Representation theory

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Contents

Chapter I

Generalities on Modules

All rings are associative and unital.

1 Modules and algebras

1.1 Module over a ring

Definition 1.1 Let R a ring. A left R-module M is an abelian group together with a map $R \times M \to M$ sending (a, m) to $a.m$ satisfying for any $a, a' \in A$, $m \in M$

- $(a + a').m = a.m + a'.m$;
- $a.(m + m') = a.m + a.m'$;
- $(aa').m = a.(a'.m)$;
- $1_A.m = m$.

Example 1.2 $R = k$ field then k-vector space. $R = \mathbb{Z}$ then abelian group.
R is an R-module over itself.
If M is an abelian group, then M is a left $\mathrm{End}_{\mathbb{Z}}(M)$-module.

Representation point of view.

Proposition 1.3
Let M be an abelian group. Then M is a left R-module if and only if there exists a ring homomorphism $\rho : R \to \mathrm{End}(M)$.

Difference between left and right modules.

If R is commutative, then a left R-module is a right R-module.

A left R-module is a right R^{op}-module.

Definition 1.4 Let R and R' be rings. A R-R'-bimodule M is an abelian group M which is a left R-module, a right R'-module, and such that $(a.m).b = a.(m.b)$ for any $a \in R$, $m \in M$ and $b \in R'$.

If R is commutative, then a R-module is automatically a R-bimodule.

Definition 1.5 Let M and N be R-modules. A morphism of R-modules (or a R-linear application) is a morphism of abelian groups such that $f : M \to N$ such that $f(xm) = xf(m)$

1.2 Algebras

In all what follows, k will be a commutative ring.

Definition 1.6 A k-algebra is a unital ring with a structure of k-module such that $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for $\lambda \in k$ and $a, b \in A$.

Example 1.7 Typical examples in this course: $k = \mathbb{Z}$ or k is a field. The algebra A could be $\mathcal{M}_n(k)$, $k[X]$, $k[X,Y]$, $\mathcal{T}_n^+(k) \subset \mathcal{M}_n(k)$ upper triangular matrices. $A = kG$ where G is a group.

A morphism of k-algebras is a map $f : A \to B$ which is a ring morphism and a k-module morphism.

Notions of subalgebras and ideals (left, right and two-sided).

Remark 1.8 A ring is always a $\mathbb{Z}$-algebra.

The map $k \to A$ sending λ to $\lambda 1_A$ is a k-algebra morphism whose image is in the center of A.

1.3 Modules over algebras

Definition 1.9 A left A-module M is a left A-module thinking of A as a ring.

Because of the map $k \to A$, a A-module is automoatically a k-module. And we have $(\lambda a)m = \lambda(am) = a(\lambda m)$. (Scalars commute with everything).

So in other words, any ring map $A \to \text{End}(M)$ factors through a k-algebra map $A \to \text{End}_k(M)$.

Proposition 1.10
Let G be a group, and k be a field. Let M be a k-vector space. It has a structure of kG-modules if and only if there exists a group morphism $\rho : G \to \mathrm{Aut}(M)$. (ρ, M) is called a representation of the group G.

1.4 Submodules, quotients and direct sums

Submodules

Definition 1.11 Let M be a left A-module. A submodule $N \subset M$ is a subgroup which is stable under A-multiplication.

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For example, the submodules of A seen as a left A-module are the left ideals of A.

Quotient

Proposition 1.12
Let M be a A-module, and $N \subset M$ a submodule, then M/N has a natural structure of A-module, and the projection $M \to M/N$ is A-linear.

If N, N' are submodules of M, so are $N + N'$ and $N \cap N'$.

Submodules and morphisms

Proposition 1.13
Let $f : M \to N$ ba a morphism of A-modules, then $\mathrm{Ker} f$ and $\mathrm{Im}\, f$ are A-modules.

If $M' \subset M$ is a submodule. Then there exists a unique $\bar{f} : M/M' \to N$ such that $\bar{f} \circ p = f$ if and only if $M' \subset \mathrm{Ker} f$.

In particular f induces an isomorphism $M/\mathrm{Ker} f \simeq \mathrm{Im}\, f$.

Definition 1.14 For $f : M \to N$ a morphism, we define $\mathrm{Coker} f := N/\mathrm{Im}\, f$ the cokernel of f. It is a A-module.

Direct sum

Proposition 1.15
Let M and N be A-modules. Then $M \times N$ has naturally a structure of A-module.

We denote it as $M \oplus N$ (external direct sum).

Note that if M_1 and M_2 are submodules of M, such that $M_1 \cap M_2 = \{0\}$, then $M_1 + M_2 \simeq M_1 \oplus M_2$. (so internal direct sums coincide with external ones).

If M and N are modules, M is naturally isomorphic to a submodule of $M \oplus N$ and its quotient is isomorphic to N. However, if $N \subset M$ is a submodule, M is not isomorphic in general to $N \oplus M/N$.

Proposition 1.16
Let X be a A-module.

If there exist $p_1, p_2 \in \mathrm{End}_A(X)$ such that

$$p_1 \circ p_2 = p_2 \circ p_1 = 0 \; p_i^2 = p_i \quad \text{and } p_1 + p_2 = \mathrm{Id}_X,$$

then X is isomorphic to $\mathrm{Im}\, p_1 \oplus \mathrm{Im}\, p_2$.

Example 1.17 Assume $1_A = e_1 + e_2$ with $e_i^2 = e_i$ (idempotent), $e_1 e_2 = e_2 e_1 = 0$ (orthogonal), then $A \simeq Ae_1 \oplus Ae_2$ as a left A-module.

For example if $A = \mathcal{M}_n(k)$, then one can prove that $A \simeq (k^n)^n$ as a left A-module.

2 Tensor products and Hom

2.1 Homomorphism module

Let M and N be A-modules. Then $\mathrm{Hom}_A(M, N)$ has a structure of $\mathrm{End}(N)$-$\mathrm{End}(M)$-bimodule (in particular it is a k-bimodule) given by right and left composition.

As a consequence, if M is a A-B-bimodule and N a A-C-bimodule. Then $\mathrm{Hom}_A(M, N)$ has a structure of B-C-bimodule, given by

$$b.f.c(m) := f(mb)c, \text{ for } m \in M,\ b \in B,\ c \in C \text{ and } f \in \mathrm{Hom}(M, N).$$

Example 2.1 If M is a left A-module, then $\mathrm{Hom}_k(M, k) = M^*$ and $M^\vee := \mathrm{Hom}_A(M, A)$ are right A-modules. $A^* = \mathrm{Hom}_k(A, k)$ is a left A-module (in fact it is a A-bimodule.)

