Geometry and combinatorics of spherical varieties.

Notes of a course taught by Guido Pezzini.

Abstract

This is the lecture notes from a mini course at the Winter School "Geometry and Representation Theory" from Erwin Schrụdinger International Institute for Mathematics and Physics in January 2017.

In these lectures we will introduce spherical varieties and discuss some of their basic properties; we will also introduce related combinatorial objects and see how they govern the geometry of such varieties.

Contents

1 Definitions, examples, first properties

We consider a connected linear reductive algebraic group G over $\mathbb{C}$, e.g. $G = GL_n$ or SL_n, and a Borel subgroup B of G.

Definition 1.1. We will call a $G-$ *variety* an irreducible $\mathbb{C}$-variety X with a G-action $G \times X \to X$ which is a morphism of $\mathbb{C}$-varieties.

Definition 1.2. If X is a G-variety, then we denote the minimal co-dimension of a B-orbit by $c(X)$, also called the *complexity* of X.

A *spherical variety* is a normal G-variety X such that $c(X) = 0$.

The complexity for a G-variety plays a role similar to the dimension of a variety. An algebraic variety of dimension 0 is determined by a combinatorial datum, which is simply the number of its points. We will see that a spherical variety is also determined by combinatorial objects, which are however much more complicated.

Example 1.3. (i) If B is a Borel of GL_n, then GL_n/B is a spherical variety. More generally if P is a parabolic subgroup of G then G/P is a spherical variety.

(ii) Symmetric spaces: If θ is an involution of G and G^θ is the closed subgroup of G consisting of the fixed points of θ, then G/G^θ is a spherical variety.

For example, SL_n/SO_n and $G \times G/\text{diag}(G)$ are symmetric spaces.

(iii) $X = \dfrac{SL_2 \times SL_2 \times SL_2}{\text{diag}(SL_2)}$ is a spherical variety.

(iv) If X is a toric variety for the action of a torus $(\mathbb{C}^*)^{\dim(X)}$, then X is a spherical variety with $G = (\mathbb{C}^*)^{\dim(X)}$.

(v) If X is a smooth Schubert sub-variety of the Grassmannian $\text{Gr}(n, d)$, then X is spherical [HL16].

Theorem 1.4. *Let X is a spherical variety.*

(i) X has a finite number of G-orbits.

(ii) X has a finite number of B-orbits.

Remark 1.5. If $c(X) = 0$, then X has a dense B-orbit and a dense G-orbit.

Sketch of proof of (i) *for X affine.* The ring $\mathbb{C}[X]$ is a rational G-module. So it is a direct sum of irreducible G-modules. We denote irreducible G-modules as $V(\lambda)$, where λ is the highest weight (so λ is in the group $\chi(B)$ of characters of B).

Exercise: $\mathbb{C}[X]$ is multiplicity-free i.e., for every λ, then $V(\lambda)$ appears at most once. (If X is normal and affine, this property is equivalent to X being spherical.)

Thus

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Gamma} V(\lambda),$$

where Γ is a set of dominant weight. Denote by $\mathbb{Z}\Gamma$ the subgroup of $\chi(B)$ generated by Γ, and let us consider Γ inside the vector space $\mathbb{Z}\Gamma \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\mathbb{C}[X]$ is a finitely generated ring over $\mathbb{C}$, then Γ is the set of integral points contained in a convex polyhedral cone C.

If Y is a G-orbit, then $\overline{Y}$ is an irreducible closed G-stable set. And if Z is an irreducible G-stable set, then we can consider the prime ideal $I(Z)$, it is a G-module. So

$$I(Z) = \bigoplus_{\lambda \in \Lambda} V(\lambda).$$

And $I(Z)$ is prime, so Λ is the set of integral points on a face of C. Since C has finitely many faces, it follows that X has finitely many G-orbits. $\square$

Exercise 1.6. Prove the statement (ii) of Theorem 1.4 for $G = SL_2$.

2 The Luna-Vust theory of embeddings

We fix an open G-orbit $X_0 \subset X$. In order to describe $X \setminus X_0$, we will use the valuations of $\mathbb{C}(X) = \mathbb{C}(X_0)$.

Example 2.1. If $G = SL_2$, then $X = \mathbb{C}^2$ is a spherical variety and $X_0 = \mathbb{C}^2 \setminus \{0\}$ is an open G-orbit.

