Singularities of (L)MMP and applications

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Introduction

These lectures are devoted to an introduction to singularities occurring in the various flavors of Minimal Model Program. In these days an enormous interest came back to the program thank to the amazing results of Birkar, Cascini, Hacon, and Mc Kernan, [**BCHM**]. The Grenoble school is therefore a lucky coincidence to attract even more interest on the subject.

The first chapter is a tribute to [**YPG**]. The wonderful paper of Miles Reid that allowed various generation of mathematician to understand singularities and how to work with them. It is not by chance that after more than twenty years this is still a milestone of the theory. Here I revise the main constructions and results of terminal and canonical 3-folds. My main aim is to give examples and geometric intuition to this technical part of MMP theory. For this reason the first and the last section are, hopefully, half way between examples and exercises one should work out by himself on his desk.

The second chapter is dedicated to singularities of pairs. Here it is difficult to put technicalities apart. I hope that the applications to birational and biregular geometry given in the last two sections will help swallow all the definitions. For this part there are various references, the ones that I mainly follow are, [**U2**], [**KM2**], and [C$^+$].

It is a pleasure to thank the organizers of this school for the great job they did and for the warm hospitality they gave to my family and myself. I want to thank all the participants for their interest and comments on both the oral and this written presentation. A special thank to Laurent Bonavero, Stephane Druel and Alex Kuronya for their comments and remarks.

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CHAPTER 1

Terminal and canonical singularities

1. Why bother with singular varieties ?

Doing algebraic geometry is quite hard. To simplify life usually one tries to consider smooth varieties. Hironaka's result [**Hi**] tells us that over a characteristic zero field we always have a smooth model birational to the one we started with. This suggests that in doing birational geometry it would be simpler to consider smooth objects. This general belief was prevailing till the '70's. A further important motivation was the successful classification of surfaces completed by the Italian school of the beginning of the XX$^{\text{th}}$ century.

Let us briefly review the theory.

DEFINITION 1.1.1. Let S be a smooth surface and $C \subset S$ a smooth rational curve. Then C is called a (-1)-curve if $C^2 = -1$ and S is called minimal if it does not contain (-1)-curves.

A central result is the following contraction theorem, you can find a modern proof in [**Ha**].

THEOREM 1.1.2 (Castelnuovo Contraction Theorem). *Let $C \subset S$ be a (-1)-curve. Then there exists a smooth surface S' and a birational map $f : S \to S'$ such that $f(C) = p$ and $f_{|S\setminus C} : S \setminus C \to S' \setminus p$ is an isomorphism.*

REMARK 1.1.3. *Note that if you look at the morphism $f : S \to S'$ from the surface S' this is just the blow up of the maximal ideal of the point p.*

In other words if you have a (-1)-curve then you can contract it and the resulting surface is still smooth. If you are interested in the birational geometry of surfaces you can always assume that your surface is minimal. Starting from this Castelnuovo and Enriques, with a lot of work, started the classification of minimal surfaces and proved that any morphism between two smooth surfaces can be factored by a series of these contractions.

For a long time an higher dimensional analog was looked for. The idea was to find elementary contractions that kept smoothness and allowed to factor any map between smooth objects. At the end of the '70's S. Mori and M. Reid explained to the world that this was not the right approach. They realized that the main point of Castelnuovo's result was not the smoothness but the negativity of the canonical class.

DEFINITION 1.1.4. Let X be a normal variety and D a Cartier divisor. We say that D is nef if $D \cdot C \geq 0$ for any curve $C \subset X$.

REMARK 1.1.5. *If you are worried to perform intersection on a singular variety consider the pull-back of D, it is a Cartier divisor!, on a resolution and then use the projection formula.*

When you consider a smooth variety X there is a unique divisor class naturally attached to X.

DEFINITION 1.1.6. Let X be a smooth n-fold, $\omega_X :=: \Omega^n_X$ is the invertible sheaf generated by $dx_1 \wedge \ldots \wedge dx_n$, where $(x_1, \ldots, x_n)$ are local coordinates. We indicate with K_X the Cartier divisor associated to the sheaf ω_X and we call it the canonical divisor or canonical class.

There are many good reasons to consider this sheaf: it is intrinsic, makes Serre duality work, vanishing theorems, adjunction formulas, the sections are a birational invariant for smooth varieties, ... see [**YPG**]

Let us consider a (-1)-curve C. Then due to adjunction formula

$$K_S \cdot C = \deg \omega_C - C^2 = -1$$

(-1)-curves have negative intersection with the canonical class of a surface. The idea is to modify Definition 1.1.1 in the following way

DEFINITION 1.1.7. Let X be a variety we say that X is a minimal model if K_X is nef.

GOING IN DEEPER 1.1.8. The two definitions are not equivalent. Think of a minimal surface S, in the old definition, that is not a minimal model.

To get a minimal model it is therefore enough to get rid of all curves that are negative with respect to the canonical class. To do this we are forced to leave smooth objects and much more.

EXAMPLE 1.1.9 (Cone over the Veronese). *Consider a Veronese surface $V \subset \mathbb{P}^5$ and let $X \subset \mathbb{P}^6$ be the cone over it, with vertex $v \in X$. It is standard to realize X by a morphism*

$$\epsilon : Y := \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(-2)) \to X \subset \mathbb{P}^6$$

that contracts the section associated to $\mathcal{O}$. The effective curves on Y are all linear combinations of f, a line in the projective bundle structure over $\mathbb{P}^2$, and l, a line in the ϵ-exceptional divisor $E \cong \mathbb{P}^2$. It is easy to check that $-K_X \cdot f = -2$ and $-K_X \cdot l = -1$. Moreover the normal bundle of E reads $E_{|E} \sim \mathcal{O}(-2)$. Note that due to Definition 1.1.6 $K_{Y \setminus E} \cong K_{X \setminus \{v\}}$. It is not difficult to see that $2K_X \sim \mathcal{O}_X(3)$ and is therefore a Cartier divisor. We can therefore express the discrepancy between K_X and K_Y in the following form

$$K_Y = \epsilon^*(K_X) + 1/2E$$

To get rid of negative curves on Y we are forced, either to consider the bundle structure or to contract the divisor E, introducing a singularity. In the latter case we then obtain a 3-fold X where the canonical divisor is not Cartier. In this way we produce a non Gorenstein 3-fold from a smooth one. It is hard to believe the amazing world we enter from here.

Note that to resolve the singularity of X it is enough to blow up the maximal ideal, one can do it directly in the embedding in $\mathbb{P}^6$. Further note that in the language of weighted projective spaces (wps) $X = \mathbb{P}(1, 1, 1, 2)$.

This is bad but not too bad. The canonical class is no longer a Cartier divisor but $2K_X$ is Cartier therefore if we allow rational intersection numbers nothing really changes.

Definition 1.1.10. Let X be a normal variety and D a Weil divisor, i.e. a combination of codimension 1 subvarieties. We say that D is $\mathbb{Q}$-Cartier if there exists an integer m such that mD is Cartier.

Remark 1.1.11. *The intersection theory works perfectly well for $\mathbb{Q}$-divisors. We can still talk of nef $\mathbb{Q}$-divisors. In particular if K_X is $\mathbb{Q}$-Cartier it is meaningful to ask whether it is nef.*

These kinds of singularities are not the worst thing can happen. The main problem comes from "small" contractions when $\dim X \geq 3$. Assume that it is possible to contract a finite number of curves, say Z such that $K_X \cdot Z < 0$. Let $f : X \to W$ the contraction. I claim that K_W cannot be $\mathbb{Q}$-Cartier. If this was the case then mK_X and f^*mK_W would agree outside a codimension 2 set and therefore they would define the same Cartier divisor. On the other hand $K_X \cdot Z < 0$ while $f^*K_W \cdot Z = 0$. This means that on W the notion of nef canonical class is meaningless. To get rid of the curve Z we therefore cannot contract it to the variety W.

One of the main point of MMP is to control the birational modifications originated by this "small" contractions. We realized that it is not allowed to contract them. Something different has to be done.

Definition 1.1.12 (heuristic definition of 3-fold flip). Let $f : X^- \to W$ be a contraction of a curve $Z = \cup C_i$ on the 3-fold X, with $-K_X$ relatively ample. The flip of f is a modification χ

$$\begin{array}{ccc} X & \overset{\chi}{\dashrightarrow} & X^+ \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & W & \end{array}$$

that makes K_{X^+} f^+-ample, in particular χ is an isomorphism outside of Z .

It is good to have at least a simple example of this both locally and in a projective setting.

Example 1.1.13 (local Francia's flip). *Let $C \subset X^+$ be an analytic neighborhood of a smooth rational curve with $N_{C/X^+} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2)$. Then we can assume that $C = S \cap T$ is a complete intersection of two smooth surfaces*

You can convince yourself that there are no other effective curves numerically proportional to l on X. The splitting of the normal bundle says that all these curves stays in the intersection of the surfaces S and T.

Following figure 1 we blow up first C, to get $F_1 \cong \mathbb{F}_1$ and then $C' := S \cap F_1$ to get $F_0 \cong \mathbb{F}_0$. Then we can contract with the morphism g, in the projective category!, the other ruling of F_0 and finally the $g_*(F_1) \cong \mathbb{P}^2$. Note that the latter has normal bundle $\mathcal{O}(-2)$, this is easily read from the picture since a line of the $\mathbb{P}^2$ has self intersection -2 inside T. This produces a curve C^- with $K_{X^-} \cdot C^- = -1/2$ and a singular point, v locally isomorphic to the one described in Example 1.1.9. We just followed the inverse modification of the Francia flip, i.e. the birational map $\chi : X^- \dashrightarrow X^+$. The modification on C' is called a flop.

The above analytic situation can be realized on projective 3-folds.

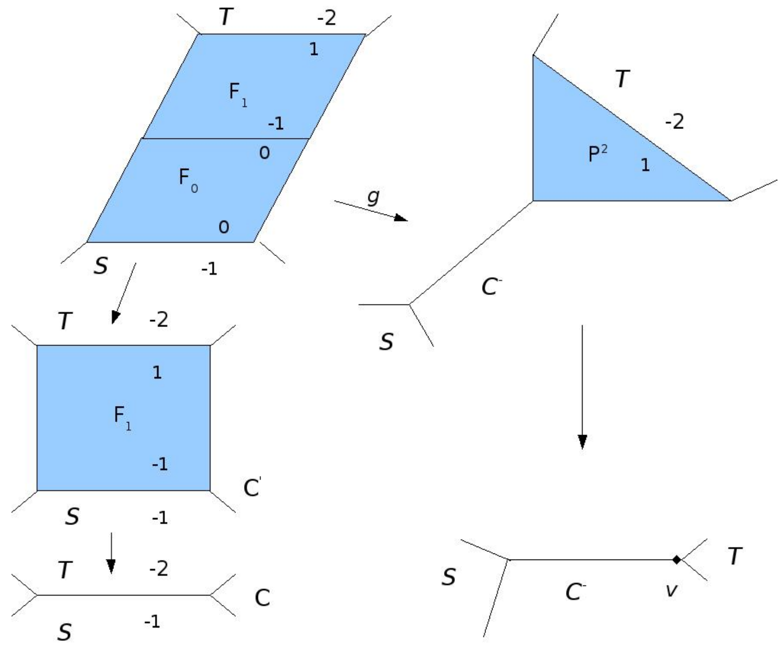


FIGURE 1. Local Francia's flip

EXAMPLE 1.1.14 (Projective Francia's flip). *Let* $x \equiv (1, 0, 0, 0) \in \mathbb{P}^3$ *and consider a weighted blow up (wbu) with weights* $(1, 2, 3)$*, say* $\epsilon : Y \to \mathbb{P}^3$*, with exceptional divisor* $E \cong \mathbb{P}(1, 2, 3)$*. You can think of it as follows.*

Following figure 2, we blow up a point, to get an exceptional divisor $F_1 \cong \mathbb{P}^2$*. Then blow up a point* $x_1 \in F_1$*, to get another exceptional divisor* $F_2 \cong \mathbb{P}^2$*, note that the coordinates with weight* ≥ 2 *are those passing through this point. Finally blow up a line* $r \subset F_2$*, to get the valuation* E*, this line gives you the weight 3 coordinate.*

To go back onto Y *one has to do an inverse Francia flip, and contract the (strict transform of) divisors* F_i*.*

Choose the coordinate in such a way that the local coordinates of the blow up are (x_1, x_2, x_3)*. Then consider the line* $l = (x_2 = x_3 = 0)$*. It is easy to check that* $l_Y = \epsilon_*^{-1}(l)$ *is on the smooth locus and* $l_Y \cdot E = 1$*. By standard formulas we have* $K_Y = \epsilon^* K_{\mathbb{P}^3} + 5E$*. Then* $K_Y \cdot l = 1$*. Let* $S = \epsilon_*^{-1}(x_2 = 0)$ *and* $T = \epsilon_*^{-1}(x_3 = 0)$*. To compute the normal bundle of* l_Y *it is enough to observe that* $l_Y = S \cap T$*. Moreover* $\epsilon_{|S}$ *is just the wbu with weights* $(1, 3)$*, while* $\epsilon_{|T}$ *is the wbu* $(1, 2)$*. Therefore* $(l_Y \cdot l_Y)_S = 1 - 3 = -2$ *while* $(l_Y \cdot l_Y)_T = 1 - 2 = -1$*. This gives* $N_{l_Y/Y} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-2)$*.*

GOING IN DEEPER 1.1.15. Do by yourself, in details:

. the flop of the curve with normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, first locally and then find a projective realization
. the flip and the contractions in the geometric description of the weighted blow up.

