Fall 2020 **Partial Exam** M2

# Representation theory

2020.11.16, 9:30-13:00

**Exercise 1.** Let $A$ and $B$ be $k$-algebras. Let $M$ be a $A$-$B$-bimodule, let $N$ be a right $B$-module and $P$ be a left $A$-module.

1. Show that

$$\begin{array}{rcl} F_{M,P} : \mathrm{Hom}_{B^{\mathrm{op}}}(M,N) \otimes_A P & \longrightarrow & \mathrm{Hom}_{B^{\mathrm{op}}}(\mathrm{Hom}_A(P,M),N) \\ \varphi \otimes p & \longmapsto & (f \mapsto \varphi \circ f(p)) \end{array}$$

   is a well defined $k$-linear map.

2. Show that the map $F_{M,P}$ is functorial in $M$ and $P$.
3. Show that if $N$ is injective and $P$ is finitely presented, then $F_{M,P}$ is an isomorphism.

For $M \in \mathrm{Mod}\, A$ we denote by $M^{\wedge} := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \in \mathrm{Mod}\, A^{\mathrm{op}}$ the Pontrijagin dual of $M$.

4. Let $X \to Y$ be a $A$-linear map. Show that it is surjective if and only if $X^{\wedge} \to Y^{\wedge}$ is injective.
5. Deduce that any finitely presented flat module is projective.

**Exercise 2.** Let $A$ be a $k$-algebra.

1. Let $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ be a short exact sequence of $A$-modules. Assume that there are two short exact sequences

$$0 \longrightarrow P_1 \xrightarrow{f} P_0 \xrightarrow{f'} Z \longrightarrow 0 \qquad 0 \longrightarrow Q_1 \xrightarrow{g} Q_0 \xrightarrow{g'} X \longrightarrow 0$$

   with $P_0, P_1, Q_0, Q_1$ projective modules.

   (a) Show that there exists a surjective map $P_0 \oplus Q_0 \to Y$.

(b) Show that there exists a map $h : P_1 \to Q_0$ such that the sequence

$$0 \longrightarrow P_1 \oplus Q_1 \xrightarrow{\begin{pmatrix} g & 0 \\ -h & f \end{pmatrix}} P_0 \oplus Q_0 \longrightarrow Y \longrightarrow 0$$

is exact.

2. Let $k$ be field, and $Q$ be a quiver without oriented cycles. For $i \in Q_0$ a vertex, denote by $S_i$ the 1-dimensional $KQ$-module associated to vertex $i$.

(a) For any $i \in Q_0$, show that there is a short exact sequence

$$0 \to \bigoplus_{a \in Q_1, s(a)=i} kQe_{t(a)} \to kQe_i \to S_i \to 0$$

(b) Deduce that if $M$ is a finite dimensional $kQ$-module, then there exists a short exact sequence of the form

$$0 \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

with $P_0$ and $P_1$ projective.

(c) Describe such a sequence for $Q$ given by the following quiver

$$4 \longleftarrow 3 \longleftarrow 2 \longrightarrow 1$$

and $M$ given by the following representation

$$0 \longleftarrow k \xleftarrow[1]{} k \longrightarrow 0$$

**Exercise 3.** Let $A$ be a $k$-algebra.

For $X$ and $Z$ in $\operatorname{Mod} A$, we denote by $\mathcal{E}xt^1_A(Z, X)$ the set of $(Y, u, v)$ where $Y$ is in $\operatorname{Mod} A$, and $u : X \to Y$ and $v : Y \to Z$ are $A$-linear maps such that

$$0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$$

is a short exact sequence. We define on $\mathcal{E}xt^1_A(Z, X)$ the following equivalence relation $(Y, u, v) \sim (Y', u', v')$ if there exists an isomorphism $\varphi : Y \to Y'$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow \varphi & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z & \longrightarrow & 0 \end{array}$$

We denote by $\operatorname{Ext}^1_A(Z, X)$ the set of equivalences classes.

1. Show that the set of split short exact sequences form a class in $\mathrm{Ext}^1_A(Z,X)$ that we will denote by $\epsilon_{ZX}$.

2. What can we say about the set $\mathrm{Ext}^1_A(Z,X)$ if $Z$ is projective ?

3. Let $0 \longrightarrow K \xrightarrow{i} P \longrightarrow Z \longrightarrow 0$ be a short exact sequence. We define a map $\delta_X : \mathrm{Hom}_A(K,X) \to \mathrm{Ext}^1_A(Z,X)$ as follows. If $f : K \to X$ be a $A$-linear map, $\delta_X(f)$ is defined to be the class of a short exact sequence defined by the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & P & \longrightarrow & Z & \longrightarrow & 0\,, \\ & & \downarrow{\scriptstyle f} & & \downarrow & & \| & & \\ 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

   where the left square is a push-out.

   Show that $\delta_X$ is well-defined.

4. Show that the composition

$$\mathrm{Hom}_A(P,X) \xrightarrow{\mathrm{Hom}_A(i,X)} \mathrm{Hom}_A(K,X) \xrightarrow{\delta_X} \mathrm{Ext}^1_A(Z,X)$$

   is the constant map to $\epsilon_{ZX}$.

5. Show that if $f, f' \in \mathrm{Hom}_A(K,X)$ satisfies $\delta_X(f) = \delta_X(f')$, then $f - f'$ is in the image of $\mathrm{Hom}_A(i,X)$.

6. Deduce that if $P$ is projective, then $\mathrm{Ext}^1_A(Z,X)$ is in natural bijection (via $\delta_X$) with the cokernel $\mathrm{Hom}_A(i,X)$ and that it induces a structure of $k$-module on $\mathrm{Ext}^1_A(Z,X)$ for which $\epsilon_{ZX}$ is the zero element.