But $X' = \mathrm{Bl}_0(\mathbb{C}^2)$ (the blow-up of 0 in $\mathbb{C}^2$), $X'' = \mathbb{P}^2$ and $X''' = \mathrm{Bl}_0(\mathbb{P}^2)$ (the llow-up of 0 in $\mathbb{P}^2$) are also spherical variety that contain an open G-orbit isomorphic to $\mathbb{C}^2 \setminus \{0\}$.

Definition 2.2. Let X be a spherical variety.

(i) $\Xi(X) = \{\lambda | \lambda$ is a B-eigenvalue of a B-eigenvector $f_\lambda \in \mathbb{C}(X)\}$.

(ii) $N(X) = \mathrm{Hom}_{\mathbb{Z}}(\Xi(X), \mathbb{Q})$.

(iii) If $D \subset X$ is a B-stable prime divisor then we define $\rho(D) \in N(X)$ as $\langle \rho(D), \lambda \rangle := \mathrm{ord}_D f_\lambda$.

Remark 2.3. Since X is spherical, λ determines f_λ up to multiplication by a constant. Because, if f_λ, $g_\lambda \in \mathbb{C}(X)$ are semi-invariant with weight λ, then f_λ / g_λ is B-invariant, thus constant.

Remark 2.4. If $Z \subset X \setminus X_0$ is of codimension strictly greater than 1, then $Z \subset D$ for a prime divisor D that is B-stable but not G-stable.

Definition 2.5. A divisor $D \subset X$ is a *color* if it is a B-stable, not G-stable prime divisor.

The set of colors of X is denoted $\Delta(X)$.

Example 2.6. We consider $G = SL_2$ and B the Borel subgroup of upper triangular matrices of determinant 1.

Then the only color of the spherical variety $X = \mathbb{C}^2$ is $\{x = 0\}$. It contains $X \setminus X_0 = \{0\}$.

Definition 2.7. Let X be a spherical variety, and $Y \subset X$ be a G-orbit.

We denote the convex cone in $N(X)$ generated by $\rho(D)$ for D a B-stable prime divisor containing Y by $\mathcal{C}_Y$.

Let $\mathcal{D}_Y = \{D | D$ is a color of X containing $Y\}$.

The *colored cone* of Y is the couple $(\mathcal{C}_Y, \mathcal{D}_Y)$.

The *colored fan* of X is the set $\mathcal{F}_X = \{(\mathcal{C}_Y, \mathcal{D}_Y) | Y \subset X$ is a G-orbit$\}$.

The colored fan of X describes what is outside of the open G-orbit. If we fix an open G-orbit X_0 then the colored fan $\mathcal{F}_X$ uniquely determines X.

Theorem 2.8 (uniqueness part of Luna-Vust Theory)**.** *Let X, Y be spherical varieties with the same open G-orbit $X_0 \cong Y_0$. Then $\mathcal{F}_X = \mathcal{F}_Y$ if and only if $X \cong Y$ as G-varieties.*

Example 2.9. We consider $G = GL_{n+1}$ and X the quadrics in $\mathbb{P}^n$. Then the smooth quadrics in X form an open G-orbit. And $X = \mathbb{P}(M)$ where M is the set of $(n+1) \times (n+1)$ symmetric matrices.

If $\alpha_1, \ldots, \alpha_n$ are simple roots, then (exercise) $\Xi(X) = \langle 2\alpha_1, 2\alpha_2, \ldots, 2\alpha_n \rangle$ (Hint: consider $p \in X$ as $p = [q]$ where q is a symmetric matrix, and define B-semi-invariant rational functions on X using minors of q.)

If $n = 2$ then X has 2 colors D_1 and D_2 such that $\rho(D_1) = \frac{1}{2}\alpha_1^\vee$ and $\rho(D_2) = \frac{1}{2}\alpha_2^\vee$.

Theorem 2.10. *Let X_0 be a spherical homogeneous space. Let $\mathcal{F}$ be a set of couples $(\mathcal{C}, \mathcal{D})$ such that $\mathcal{C}$ is a strictly convex polyhedral cone in $N(X_0)$ and $\mathcal{D}$ is a set of colors of X_0.*

Then $\mathcal{F}$ is $\mathcal{F}(X)$ for a spherical variety X with open G-orbit X_0 if and only if the following conditions are verified for all $(\mathcal{C}, \mathcal{D})$ in $\mathcal{F}$:

(i) $\mathcal{C}$ is generated by $\rho(D)$ for $D \in \mathcal{D}$ and by finitely many elements in the valuation cone of X_0 (denoted by $V(X_0)$ [1]);

