\delta}(a_{\delta,d}-a_\delta)\,P_\delta\xi\cdot P_\delta\xi dx+|D_\delta|^{-1}\int_O a_{\delta,d}\nabla V_{\delta,d}\cdot\nabla V_{\delta,d}\,dx\right),$$

where $V_{\delta,d}\in H_0^1(\Omega)$ is the solution of

$$\nabla\cdot(a_{\delta,d}\nabla V_{\delta,d})=\nabla\cdot((a_\delta-a_{\delta,d})\,P_\delta\xi)\,. \tag{9.2}$$

Proof. Collecting the various elements introduced previously, we have defined so far the polarization tensor M_* by the formula (4.6):

$$\begin{aligned} M_* &= \lim_{|D_\delta|\to 0}\left(|D_\delta|^{-1}\int_{D_\delta}(a_{\delta,d}-a_\delta)\,P_\delta^T P_\delta dx\right)+P_M \\ &= |D_\delta|^{-1}\int_{D_\delta}(a_{\delta,d}-a_\delta)\,P_\delta^T P_\delta dx+P_M+o(1), \end{aligned}$$

where P_M is in turn defined in (8.5) by

$$P_M\xi\cdot\zeta f(0)=\lim_{|D_\delta|\to 0}|D_\delta|^{-1}\int_{D_\delta}(a_\delta-a_{\delta,d})\,\nabla(V_{\delta,d}\xi)\cdot P_\delta\zeta f(x)dx$$

for all ξ,ζ in $\mathbb{R}^2$ and all $f\in\mathcal{C}_c^0(B_1)$, with the convention that $V_{\delta,d}\xi:=V^1_{\delta,d}\xi^1+V^2_{\delta,d}\xi^2$, and where $V^i_{\delta,d}\in H^1_0(O)$ satisfy (8.7). Introducing similarly $V_{\delta,d}\zeta=V^1_{\delta,d}\zeta^1+V^2_{\delta,d}\zeta^2$ and testing (8.7) against $V_{\delta,d}\xi$, we obtain

$$\int_O a_{\delta,d}\nabla V_{\delta,d}\zeta\cdot\nabla V_{\delta,d}\xi dx=\int_{D_\delta}(a_\delta-a_{\delta,d})\,\nabla\left(\sum_{i=1}^2 v^i_d\zeta^i\right)\cdot\nabla V_{\delta,d}.$$

Approximating $\sum_{i=1}^{2} v_d^i \zeta^i$ by $P_\delta \zeta$ we obtain

$$\int_O a_{\delta,d} \nabla V_{\delta,d}\zeta \cdot \nabla V_{\delta,d}\xi dx = P_M \xi\zeta + \text{err},$$

with

$$\begin{aligned} |\text{err}| &\leq \|a_\delta - a_{\delta,d}\|_{0,\infty,\Omega} |D_\delta|^{1/2} \|\nabla V_{\delta,d}\|_{0,2,D_\delta} \left\| \sum_{i=1}^{2} v_d^i \zeta^i - P_\delta \zeta \right\|_{0,\infty,D_\delta} \\ &\leq C |D_\delta| o(1). \end{aligned}$$

We have therefore obtained that

$$M_* \xi \cdot \zeta = |D_\delta|^{-1} \int_{D_\delta} (a_{\delta,d} - a_\delta) P_\delta \xi \cdot P_\delta \zeta dx + |D_\delta|^{-1} \int_O a_{\delta,d} \nabla V_{\delta,d}\xi \cdot \nabla V_{\delta,d}\zeta \, dx + o(1),$$

where $o(1)$ tends to zero as $|D_\delta|$ (and δ) tend to zero. Under this form, it is very clear that M_* is symmetric. Comparing (8.7) and (9.2) we see that

$$\nabla \cdot (a_{\delta,d} \nabla (V_{\delta,d} - V_{\delta,d}\xi)) = \nabla \cdot \left((a_\delta - a_{\delta,d}) \left(P_\delta \xi - \sum_{i=1}^{2} v_d^i \zeta^i \right) \right),$$

and the right-hand side is small, namely,

$$\left\| (a_\delta - a_{\delta,d}) \left(P_\delta \xi - \sum_{i=1}^{2} v_d^i \zeta^i \right) \right\|_{-1,2,O} \leq C |D_\delta| o(1) = o(|D_\delta|).$$

Hence, we conclude from Proposition 3.1 that

$$\frac{1}{|D_\delta|} \|V_{\delta,d} - V_{\delta,d}\xi\|_{1,2,O} = o(1),$$

and, in turn, we have proved (9.1). ∎

To conclude this section, we turn to positivity properties of the polarization term M^*.