If $A = kG$, since $g \mapsto g^{-1}$ is an isomrophism $kG \to kG^{\mathrm{op}}$, then is V is a G-representation, then V^* is naturally a kG^{op}-module, hence a kG-module.

If B is a sublagebra of A, then A is naturally a B-module. Then for a A-module M, we have

$${}_BM \simeq \mathrm{Hom}_A(A_B, M)\text{as } B\text{-modules}$$

Proposition 2.2

1. For each A-module M, there is an isomorphism of A-module $\mathrm{Hom}_A(A, M) \simeq M$.
2. There is an algebra isomorphism $\mathrm{End}_A(A) \simeq A^{\mathrm{op}}$.

$$\mathrm{Hom}(M \oplus M', N \oplus N') \simeq \begin{bmatrix} \mathrm{Hom}(M, N) & \mathrm{Hom}(M', N) \\ \mathrm{Hom}(M, N') & \mathrm{Hom}(M', N') \end{bmatrix} \text{ as } k\text{-module.}$$

Proposition 2.3
Let M and N be modules. Then we have an isomrophism $\mathrm{End}(M \oplus N) \simeq \begin{bmatrix} \mathrm{End}(M) & \mathrm{Hom}(N, M) \\ \mathrm{Hom}(M, N) & \mathrm{End}(N) \end{bmatrix}$ as a k-algebra.

2.2 Tensor product

Let M be a right A-module and N be a left A-module. We define the space $M \otimes_A N$ as the k-free module generated by the $m \otimes n$ mod out by the submodule generated by

- $(m_1 + m_2) \otimes n - m_1 \otimes n - m_2 \otimes n$

- $m \otimes (n_1 + n_2) - m \otimes n_1 - m \otimes n_2$
- $(ma) \otimes n - m \otimes (an)$, $a \in A$, $m \in M$, $n \in N$;

If M is a B-A-bimodule, and if N is a A-C-bimodule, then $M \otimes_A N$ is a B-C-bimodule.

Proposition 2.4

1. there is a canonical isomorphism $A \otimes_A X \simeq X$.
2. there is a unique isomorphism $(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z)$ sending $(x \otimes y) \otimes z$ to $x \otimes (y \otimes z)$
3. If A is commutative then there is a unique isomorphism $X \otimes Y \simeq Y \otimes X$ sending $x \otimes y$ to $y \otimes x$.
4. There is a canonical isomorphism
$$(M \oplus M') \otimes_A N \simeq (M \otimes_A N) \oplus (M \otimes_A N)$$
5. If $f : X_1 \to X_2$ and $g : Y_1 \to Y_2$ are module morphisms, then there exists a unique module morphism $f \otimes g : X_1 \otimes Y_1 \to X_2 \otimes Y_2$ sending $x \otimes y$ on $f(x) \otimes g(y)$.

Extension of scalars

If $A \to B$ is a morphism of algebras, it makes B a A-module, hence we can define $B \otimes_A M$ for any A-module M.

Example 2.5 $B \otimes_A A[X] \simeq B[X]$.

If G is a group and H is a subgroup. We have an injection $kH \to hG$. So for any kH-module M, there is a KG-module defined by $\mathrm{Ind}_G^H(M) := kG \otimes_{kH} M$.

Tensor product of algebras

Theorem 2.6
$A \otimes_K B$ is an algebra.

The data of a A-B-bimodule is the same of a $A \otimes B^{\mathrm{op}}$-module. And same for the morphisms.

2.3 Adjunction formula

Theorem 2.7
Let A and B be algebras, let X be a A-module, let Y be a B-A-bimodule and let Z be a B-module. Then there is a canonical isomorphism

$$\mathrm{Hom}_A({}_AX, \mathrm{Hom}_B({}_BY_A, {}_BZ)) \simeq \mathrm{Hom}_B({}_BY \otimes_A X, {}_BZ).$$

Example 2.8 Let H be a subgroup of G. Let M be a representation of H and N bea representation of NThen we have

$$\mathrm{Hom}_{kG}(\mathrm{Ind}_G^H(M), N) \simeq \mathrm{Hom}_{kH}(M, \mathrm{Res}_H^G(N)).$$

3 Finite and infinite modules

3.1 Product and sums

Let I be a set and $(M_i)_{i\in I}$ be a collection of A-modules. Then

$$\prod_{i\in I} M_i := \{(m_i, i \in I), m_i \in M_i\}$$

is naturally a A-module.

We define $\bigoplus_{i\in I} M_i$ the subset of $\prod_i M_i$ consisting of finitely supported I-uples. It is a A-submodule.

Proposition 3.1
For any sets I and J, and modules (M_i), (N_j), there is an isomorphism

$$\mathrm{Hom}_A(\bigoplus_i M_i, \prod_j N_j) \simeq \prod_{(i,j)} \mathrm{Hom}(M_i, N_j).$$

3.2 Free modules

If I a set, define $A^I := \{f : I \to A\}$ and $A^{(I)} := \{f : I \to A \text{ with finite support }\}$.

Definition 3.2 A A-module is called free if it admits a basis, that is a family of elements $(x_i)_{i\in I}$ that is linerarily independant (every finite linear combination...) that generates it.

Theorem 3.3
Any free A-module is of the form $A^{(I)}$.

Theorem 3.4
Every A-module is a quotient of a free A-module.

As a corollary, any A-module M is the cokernel of a A-module morphism between free modules.

3.3 Finite modules

Definition 3.5 A finitely generated A-module (or module of finite type) M is a module of the form $\langle X\rangle$ for X a finite subset of M.

A module M is of finite type if and only if there exists a map $A^n \to M$. However, in general it could happen that the kernel of this map is not finitely generated. If it is, M is called finitely presented and there exists a map $A^m \to A^n$ such that M is isomorphic to the cokernel.

Particular cases:

If A is a finite dimensional k-algebra, then

module of finite type= module of finite dimension= module of finite presentation

In this case, it is clearly closed under kernel and cokernel.

The same is true if $A = \mathbb{Z}$. Any subgroup of a finitely generated abelian group is finitely generated.

Moreover any subgroup of a finitely generated free abelian group is a free abelian group.

We know well the structure of finitely generated abelian groups (built from $\mathbb{Z}$ and $\mathbb{Z}_{p^\alpha}$) but non finitely generated abelian groups are much more complicated: $\mathbb{R}$, $\mathbb{Q}$, ...