(ii) For every $D \in \mathcal{D}$, we have $\rho(D) \neq 0$;

(iii) $\mathcal{C}^0 \cap V(X_0) \neq \emptyset$, where $\mathcal{C}^0$ is the relative interior of $\mathcal{C}$;

(iv) Every face of $(\mathcal{C}, \mathcal{D})$ belongs to $\mathcal{F}$, where $(\mathcal{C}', \mathcal{D}')$ is called a face of $(\mathcal{C}, \mathcal{D})$ if $\mathcal{C}'$ is a face of $\mathcal{C}$ and $\mathcal{D}' = \mathcal{D} \cap \rho^{-1}(\mathcal{C}')$;

(v) For all $x \in V(X_0)$, there is at most one $(\mathcal{C}, \mathcal{D}) \in \mathcal{F}$ such that $x \in \mathcal{C}^0$.

3 Local structure theorem

The colored fan determines uniquely a spherical variety among those having the same open G-orbit. Nevertheless, it is not trivial to deduce geometric properties from the colored fan. For example, using the colored fan it is relatively easy to determine whether a variety is complete, or projective, and there is an explicit description of the Picard group. Other questions are more difficult, for example whether the variety is smooth.

In this section we will see a powerful tool to investigate the geometry of a spherical variety in the neighborhood of (points on) a G-orbit.

Example 3.1. Let G be GL_{n+1} and let X be the quadric in $\mathbb{P}^n$. Then $p = [x_0^2]$ lies on the closed G-orbit.

Exercise: no neighbourhood of $G.p$ in X retracts G-equivalently to $G.p$ (Hint: compare the stabilizers of a general point of X and of p).

Let us consider the neighbourhood $Y = \{q(1, 0, \ldots, 0) \neq 0\}$ of p, where q denotes a quadratic form representing a point $[q]$ on X. If $[q] \in Y$ then $q = c.(x_0 + a_1x_1 + \cdots + a_nx_n)^2 + q'(x_1, \ldots, x_n)$ where q' is also a quadratic form and $c \in \mathbb{C} \setminus \{0\}$, $a_i \in \mathbb{C}$ for all $i \in \{1, \ldots, n\}$.

Let $P = \mathrm{Stab}_G(Y)$. Then $P = \mathbb{C}^n \rtimes L$, where L is a Levi subgroup of P, and we can choose $L = GL_n$ acting linearly on $x_1, \ldots, x_n$.

The unipotent radical $P^u \cong \mathbb{C}^n$ of P acts by translation on $a_1, \ldots, a_n$.

We consider $S = \big\{[c.x_0^2 + q'(x_1, \ldots, x_n)]\big\}$. The above expression $q = c.(x_0 + a_1x_1 + \cdots + a_nx_n)^2 + q'(x_1, \ldots, x_n)$ for $[q] \in Y$ shows that $(u, [q]) \in P^u \times S \to u.[q] \in Y$ is an isomorphism. Moreover this isomorphism is P-equivariant, where for every $p = ul$ ($u \in P^u$ and $l \in L$) we set $p \cdot (g, [q]) = \big(ulgl^{-1}, l[q]\big)$.

Theorem 3.2. *We consider X a spherical variety and Z a closed G-orbit. Let us consider*

$$X_Z = X \setminus \bigcup_D D,$$

where the union is over the B-stable prime divisors D not containing Z.

Then X_Z is open, affine and intersects Z. If $P = Stab_G(X_Z)$, then $B \subseteq P$, and there is $S_Z \subseteq X_Z$ such that:

[1] For the definition of $V(X_0)$ see Section 4, Definition 4.5

(i) S_Z is closed in X_Z, it is L-stable (where L is a Levi of P) and $P^u \times S_Z \to X_Z$ is a P-equivariant isomorphism.

(ii) S_Z is a spherical L-variety.

Remark 3.3. (i) X_Z is smooth if and only if X is smooth along Z if and only if S_Z is smooth.

(ii) S_Z is a toric variety (i.e. (L, L) acts trivially) for all G-orbits Z if and only if no color of X contains a G-orbit.

In this case X is call toroidal.

Exercise 3.4. Find S_Z for $X = \mathbb{P}^1 \times \mathbb{P}^1$, $G = SL_2$ and $Z = \operatorname{diag}\left(\mathbb{P}^1\right)$.