PROPOSITION 9.2. *Assume that a_δ and $a_{\delta,d}$ are real valued.*

If $a_\delta \geq a_{\delta,d}$ a.e. in D_δ and $a_\delta - a_{\delta,d} > c_0 > 0$ on $\tilde{D}_\delta \subset D_\delta$ with $|\tilde{D}_\delta| > 0$ for some constant c_0, then M^* is positive definite.

If $a_{\delta,d} \geq a_\delta$ a.e. in D_δ and $a_{\delta,d} - a_\delta > c_0 > 0$ on $\tilde{D}_\delta \subset D_\delta$ with $|\tilde{D}_\delta| > 0$ for some constant c_0, then M^* is negative definite.

Proof. Let us suppose that $a_{d,\delta}$ and a_d are real. Testing (9.2) against $V_{\delta,d}$ we obtain

$$\begin{aligned} \int_O a_{\delta,d} \nabla V_{\delta,d} \cdot \nabla V_{\delta,d} &= \int_{D_\delta} (a_\delta - a_{\delta,d}) P_\delta \xi \cdot \nabla V_{\delta,d} \\ &\leq \left(\int_{D_\delta} (a_\delta - a_{\delta,d})^2 \frac{1}{a_{\delta,d}} P_\delta \xi \cdot P_\delta \xi \right)^{1/2} \left(\int_{D_\delta} a_{\delta,d} V_{\delta,d} \cdot V_{\delta,d} \right)^{1/2}, \end{aligned}$$

which shows that

$$\int_O a_{\delta,d} \nabla V_{\delta,d} \cdot \nabla V_{\delta,d} \leq \int_{D_\delta} (a_\delta - a_{\delta,d})^2 \frac{1}{a_{\delta,d}} P_\delta \xi \cdot P_\delta \xi.$$

We have obtained that M_* satisfies two bounds, namely,

$$|D|^{-1} \int_D (a_{\delta,d} - a_\delta) P_\delta^T P_\delta \leq M_* + o(1) \tag{9.3}$$

and

$$M_* \leq |D_\delta|^{-1} \int_{D_\delta} (a_{\delta,d} - a_\delta) \frac{a_\delta}{a_{\delta,d}} P_\delta^T P_\delta + o(1). \tag{9.4}$$

If $a_{\delta,d} \geq a_\delta$ a.e. in D_δ and $a_{\delta,d} - a_\delta > c_0 > 0$ on $\tilde{D_\delta}$, with $|\tilde{D_\delta}| > 0$, then the bound (9.3) yields

$$M_* \geq \min\left(\frac{a_{\delta,d}}{a_\delta} - 1\right) |D_\delta|^{-1} \int_{\tilde{D_\delta}} a_\delta P_\delta^T P_\delta.$$

If $[-\delta/2, \delta/2]^2 \subset \tilde{D_\delta}$, that is, if (at least) one unit cell is contained in $\tilde{D_\delta}$, then we find

$$\begin{aligned} M_* &\geq \min_{\tilde{D_\delta}} \left(\frac{a_{\delta,d}}{a_\delta} - 1\right) |D_\delta|^{-1} \int_{\delta Y} a_\delta P_\delta^T P_\delta \\ &\geq \min_{\tilde{D_\delta}} \left(\frac{a_{\delta,d}}{a_\delta} - 1\right) \frac{\delta^2}{|D_\delta|} a^*. \end{aligned}$$

When the grid size is larger than D_δ, or $\tilde{D_\delta}$, a result of Alessandrini and Nesi [2, Theorems 2 and 6] ensures that for every $Y' \subset Y$

$$\log(\det P) \in BMO\,(Y')\,.$$

Rewriting (9.3) we therefore obtain

$$M_* \geq \min_{\tilde{D_\delta}} (a_{\delta,d} - a_\delta) |D_\delta|^{-1} \int_{\tilde{D_\delta}} P_\delta^T P_\delta > 0,$$

which is our claim. The case when $a_{\delta,d} \leq a_\delta$ a.e. in D_δ and $a_\delta - a_{\delta,d} > c_0 > 0$ on $\tilde{D_\delta}$ is similar, using (9.4) instead of (9.3). $\square$

10. Concluding remarks. We first defined a notion of resolution limit for anomaly detection. We then showed that the presence of a microstructure around the anomaly modifies this resolution limit.

Our results up to section 9 are valid in the three-dimensional case. In section 9, we rely on the result of Alessandrini and Nesi [2] which is known to be false in three dimensions. We do not know if this result could be extended to the complex case in two dimensions.

For the sake of simplicity, we have considered a periodic microstructure. Our arguments rely on only two key ingredients, namely, H-convergence and local elliptic regularity. Therefore, our approach can be extended to the general case when the scatterers have a smooth boundary, but the microstructure has a limit in the sense of H-convergence, provided that the local regularity property still holds.

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