Chapter II

Categories of modules

1 Linear categories and functors

1.1 Definition

Definition 1.1 A k-linear category $\mathcal{C}$ is a collection of objects (also denoted by $\mathcal{C}$) and for each X, Y a k-module $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ together with a k-bilinear map

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(X,Y) \times \mathrm{Hom}_{\mathcal{C}}(Y,Z) & \to & \mathrm{Hom}_{\mathcal{C}}(X,Z) \\ (f,g) & \mapsto & g \circ f \end{array}$$

satisfying $h \circ (g \circ f) = (h \circ g) \circ f$ and with the following properties

- for each $X \in \mathcal{C}$, there is $1_X \in \mathrm{End}_{\mathcal{C}}(X)$ such that $f \circ 1_X = f$ and $1_X \circ g = 1_X$ for any $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(Y, X)$;
- there is an object $0 \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(X, 0) = \mathrm{Hom}_{\mathcal{C}}(0, X) = 0$ for all X;
- for each X, Y in $\mathcal{C}$ there is an object $X \oplus Y$ such that

$$\mathrm{Hom}(X \oplus Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Z) \oplus \mathrm{Hom}_{\mathcal{C}}(Y, Z) \text{ and}$$

$$\mathrm{Hom}(Z, X \oplus Y) \simeq \mathrm{Hom}_{\mathcal{C}}(Z, X) \oplus \mathrm{Hom}_{\mathcal{C}}(Z, Y) \text{ (as } k\text{-modules).}$$

The category of A-modules $\mathrm{Mod}\, A$ is such a category. The category of finitely generated A-modules $\mathrm{mod}\, A$ is also such a category.

Note that in $\mathrm{Mod}\, A$, the ismorphisms above are given by

$$\begin{array}{rcl} \mathrm{Hom}_A(X \oplus Y, Z) & \simeq & \mathrm{Hom}_A(X, Z) \oplus \mathrm{Hom}_A(Y, Z) \\ f & \mapsto & (f \circ i_X, f \circ i_Y) \\ f \circ p_X + g \circ p_Y & \leftarrow & (f, g) \end{array}$$

1.2 Linear functors

Definition 1.2 A k-linear covariant (resp. contravariant) functor F between two k-linear categories $\mathcal{C}_1$ and $\mathcal{C}_2$ is the data of an object $FX \in \mathcal{C}_2$ for each object $X \in \mathcal{C}_1$, and a k-linear map

$$F_{X,Y} : \mathrm{Hom}_{\mathcal{C}_1}(X, Y) \to \mathrm{Hom}_{\mathcal{C}_2}(FX, FY)$$

such that

- $F(f \circ g) = F(f) \circ F(g)$ (resp. $F(f \circ g) = F(g) \circ F(f)$);
- $F(1_X) = 1_{FX}$ for each $X \in \mathcal{C}_1$;
- $F(0) = 0$;
- $F(X \oplus Y) \simeq FX \oplus FY$ and these maps are compatible with the isomorphisms for the Hom, i.e. the following commutes

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{C}_1}(X \oplus Y, Z) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}_2}(F(X \oplus Y), Z) \\
 & & \downarrow \sim \\
\downarrow \sim & & \mathrm{Hom}_{\mathcal{C}_2}(FX \oplus FY, Z) \\
 & & \downarrow \sim \\
\mathrm{Hom}_{\mathcal{C}_1}(X, Z) \oplus \mathrm{Hom}_{\mathcal{C}_1}(Y, Z) & \xrightarrow{\sim} & \mathrm{Hom}_{\mathcal{C}_2}(FX, FZ) \oplus \mathrm{Hom}_{\mathcal{C}_2}(FY, FZ)
\end{array}$$

A composition of linear functors is clearly a linear functor.

Definition 1.3 Let $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a k-linear functor. If for any $Y \in \mathcal{C}_2$ there exists $X \in \mathcal{C}_1$ such that FX is isomorphic to Y, we say that F is dense. If for any X, Y, the map $F_{X,Y}$ is an isomorphism, we way that F is fully faithful. A functor which is dense and fully faithful is called an equivalence, and the categories $\mathcal{C}_1$ and $\mathcal{C}_2$ are said to be equivalent k-linear categories.

A natural transformation $\eta : F \to G$ between two functors $F : \mathcal{C}_1 \to \mathcal{C}_2$ and $G : \mathcal{C}_1 \to \mathcal{C}_2$ assigns $\eta_X \in \mathrm{Hom}_{\mathcal{C}_2}(FX, GX)$ for each $X \in \mathcal{C}_1$ such that $\eta_Y \circ F_{X,Y}(f) = G_{X,Y}(f) \circ \eta_X$ for any $f \in \mathrm{Hom}_{\mathcal{C}_1}(X, Y)$.

If moreover each η_X is invertible, we say that there is a functorial isomorphism between F and G.

For example if $B \to A$ is a morphism of algebras, then ${}_AM \mapsto {}_BM$ from $\mathrm{Mod}\, A \to \mathrm{Mod}\, B$ is a functor. For example, $\mathrm{Mod}\, A \to \mathrm{Mod}\, k$, or $\mathrm{Mod}\, A \to \mathrm{Mod}\, \mathbb{Z}$ are functors called forgetful functors.

1.3 Functors Hom and $\otimes$

Hom and $\otimes$ are the main examples of functors in representation theory.

Theorem 1.4
Let ${}_AM_B$ be a A-B-bimodule, then

- $\mathrm{Hom}_A(M, -)$ is a covariant functor $\mathrm{Mod}\, A \to \mathrm{Mod}\, B$;
- $\mathrm{Hom}_A(-, M)$ is a contravariant functor $\mathrm{Mod}\, A \to \mathrm{Mod}\, B$;
- $- \otimes_A M$ is a covariant functor $\mathrm{Mod}\, A^{\mathrm{op}}$ to $\mathrm{Mod}\, B^{\mathrm{op}}$
- $M \otimes_B -$ is a covariant functor $\mathrm{Mod}\, B \to \mathrm{Mod}\, A$.

Proposition 1.5
All the isomorphisms described in the previous chapter subsections 2.1, 2.2 and 2.3 are functorial isomorphisms.

For example the functor $\mathrm{Hom}_A(A, -)$ is isomorphic to $\mathrm{Id} : \mathrm{Mod}\, A \to \mathrm{Mod}\, A$.

The contravariant functors (from $\mathrm{Mod}\, A$ to $\mathrm{Mod}\, k$) $\mathrm{Hom}_A(-, \mathrm{Hom}_B({}_BY_A, {}_BZ))$ and $\mathrm{Hom}_B({}_BY \otimes_A -, {}_BZ)$ are isomorphic.

2 Short exact sequences

2.1 Abelian category

Definition 2.1 Let X, Y and Z be A-modules. A sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ of morphisms is called exact if $\mathrm{Ker} g = \mathrm{Im}\, f$.

A short exact sequence is an exact sequence of the form

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0;$$

so equivalently, f is injective, $\mathrm{Ker} g = \mathrm{Im}\, f$ and g is surjective.