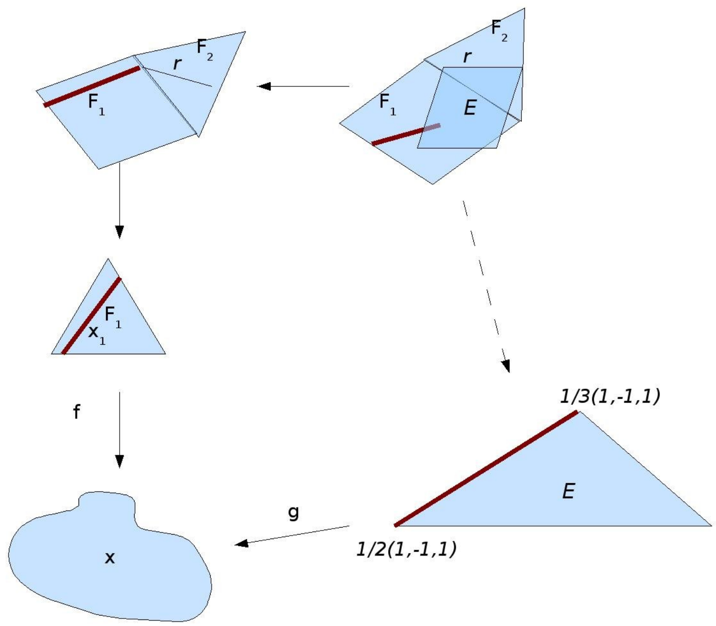


FIGURE 2. The wbu $(1, 2, 3)$

2. Terminal and Canonical singularities

It is time to define the singularities we are going to study.

DEFINITION 1.2.1. A variety X is terminal (canonical) if the following conditions are fulfilled:

. X is normal
. K_X is $\mathbb{Q}$-Cartier
. there exists a resolution $f : Y \to X$ such that

$$K_Y = f^* K_X + \sum a_i E_i$$

with $a_i > 0$ ($a_i \geq 0$)

The rational numbers a_i are called discrepancies and are associated to the valuation E_i. To stress this dependence we write $a(E_i, X) := a_i$. Note that the existence of a resolution with this property forces all resolutions to enjoy the same bounds.

This definition needs some explanations, [**Re1**]. On a normal variety X there is a one to one correspondence between

$$\{\text{rk1 reflexive sheaves}\} \longleftrightarrow \{\text{Weil divisors D} \subset \text{X}\}/\text{rational equivalence}$$

Note that these are codimension 1 objects. That is they are defined outside a codimension 2 set. On a normal projective variety the dualizing sheaf ω_X (i.e. the sheaf that gives the pairing $Hom(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \cong \mathbb{C}$) is reflexive of rank 1. In particular to such a sheaf we can associate a Weil divisor, K_X. It is therefore meaningful to say it is $\mathbb{Q}$-Cartier. A different viewpoint is to consider the smooth locus $X^0 \subset X$. The variety X is normal therefore $(X \setminus X^0)$ is of codimension at least 2. Let $j : X^0 \hookrightarrow X$ the natural inclusion then we can, equivalently, define $\omega_X := j_*(\Omega^n_{X^0})^{**}$.

EXAMPLE 1.2.2 (The surface case). *It is interesting and a good training to study the surface case. Consider a terminal surface S and a resolution $f : Z \to S$, then*

$$K_Z = f^*K_S + \sum a_iE_i \quad \text{with } a_i > 0$$

The normality of S ensures that the fibers are connected. A result of Mumford, Grauert, see [**BPvV**]*, says that that the intersection matrix is negative definite. In particular there is a j such that*

$$E_j \cdot (\sum a_iE_i) < 0$$

This gives $K_Z \cdot E_j < 0$ and $E_j^2 < 0$. Then E_j is a (-1)-curve. By Castelnuovo we can contract E_j and recursively prove that S is smooth. Terminal surfaces are smooth!

Let us now consider canonical surfaces. Let S be a canonical surface and $f : Z \to S$ a relatively minimal resolution, i.e. there are no (-1)-curves contracted by f. The definition and the above calculation show that

$$K_Z = f^*K_S$$

Let E be a curve contracted by f. Then $K_Z \cdot E = 0$ and $E^2 < 0$. Then $(K_Z + E)_{|E}$ is not effective and E is a smooth rational curve with $E^2 = -2$. Moreover the negativity of the intersection matrix shows $(E_i + E_j)^2 < 0$ for any pair of curves therefore $E_i \cdot E_j \leq 1$ and any connected fiber of f is a tree of rational curves. The negativity of the intersection matrix also forces the tree to be one of the Dynkin diagrams A_n, D_n, E_6, E_7, and E_8. This class of singularities are well known. They are called Rational Double Points, Du Val singularities, ADE singularities, in particular they are the singularities of canonical models of surfaces of general type, see definition 1.3.1 for the equations

Note that slicing with a surface section you get that for a resolution $f : Y \to X$ of a terminal variety $-K_Y$ is not f-nef. One can even get a refined version of it via Negativity Lemma [**U2**, Lemma 2.19].

The singularities we are interested in have a nice behavior when we pass from X to a general hyperplane section. Fix a resolution $f : Z \to X$. Let D be a general Cartier divisor on X and $D_Z = f_*^{-1}D$. Then the adjunction formulas read

$$(K_X + D)_{|D} = K_D$$

and

$$K_Z + D_Z = f^*(K_X + D) + \sum a_iE_i$$

Then the singularities of D are not worse than those of X, see Proposition 2.2.2 for a more detailed account. The other direction, that is determine the discrepancies of X from that of a section is called inversion of adjunction and is a central problem in MMP theory, we will discuss some of its instances in Chapter 2. We have thus proved that terminal singularities are smooth in codimension 2, while canonical singularities are Du Val in codimension 2.

Before exploiting other interesting features of the definition let us observe the following.

LEMMA 1.2.3. *Let $f : Y \to X$ be a birational map between normal varieties and E_i exceptional divisors. Then $f_*(\sum a_i E_i) = \mathcal{O}_X$ if and only if $a_i \geq 0$ for all i.*

PROOF. Consider the sheaf $\mathcal{O}_Y(D)$ as rational functions in $K(Y)$ with at worst poles along D. Any divisor E_i correspond to some valuation with a center on X of codimension at least two. Therefore to get $\mathcal{O}_X$ we cannot have zero's along any E_i. $\square$

It is also interesting to observe that canonical singularities are preserved by étale morphism in codimension 1, [**Re1**] (square a resolution with the etale morphism and consider the equivalent definition of canonical with $f_*\omega_Y = \omega_X$ that is the pull back of a generator is a generator).

DEFINITION-THEOREM 1.2.4 (Reid). Let $f : Y \dashrightarrow X$ be a birational map between canonical varieties, then

$$p_n(X) := h^0(X, nK_X) = h^0(Y, nK_Y) =: p_n(Y)$$

for all n. In other words the canonical rings $R(X, K_X) := \bigoplus H^0(X, nK_X)$ and $R(Y, K_Y)$ are the same.

PROOF. By considering a resolution of the map f we can assume that f is a morphism. Then by hypothesis

$$nK_Y = f^* nK_X + \sum n a_i E_i$$

where the E_i are f-exceptional and $a_i \geq 0$. By Lemma 1.2.3 there is a bijection between the sections of nK_Y and nK_X. $\square$

The following example shows, if you are not convinced yet, the need of singularities in the MMP.

EXAMPLE 1.2.5. *Let $X := X_{14} \subset \mathbb{P}(1,1,2,2,7)$ be an hypersurface in the weighted projective space. The general such has terminal singularities and $K_X \sim \mathcal{O}(1)$. In particular $K_X^3 = 1/2$ and therefore*

$$p_n(X) \sim \frac{1}{3!}\frac{1}{2}n^3$$

as $n \to \infty$. No smooth minimal model can have this behavior in plurigenus. By Kawamata–Viehweg vanishing theorem and Riemann–Roch theorem a minimal 3-fold of general type has $h^0(X, mK_X) = \chi(X, mK_X) \sim 1/3! K_X^3 m^3$. This shows that X is not birational to any smooth minimal model.

Whenever you have a graded ring consider its Proj.

DEFINITION 1.2.6. Let X be a canonical n-fold of general type, that is $\kappa(X) = n$ or K_X big. Assume that $R(X, K_X)$ is finitely generated, this is now a Theorem [**BCHM**]. Then $Proj(R(X, K_X))$ is the canonical model of X.

REMARK 1.2.7. *You can think of the canonical model as the image of X via sections of a pluricanonical system. In case K_X is ample it is an isomorphism and X is itself a canonical model. When K_X is free then you maybe contract some classes of curves. In general mK_X has base locus and the map is rational.*

An important feature of canonical singularities is the following higher dimensional analog of what we observed for surfaces.

THEOREM 1.2.8 ([**YPG**], [**Re1**]). *The canonical model has canonical singularities.*

PROOF. By the birational invariance we can assume that X is smooth and $|kK_X| = F + \mathcal{M}$ where F is a fixed part and $\mathcal{M}$ is a free linear system. Let $g : X \to Y \subset \mathbb{P}^N$ be the map given by $\mathcal{M}$. By finite generation I can, and will, assume that Y is independent of k. In particular Y is normal, this is the geometric Stein factorization, [**YPG**]. Let us prove that $\operatorname{codim} g(F) \geq 2$. Assume that there is a divisorial component $G \subset F$ that is mapped to a divisor. This means that for any divisor D on G

$$h^0(G, mM + D) \sim m^{n-1}$$

for $m \gg 0$ and $M \in \mathcal{M}$.

CLAIM 1. $G \subset \operatorname{Bs}|mM + G|$ for all $m \gg 0$.

PROOF FO THE CLAIM. Assume that $G \not\subset |mM + G|$. Then we have a section $A \in |mM + mF|$ with $\operatorname{mult}_G A < m$. On the other hand by finite generation we have the equality

$$H^0(X, m(M + F)) = \otimes^m H^0(X, M + F)$$

The contradiction proves the claim. □

By the claim we have that

$$H^0(X, mM + G) \to H^0(G, (mM + G)_{|G})$$

is the zero map. Then, from the structure sequence, we have

$$h^1(X, mM) \sim m^{n-1}$$

On the other hand by Serre's vanishing $h^i(\mathcal{O}_Y(m)) = 0$ for $i = 1, 2$ and $m \gg 0$. Playing a bit with Leray's spectral sequence gives

$$H^1(X, mM) \cong H^0(Y, R^1 g_* mM)$$

The latter is supported on the image of the g-exceptional locus, then

$$h^1(X, mM) \sim m^{n-2}$$

This contradiction proves that $\operatorname{codim} g(F) \geq 2$.

It is now enough to consider the codimension 2 open set $Y^0 \subset Y$ where g is an isomorphism. Keep in mind that $g^{-1}(Y^0) \not\supset F$

$$\mathcal{O}_{Y^0}(1) \cong \mathcal{O}_{g^{-1}(Y^0)}(mM) \cong \mathcal{O}_{g^{-1}(Y^0)}(mK_X) \cong \mathcal{O}_{Y^0}(mK_Y)$$

Therefore K_Y is ample and

$$K_X = g^* K_Y + \text{ effective}$$

□

REMARK 1.2.9. *A crucial step in the finite generation proof of* [**BCHM**] *is to prove that the map given by* $|m(K_X + \Delta)|$ *contracts all the stable base locus.*

3. Three dimensional terminal singularities

We already proved that 3-fold terminal singularities are isolated. These singularities have many interesting properties that allow a complete classification and gave rise to the first proof of MMP for 3-fold, [**Mo3**]. The best reference is again, [**YPG**]. In this section we draw the main lines of this classification keeping [**YPG**] as a constant reference.

DEFINITION 1.3.1. A *cDV* singularity is a 3-fold hypersurface singularity

$$(P \in X) \cong (0 \in (F=0) \subset \mathbb{C}^4)$$

given by an equation of the form

$$F(x,y,z,t) = f(x,y,z) + tg(x,y,z,t)$$

where f is the equation of a Du Val singularity, and g is arbitrary.

$$\begin{array}{lcll} A_n & : & x^2+y^2+z^{n+1} & \text{for } n \geq 1 \\ D_n & : & x^2+y^2z+z^{n+1} & \text{for } n \geq 4 \\ E_6 & : & x^2+y^3+z^4 & \\ E_7 & : & x^2+y^3+yz^3 & \\ E_8 & : & x^2+y^3+z^5 & \end{array}$$

Note that cDV singularities can be interpreted as deformations of Du Val points and have hyperplane sections with Du Val singularities, see also Corollary 2.2.7.