4 Spherical roots and the multiplication of regular functions

4.1 Spherical roots

Remark 4.1. If X is a quasi-affine spherical variety, then

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \Gamma} V(\lambda)$$

for a set of dominant weights Γ.

In general $\mathbb{C}[X]$ is not graded by Γ, i.e., $V(\lambda)V(\mu) \supseteq V(\lambda + \mu)$ but in general the inclusion is strict. (Here we denote by $V(\lambda)V(\mu)$ the vector spaced spanned by the products fg where $f \in V(\lambda)$ and $g \in V(\mu)$. It is a G-submodule.)

Example 4.2. We consider $G = SL_2$ and $X = \mathbb{C}^2$. Let ω be the fundamental dominant weight, then

$$\mathbb{C}[X] = \bigoplus_{\lambda \in \mathbb{N}\omega} V(\lambda)$$

is graded.

But if $Y = SL_2/T$, then

$$\mathbb{C}[Y] = \bigoplus_{\lambda \in \mathbb{N}2\omega} V(\lambda)$$

is not graded.

Even for well-known varieties, it is extremely difficult to describe the set of (λ, μ, ν) satisfying $V(\nu) \subseteq V(\lambda)V(\mu)$.

However, we can extract some "global data" from the set of differences $\lambda + \mu - \nu$, and remarkably this turns out to be of fundamental importance for studying the geometry of X.

Definition 4.3. If X is a quasi-affine spherical variety we define:

(i) $\tau(X) = \{\lambda + \mu - \nu | V(\nu) \subseteq V(\lambda)V(\mu)\}$.

(ii) We denote by $\Sigma(X)$ the set of primitive elements in $\Xi(X)$ on the extremal rays of the cone $\operatorname{cone}(\tau(X))$ generated by $\tau(X)$.

The elements of $\Sigma(X)$ are called the *spherical roots* of X.

Example 4.4. (i) $\Sigma(\mathbb{C}^2) = \emptyset$

(ii) $\Sigma(SL_2/T) = \{\alpha\}$ (Exercise: Find λ, μ, ν such that $\nu = \lambda + \nu - 2\alpha$)

(iii) $\Sigma(\text{quadrics in } \mathbb{P}^n) = \{2\alpha_1, \dots, 2\alpha_n\}$

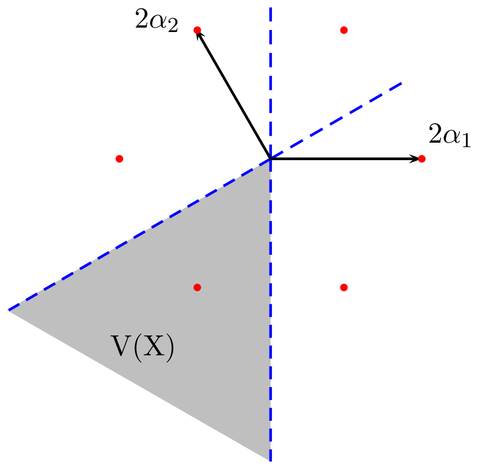
(iv) $\Sigma(SL_{n+1}/GL_n) = \{\alpha_1 + \cdots + \alpha_n\}$

If X is not a quasi-affine, then we observe that $X_0 \subseteq \mathbb{P}(V)$ where V is a G-module.

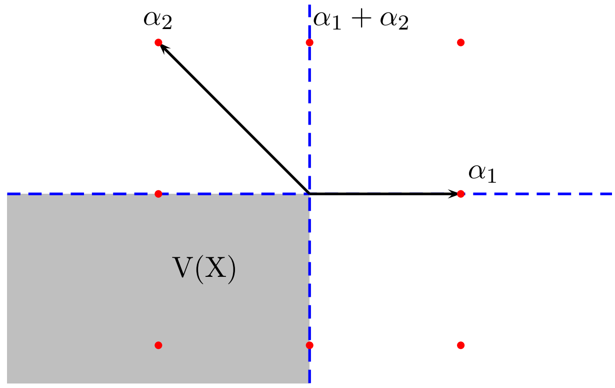
We define in this case $\tau(X) = \{\lambda + \mu - \nu | V(\nu) \subseteq V(\lambda)V(\mu) \in \mathbb{C}[\widehat{X_0}]\}$, where $\widehat{X_0}$ is the cone in V over X_0. (One shows this doesn't depend on V, and that $\Sigma(X) \subset \Xi(X)$).