For example if $N \subset M$ is a submodule, there is a natural short exact sequence of the form

$$0 \longrightarrow N \xrightarrow{i} M \xrightarrow{p} M/N \longrightarrow 0.$$

Here is the fundamental property of the module category:

Proposition 2.2
Let $f : M \to N$, then there exist two short exact sequences:

$$0 \longrightarrow K \longrightarrow M \xrightarrow{p} I \longrightarrow 0 \qquad 0 \xrightarrow{i} I \longrightarrow N \longrightarrow C \longrightarrow 0$$

such that $i \circ p = f$.

This is the property that makes the category $\operatorname{Mod} A$ an abelian category.

NB: if k is a field, and A is a finite dimensional k-algebra, then $\operatorname{mod} A$ is also an abelian category. Indeed, if M and N are finitely generated so are K, I and C. The same is true for finitely generated abelian groups.

2.2 Monomorphisms and epimorphisms

Definition 2.3 A morphism $f : X \to Y$ is called a monomorphism if $f \circ g = f \circ h \Rightarrow g = h$ (or equivalently $f \circ g = 0 \Rightarrow g = 0$).

A morphism $f : X \to Y$ is called an epimoprhism if $g \circ f = h \circ f \Rightarrow g = h$ (or equivalently $g \circ f = 0 \Rightarrow g = 0$).

Proposition 2.4

1. A A-linear map $f : X \to Y$ is a monomorphism if and only if f is injective.
2. A A-linear map $f : X \to Y$ is an epimorphism if and only if f is surjective.

Proposition 2.5
Let $f : X \to Y$ be morphism in $\operatorname{Mod} A$. Then

1. for any morphism $g : U \to X$ such that $f \circ g = 0$ there exists a unique morphism $h : U \to \mathrm{Ker} f$ such that $g = i \circ h$ where $i : \mathrm{Ker} f \to X$.
2. for any morphism $g : Y \to Z$ such that $g \circ f = 0$ there exists a unique morphism $h : \mathrm{Coker} f \to Z$ such that $g = f \circ p$ where $p : Y \to \mathrm{Coker} f$.

Corollary 2.6
Let $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} \\ X' & \xrightarrow{f'} & Y' \end{array}$ be a commutative square (that is $\psi \circ f = f' \circ \varphi$), then it can be completed into a commutative diagram

$$\begin{array}{ccccccc} \mathrm{Ker} f & \longrightarrow & X & \xrightarrow{f} & Y & \longrightarrow & \mathrm{Coker} f \\ \downarrow & & {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\psi} & & \downarrow \\ \mathrm{Ker} f' & \longrightarrow & X' & \xrightarrow{f'} & Y' & \longrightarrow & \mathrm{Coker} f' \end{array}$$

2.3 Split short exact sequences

Definition 2.7 A short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ is a split short exact sequence if there exists an isomorphism $h : Y \to X \oplus Z$ such that there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle h} & & \| & & \\ 0 & \longrightarrow & X & \xrightarrow{i} & X \oplus Z & \xrightarrow{p} & Z & \longrightarrow & 0 \end{array}$$

A morphism $f : X \to Y$ is said to be a section if there exists $f' : Y \to X$ with $f' \circ f = 1_X$.

A morphism $g : Y \to Z$ is said to be a retraction if there exists $g' : Z \to Y$ such that $g \circ g' = 1_Z$.

Proposition 2.8
A short exact sequence $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$ if and only if one of the following occurs:

- f is a section;
- g is a retraction.

Example 2.9 In the category Mod k, every short exact sequence splits.

As we will see later, it is also the case in Mod $\mathbb{C}G$ where G is a finite group.

It is not the case in Mod $\mathbb{Z}$, for instance $0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_4 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$ does not split.

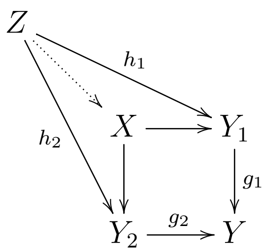
2.4 Push forward and pull back

Pull back

Definition 2.10 Let $g_1 : Y_1 \to Y$ and $g_2 : Y_2 \to Y$ be morphisms. Then a pull back of g_1 anf g_2 is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f_1} & Y_1 \\ \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle g_1} \\ Y_2 & \xrightarrow{g_2} & Y \end{array} \quad ;$$

such that for any commutative square



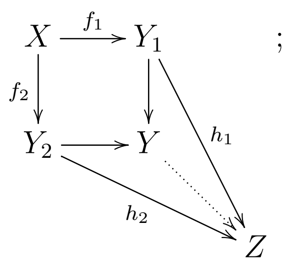
;

there exists a unique $h : Z \to X$ such that $h_1 = f_1 \circ h$ and $h_2 = f_2 \circ h$.

Definition 2.11 Let $f_1 : X \to X_1$ and $f_2 : X \to X_2$ be morphisms. Then a push forward of f_1 anf f_2 is a commutative square

$$\begin{array}{ccc} X & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\downarrow & & \downarrow{\scriptstyle g_1} \\ X_2 & \xrightarrow{g_2} & Y \end{array} \quad ;$$

such that for any commutative square



there exists a unique $h : Y \to Z$ such that $h_1 = h \circ g_1$ and $h_2 = h \circ g_2$.

Example 2.12 Let $f : X \to Y$ be a morphism. The commutative square $\begin{array}{ccc} \mathrm{Ker} f & \longrightarrow & 0 \\ {\scriptstyle i}\downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$ is a pull back.

The commutative square $\begin{array}{ccc} X_1 \times X_2 & \xrightarrow{p_1} & X_1 \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ X_2 & \longrightarrow & 0 \end{array}$ is a pull back.

Proposition 2.13
There exist pull backs and push outs in the category $\mathrm{Mod}\, A$.

Proof : The pull back of $(g_1 : Y_1 \to Y, g_2 : Y_2 \to Y)$ is given by

$$X := \mathrm{Ker}(g_1 - g_2 : Y_1 \oplus Y_2 \to Y).$$

The push-out of $(f_1 : X \to X_1, f_2 : X \to X_2)$ is given by

$$Y = \mathrm{Coker}((f_1, -f_2) : X \to X_1 \oplus X_2).$$

$\square$

Theorem 2.14
Let $0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} 0$ be a short exact sequence.