An important tool to study terminal and canonical singularities is the cyclic covering trick. Let $P \in X$ be a point in a normal variety and $D \subset X$ a Weil $\mathbb{Q}$-Cartier divisor. Assume that rD is Cartier and $r'D$ is not Cartier for any $0 < r' < r$. This r is called the index of D. Fix a local basis $s \in \mathcal{O}_X(-rD)$, maybe shrinking X if necessary. We can then view s as an isomorphism

$$s : \mathcal{O}_X(rD) \xrightarrow{\sim} \mathcal{O}_X$$

Let X^0 be the open where D is Cartier. Then over X^0 we have an invertible sheaf $\mathcal{O}_{X^0}(-D)$ that correspond to a line bundle $L^0 \to X^0$. Let $z \in \mathcal{O}_{X^0}(-D)$ be a local generator. Next we consider $Y^0 := (z^r = s) \subset L^0$. Since s is nowhere vanishing then the projection $\pi^0 : Y^0 \to X^0$ is étale and with a bit of work one can extend this to the whole of X. With the isomorphism s one can build up a sheaf of $\mathcal{O}_X$-algebras

$$\mathcal{O}_X \oplus \mathcal{O}_X(D) \oplus \ldots \oplus \mathcal{O}_X((r-1)D)$$

to get the required extension.

PROPOSITION 1.3.2 ([**YPG**]). *There exists a cover $\pi : Y \to X$ which is Galois with group μ_r, and such that the sheaves $\mathcal{O}_X(iD)$ are the eigensheaves of the group action on $\pi_*\mathcal{O}_Y$, that is,*

$$\mathcal{O}_X(iD) = \{f \in \pi_*\mathcal{O}_Y | \epsilon(f) = \epsilon^i \cdot f \text{ for all } \epsilon \in \mu_r\}$$

Also, Y is normal, π is étale over X^0 and $\pi^{-1}P = Q$ is a single point outside of X^0. The $\mathbb{Q}$-divisor $\pi^*D = E$ is Cartier on Y.

Apply Proposition 1.3.2 to the canonical class to get.

PROPOSITION 1.3.3. *Let $P \in X$ be a canonical singularity then there is a local μ_r-cover $\pi : X' \to X$ with $K_{X'}$ Cartier and $K_{X'} = \pi^*K_X$.*

This reduces the study of canonical, and terminal singularities, to that of canonical (terminal) singularities with invertible canonical class.

A further crucial step in the classification is the following

THEOREM 1.3.4 ([**El**][**Fl**]). *Canonical singularities are rational. That is there exists a resolution $f : Y \to X$ such that $R^i f_* \mathcal{O}_Y = 0$ for all $i > 0$.*

There is a very nice and geometric proof of the above statement for 3-folds in [**YPG**].

Next one studies the general hyperplane section of a rational Gorenstein singularity and realizes that the section is either rational or elliptic, [**YPG**]. With this and a subtle analysis of surface elliptic singularities (i.e. for a resolution $f : T \to S$ we have $f_* \omega_t = m_x \cdot \omega_S$) Reid proved the following

THEOREM 1.3.5 ([**YPG**],[**Re2**]). *For a 3-fold X terminal Gorenstein is equivalent to isolated cDV.*

REMARK 1.3.6. *It is important to note that this analysis, crucial for the classification of terminal singularities, works only in the 3-fold case. No analog is possible in higher dimensions.*

Putting all together we have

THEOREM 1.3.7 ([**Re2**]). *Let $p \in X$ be a terminal 3-fold point. Then*

$$P \in X \cong Y/\mu_r$$

where $Q \in Y$ is either a cDV singularity or smooth, and μ_r acts on Y freely outside Q and such that on a generator $s \in \omega_Y$,

$$\mu_r \ni \epsilon : s \mapsto \epsilon s$$

To conclude the classification it is therefore enough to understand which actions give rise to terminal quotients. Assume Y is smooth then $Y \cong \mathbb{C}^3$ and the action can be diagonalized as

$$\mu_r \ni \epsilon : (x_1, x_2, x_3) \mapsto (\epsilon^{a_1} x_1, \epsilon^{a_2} x_2, \epsilon^{a_3} x_3)$$

for a triple $a_1, a_2, a_3 \in \mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$. It is therefore quite natural to indicate the type of the singularity as

$$\frac{1}{r}(a_1, a_2, a_3)$$

cDV are hypersurface singularities, therefore we can slightly modify the above to consider an action on $\mathbb{C}^4$ and indicate the type as

$$\frac{1}{r}(a_1, a_2, a_3, a_4), \text{ together with an invariant hypersurface } Y = (f = 0) \subset \mathbb{C}^4$$

Understand the terminality is now a "game" to relate the a_i's and f to the discrepancies of X, [**YPG**]. The output is

THEOREM 1.3.8. *A 3-fold cyclic singularity is terminal if and only if it is of type*

$$\frac{1}{r}(a, -a, 1), \text{ with } a \text{ coprime to } r$$

REMARK 1.3.9. *Note that the first example we met, the singularity of the cone over the Veronese surface is just $1/2(1, -1, 1)$. Note further that for canonical singularities we have the following characterization, assuming the a_i are minimal,* [**YPG**]. *A cyclic quotient is canonical if and only if $\sum a_i \geq r$,* [**Re1**] *and* [**YPG**].

The end of this story is the complete classification of 3-fold hypersurface quotients given by Mori, [**Mo2**]. The result is a small list made up of infinite cases that lie in a main series, namely

$$\frac{1}{r}(a,-a,1,0) \ \ (xy+g(z^r,t)=0), \text{ with } a \text{ coprime to } r$$

and 5 special families with fixed $r \leq 4$. For all these families it is possible to explicitly describe a section of $|-K_X|$ with only Du Val singularities. This together with inversion of adjunction says that the singularity is terminal, see Remark 2.2.8. For instance in the smooth case

$$\frac{1}{r}(a,-a,1) \ \ \mathbb{C}^3, \text{ with } a \text{ coprime to } r$$

consider $(z=0)$, keep in mind that the (anti)canonical class correspond to the eigenvalue 1. Then we come out with

$$\frac{1}{r}(a,-a) \ \ \mathbb{C}^2, \text{ with } a \text{ coprime to } r$$

and since $(a,r)=1$ this is the same as

$$\frac{1}{r}(1,-1)\ \mathbb{C}^2$$

then the singularity is canonical by Remark 1.3.9, simply because $1+(r-1)\geq r$. In this case it is nothing else than A_{r-1}.

Similarly one can compute

$$\frac{1}{r}(a,-a,1,0) \ \ (xy+g(z^r,t)=0), \text{ with } a \text{ coprime to } r$$

consider $(z=0)$. Then we come out with

$$\frac{1}{r}(a,-a,0) \ \ (xy+t^k=0), \text{ with } a \text{ coprime to } r$$

for some k and since $(a,r)=1$ this is the same as

$$\frac{1}{r}(1,-1,0) \ \ (xy+t^k=0)$$

also in this case the singularity is canonical by a generalization of the above inequality, see [**Re1**] and [**YPG**], this can be worked out as a A_{rk-1} singularity.

In all cases one can define a $\mathbb{Q}$-smoothing, that is a deformation X_λ of the singularity $X=X_0$ such that X_λ has only terminal quotients, this is the reason for the $\mathbb{Q}$ in front of smoothing. The latter allows to compute Riemann–Roch formula for terminal 3-folds, [**YPG**].

With this framework it is also possible to study canonical singularities. Canonical singularities could be singular in codimension 2 but of a very special type. Slicing with a general hyperplane one sees immediately that on a general point of a codimension 2 singular locus the singularity is analytically of type

$$(\text{Du Val})\times\mathbb{C}^{n-2}$$

Definition-Theorem 1.3.10 ([**Re2**]). If S is a 3-fold with canonical singularities then there exists a partial resolution $f:Y\to X$ where Y has only terminal singularities and $K_Y=f^*K_X$. Such a resolution is called *crepant*.

We conclude with different applications of the covering trick to terminal $\mathbb{Q}$-factorial singularities. For 3-dimensional terminal singularities the canonical class locally generates all divisors. This is a topological fact related to Reid's theorem.

LEMMA 1.3.11. *Let $P \in X$ be a terminal point in a Gorenstein 3-fold, and D a $\mathbb{Q}$-Cartier divisor. Then D is Cartier.*

PROOF. Let r be the index of D and $\pi : Y \to X$ the cyclic cover associated to $s \in \mathcal{O}_X(-rD)$. Then $\pi : Y \setminus P \to X \setminus P$ is étale. On the other hand X is an isolated hypersurface singularity therefore $\pi_1(X \setminus P) = \{1\}$, by the result of Milnor [**Mi**]. This proves that π is trivial and $r = 1$. □

REMARK 1.3.12. *Let X be a terminal 3-fold of canonical index r, i.e. rK_X is Cartier and r_1K_X is not Cartier for any $0 < r_1 < r$. Then all the discrepancies are in $\frac{1}{r}\mathbb{Z}$. It is possible to prove that actually the minimal discrepancy, namely $1/r$ is always attained by some valuation,* [**Sh2**, Appendix from Kawamata], [**Ma**].

We can use a similar argument to get the following

LEMMA 1.3.13. *Let X be a terminal $\mathbb{Q}$-factorial 3-fold and E a smooth divisor on X. Then E is out of the singularities of X.*

REMARK 1.3.14. *Note that this is not the case even for canonical singularities of surfaces. Take a line on the quadratic cone. The $\mathbb{Q}$-factoriality assumption is crucial, think of a quadratic cone in $\mathbb{P}^4$.*

The notion of $\mathbb{Q}$-factoriality is quite tricky. It is not difficult to give examples of non $\mathbb{Q}$-factorial analytic singularities that live in $\mathbb{Q}$-factorial varieties. The ordinary double point $0 \in (xy + zt = 0) \subset \mathbb{C}^4$ is not analytically $\mathbb{Q}$-factorial. Note that the two planes $(x = z = 0)$ and $(y = t = 0)$ only intersect at the point $0 \in \mathbb{C}^4$. On the other hand a quartic hypersurface $X_4 \subset \mathbb{P}^4$, with a unique ordinary double point is always $\mathbb{Q}$-factorial. This is just Lefschetz theorem.

PROOF. Let $x \in X$ be a point of canonical index i. Let $p : Y \to X$ be the canonical cover. Then p is etale outside of x, and Y is C-M and $\mathbb{Q}$-factorial. Let $p^*(E) = E_1 \cup \ldots \cup E_r$. The morphism p is etale in codimension 2 therefore $r = 1$. Then

$$E_1 \setminus p^{-1}(x) \to E \setminus \{x\}$$

is etale and $E \setminus \{x\}$ is simply connected. Therefore $i = 1$. The divisor E is Cartier by Lemma 1.3.11 and $x \in X$ is smooth. □

4. Explicit examples on 3-folds

Here we are concerned in giving explicit examples of contractions from and to terminal 3-folds. If one starts from a smooth 3-fold the classification is due to Mori and this has been the starting point of the whole MMP.

THEOREM 1.4.1 ([**Mo1**]). *Let $f : Y \to X$ be a birational morphism from a smooth 3-fold Y to a terminal 3-fold X with an irreducible exceptional divisor E. Then one of the following occurs:*

- *f is the blow up of a smooth point, $E \cong \mathbb{P}^2$, $a(E, X) = 2$*
- *f is the blow up of a smooth curve, $E = \mathbb{P}(\mathcal{I}_C/\mathcal{I}_C^2)$, $a(E, X) = 1$*
- *f is the blow up of the maximal ideal of $0 \in (xy + z^2 + t^k) \subset \mathbb{C}^4$, with $k \leq 3$, E is an irreducible quadric and $a(E, X) = 1$.*

- *f is the blow up of the maximal ideal of* $0 \in 1/2(1,-1,1)$, $E \cong \mathbb{P}^2$ *and* $a(E,X) = 1/2$.

REMARK 1.4.2. *A similar statement has been proved by Cutkosky for Gorenstein terminal 3-folds,* [**Cu**]. *The last case is the unique that lead to non Gorenstein 3-folds. It is almost unbelievable that from this an explosion of wild birational modification enter in the scene.*

DEFINITION 1.4.3. A terminal extraction is a birational map $f : Y \to X$ between terminal 3-folds with irreducible exceptional divisor $E \subset Y$ and $-K_Y$ f-ample.

You may think of a terminal extraction as a way to realize a unique valuation on X as a divisor, staying in the category of terminal models. The aim of this section is to give some examples and working material of 3-fold geometry.

Here we collect some examples

EXAMPLE 1.4.4. *Let* $x \in X \cong (0 \in (xy + z^2 + t^k = 0) \subset \mathbb{C}^4$ *and consider* $f : Y \to X$ *the blow up of the maximal ideal, with exceptional divisor* E_k. *The exceptional divisors are*

$$E_2 \cong (xy + z^2 + t^2 = 0) \subset \mathbb{P}^3 \quad E_k \cong (xy + z^2 = 0) \subset \mathbb{P}^3 \text{ for } k \geq 3$$

Then $E_2 \cong \mathbb{Q}^2$, *while* $E_k \cong \mathbb{P}(1,1,2)$ *for* $k \geq 3$ *and* Y *is not smooth for* $k > 3$, $a(E,X) = 1$. *Note that* Y *is always Gorenstein since* X *is Gorenstein and the discrepancy is an integer.*

One could expect that to extract a single valuation the discrepancy has to be small. Here are a couple of examples in the opposite directions.