Definition 4.5. $V(X) = \{\eta \in N(X) | \forall \sigma \in \Sigma(X),\ \langle \sigma, \eta \rangle \leqslant 0\}$

Example 4.6. (i) We consider $G = GL_3$ and X the conics in $\mathbb{P}^2$.



(ii) Let us consider $G = SO_5$ and $X = SO_5/GL_2$. Then $\Sigma(X) = \{\alpha_1, \alpha_1 + \alpha_2\}$.



Question 4.7. *Why does $V(X)$ appear in the Luna-Vust theorem?*

In order to answer this question, we consider $\nu : \mathbb{C}(X)^* \to \mathbb{Q}$ discrete G-invariant valuation.

Then ν induces an element $\eta \in N(X)$ similarly as above: for every B-semi-invariant $f_\lambda \in \mathbb{C}(X)$ of B-eigenvalue λ, we set $\nu(f_\lambda) = \langle \eta, \lambda \rangle$.

Proposition 4.8. *$V(X)$ is the set of $\eta \in N(X)$ such that η is induced by some G-invariant valuation.*

Exercise 4.9. If X is affine, prove that: if $\nu : \mathbb{C}(X)^* \to \mathbb{Q}$ is a G-invariant discrete valuation, and $V(\nu) \subseteq V(\lambda)V(\mu)$, then $\eta(\lambda + \mu - \nu) \leqslant 0$.

4.2 Classification of homogeneous spherical varieties

The following Theorem is due to M. Brion [Bri90] in characteristic 0 and F. Knop [Kno14] in characteristic $p > 2$. It is false in characteristic 2 [Sch11].

Theorem 4.10. *$V(X)$ is a fundamental chamber of a Weyl group $W(X)$ of a root system with simple roots $\Sigma(X)$.*

We have the following analogy: $\Xi(X)$: Lattice; $\Sigma(X)$: Simple roots of a root system; $\Delta(X)$: "Simple coroots".

This analogy is precise for $\Sigma(X)$, which is indeed the set of simple roots of a root system, but not for $\Delta(X)$, which is not the corresponding set of simple coroots. However it possesses some quite strict combinatorial properties, motivating the idea that (Ξ, Σ, Δ) is in some sense a generalization of a root datum.

Luna has given combinatorial axioms in [Lun01] of this generalization, calling the resulting objects *homogeneous spherical data*. They also include a fourth object, which is a parabolic subgroup of G (see the theorem below) and is actually somewhat less relevant, since in most interesting cases it can be deduced from the other three data.

Example 4.11. We consider $G = GL_{n+1}$, and we write $B = TU$ where T is a maximal torus and U the unipotent radical. Set $H = T \cdot (U, U)$.

Then $\Sigma(G/H) = \{\alpha_1, \ldots, \alpha_n\}$, and $\Delta(G/H) = \{D_1^+, D_1^-, \ldots, D_n^+, D_n^-\}$ (so it has $2n$ elements). Then $\Delta(G/H)$ has too many elements to be the set of simple coroots of $\Sigma(G/H)$, but notice that it holds $D_i^+ + D_i^- = \alpha_i^\vee$ for all i.

Theorem 4.12. *[BP16, Th. 1.2.3] Let X_0 be a homogeneous spherical G-variety, and let $P(X_0)$ be the stabilizer in G of the open B-orbit X_0.*

The map:

$$X_0 \to (\Xi(X_0), \Sigma(X_0), \Delta(X_0), P(X_0))$$

is a bijection between homogeneous spherical varieties and homogeneous spherical data.

5 References

[BP16] P. Bravi and G. Pezzini. Primitive wonderful varieties. *Mathematische Zeitschrift*, 282:1067–1096, 2016.

[Bri90] M. Brion. Vers une généralisation des espaces symétriques. (Towards a generalization of symmetric spaces). *J. Algebra*, 134(1):115–143, 1990.

[HL16] R. Hodges and V. Lakshmibai. Levi Subgroup Actions on Schubert Varieties, Induced Decompositions of their Coordinate Rings, Sphericity and Singularity Consequences. *ArXiv e-prints*, September 2016.

[Kno14] F. Knop. Racines sphériques de variétés sphériques. *Ann. Inst. Fourier*, 64(6):2503–2526, 2014.

[Lun01] D. Luna. Variétés sphériques de type a. *Publications Mathématiques de l'IHÉS*, 94:161–226, 2001.

[Sch11] B. Schalke. Sphärische Einbettungen in positiver Charakteristik, 2011.