1. For any $z : Z' \to Z$, there exists a commutative diagram where the horizontal lines are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle z} & & \\ 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \end{array}$$

2. For any $x : X \to X'$, there exists a commutative diagram where horizontal lines are exact

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle x} & & \downarrow & & \| & & \\ 0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z & \longrightarrow & 0 \end{array}$$

3 Short exact sequences and functors

3.1 Exact functors

Definition 3.1
1. A covariant functor F is left exact if for any $0 \to X \to Y \to Z$ the sequence $0 \to FX \to FY \to FZ$ is exact
2. A covariant functor is called right exact if for any exact sequence $X \to Y \to Z \to 0$, the sequence $FX \to FY \to FZ \to 0$ is exact.
3. A contravariant functor F is left exact if for any $X \to Y \to Z \to 0$, the sequence $0 \to FZ \to FY \to FX$ is exact.
4. A contravariant functor F is right exact if for any $0 \to X \to Y \to Z$, the sequence $FZ \to FY \to FX \to 0$ is exact.
5. A functor is called exact if it is both left and right exact. So it sends any short exact sequence to a short exact sequence.

Theorem 3.2
Let M be a A-B-bimodule. We have the following:

1. the functors $\mathrm{Hom}_A(M, -)$ and $\mathrm{Hom}_A(-, M)$ are left exact;
2. the functors $M \otimes_B -$ and $- \otimes_A M$ are right exact.

Proof : Here we need to show a statement a bit more precise. We will show that a sequence $0 \to X \to Y \to Z$ is exact if and only if for all $M \in \mathrm{Mod}\, A$ the sequence $0 \to \mathrm{Hom}(M, X) \to \mathrm{Hom}(M, Y) \to \mathrm{Hom}(M, Z)$ is exact, and the similar statement for the other functors. □

3.2 Projective, injective and flat modules

Definition 3.3 An A-module P is said to be projective if the functor $\mathrm{Hom}_A(P, -)$ is exact.

An A-module I is said to be injective if $\mathrm{Hom}_A(-, I)$ is exact.

An A-module F is said to be flat if $F \otimes_A -$ is exact.

The following is clear from the definition.

Proposition 3.4

1. A A-module P is projective if and only if for any epimorphism $f : X \to Y$ and morphism $u : P \to Y$, there exists a morphism $v : P \to X$ such that $f \circ v = u$.
2. A A-module I is injective if and only if for any monomorphism $f : X \to Y$, and any morphism $u : X \to I$, there exists a morphism $v : Y \to I$ such that $v \circ f = u$.

Lemma 3.5

1. Let $(M_i)_{i \in E}$ be a family of A-modules. Then $\bigoplus_{i \in E} M_i$ is projective if and only if M_i is projective for any $i \in E$.

2. Let $(M_i)_{i \in E}$ be a family of A-modules. Then $\prod_{i \in E} M_i$ is injective if and only if M_i is injective for any $i \in E$.
3. Let $(M_i)_{i \in E}$ be a family of A-modules. Then $\bigoplus_{i \in E} M_i$ is flat if and only if M_i is flat for any $i \in E$.

3.3 Existence of projective and flat modules

Theorem 3.6
A module M is projective if and only if it is a direct summand of a free module.

Proof : The proof here comes from the fact that $\mathrm{Hom}_A(A, M) \simeq M$, it is then clear that A is projective. Then by the previous lemma we clearly have that any free module is projective and so is any direct summand of a free module.

Now given a projective module P, we can take a free cover $F \to P \to 0$ of P. Then since P is projective, the map $F \to P$ is a retraction therefore P is isomrophic to a direct summand of F.

□

Theorem 3.7
Free $\Rightarrow$ projective $\Rightarrow$ flat.

Proof : This is an easy consequence of the previous lemma, and of the fact that A is flat. □

We will wee later that for certain nice rings (Noetherian) finitely generated projective modules coincide with finitely generated flat modules.

3.4 Existence of injective modules

Case where k is a field

Lemma 3.8
If k is a field, then k is injective in $\operatorname{Mod} k$.

Proof : This comes from the fact that all short exact sequences splits in $\operatorname{Mod} k$. □

As a consequence, and using the natural embedding $M \to M^{**}$ we obtain the following.

Theorem 3.9
Let k be a field and A be a k-algebra. Then we have

$$M \in \operatorname{Mod} A \text{ is projective} \Rightarrow \; M^* = \operatorname{Hom}_k(M, k) \in \operatorname{Mod} A^{\mathrm{op}} \text{ is injective.}$$

As an immediate corollary, we obtain that A^* is naturally a left A-module injective.

Note that in the case of where A is finite dimensional, the k-duality induces a bijection between projective and injective objects in $\operatorname{mod} A$ (finite dimensional A-modules).

Case of abelian groups

The general case is much more complicated. Already for $A = \mathbb{Z}$ it is difficult to exhibit injective $\mathbb{Z}$-modules. For example, using the embedding $\mathbb{Z} \to \mathbb{Q}$, one can see that $\mathbb{Z}$ is not an injective object.

However, the aim here is to prove that $\mathbb{Q}$ is injective. To prove this, we will use the following criterion.

Theorem 3.10 (Baer's criterion)
Let A be a k-algebra. Then a A-module M is injective if and only if for any submodule $J \subset A$, the map $\mathrm{Hom}_A(A, M) \to \mathrm{Hom}_A(J, M)$ is surjective.

Necessity is clear. The converse direction is more involved and uses Zorn lemma, we refer to Assem (Theorem 3.4 in Chapter IV) for a complete proof.

But this lemma implies easily the following:

Proposition 3.11
$\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are injective as $\mathbb{Z}$-modules.

General case

This leads us to introduce an other notion of dual.

Definition 3.12 Let $M \in \mathrm{Mod}\, A$, we define $M^\wedge := \mathrm{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \in \mathrm{Mod}\, A^{\mathrm{op}}$ the Pontryagin dual of M.

We have then the same kind of properties that for the k-dual.

Lemma 3.13
A map $X \to Y$ in $\mathrm{Mod}\, A$ is injective if and only if the corresponding map $Y^\wedge \to X^\wedge$ in $\mathrm{Mod}\, A^{\mathrm{op}}$ is surjective.

Theorem 3.14
A right A-module X is flat if and only if the A-module $X^\wedge$ is injective.

As a corollary, we then obtain that $A^\wedge$ is an injective left A-module. It is unfortunately in general not finitely generated.

Chapter III

Decomposition theorems

1 Noetherian and Artinian

1.1 Noetherian and Artinian modules

Definition 1.1

1. A A-module M is said to be Artinian if for any decraesing sequence $M_0 \supseteq M_1 \supseteq \cdots$ of submodules there exists n such that $M_j = M_n$ $\forall j \geqslant n$.
2. A A-module M is said to be Noetherian if for any increasing sequence $M_0 \subseteq M_1 \subseteq \cdots$ of submodules there exists n such that $M_j = M_n$ $\forall j \geqslant n$.

Example 1.2 If k is a field, then any finite dimensional A-module is both Artinian and Noetherian.