EXAMPLE 1.4.5. *Let* $x \in X$ *be a smooth point, and* $f : Y \to X$ *be the weighted blow up with weights* $(1,a,b)$, *with* $(a,b) = 1$. *Then the exceptional divisor* $E \cong \mathbb{P}(1,a,b)$, $a(E,X) = a + b$ *and* Y *is terminal.*

Consider $x \in X \cong (0 \in (xy + z^k + t^{km} = 0) \subset \mathbb{C}^4$, *and let* $f : Y \to X$ *be a weighted blow up with weights* $(a,b,m,1)$, *where* $a + b = mk$ *and* $(a,b) = 1$. *Then one can compute that* Y *is terminal and* $a(E,X) = (a+b+m) - (a+b) = m$. *It is therefore possible to extract a unique valuation with arbitrarily high discrepancy.*

EXAMPLE 1.4.6. *Consider* $x \in X \cong (0 \in (xy + z^3 + t^3 = 0) \subset \mathbb{C}^4$ *and blow up the maximal ideal. Then we get* $f : Y \to X$, *with exceptional divisor* $E = F_1 + F_2$, *where:* $F_i \cong \mathbb{P}^2$, *and, with a slight abuse of language,* Y *is singular at* $(x = 0) \cap (y = 0) \cap (z^3 + t^3 = 0) \subset E$. *In this case the exceptional divisor is not irreducible. This means that we have extracted two valuations from the point* $x \in X$. *Note that* E *is Cartier but* F_i *are not* $\mathbb{Q}$*-Cartier, keep in mind Lemma 1.3.11. To extract a single valuation consider a weighted blow up* $g : Z \to X$, *with weights* $(2,1,1,1)$. *Then the exceptional divisor* $F_2 \cong (xy + z^3 + t^3 = 0) \subset \mathbb{P}(2,1,1,1)$ *and* $a(F_2,X) = 1$. *Geometrically the weight on the first coordinate allow to contract the other* $\mathbb{P}^2$ *to the singular point* $1/2(1,-1,1)$ *by a small resolution of the 3 singular points on the line of intersection.*

We can follow this wbu in figure 3. The second blow up h *is a* $\mathbb{Q}$*-factorization of* $(x = 0) = F_2$, *in particular it extracts a* $\mathbb{P}^1$ *from each singular point. Then we blow down with* b *the other exceptional divisor to the singularity* $1/2(1,-1,1)$. *Note that* h *is an isomorphism on* F_1 *while* $b_*(F_2)$ *is the anticanonical model of* $\mathbb{P}^2$

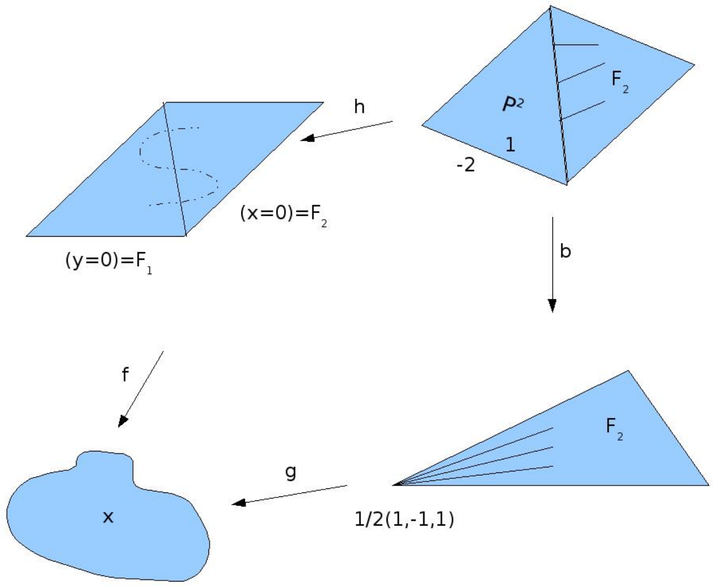


FIGURE 3. The wbu $(2,1,1,1)$, (the dotted line is $(z^3+t^3=0)\cap E$).

blown up along three collinear points. One can show that this and the symmetric wbu $(1,2,1,1)$ *are the unique terminal extractions from this singular point,* [**CM**[u]].

In few cases there is a complete classification of terminal extractions.

THEOREM 1.4.7 ([**Ka2**]). *The unique terminal extraction with a center that contains a cyclic singularity of type* $1/r(a,-a,1)$ *is the wbu with weights* $(a,-a,1)$.

The idea of the proof is quite simple: the candidate is the valuation with minimal discrepancy. When you try to extract something else this valuation introduces non-terminal singularities. This is very special of this toric situation. We already encountered extractions with arbitrarily high discrepancies.

REMARK 1.4.8. *Note that Kawamata's results says that it is not possible to blow up a curve, passing through a cyclic singularities and keep terminality.*

Kawakita did an enormous work for extractions from points. Beside proving the following theorem, classified many cases and gave structure theorems for almost all possible situations, [**Kw2**], [**Kw3**], [**Kw4**].

THEOREM 1.4.9 ([**Kw1**]). *The unique terminal extractions from smooth points are the wbu* $(1,a,b)$ *with* $(a,b)=1$.

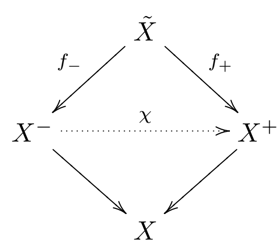
The case of curves is of a totally different nature. It is immediate that on the general point one has to blow up the maximal ideal of the curve. It is enough to consider the slice with a general surface section. It is not difficult to show that if an extraction exists then it is unique. Assume that $f:Y\to X$ and $g:Z\to X$ are terminal extractions with exceptional divisor E_Y and E_Z. Then Y and Z are isomorphic in codimension 2 and as X-schemes the anticanonical class is ample. This is enough to prove that $Y \cong Z$. A completely different story is to understand which sheaves one has to blow up and which curves admit a terminal extraction, keep in mind Kawamata's result. In this realm Tziolas has many results, [**Tz1**], [**Tz2**], [**Tz3**].

Example 1.4.10. *It is possible to "extract" a curve $\Gamma \subset X$ with an ordinary r-tuple point p on a smooth 3-fold if $r \leq 3$. To see this consider the blow up of p, with exceptional divisor $E \cong \mathbb{P}^2$. Then blow up the (strict transform of the) curve Γ , with exceptional divisor E_Γ. Let $f : Y \to X$ the resulting map. By construction* $\operatorname{rk}\operatorname{Pic}(Y/X) = 2$*. It is easy to see that any line in E intersecting E_Γ in two points can be flopped. After the flop of these lines the (strict transform of the) divisor E is either a smooth quadric, if $r = 2$, or a $\mathbb{P}^2$, if $r = 3$. Then we can blow it down to get the required terminal extraction.*

Note that for a 4-tuple point this is not possible. In this case the divisor $E \subset Y$ is covered by conics C such that $K_Y \cdot C = 0$. It is therefore possible to contract E from Y onto a not terminal singularity.

Flips in dimension three has been classified, in some sense. There are two fundamental, very technical, papers [**Mo3**] [**KM1**] that describe almost everything of a neighborhood of a flipping curve. On the other hand it is difficult to describe the birational modification out of this local description. We try to give a simpler description of special classes, inspired by M. Reid.

Consider the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ as a quadric Q of equation $(xy + zt = 0) \subset \mathbb{P}^3$. Let $X \subset \mathbb{P}^4$ be the cone over Q. We already noticed that the blow up of the vertex, $\tilde{X}$ has an exceptional divisor $E \cong Q$. Then considering the strict transform of the plane $(x = z = 0)$ we realize that the curves in E are not numerical proportional and we can blow down the two rulings of E independently, this is the geometric reason for the non $\mathbb{Q}$-factoriality of the quadric cone. This is equivalent to consider the two ratios $x/z = -t/y$ and $x/t = -z/y$.



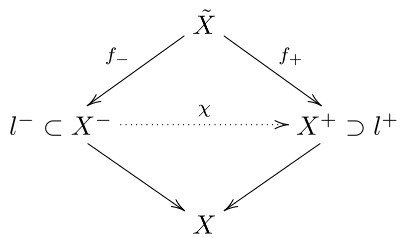
The map χ is the famous Atiyah flop.

Miles Reid interpreted this construction in a different way. Consider the $\mathbb{C}^*$ action on $\mathbb{C}^4$, with coordinates (x_1, x_2, y_1, y_2) given by

$$(x_1, x_2, y_1, y_2) \mapsto (\lambda x_1, \lambda x_2, \lambda^{-1} y_1, \lambda^{-1} y_2)$$

It is clear that the ring of invariant polynomials is generated by the products $x_i y_j$ and with a bit more work one can prove that the the GIT quotient of $\mathbb{C}^4 \setminus 0$ is exactly the cone X. The vertex comes from the non closed orbits originated by points in either $B^- := \{(x_1, x_2, 0, 0)\}$ or $B^+ := (0, 0, y_1, y_2)$. Then I can produce $X^\pm$ has GIT quotient of $\mathbb{C}^4 \setminus B^\pm$. One can extend this construction to different $\mathbb{C}^*$ action and varieties and describe many classes of flips in terms of GIT. This has been done by G. Brown, [**Br**]. A very interesting interpretation and generalization of Reid ideas has been given by Michael Thaddeus, [**Th1**], [**Th2**]

EXAMPLE 1.4.11. *Consider* $\mathbb{C}^4$ *and a* $\mathbb{C}^*$ *action with weights* $(m,1;-1-1)$. *We have bad locus* $B^- = (x_1 = x_2 = 0)$ *and* $B^+ = (y_1 = y_2 = 0)$. *The flipping diagram is again as follows*



The invariants of the action are

$$(x_1 y_1^m, x_1 y_1^{m-1} y_2, \ldots, x_2 y_1, x_2 y_2) = (u_0, u_1, \ldots, u_m, v_0, v_1)$$

Then computing the relation we get that X *is the cone over the scroll* $S(m,1)$, *that is the Hirzebruch surface* $\mathbb{F}_{m-1}$

$$\mathrm{rk}\begin{pmatrix} u_0 & \ldots & u_{m-1} & v_0 \\ u_1 & \ldots & u_m & v_1 \end{pmatrix} = 1$$

To get $X^- := \mathbb{C}^4 \backslash B^- / \mathbb{C}^*$ *we introduce the ratio* $t = x_2^m / x_1$ *this gives new invariants*

$$u_i t - v_0^{m-i} v_1^i$$

Let us consider the affine chart $x_2 \neq 0$, *say* X_2^-. *There we can divide by* x_2 *to get* $u_i = v_0^{m-i} v_1^i t^{-1}$ *Therefore*

$$X_2^- = Spec\mathbb{C}[u_0, \ldots, u_m, v_0, v_1, t^{-1}]/(u_i = v_0^{m-i} v_1^i t^{-1}) = \mathbb{C}^3$$

For the chart $X_1^- = (x_1 \neq 0)$, *it is not difficult to convince yourself that the ring is*

$$\mathbb{C}[u_0, \ldots, u_m, v_0, v_1, t]/(\mathrm{rk}\, M \leq 1)$$

where

$$M = \begin{pmatrix} u_0 & u_1 & \ldots & u_{m-1} & v_0 \\ u_1 & u_2 & \ldots & u_m & v_1 \\ v_0^{m-1} & v_0^{m-2} v_1 & \ldots & v_1^{m-1} & t \end{pmatrix}$$

and with a bit of calculation one realizes it as the equation of the singularity $1/m(1,-1,1)$. *Note that for* $m = 2$ *this is just the cone over the Veronese surface.*

Similarly we can realize that $X^+ := \mathbb{C}^4 \setminus B^+ / \mathbb{C}^*$, *obtained with* $s = y_1 / y_0$ *is smooth and* $N_{l^+} = \mathcal{O}(-m) \oplus \mathcal{O}(-1)$. *This is a generalization of Francia's flip.*

We have a geometric interpretation of this construction. The scroll structure on $S(m,1)$ *extend to a rational map* $\varphi : X \dashrightarrow P^1$, *not defined on the vertex. To resolve this indeterminacy we can consider the closure of the graph of* φ *in* $X \times \mathbb{P}^1$.

We can do something more complicate by considering invariant hypersurfaces in $\mathbb{C}^5$.

EXAMPLE 1.4.12. *Consider* $A = (x_1y_1 + g_{m-1}(x_2, x_3) = 0) \subset \mathbb{C}^5$ *and a* $\mathbb{C}^*$ *action with weights* $(m, 1, 1; -1 - 1)$. *Then* A *is invariant and we can consider the quotient* $A//\mathbb{C}^*$. *We have bad locus* $B^+ = (x_1 = x_2 = x_3 = 0)$ *and* $B^- = (y_1 = y_2 = 0)$. *This time there are cDV singularities on* X^+ *and the singularity on* X^- *are quotients of cDV.*

CHAPTER 2

Log Pairs

Singularities of pairs are a technical subject. The number of different definitions and bounds on the discrepancies tend to confine them to experts. On the other hand the outstanding result that came out of this technicalities should convince anyone to spend some time on them.