$\mathbb{Z}$ or more generally any principal ring is Noethrian. But $\mathbb{Z}$ is not Artinian. The ring $\mathbb{Z}/n\mathbb{Z}$ is Artinian and Noetherian.

Proposition 1.3
Let $0 \to X \to Y \to Z \to 0$ be a short exact sequence of A-modules. Then we have

1. X and Z are Artinian if and only if so is Y.
2. X and Z are Noetherian if and only if so is Y.

Proof : If Y is Artinian, then so is X since it is a submodule of Y. If $Z_0 \supseteq Z_1 \supseteq \cdots$ is a dcreasing chain of submodules of Z, then $p^{-1}(Z_0) \supseteq p^{-1}(Z_1) \supseteq$

$\cdots$ is a decreasing chain of submodules of Y. So $p^{-1}(Z_\ell) = p^{-1}(Z_{\ell+1})$ which implies $Z_\ell = Z_{\ell+1}$.

Conversely, let $Y_0 \supseteq Y_1 \supseteq \cdots$ be a decreasing chain of submodules of Y. Then we obtain the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X \cap Y_{\ell+1} & \longrightarrow & Y_{\ell+1} & \longrightarrow & Z \cap Y_{\ell+1} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & X \cap Y_\ell & \longrightarrow & Y_\ell & \longrightarrow & Z \cap Y_\ell & \longrightarrow & 0 \end{array}$$

If both left and right maps are equalities, then so is the middle one. □

Theorem 1.4
Let M be a A-module. Then M is Noetherian if and only if every submodule of M is finitely generated.

Proof:Let N be a submodule of M, and consider the set $\mathcal{E}$ of submodules of N which are finitely generated. It is non empty since $\{0\}$ is finitely generated. Since M is Noetherian, any incraesing chain has an upper bound. So by Zorn's lemma, it has a maximal element L. If $L \neq N$, then there exists $x \in N \setminus L$, and $\langle L, x \rangle$ is a finitely generated submodule of N, which contradicts maximality. So N is maximal, and so N is finitely generated.

Conversely, let $M_0 \subseteq M_1 \cdots$ be a increasing chain of submodules in M. Then the union of the M_i is a submodule of M. It has a finite set of generators, so there exists n such that every generator is in M_n, and so the union of the M_i equals M_n. □

Corollary 1.5
If A Noetherian as left module, then any finitely generated A-module is finitely presented.

1.2 Noetherian and Artinian algebras

Definition 1.6
1. An algebra A is called left Artinian if the module ${}_AA$ is Artinian.
2. An algebra A is called left Noetherian if the module ${}_AA$ is Noetherian.

Theorem 1.7
Let A be a k-algebra.

1. If A is left Artinian, then any A-module of finite type is Artinian.
2. If A is left Noetherian, then any A-module of finite type is Noetherian.

Proof : It follows directly from Proposition 1.3.

□

Corollary 1.8
If A is Artinian or Noetherian, then the category $\operatorname{mod} A$ of finitely generated A-modules is an Abelian category.

2 Indecomposable modules and algebras

2.1 Idempotents

Definition 2.1 An algebra A is said to be connected if it is not isomorphic to the product of two non trivial algebras.

A A-module M is said to be indecomposable if it is not isomorphic to the direct sum of two proper submodules.

The key notion here is the notion of idempotents. Indeed, if $A = \prod_i A_i$, then denoting by $e_i := (0, \ldots, 1_{A_i}, 0, \ldots)$ we have the following relations

$$e_i^2 = e_i,\ \ e_ie_j = 0 \text{ for } i \neq j,\ \ 1_A = \sum_i e_i \text{ and } e_i \in \mathrm{Z}(A).$$

Moreover $A_i \simeq e_i A e_i$.

Similarly, let $M = \bigoplus_i M_i$ be decomposable. Denote by p_j and i_j the projections and injections, and set$e_j := p_j \circ i_j \in \mathrm{End}_A(M)$. Then we have

$$e_i^2 = e_i,\ e_i e_j = 0 \text{ for } i \neq j,\ \mathrm{Id}_M = \sum_i e_i$$

So roughly speaking, an algebra will be connected if it has very few idempotent, and a module M will be indecomposable if its endomorphism algebra has also very few idempotents.

2.2 Algebras and bloc decomposition

The first result show that the conditions above on the idempotents is sufficiant to decompose an algebra.

Theorem 2.2
Let A be a k algebra. Assume that $1_A = \sum_{i=1}^{s} e_i$ with the properties

$$e_i^2 = e_i,\ e_i e_j = 0 \text{ for } i \neq j, \text{ and } e_i \in \mathrm{Z}(A),$$

then A is isomorphic to $\prod_{i=1}^{s} A_i$ with $A_i = Ae_i$.

Proof : First note that $A_i = Ae_i = e_i A e_i$ is an algebra, and a tw-sided ideal of A.

The isomorphisms $A \to \prod_{i=1}^{s} A_i$ and $\prod_{i=1}^{s} A_i \to A$ are given by

$$a \mapsto (ae_1, \ldots, ae_s) \quad \text{ and } (b_1, \ldots, b_s) \mapsto \sum_i b_i.$$

One easily checks that these are isomorphisms of algebras and inverse one of each other. □

Theorem 2.3
Let A be a k-algebra which is Noetherian or Artinian. Then A is isomorphic to a finite product of connected algebras which are uniquely determined.

Proof : The existence comes from Noetherianity ar Artinianity.

Unicity is quite easy, using the fact that the product of two two-sided ideal is a two-sided ideal. So if we have

$$A_1 \times \cdots \times A_s = B_1 \times \cdots \times B_t,$$

then we have $A_i = AA_i = \prod_j (B_j A_i)$ and since A_i is connected we obtain $A_i = B_j A_i$. Using then $B_j = B_j A$, we obtain $A_i = B_j$. Finally we use the fact that

$$\prod_{\ell \neq i} A_\ell = A/A_i = A/B_j = \prod_{k \neq j} B_k$$

and conclude by induction. □

Theorem 2.4
Let $A = A_1 \times A_2$ be the direct product of two k algebras. Then there is an equivalence of categories

$$\mathrm{Mod}\, A \simeq \mathrm{Mod}\, A_1 \times \mathrm{Mod}\, A_2.$$

Proof : The functor is given by $M \mapsto (e_1 M, e_2 M)$ where the e_i are the idempotents defined above. One then needs to show that for $M, N \in \mathrm{Mod}\, A$, then

$$\mathrm{Hom}_A(e_1 M, e_2 N) = 0 \quad \text{and} \quad \mathrm{Hom}_A(e_1 M, e_1 N) = \mathrm{Hom}_{A_1}(e_1 M, e_1 N).$$

□

2.3 Indecomposable modules and local rings

Definition 2.5 A k-algebra is said to be local if it has a unique maximal left ideal.