1. Definitions and flavors

Let us consider a $\mathbb{Q}$-Weil divisor $D = \sum d_i D_i$ on a normal variety X. We always assume that the D_i are distinct. Our aim is to give a reasonable sense to a notion of singularities of the pair (X, D). The first requirement is that $K_X + D$ is $\mathbb{Q}$-Cartier. Then for a resolution $f : Y \to X$ we can write the discrepancy formula

$$K_Y = f^*(K_X + D) + \sum a_i E_i$$

where the E_i are either f-exceptional or strict transforms of D_i. Thinking of a smooth X with a badly singular D one realizes that a resolution of X is meaningless for the pair (X, D). We have to ask for a "resolution" of D as well. To do this we have first to decide what it means to resolve a divisor $D = \sum d_i D_i \subset X$.

Definition 2.1.1. A Weil divisor $D = \sum d_i D_i \subset X$ on a smooth X is *simple normal crossing* (snc) if for every point $p \in X$ a local equation of $\sum D_i$ is $x_1 \cdot \ldots \cdot x_r$ for independent local parameters x_i in $\mathcal{O}_{p,X}$. A log resolution of the pair (X, D) is a birational morphism $f : Y \to X$ such that Y is smooth and $f^{-1}D \cup \mathrm{Exc}(f)$ is snc

Remark 2.1.2. *This is sometimes called normal crossing. Every irreducible component of a snc divisor is smooth. This is the strictest possible notion. One could relax the hypothesis allowing self intersections of the D_i. Sometimes it is better to have more room, on the other hand under this hypothesis everything works. Note that thanks to Hironaka,* [**Hi**]*, a log resolution always exists.*

As usual our definitions will be related to bounds on the discrepancies a_i. Before stating the definitions of singularities for pairs (X, D) let us do this simple calculation. Assume that X is smooth and consider $(1+\epsilon)D$ with D smooth. Then $id_X : X \to X$ is a log resolution and

$$K_X = id^*(K_X + (1+\epsilon)D) - (1+\epsilon)D$$

In other terms we are assuming that there is a valuation with discrepancy $-(1+\epsilon)$. Let $f_1 : Y_1 \to X$ be the blow up of a codimension 2 smooth $Z \subset D$, with exceptional divisor E_1. Then we have

$$K_{Y_1} = f_1^*(K_X + (1+\epsilon)D) + (1 - (1+\epsilon))E_1 - (1+\epsilon)D_1$$

where D_1 is the strict transform of D on Y_1. Let $g : Y_2 \to Y_1$ be the blow up of $D_1 \cap E_1$ and $f_2 = (g \circ f_1)$ then

$$K_{Y_2} = f_2^*(K_X + (1+\epsilon)D) + (1 - (1+\epsilon) - \epsilon)E_2 - \epsilon E_1 - (1+\epsilon)D_2$$

Iterating this procedure we get arbitrarily negative discrepancies. That is a single discrepancy less then -1 annihilates the notion of minimal discrepancy. With this in mind we are ready to swallow the following definition.

Definition 2.1.3. Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-Weil divisor. Assume that $(K_X + D)$ is $\mathbb{Q}$-Cartier. Let $f : Y \to X$ be a log resolution of the pair (X, D) with

$$K_Y = f^*(K_X + D) + \sum a(E_i, X, D)E_i$$

We call

$$\text{discrep}(X, D) := \min_{E_i}\{a(E_i, X, D) | \\ E_i \text{ is an } f\text{-exceptional divisor for some log resolution}\}$$

$$\text{totaldiscrep}(X, D) := \min_{E_i}\{a(E_i, X, D) | \\ E_i \text{ is a divisor in some log resolution}\}$$

Then we say that (X, D) is

$$\left.\begin{array}{l} \textit{terminal} \\ \textit{canonical} \\ \textit{klt} \\ \textit{plt} \\ \textit{lc} \end{array}\right\} \text{ if } \operatorname{discrep}(X, D) \quad \left\{\begin{array}{l} > 0 \\ \geq 0 \\ > -1 \text{ and } d_i < 1 \\ > -1 \\ \geq -1 \end{array}\right.$$

Here klt (plt, lc) stands for Kawamata log terminal (purely log terminal, log canonical).

Remark 2.1.4. *In case $D = 0$ plt and klt are simply called lt (log terminal). All these classes have a precise meaning. When $D = 0$ we already encountered terminal and canonical as the singularities of minimal and canonical models.*

Klt is the class of singularities where proofs work better. The result for canonical singularities generalize "almost" completely to this wider setting. Klt singularities are rational, and therefore CM. The main disadvantage is that one cannot use it to study open varieties, and induction arguments do not work very well since the coefficients of the boundary cannot be 1.

Plt are powerful in inductive arguments. lc is the widest class where all the theory should in principle work, but no one knows how to do it.

lc singularities, even if harder then the other flavors, enjoy beautiful numerical properties and have many natural applications outside the MMP. The presence of more variants, we will introduce one more in a while and forget about few others, for log terminal is probably due to the lack of the right definition.

It is common to define the log discrepancies as

$$a_l(E, X, D) := 1 + a(E, X, D)$$

This is sometimes useful in formulas.

I advise to read Fujino's beautiful paper, [C$^+$, Chapter 3]*, to appreciate all the colors of definitions of log pairs.*

REMARK 2.1.5. *Instead of considering a fixed Weil divisor D one can consider a movable linear system, $\mathcal{H}$, and a general divisor in this linear system. This notion of singularities measure the singularities of X and those of $\operatorname{Bs}\mathcal{H}$. This turns out to be very fruitful in birational geometry of 3-folds after Sarkisov.*

A further important notion of log terminal useful in MMP is the following

DEFINITION 2.1.6 ([**KM2**]). Let X be a normal variety and $D = \sum d_i D_i$ a $\mathbb{Q}$-Weil divisor with $0 \leq d_i \leq 1$. Assume that $(K_X + D)$ is $\mathbb{Q}$-Cartier. We say that (X, D) is *dlt* (divisorially log terminal) if there exists a closed subset $Z \subset X$ such that

- $X \setminus Z$ is smooth and $D_{|X \setminus Z}$ is a snc divisor
- If $f : Y \to X$ is birational and $E \subset Y$ is an irreducible divisor such that $\operatorname{center}_X E \subset Z$ then $a(E, X, D) > -1$

REMARK 2.1.7. *The snc assumption is crucial in the definition, see* [C$^+$, Chapter 3]. *The definition seems rather different from the previous but by Szabó's work,* [**Sz**] *we can reformulate it as follows. (X, D) is dlt if there exists a log resolution $f : Y \to X$ such that $a(E, X, D) > -1$ for every f-exceptional divisor E and $Exc(f)$ of pure codimension 1. This could be not true for other log resolutions. The purity is crucial, see* [C$^+$, Example 3.8.4]. *This clearly gives klt $\Longrightarrow$ plt $\Longrightarrow$ dlt.*

It is time to work out a few examples.

EXAMPLE 2.1.8. *A nodal curve $D \subset \mathbb{C}^2$ is not dlt but is lc. The identity is not a log resolution, it is not snc! The discrepancy of the node is -1. The divisor D is not normal one should compare it with Corollary 2.2.10.*

A cuspidal curve in $\mathbb{C}^2$ is not lc. To have a log resolution we have to blow up three times and the final valuation has discrepancy -2. If we consider $(x^2 = y^3) \subset \mathbb{C}^2$ we can do a wbu with weights $(3, 2)$ to have $f : Y \to \mathbb{C}^2$ with exceptional divisor E and $a(E, \mathbb{C}^2, D) = 4 - 6$. It is easy to see that (Y, C_Y) is lc.

It is clear that a snc divisor $D = \sum d_i D_i$, with $1 \geq d_i \geq 0$ on a smooth X is dlt (and is klt if $d_i < 1$) take $Z = \emptyset$.

A reducible connected snc integral divisor D is dlt but is not plt, think for instance to $(\mathbb{C}^2, (xy = 0))$. Blowing up the intersection of two components gives rise to LC singularities. This shows that to check plt you have really to go through all possible log resolutions, this is not the case for klt and lc.

An effective klt pair is always dlt, consider $Z = X$. A dlt pair (X, D) is klt if $d_i < 1$. This is immediate once you notice that there exists a log resolution that is an isomorphism on the open set where X is smooth and D is snc, [C$^+$, Chap. 3]. *For surfaces there is a complete classification of all these singularities, see* [**U2**]. *The case of $D = 0$ is enlightening:*

terminal is smooth
canonical is Du Val i.e. $\mathbb{C}^2/($finite subgroup of $SL(2, \mathbb{C}))$
lt is $\mathbb{C}^2/($finite subgroup of $GL(2, \mathbb{C}))$
lc is simple elliptic, cusp, smooth or a quotient of these

To better understand the differences between these notions it is interesting to note the following

LEMMA 2.1.9. *Let Y be a smooth variety, $F = \sum_1 f_i F_i$ a snc divisor, and W a smooth subvariety. Assume that* $\operatorname{codim} W \geq 2$, *$W \subseteq F_1 \cap \ldots \cap F_k$ and $W \not\subset F_i$ for $i > k$. Then we have*

$$a_l(W, Y, F) = \sum_1^k a_l(F_i, Y, F) + \operatorname{codim} W - k$$

In particular klt singularities have finitely many valuations with negative discrepancy, and a dlt pair (X, D) with $\lfloor D \rfloor =: S$ is an irreducible integer divisor is plt.

PROOF. Let $W \subset Y$ be as in the hypothesis and E_W the valuation associated to its maximal ideal. Note that by snc $k \leq \operatorname{codim} W$. Let $\mu : Y_W \to Y$ be the blow up of W, with exceptional divisor E_W. Then

$$a_l(E_W, Y, F) = \operatorname{codim} W - \sum_1^k f_i = \sum_1^k a_l(F_i, Y, F) + \operatorname{codim} W - k \tag{1}$$

Assume now that (X, D) is klt. Let $f : Y \to X$ be a log resolution of the pair (X, D), with $K_Y = f^*(K_X + D) + \sum a_i(E_i, X, D)E_i$. Then $(Y, -\sum a_i(E_i, X, D)E_i)$ is klt and the identity is a log resolution. It is enough to prove the statement for $(Y, -\sum a_i(E_i, X, D)E_i) =: (Y, F)$. Fix $\epsilon = \min\{a_l(E_i, X, D)\} > 0$. By equation (1) a valuation E has negative discrepancy for (Y, F) only if the center of E on Y is an irreducible component of a complete intersections of the E_i's. By definition and equation (1) we have

$$a_i(E_W, Y, F) \geq a_i(E_i, X, D) + \epsilon$$

for any $E_i \supset W$. By Hironaka we can realize any valuation of $K(Y)$ with a succession of smooth blow ups and therefore after finitely many blow ups no more negative valuations can be introduced.

Assume now that (X, D) is dlt and $\lfloor D \rfloor = S$ is irreducible. We have to check only the discrepancies outside of Z, with center of codimension at least 2. There (X, D) is snc and the assumption on the round down and equation (1) tell us that $a_l(E, X, D) \geq 0$ with equality if and only if $E = S$. □

REMARK 2.1.10. *Note that a plt pair can have infinitely many negative discrepancies. We can blow up a codimension 2 intersection of an integer Weil divisors D_1 in D and any other divisor D_2 in D.*

2. Inversion of adjunction

The first intuition in this direction is due to Shokurov, [**Sh2**]. Here we follow [**KM2**] and [**U2**]. Let $(X, S + D)$ be a log pair with S a reduced Weil divisor and D a $\mathbb{Q}$-divisor. Our aim is to compare discrepancies of $(X, S + D)$ to those of $(S, D_{|S})$. Clearly every result in this direction is important in induction statements and many questions of lifting properties of divisors to the ambient variety. For simplicity, following [**KM2**], I assume that S is Cartier in codimension 2. Then adjunction formula for S is the usual one, [**U2**, §16]. To avoid this assumption one has to consider a correction to the adjunction formula which is called Different, [**U2**, §16]. With this in mind everything work under this wider hypothesis.

DEFINITION 2.2.1. Let (X, D) be a pair and $Z \subset S \subset X$ closed subschemes. Define

$$\operatorname{discrep}(\text{center} \subset Z, X, D) := \inf\{a(E, X, D) | \\ E \text{ is exceptional and } \operatorname{center}_X(E) \subset Z\}$$

and

$$\operatorname{discrep}(\text{center} \cap S \subset Z, X, D) := \inf\{a(E, X, D) | \\ E \text{ is exceptional and } \operatorname{center}_X(E) \cap S \subset Z\}$$

and similarly for totaldiscrep.

This is the easy way to relate the discrepancies.

PROPOSITION 2.2.2 ([**U2**, 17.2]). *Let X be a normal variety, S a normal Weil divisor which is Cartier in codimension 2, $Z \subset S$ a closed subvariety and $D = \sum d_i D_i$ a $\mathbb{Q}$-divisor. Assume that $K_X + S + D$ is $\mathbb{Q}$-Cartier. Then*

$$\begin{aligned} \operatorname{totaldiscrep}(\textit{center} \subset Z, S, D_{|S}) &\geq \operatorname{discrep}(\textit{center} \subset Z, X, S + D) \\ &\geq \operatorname{discrep}(\textit{center} \cap S \subset Z, X, S + D) \end{aligned}$$

PROOF. Let $f : Y \to X$ be a log resolution, with E_i f-exceptional divisors and S_Y, D_Y strict transform of S and D on Y. I assume that $S_Y \cap D_Y = \emptyset$. Therefore we have if $E_i \cap S_Y \neq \emptyset$ then $\operatorname{center}_X(E_i) \subset S$. Write

$$K_Y + S_Y \equiv f^*(K_X + S + D) + \sum e_i E_i$$

apply adjunction formula to both sides to get

$$K_{S_Y} \equiv f^*(K_S + D_{|S}) + \sum e_i (E_i \cap S_Y)$$

The divisor S_Y is disjoint from D_Y therefore if $E_i \cap S_Y \neq \emptyset$ then E_i is f-exceptional and $\operatorname{center}_X(E_i) \subset S$. This is enough to prove that every discrepancy that enter in $S_Y \to S$ comes from f-exceptional divisors. Note that $S_Y \cap E_i$ could well be non $f_{|S_Y}$-exceptional. □

To go in the other direction one has to avoid this consideration. In general there are divisors E_i that do not intersect S_Y and there is no good reason to expect that equality holds. Nonetheless this was the conjecture. After many special cases checked, see [**Kw5**], and [**La**] for a detailed account. This has been proved by [**BCHM**] as a byproduct of finite generation following arguments in [**U2**, §17].

The main theorem here is Shokurov connectedness result, [**Sh2**]. Here we present the n-dimensional proof of [**U2**].

THEOREM 2.2.3. [**U2**, 17.4] *Let $g : Y \to X$ be a proper birational morphism, Y smooth and X normal. Let $D = \sum d_i D_i$ be a snc $\mathbb{Q}$-divisor on Y such that $g_* D$ is effective and $-(K_Y + D)$ is g-nef. Write*

$$A = \sum_{i : d_i < 1} d_i D_i \text{ and } F = \sum_{i : d_i \geq 1} d_i D_i$$

Then $\operatorname{Supp} F = \operatorname{Supp} \lfloor F \rfloor F$ *is connected in a neighborhood of any fiber of g.*

REMARK 2.2.4. *One of the most useful application of the theorem is the following. Consider a log resolution $g : Y \to X$ of a pair (X, D_X) with*

$$K_Y = f^*(K_X + D_X) + \sum a_i E_i$$

Then $-f^*(K_X + D_X)$ is f-nef! and we let $D = -\sum a_i E_i$. Let $Z \subset Y$ be the locus where (Y, D) is not klt. Then Z is connected in a neighborhood of any fiber of g.

PROOF. Consider the following divisor

$$\lceil -A \rceil - \lfloor F \rfloor = K_Y - (K_Y + D) + \text{fractional effective}$$

The morphism g is birational therefore g-nef is equivalent to g-nef and big. By Kawamata–Viehweg relative vanishing I get

$$R^1 g_*(\mathcal{O}_Y(\lceil -A \rceil - \lfloor F \rfloor)) = 0$$

Apply g_* to the structure sequence of $\lfloor F \rfloor$ to get

$$g_*\mathcal{O}_Y(\lceil -A \rceil) \to g_*\mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil) \text{ is surjective}$$

Note that $\lceil -A \rceil$ is g-exceptional and effective therefore $g_*\mathcal{O}_Y(\lceil -A \rceil) \sim \mathcal{O}_X$. Assume that $\lfloor F \rfloor$ has at least two connected components, say $F_1 \cup F_2$ in a neighborhood of $g^{-1}(x)$ for some $x \in X$. Then

$$g_*(\mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil)_{(x)} \cong \mathcal{F}_1 + \mathcal{F}_2$$

and neither of the summand is zero. Thus $g_*(\mathcal{O}_{\lfloor F \rfloor}(\lceil -A \rceil)_{(x)}$ is not a quotient of the cyclic module $\mathcal{O}_{X,x}$. □

A corollary of connectedness theorem is a weak version of Inversion of adjunction

THEOREM 2.2.5 ([**U2**, 17.6]). *Let X be normal and $S \subset X$ a normal Weil divisor which is Cartier in codimension 2. Let D be an effective $\mathbb{Q}$-divisor and assume that $K_X + S + D$ is $\mathbb{Q}$-Cartier. Then $(X, S + D)$ is plt near S if and only if $(S, D_{|S})$ is klt.*

PROOF. We have to prove the if part. Let $g : Y \to X$ be a log resolution of $(X, S + D)$ and write

$$K_Y = g^*(K_X + S + D) - A - F$$

with the notation of the proof of theorem 2.2.3. Let $S_Y = g_*^{-1}S$ and $F = S_Y + F'$. By adjunction formula

$$K_{S_Y} = g^*(K_S + D_{|S}) + (A - F')_{|S_Y}$$

We have to prove that $F' = \emptyset$ if $F' \cap S_Y = \emptyset$. By Theorem 2.2.3 $F = S_Y \cup F'$ is connected in a neighborhood of $g^{-1}(x)$ for any $x \in X$, therefore $F' = \emptyset$ if and only if $F' \cap S_Y = \emptyset$ in a neighborhood of S. □

REMARK 2.2.6. *Note that we used a property of lc singularities to prove plt singularities of a pair. This is not unusual but always amazing. We will see other applications of Theorem 2.2.3 in the following sections.*

As a byproduct of this weak inversion of adjunction we can easily prove that isolated cDV on a 3-fold are terminal.

COROLLARY 2.2.7. *Let $p \in X$ be an isolated cDV singularity of a 3-fold Then $p \in X$ is terminal.*

PROOF. Let D be a hyperplane section with Du Val singularities, in the notation of Definition 1.3.1 $(t=0)\subset\mathbb{C}^4$. Consider the log pair (X,D). By construction $(D,0)$ is canonical hence klt. Then inversion of adjunction gives (X,D) plt. On the other hand D is a Cartier divisor hence the contribution of D to the discrepancy of any valuation E centered in p is at least 1. This shows that $a(E,X)>0$. $\square$

REMARK 2.2.8. *With the full inversion of adjunction we conclude that p is terminal even if D is a Weil divisor. Indeed the hypothesis give (X,D) canonical and since D has a positive discrepancy on any valuation centered at p we conclude.*

A technical but important property of dlt is that a dlt pair is the limit of klt pairs with an arbitrary perturbation. With this I mean the following. Let (X,D) be dlt. Then there exist $\mathbb{Q}$-divisors D_1 such that $\mathrm{Supp}(D_1)\supseteq\mathrm{Supp}(D)$, and for any sufficiently small positive ϵ, $(X,D-\epsilon(D_1-\Delta))$ is klt, [**KM2**, Proposition 2.43], for any divisor Δ. The same property is immediate for plt.

With argument as in the proof of Theorem 2.2.3 one can prove.

PROPOSITION 2.2.9 ([**KM2**, Proposition 5.51]). *Assume that (X,D) is dlt then the following are equivalent:*

- *(X,D) is plt*
- *$\lfloor D\rfloor$ is normal*
- *$\lfloor D\rfloor$ is the disjoint union of its irreducible components.*

Putting everything together we get.

COROLLARY 2.2.10. *Let (X,D) be dlt. Then every irreducible component of $\lfloor D\rfloor$ is normal.*

PROOF. Let $S\subset\lfloor D\rfloor$ be an irreducible component and set $D_1:=D-S$. Then $(X,D-\epsilon D_1)$ is dlt and $\lfloor D-\epsilon D_1\rfloor=S$. Therefore by Proposition 2.2.9 S is normal. $\square$

This gives us an interesting inductive behavior of dlt with respect to adjunction formula.

Let (X,D) be a dlt pair where $D=S+\sum d_iD_i$ and $f:Y\to X$ a log resolution. Then

$$K_Y+S_Y=f^*(K_X+S+\sum d_iD_i)+\sum a_i(E_i,X,D)$$

Note that $f_{|S_Y}:S_Y\to S$ is a log resolution for $(S,(\sum d_iD_i)_{|S})$ Then S is normal by Corollary 2.2.10 and $(S,(\sum d_iD_i)_{|S})$ is dlt by taking the same Z.

As a further application of Theorem 2.2.3 we will classify terminal extractions from the ordinary double point. It has to be said that Kawakita's algebraic approach gives much stronger results. But still I believe that the geometric beauty of connectedness theorem is worthwhile.

THEOREM 2.2.11 ([**Co2**]). *Let $P\in X$ be a 3-fold germ analytically isomorphic to an ordinary node*

$$xy+zt=0$$

and $f\colon(E\subset Y)\to(P\in X)$ a terminal extraction; assume in addition that $f(E)=P$. Then f is the blow up of the maximal ideal at P.

Proof. Let n be a sufficiently large and divisible positive integer; fix a finite dimensional very ample linear system

$$\mathcal{H}_Y \subset |-nK_Y|$$

Denote $\mathcal{H} = f_*(\mathcal{H}_Y)$ the image of $\mathcal{H}_Y$ in X, so that

$$K_Y + \frac{1}{n}\mathcal{H}_Y = p^*\Big(K_X + \frac{1}{n}\mathcal{H}\Big)$$

By construction

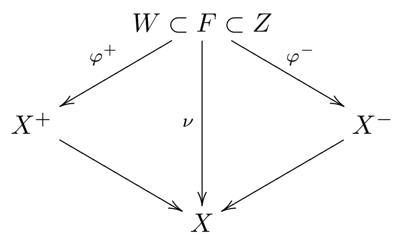
$$\mathrm{mult}_E(\mathcal{H}) = na(E, X)$$

while

$$\mathrm{mult}_G(\mathcal{H}) < na(G, X) \tag{2}$$

for all valuations $G \neq E$.

Let $\nu : Z \to X$ be the blow up of the maximal ideal, with exceptional divisor a quadric F. Let $W = \mathrm{center}_Z\, E \subset F$ and $S \subset X$ a general Cartier divisor through x. Then $(X, 1/n\mathcal{H} + S)$ is not klt both on S and on E. By Theorem 2.2.3, applied to the morphism ν this forces the existence of a non klt curve $C \subset F$ for this pair.



Let $\varphi^{\pm} : Z \to X^{\pm}$ be the two contraction on the small resolutions of X. Applying again Theorem 2.2.3 to $\varphi^{\pm}$ we conclude that $a(F, X, 1/n\mathcal{H}, S) \leq -1$. On the other hand by equation (2) we have that $a(G, X, 1/n\mathcal{H}, S) > -1$ for any valuation $G \neq E$. Therefore we conclude that $E = F$. ☐

3. Birational geometry

Here we apply some of the theory of pairs to the birational geometry of 3-folds. This is one of the few places where the philosophy of MMP comes down to earth and therefore get filled by mud.

We start with a definition

Definition 2.3.1. A Mori fiber Space (MfS) is a morphism $\pi : X \to S$ satisfying the following conditions

. X is terminal $\mathbb{Q}$-factorial
. $-K_X$ is π ample
. $\dim S < \dim X$
. $\mathrm{rk}\,\mathrm{Pic}(X/S) = 1$

MfS's are the output of MMP applied to uniruled varieties. It is quite clear that one can have different MfS's attached to a variety X. Think of $\mathbb{P}^n$ and the blow up of $\mathbb{P}^n$ along a linear space. It is much less clear when two MfS's are birational to each other. The aim of the maximal singularities method and Sarkisov Theory in to put some order in this. The former tries to exclude the existence of birational maps between MfS's while the latter is a way to factor birational maps between MfS's in "elementary" steps, called links.

REMARK 2.3.2. *Note that if* $\dim X = 2$ *MfS's are quite simple. The only possibilities are either ruled surfaces when dimW = 1, this is quite simple if you assume Tsen's theorem (i.e. the existence of a section of the fibration), and* $\mathbb{P}^2$ *when* $\dim W = 0$*, this is not immediate. Already in the 3-fold case this simplicity is completely lost. There are few hundred cases with* $\dim W = 0$*, del Pezzo fibrations and conic bundles without sections. A good point in between, to train your intuition, are surfaces over non algebraically closed fields.*

3.1. The Sarkisov category.

DEFINITION 2.3.3. (1) The *Sarkisov category* is the category whose objects are Mori fiber spaces and whose morphisms birational maps (regardless of the fiber structure).

(2) Let $X \to S$ and $X' \to S'$ be Mori fiber spaces. A morphism in the Sarkisov category, that is, a birational map $f\colon X \dashrightarrow X'$, is *square* if it fits into a commutative square

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & X' \\ \downarrow & & \downarrow \\ S & \overset{g}{\dashrightarrow} & S' \end{array}$$

where g is a birational map (which thus identifies the function field L of S with that of S') and, in addition, the induced birational map of generic fibers $f_L\colon X_L \dashrightarrow X'_L$ is biregular. In this case, we say that $X \to S$ and $X' \to S'$ are *square birational*, or *square equivalent*.