The link between these two notions is given by the Theorem below.

Theorem 2.6
A module M which is Artinian and Noetherian is indecomposable if and only if $\mathrm{End}_A(M)$ is local.

In order to prove this result, we need two lemmas.

Lemma 2.7
Let A be a k-algebra. Then A is local if and only if for any $x \in A$, x or $1 - x$ is invertible.

Proof : One direction is clear, since if both x and $1 - x$ are non invertible, then they are both in the maximal ideal J, which implies $J = A$.

For the other direction, we prove that the set J of non left invertible elements of A is an ideal. It statisfies clearly $AJ \subset J$. Now let x and y be in J such that $x - y$ has a left inverse a. Since ax is in J, then $1 - ax = ay$ is invertible which is a contradiction. Finally J clearly contains all proper ideals of A, so A is local. □

Lemma 2.8 (Fitting's lemma)
Let $f \in \mathrm{End}_A(M)$ where M is Noetherian and Artinian, then there exists $n \geqslant 1$ such that
$$M = \mathrm{Ker} f^n \oplus \mathrm{Im}\, f^n$$

Proof: By Artinianity and Noetherianity, there exists n suxh that $\mathrm{Ker} f^{n+1} = \mathrm{Ker} f^n$ and $\mathrm{Im}\, f^{n+1} = \mathrm{Im}\, f^n$. One easily checks that $\mathrm{Ker} f^n \cap \mathrm{Im}\, f^n = \{0\}$. And if $y \in M$, taking x such that $f^n(x) = f^{2n}(y)$ we obtain
$$y = (x - f^n(y)) + f^n(y) \in \mathrm{Ker} f^n \oplus \mathrm{Im}\, f^n.$$
□

2.4 Krull-Schmidt decomposition

Theorem 2.9 (Azumaya-Krull-Remak-Schmidt)
Let A be a k-algebra and let $M \in \operatorname{Mod} A$. If M is Noetherian or Artinian, then M is isomorphic to a finite direct sum of indecomposable modules.

If M is both Artinian and Noetherian, then the decomposition is essentially unique.

Proof : The proof of exsitence is the same as for the bloc decomposition of algebras.

Assume that $\bigoplus_{i=1}^{m} M_i = \bigoplus_{j=1}^{n} N_j$. We proceed on induction on m. Using that $\operatorname{End}(M_1)$ is local, we obtain a j such that $p_{M_1} \circ i_{N_j} \circ p_{N_j} \circ i_{M_1}$ is invertible. Then using the fact that $\operatorname{End}(N_j)$ is local, we prove that both $p_{M_1} \circ i_{N_j}$ and $p_{N_j} \circ i_{M_1}$ are invertible. So N_j is isomorphic to M_1. Finally we need to construct an isomorphism $\varphi : M \to M$ such that there exists a commutative diagram

$$\begin{array}{ccc} M_1 & \xrightarrow{i_1} & M \\ \downarrow{\scriptstyle p_{N_j} \circ i_{M_1}} & & \downarrow{\scriptstyle \varphi} \\ N_j & \xrightarrow{i_j} & M \end{array} .$$

And we apply the induction hypothesis.

□

Corollary 2.10
If k is a field and if A is a k-algebra, then any finite dimensional A-module can be decomposed into a unique finite direct sum of indecomposable modules.

3 Simple and Semi-simple

3.1 Simple modules

Definition 3.1 A A-module $S \neq 0$ is called simple if it has no non zero proper submodule.

Example 3.2 If k is a field, then any 1-dimensional A-module is simple. In $\operatorname{mod} k$, there is only one simple module up to isomorphism k.

The $\mathcal{M}_n(k)$-module k^n is simple.

Any division k-algebra D over k is a simple D-module. So there might be infinite dimensional simple module (e.g. $D = \mathbb{C}(X)$).

$\mathbb{Z}_n$ is simple if and only if n is prime.

For $G = \mathbb{D}_4$ the dihedral group, consider the 2-dimensional representation given by

$$\begin{array}{rccc} \rho: & G & \longrightarrow & \mathrm{GL}_2(\mathbb{C}) \\ & r & \longmapsto & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ & s & \longmapsto & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{array}$$

is simple.

We have the following characterization for simple modules.

Lemma 3.3
For a A-module S the following are equivalent:

1. S is simple;
2. $\forall x \neq 0 \in S$, $Ax = S$;
3. $\forall x \neq 0 \in S$, $S \simeq A/\mathrm{Ann}(x)$;

Lemma 3.4 (Schur's lemma)
Let S be a simple A-module.

1. then the k-algebra $\mathrm{End}_A(S)$ is a division algebra.
2. if moreover $k = \bar{k}$ is an algebraically closed field and S is finite dimensional, then $\mathrm{End}_A(S) \simeq k$.

Proposition 3.5
Let $k = \bar{k}$ be an algebraically closed field, and A be a commutative k-algebra. Then any finite dimensional simple A-module is 1-dimensional.

3.2 Composition series

Definition 3.6 Let M be a non zero A-module. A sequence of submodules

$$0 = M_0 \subset M_1 \subset \cdots M_m = M$$

is called a composition series for M if M_{i+1}/M_i is simple for any i. These quotients are called composition factors.

Example 3.7 $\mathbb{Z}_{p^m}$ has a unique composition series:

$$p^m \mathbb{Z}_{p^m} \subset p^{m-1} \mathbb{Z}_{p^m} \subset \cdots \subset \mathbb{Z}_{p^m}.$$

$\mathbb{Z}_2 \oplus \mathbb{Z}_2$ has three composition series.

Let $M_\ell = k^\ell$ be the $\mathcal{T}_n(k)$-module given as in the exercises. Then M_n has a unique composition series

$$M_0 \subset M_1 \subset \ldots M_n.$$

Proposition 3.8
A A-module has a composition series if and only if it is Artinian and Noetherian.

Proof : We use Artinianity to prove that any module has a simple submodule. We then construct a composition series inductively, that stops by Noetherianity.

The converse can be easily shown by induction on the length of the composition series using Proposition 1.3. □

Theorem 3.9 (Jordan Hölder)
If A-module M admits two composition series

$$0 = M_0 \subset M_1 \subset \cdots M_m = M, \quad 0 = N_0 \subset M_1 \subset \cdots N_n = M,$$

then $m = n$, and there exists a permutation $\sigma \in \mathfrak{S}_m$ such that $M_{\sigma(i)+1}/M_{\sigma(i)} \simeq N_{i+1}/N_i$.