(3) A *Sarkisov isomorphism* is a birational map $f\colon X \dashrightarrow X'$ which is biregular and square.

(4) If X is an algebraic variety, we define the *pliability* of X to be the set

$$\mathcal{P}(X) = \big\{\text{MfS } Y \to T \mid Y \text{ is birational to } X\big\}/\text{square equivalence.}$$

We say that X is *birationally rigid* if $\mathcal{P}(X)$ consists of one element.

To say that X is rigid means that it has an essentially unique model as a Mori fiber space; this implies in particular that X is nonrational, but it is much more precise than that. For example, it is known that a nonsingular quartic 3-fold is birationally rigid [**IM**], [**Pu2**], [**Co2**]; a quartic 3-fold with only ordinary nodes is birationally rigid [**Me3**]. Note that a general determinantal quartic has only ordinary nodes it is not $\mathbb{Q}$-factorial and it is rational. Similar results are also known in arbitrary dimensions, see [**Pu3**] for a survey.

3.2. Birational geometry, classification theory and commutative algebra. It is not difficult, and fun, for someone experienced in the use of the known lists of Fano 3-folds, and aware of certain constructions of graded rings, to generate many examples of birational maps between Fano 3-folds. In particular many Fano 3-fold codimension 2 weighted complete intersections are birational to special Fano 3-fold hypersurfaces. To name but a few, a general $Y_{6,7} \subset \mathbb{P}(1,1,2,3,3,4)$ is birational to a special $X_7 \subset \mathbb{P}(1,1,1,2,3)$ with a singular point $y^2+z^2+x_1^6+x_2^6$, a general $Y_{14,15} \subset \mathbb{P}(1,2,5,6,7,9)$ is birational to a special $X_{15} \subset \mathbb{P}(1,1,2,5,7)$ with a singular point $u^2+z^2y+y^7+x^{14}$, etc. It is remarkable that a significant part of the list of Fano weighted complete intersections can be generated in this way, starting from singular hypersurfaces. Note that here $\mathbb{Q}$-factoriality of MfS plays a crucial role. These singular hypersurfaces are in general non $\mathbb{Q}$-factorial and do not contribute to the pliability.

Let us work out a simple case in some details. Let $X\subset\mathbb{P}^4$ be a quartic with a unique ordinary double point at $p_0\equiv(1,0,0,0)$ and equation

$$((x_1x_2-x_3x_4)x_0^2+Cx_0+D=0)\subset\mathbb{P}^4$$

Let $\nu:Y\to X$ be the blow up of p_0 with exceptional divisor E. The information on X and Y allow to compute the anticanonical ring of Y, namely $R(Y,-K_Y)$. It is not difficult to realize that there are four generators in degree 1 and one in degree 3. The nice part is that we can write them down easily as follows. Let me abuse the notation and call x_i the strict transform on Y of $(x_i=0)\cap X$. Then

$$\{x_1,x_2,x_3,x_4\}\in H^0(Y,-K_Y)$$

indeed $-K_Y=-\nu^*K_X-E=\nu^*\mathcal{O}_X(1)-E$. Let us look for the other generator. We have to find a cubic with multiplicity at least three along E and not in the tensor of (x_1,x_2,x_3,x_4). The best candidate is

$$y:=x_0(x_1x_2-x_3x_4)\in H^0(Y,-3K_Y)$$

This allows to realize the anticanonical model of Y as an hypersurface

$$Z\subset\mathbb{P}(1,1,1,1,3)$$

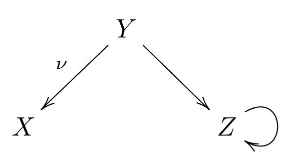
To write it down we have to eliminate x_0 from the equation of X and introduce y. Consider

$$(x_1x_2-x_3x_4)[(x_1x_2-x_3x_4)x_0^2+Cx_0+D]=y^2+yC+D(x_1x_2-x_3x_4)$$

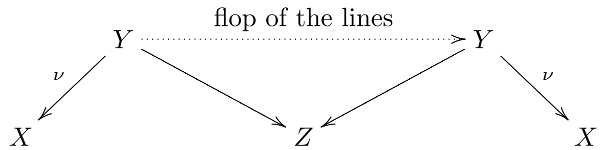
then Z is a hypersurface of degree 6 in $\mathbb{P}(1,1,1,1,3)$ with equation

$$y^2+yC+D(x_1x_2-x_3x_4)$$

The projection from $q\equiv(0,0,0,0,1)\subset\mathbb{P}(1,1,1,1,3)$ induces a biregular map onto Z and then we can go back to Y and X to produce a birational not biregular map.



Note that geometrically the birational map involved here is just the reflection trough the singular point p_0 and the birational involution on X becomes biregular on Z because in going from Y to Z we contracted the lines through p_0. That is exactly the locus where the involution is not defined. This can also be accomplished by flopping the lines on Y. Further note that Z is not $\mathbb{Q}$-factorial because the birational map from Y to Z contracts the lines of X through p_0. In particular Z is not a MfS and does not contribute to the pliability of X. Here is the diagram of the construction



This is a fairly general phenomenon. When trying to classify Fano 3-folds, the problem is often to construct a variety Y with a given Hilbert function. Usually Y has high codimension; in the absence of a structure theory of Gorenstein rings, one method to construct Y starts by studying a suitable projection $Y \dashrightarrow X$ to a Fano X in smaller codimension (the work of Fano, and then Iskovskikh, is an example of this). The classification of Fano 3-folds involves the study of the geometry of special members of some families (like our special singular quartics), as well as general members of more complicated families; the two points of view match like the pieces of a gigantic jigsaw puzzle.

The ideas here are due to Miles Reid, see for example [**Re4**]; for these and other issues, I also refer to [**Re5**].

3.3. Pliability and rationality. Traditionally, we like to think of Fano 3-folds as being "close to rational". We are now confronted with a view of 3-fold birational geometry of great richness, on a scale much larger than accessible with the methods of calculation and theoretical framework prior to Mori theory.

The notion of pliability is more flexible; a case division in terms of the various possibilities for $\mathcal{P}(X)$ allows to individuate a wider spectrum of behavior ranging from birationally rigid to rational.

If X is a quartic with only ordinary nodes as singularities, then X is $\mathbb{Q}$-factorial if and only if the nodes impose independent linear conditions on cubics. Indeed if $\tilde{X}$ is the blowup of the nodes, $H^1(\tilde{X}, \Omega^2_{\tilde{X}})$ arises from residuation of 3-forms

$$\frac{P\Omega}{F^2}$$

on $\mathbb{P}^4$ with a pole of order two along $X = \{F = 0\}$. Here $\Omega = \sum x_i \, dx_0 \cdots \widehat{dx_i} \cdots dx_4$ and P is a cubic containing all the nodes of X, see e.g. [**Cl**].

Consider a quartic 3-fold Z containing the plane $x_0 = x_1 = 0$. The equation of Z can be written in the form $x_0 a_3 + x_1 b_3 = 0$ and, in general, Z has 9 ordinary nodes $x_0 = x_1 = a_3 = b_3 = 0$. The linear system $|a_3, b_3|$ defines a map to $\mathbb{P}^1$; blowing up the base locus gives a Mori fiber space $Z \to \mathbb{P}^1$ with fibers cubic Del Pezzo surfaces. The quartic Z is not $\mathbb{Q}$-factorial. The plane $\{x_0 = x_1 = 0\} \subset Z$ is not a Cartier divisor and then by Lemma 1.3.11 we conclude that Z is not $\mathbb{Q}$-factorial. Thus Z is not a Mori fiber space; it doesn't even make sense to say that it is rigid. (Note in passing: introducing the ratio $y = a_3/x_1 = b_3/x_0$, gives a birational map $Z \dashrightarrow Y_{3,3} \subset \mathbb{P}(1^5, 2)$ to a Fano 3-fold $Y_{3,3}$, the complete intersection of two cubics in $\mathbb{P}(1^5, 2)$, a Mori fiber space birational to Z. In the language of the Sarkisov program, Z is the *midpoint* of a *link* $X \dashrightarrow Y_{3,3}$.) However, a quartic 3-fold with 9 nodes is factorial in general, and it is birationally rigid, [**Me3**]. The factoriality of projective hypersurfaces is the subject of a lovely papers by C. Ciliberto and V. Di Gennaro [**CDG**], and Cheltsov [**Ch**].

3.4. How it works. How one can study all birational map between a fixed Mori space, X/S, and any other Mori spaces. The idea, that goes back to Noether and Fano, and has been reformulated and improved by Iskovskikh, Sarkisov, Reid and Corti is quite simple. A birational, not biregular, map has a base locus in codimension at least 2. When the map is between Mori spaces this base locus has quite nice properties when we look at it from the point of view of singularities of pairs.

THEOREM 2.3.4 (Noether–Fano–Iskovskikh Inequality [**Co2**]). *Let $\chi : X/S \dashrightarrow Y/W$ be a birational not biregular map between Mori spaces. Let $\mathcal{H}_Y$ be a base point free linear system on Y and $\mathcal{H} = \chi_*^{-1}\mathcal{H}_Y$. Then $K_X + 1/\mu\mathcal{H} \equiv_\pi 0$ and either $(X, 1/\mu\mathcal{H})$ has not canonical singularities or $K_X + 1/\mu\mathcal{H}$ is not nef.*

PROOF WHEN $X = Y = \mathbb{P}^n$, (THE ORIGINAL CASE OF NOETHER).
Let $\mathcal{H}_Y = \mathcal{O}_{\mathbb{P}^n}(1)$ then $\mathcal{H} \subset |\mathcal{O}(d)|$ for some $d > 1$. Take a resolution of χ

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{q} \\ \mathbb{P}^n & \overset{\chi}{\dashrightarrow} & \mathbb{P}^n \end{array}$$

and pull back the divisor $K_{\mathbb{P}^n} + ((n+1)/d)\mathcal{H}$ and $K_{\mathbb{P}^n} + ((n+1)/d)\mathcal{O}(1)$ via p and q respectively.

We have

$$\begin{aligned} K_W + ((n+1)/d)\mathcal{H}_W &= \\ p^*\mathcal{O}_{\mathbb{P}^n} + \sum_i a_i E_i &= q^*\mathcal{O}_{\mathbb{P}^n}((n+1)(1/d-1)) + \sum_i b_i E_i \end{aligned}$$

where the E_i are either p or q exceptional divisors.

Let $l \subset \mathbb{P}^n$ be a general line in the right hand side $\mathbb{P}^n$. In particular q is an isomorphism on l and therefore $b_i E_i \cdot q^* l = 0$ for all i.

The crucial point is that on the right hand side we have some negativity coming from the non effective divisor $K_{\mathbb{P}^n} + ((n+1)/d)\mathcal{O}(1)$ that has to be compensated by some non effective exceptional divisor on the other side.

More precisely, since $d > 1$, we have on one hand that

$$(K_W + ((n+1)/d)\mathcal{H}_W) \cdot q^* l = (q^*\mathcal{O}_{\mathbb{P}^n}((n+1)(1/d-1)) + \sum_i b_i E_i) \cdot q^* l < 0,$$

so that

$$0 > (K_W + ((n+1)/d)\mathcal{H}_W) \cdot q^* l = (p^*\mathcal{O}_{\mathbb{P}^n} + \sum_i a_i E_i) \cdot q^* l.$$

and $a_h < 0$ for some h, that is $(\mathbb{P}^n, ((n+1)/d)\mathcal{H})$ has not canonical singularities. $\square$

EXAMPLE 2.3.5. *Observe that in the case of $\mathbb{P}^2$ this forces the existence of a point with $\operatorname{mult}_p \mathcal{H} > d/3$. Out of this, with some work, one can prove Noether Castelnuovo Theorem on factorization of birational self maps of $\mathbb{P}^2$,* [**AM**].

Note that if X is Fano and $\chi : X \dashrightarrow Y/W$ is a birational map then this forces the existence of linear systems with $(X, 1/\mu\mathcal{H})$ with non canonical singularities.

Our next task is to show how it is possible to exclude the existence of these linear systems and thus prove that no birational map exists. We will do it on a special foundational case.

THEOREM 2.3.6 ([**IM**]). *Let $X_4 \subset \mathbb{P}^4$ be a smooth quartic 3-fold, then any birational map $X_4 \dashrightarrow Y/W$ is biregular and an automorphism.*

Today a very simple and neat proof is available. We will use it to describe the philosophy and the main technical tools of the maximal singularities method on 3-folds.

Fix a birational map $\chi : X_4 \to Y/W$ if it not biregular then there exists $\mathcal{H} \subset |\mathcal{O}(d)|$ with $(X, 1/d\mathcal{H})$ with non canonical singularities. Note that by taking a log resolution and running a relative MMP one concludes that there exists a terminal extraction $\epsilon : Y \to X$ with exceptional divisor E and $a(E, X, 1/d\mathcal{H}) < 0$. We want to study $\mathrm{center}_X(E)$.