Proof : The proof is is done by induction on m. The idea is to extract from

$$0 \subseteq M_{m-1} \cap N_1 \subseteq \cdots \subseteq M_{m-1} \subseteq \cdots \subseteq M_{m-1} + N_{n-1} \subseteq M$$

a composition series of M of the form

$$0 = M_0' \subset M_1' \subset \cdots M_{n-2}' \subseteq M_{m-1} \subset M,$$

such that each quotient is of the form N_{i+1}/N_i for some i, and then apply induction. □

3.3 Semi-simple algebras

Definition 3.10 A A-module is called semi-simple if it is the direct sum (possibly infinite) of simple modules.

A k-algebra is called semi-simple if it is semi-simple as a left module over itself.

Theorem 3.11 (Mashke's theorem)
Let G be a finite group and k be a field such that $|G|$ is invertible in k. Then any finite dimensional kG-module is semi-simple. In particular kG is semi-simple.

Proof : If $W \subset V$ is a submodule, the idea is first to construct a k-linear retraction $V \to W$, and to modify it, using the fact that $|G|$ is invertible to obtain a kG-linear retraction. □

Theorem 3.12 (Artin Wedderburn)
A k-algebra A is semi-simple if and only if it is isomorphic to

$$\mathcal{M}_{n_1}(D_1) \times \dots \mathcal{M}_{n_s}(D_s)$$

where D_s are division k-algebras.

Proof : It is not hard to see that $\mathcal{M}_n(D)$ is semi-simple since D^n is a simple $\mathcal{M}_n(D)$-module.

Conversely, decomposing A into a sum of simple, we first show that the sum is finite, since 1_A is a finite sum of elements in this decomposition. Then we conclude using Schur's lemma 3.4, the fact that $\mathrm{End}_A(A) = A^{\mathrm{op}}$, and that

$$\mathrm{End}_A(S^n) \simeq \mathcal{M}_n(\mathrm{End}(S)) \text{ and } \mathrm{Hom}_A(S, S') = 0 \text{ if } S \neq S'$$

□

Corollary 3.13
Let $k = \bar{k}$ be an algebraically closed field. A finite dimensional k-algebra A is semi-simple if and only if it is isomorphic to

$$\mathcal{M}_{n_1}(k) \times \dots \mathcal{M}_{n_s}(k).$$

Moreover, there exist exactly s isomorphism classes of simple modules, of dimension $n_1, \dots, n_s$ respectively.

3.4 The module category for a semi-simple algebra

Proposition 3.14
A module M is semi-simple if and only if every submodule of M is a direct summand of M (in other words, any inclusion $N \subset M$ is a section).

Proof : Assume first that $M = \bigoplus_{i\in I} S_i$ a semi-simple module, and $N \subset M$ be a submodule. Define

$$\mathcal{E} := \{J \subset I \text{ s.t. } N \oplus \bigoplus_{i\in J} S_j \text{ is in direct sum}\}.$$

We have $\emptyset \in \mathcal{E}$, so $\mathcal{E} \neq \emptyset$. We now show that $\mathcal{E}$ is an inductive set. Assume we have a increasing chain J_λ of subsets in $\mathcal{E}$. Then if x is an element in $N + \sum_{i\in\cup J_\lambda} S_i$, then there exists λ such that $x \in N + \sum_{i\in J_\lambda} S_j$. But then the sum is direct, so the decomposition of x is unique. So we have

$$N + \sum_{i\in\cup J_\lambda} S_i = N \oplus \bigoplus_{i\in\cup J_\lambda} S_i.$$

Therefore the set $\mathcal{E}$ s an inductive set. By Zorn's lemma, this set has a maximal element I_0. We would like to show that $N_0 := N \oplus \bigoplus_{j\in I_0} S_i = M$. Let $i \in I$, then $S_i \cap N_0$ is either 0 or S_i since S_i is simple. If it is zero, then S_i is in direct sum with N_0 which is a contradiction with maximality of I_0. Thus $S_i \cap N_0 = S_i$, meaning that $M = \sum_{i\in I} S_i = N_0$. Hence we have $N \oplus L = M$ as required.

Let M be such that any submodule is a direct summand. First note that any submodule of M satisfies also this property. Indeed if $L \subset N \subset M$, then there exists a map $p : M \to L$ such that the composition $L \subset M \to L$ is 1_L Now, define p' to be the composition

$$p' : N \subset M \to L.$$

We have then $L \subset N \subset M \to L = 1_L$, thus $L \subset N$ is a section.

Now, we would like to show that M admits a submodule which is simple. Let $x \in M$ with $x \neq 0$. Then define $N = \langle x\rangle$, and $\mathcal{E} = \{L \subset N \text{ s.t. } L \neq N\}$. This set contains the zero module. If (N_λ) is a ascending chain of submodules in $\mathcal{E}$, then if $N = \bigcup_\lambda N_\lambda$, then there exists λ such that $x \in N_\lambda$ and then $\langle x\rangle = N = N_\lambda$ which is not true. So the submodule $\bigcup_\lambda N_\lambda$ is a strict submodule of N, so is in $\mathcal{E}$. Hence the set $\mathcal{E}$ is an inductive set, and by Zorn's lemma, it has a maximal element N_0. Then by hypothesis (and the first remark), we can write $N = N_0 \oplus S$. We want to check that S is simple. If not, then $S = T \oplus T'$ (again by the first remark), and then $N_0 \oplus T'$ is a strict submodule of N containing strictly N_0, which contradicts the maximality of N_0. Therefore, we have that N (hence M) has a submodule which is simple.

Now, we denote by I the set of simple submodules of M, and denote by $M' = \sum_{S\in I} S$ which is a submodule of M. If $M' \neq M$, then we can write $M = M' \oplus M''$, but since M'' is a submodule of M, it admits a submodule which is simple, which is a contradiction. Hence, we have $M = M'$. The last thing to show is the fact that there exists $J_0 \subset I$ such that

$$\sum_{S \in J_0} S = \bigoplus_{S \in J_0} S = \sum_{S \in I} S = M.$$

We denote by $\mathcal{E} := \{J \subset I \mid \sum_{S \in J} = \bigoplus_{S \in J} S\}$. It is clearly non empty, and it is inductive (see the argument above for the other direction). Denote by J_0 its maximal element, and by $M_0 := \bigoplus_{S \in J_0} S$. Let $S' \in I$, then $S' \cap M_0$ is either zero or S' since S' is simple. If it is zero, then $S' + M_0 = S' \oplus M_0$ which contradicts the maximality of M_0. So $S' \subset M_0$, and then $M_0 = M$, and M can be written as a direct sum of simple submodules.

□

Theorem 3.15
For an algebra A, the following are equivalent:

1. A is semi-simple;
2. every A-module is semi-simple;
3. every short exact sequence in $\operatorname{Mod} A$ splits;
4. every A-module is projective;
5. every A-module is injective.