Assume that $\mathrm{center}_X(E) = \Gamma$ is a curve. Let $\gamma := \mathrm{mult}_\Gamma \mathcal{H}$ then the inequality $a(E, X, 1/d\mathcal{H}) < 0$ translates into $\gamma > d$. Intersect a general curve section with $\mathcal{H}$

$$4d = \mathcal{O}_X(1)^2 \cdot \mathcal{H} \geq \gamma \deg \Gamma > d \deg \Gamma$$

to get $\deg \Gamma < 4$. Assume first that Γ is not planar. Then Γ is a twisted cubic and $\mathcal{I}_\Gamma(2)$ is generated. Let $T \in |\mathcal{I}_\Gamma(2)|$ be a general element and $S = T \cap X$, then we can assume that S is smooth and by adjunction formula $(\Gamma \cdot \Gamma)_S = -5$ Let $\mathcal{H}_{|S} = \mathcal{M} + \gamma\Gamma$ for $\mathcal{M}$ a linear system without base curves. This gives

$$0 \leq \mathcal{M}^2 = (\mathcal{H}_{|S} - \gamma\Gamma)^2 \leq 8d^2 - 6d\gamma - 5\gamma^2 < 0$$

and proves that Γ cannot be the center of ϵ. With similar arguments we can exclude all the plane curve of degree ≤ 3.

Assume that $\mathrm{center}_X(E) = x$ is a point. Then by Kawakita's result, Theorem 1.4.9, ϵ is a wbu with weight $(1, a, b)$, and $a(E, X, 1/d\mathcal{H}) < 0$. That is $\epsilon^*\mathcal{H} = \mathcal{H}_Y + mE$ with

$$m > d(a + b)$$

Keep in mind the numerology $E^3 = 1/ab$ and $a(E, X) = a + b$. Fix a general hyperplane section $S \ni x$, then $\epsilon^*S = S_Y + E$. By assumption

$$0 \leq S_Y \cdot \mathcal{H}_Z^2 = (\epsilon^*S - E)(\epsilon^*\mathcal{H} - mE)^2 = 4d^2 - m^2E^3 < d^2(4 - \frac{(a+b)^2}{ab}) < 0$$

Therefore no center of this kind exists on X and we proved Iskovskikh–Manin theorem.

Here we singled out a pair of technical points. First it is important to know all possible terminal extractions from a center. Second there are special surfaces on X where the numerology work. The maximal singularity method is a mixture of these two ingredients.

4. Biregular geometry

The main references are [**Re3**], [**Ko1**], [**Ka4**], [C$^+$, Chapter 8]. In this section we always assume that X is lt. All results stay true, with an heavier notation for (X, Δ) klt.

DEFINITION 2.4.1 ([**Ka4**]). Let X be a normal variety and $D = \sum d_i D_i$ an effective $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. A subvariety W of X is said to be a non klt center, or simply a center, for the pair (X, D), if there is a birational morphism from a normal variety $\mu : Y \to X$ and a prime divisor E on Y, not necessarily μ-exceptional, with the discrepancy coefficient $a(E, X, D) \leq -1$ and such that $\mu(E) = W$. For another such $\mu' : Y' \to X$, if the strict transform E' of

E exists on Y', then we have the same discrepancy coefficient for E'. The divisor E' is considered to be equivalent to E, and the equivalence class of these prime divisors is called a place of non klt singularities for (X, D). The set of all centers of non klt singularities is denoted by $CLC(X, D)$, the locus of all centers of non klt singularities is denoted by $LLC(X, D)$.

Let (X, D) be a log variety and $W \in CLC(X, D)$ a center. This means that we have a log resolution $\mu : Y \to X$ with

$$K_Y = \mu^*(K_X + D) + \sum e_i E_i,$$

and $\mu(E_0) = W$ with $a(E_0, X, D) \leq -1$. In general there is more than one valuation with low discrepancy and more than one center in $CLC(X, D)$. On the other hand it is quite clear that if we have both a unique center and a unique valuation the situation is nicer.

Our first goal is to explore how much one can say in this direction by perturbing the divisor D. Let us start with the following properties of centers.

PROPOSITION 2.4.2 ([**Ka4**]). *Let X be a normal variety and D an effective $\mathbb{Q}$-Cartier divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Assume that X is lt and (X, D) is lc. If $W_1, W_2 \in CLC(X, D)$ and W is an irreducible component of $W_1 \cap W_2$, then $W \in CLC(X, D)$. In particular, there exist minimal elements in $CLC(X, D)$ with respect to inclusion.*

PROOF. Let D_i be general members among effective Cartier divisors which contain W_i. Let $f : Y \to X$ be a log resolution of the pair $(X, D + D_1 + D_2)$. Let $e_i := \operatorname{mult}_{W_i} D$, note that $\operatorname{mult}_{W_i} D_j = \delta_{ij}$.

Since (X, D) is lc for any rational $0 < \epsilon \ll 1$ we have

$$LLC(X, (1 - \epsilon)D + \epsilon e_1 D_1 + \epsilon e_2 D_2) = W_1 \cup W_2$$

By the connectedness Theorem 2.2.3, there exist f-exceptional divisors $F_i = F_i(\epsilon)$ such that $f(F_i) \subseteq W_i$, $a(F_i, X, (1 - \epsilon)D + \epsilon e_1 D_1 + \epsilon e_2 D_2) \leq -1$ and $F_1 \cap F_2 \neq \emptyset$. Since there are only a finite number of f-exceptional divisors then for a convergent sequence $\epsilon \to 0$ the F_i are fixed, $f(F_i) = W_i$, and

$$f(F_1 \cap F_2) = W_1 \cap W_2 \in LLC(X, D)$$

□

One of the most important feature of minimal centers is the possibility to perturb the boundary D in order to accomplish special features. Fix a minimal center $W \in CLC(X, D)$. Our aim is to manipulate D to an effective D' in such a way that (X, D') is lc and W is the unique center of non klt singularities for (X, D'). This is called the "tie breaking", [C$^+$, Chapter 8], or perturbation argument, and was first explained by Reid, [**Re3**], see also [**Ka4**].

DEFINITION 2.4.3. Let (X, D) be lc. Then W is an isolated center if $W \cap Z = \emptyset$ for all centers $Z \neq W$ of the pair (X, D). An isolated center is called exceptional if there is a unique valuation E with $a(E, X, D) = -1$ and $\operatorname{center}_X E = W$.

The assumption (X, D') lc is not a big problem for any pair (X, D) one can define

$$lc_0(X, D) := \max\{t \in \mathbb{Q} | (X, tD) \text{ is lc}\}$$

to get a lc pair. This could of course erase the center W. In this case it is enough to consider $lc_0(X, D + \alpha A)$ where A is a general divisor vanishing along W and α a suitable rational number.

We can therefore assume that (X, D) is lc and W is a minimal center for the pair. The next step is to make W isolated, by erasing all other centers. Choose a generic very ample M such that $W \subset Supp(M)$ and no other $Z \in CLC(X, D)\setminus\{W\}$ is contained in $Supp(M)$, this is always possible since W is minimal in a dimensional sense. Note that $(X, (1-\epsilon)D)$ is klt for any $0 < \epsilon$, therefore it is enough to consider $D_1 := (1 - \epsilon)D + \delta M$, for some appropriate $\delta = \delta(\epsilon)$. To have (X, D_1) lc and W the only center of non klt singularities.

To get an exceptional center is a bit more subtle. We sketch it here and refer to [C$^+$, Chapter 8] for a detailed analysis. We want to have a unique discrepancy with coefficient -1 for the log pair (X, D').

Keep in mind that $D_1 := (1 - \epsilon_1)D + \delta M$,and fix a log resolution $\mu : Y \to X$, then

- (X, D_1) is LC
- $CLC(X, D_1) = W$
- $\mu^*\delta M = \sum m_i E_i + P$, with P ample; this is possible by Kodaira Lemma
- $K_Y + \sum_{j=0}^{k} E_j + \Delta - A = \mu^*(K_X + D_1) - P$ where
  - the E_j's are integer irreducible divisors with $\mu(E_j) = W$,
  - A is a μ-exceptional integral divisor
  - $\lfloor\Delta\rfloor = 0$
  - A, Δ and the E_i's have no common components

It is now enough to use the ampleness of P to choose just one of the E_j. Indeed for small enough $\delta_j > 0$ $P' := P - \sum_{j=1}^{k} \delta_j E_j$ is still ample. In this way one can produce the desired resolution

$$K_Y + E_0 + \Delta' - A = \mu^*(K_X + D') - P'; \tag{3}$$

Remark 2.4.4. *If instead of an ample M we choose a nef and big divisor, we can repeat the above argument with Kodaira Lemma, but this time we cannot choose the center $\mu(E_0)$ like before, and in particular we cannot assume that at the end we are on a minimal center for (X, D). To do this one has to be able to control the effective part of the decomposition of μ^*M.*

The study of these objects has been developed by Kawamata and we can summarize the main results in the following Theorem.

Theorem 2.4.5 ([**Ka4**],[**Ka3**]). *Let X be a normal variety and D an effective $\mathbb{Q}$-Cartier divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. Assume that X is lt and (X, D) is lc.*

- i) *If $W \in CLC(X, D)$ is a minimal center then W is normal*
- i) *(subadjunction formula)Let H be an ample Cartier divisor and ϵ a positive rational number. If W is a minimal center for $CLC(X, D)$ then there exists an effective $\mathbb{Q}$-divisor D_W on W such that $(K_X + D + \epsilon H)_{|W} \equiv K_W + D_W$ and (W, D_W) is KLT.*

Remark 2.4.6. *The first statement is essentially, a consequence of Vanishing theorem as in Shokurov Connectedness Lemma. The subadjunction formula is quite of a different flavor and is related to semipositivity results for the relative dualizing sheaf of a morphism, see also* [C$^+$, Chapter 8].

One should compare this with Fujino's [C$^+$, Proposition 3.9.2] *where the CLC of a dlt pair are proved to be dlt.*

EXAMPLE 2.4.7. *It is in fact not so difficult to work out all possible minimal centers $W \in CLC(X, D)$, where X is a smooth surface and D any divisor (i.e a curve). The same, a little harder, if X is a smooth threefold, see* [**Ka4**]. *Keep in mind that klt singularities are rational singularities.*

4.1. How to get base point free-type theorems. (i.e. the multiplier ideal method for everybody)

Assume now that X is a variety with log terminal and $\mathbb{Q}$-Gorenstein singularities and let L be an ample line bundle on X.

Let D be an effective $\mathbb{Q}$-Cartier divisor such that $D \equiv tL$ for a rational number $t < 1$. Let $W \in CLC(X, D)$ be a minimal center. Up to a perturbation one can always assume that W is exceptional and (X, D) is lc.

Let $f : Y \to X$ be a log resolution of the pair (X, D) with

$$K_Y + E = \mu^*(K_X + D) + F$$

where E is an irreducible divisor such that $\mu(E) = W$ and $\lceil F \rceil$ is effective and exceptional. Then we have

$$K_Y + (1 - t)\mu^* L \equiv \mu^*(K_X + L) - E + F$$

and by the vanishing

$$H^1(Y, \mu^*(K_X + L) - E + \lceil F \rceil) = 0$$

In particular there is the following surjection

$$H^0(Y, \mu^*(K_X + L) + \lceil F \rceil) \to H^0(E, \mu^*(K_X + L) + \lceil F \rceil).$$

The divisor $\lceil F \rceil$ is effective and exceptional therefore

$$H^0(X, K_X + L) = H^0(Y, \mu^*(K_X + L) + \lceil F \rceil)$$

and we also have

$$H^0(X, K_X + L) \to H^0(E, \mu^*(K_X + L) + \lceil F \rceil) \to 0.$$

Thus to find a section of $K_X + L$ not vanishing on W it is sufficient to find a non zero section in $H^0(E, \mu^*(K_X + L) + \lceil F \rceil)$, or equivalently in

$$H^0(W, (K_X + L + \mu_{|E} * (\lceil F \rceil))_{|W})$$

The ideal case happens when $W = x$ is a point. In this case $H^0(W, \mu^*(K_X + L)) = \mathbb{C}$ and therefore x is not in $\mathrm{Bs}\,|K + L|$. In this way one can produce base point free statements like in Fujita's conjecture. The hard part is clearly to be able construct a log pair with a point as exceptional center.

Another nice situation occurs when X is a Fano variety. In this case one has good control on possible minimal centers.

DEFINITION 2.4.8. Let (X, Δ) be a klt pair. We say that it is a log Fano variety if $-(K_X + \Delta) \equiv iH$ for some $H \in Pic(X)$ and $i > 0$. The largest possible such i is called the index of (X, Δ) and H is called the fundamental divisor.

LEMMA 2.4.9. *Let X be a lt Fano variety with $-K_X \equiv \gamma H$, for some ample Cartier H on X. Let $D \equiv tH$ be an effective divisor, with $t < \gamma$, and assume that W is an exceptional center of (X, D). Let $D_W = (D + \epsilon H)_{|W}$ for $0 < \epsilon \ll 1$. Then (W, D_W) is a klt log-Fano with $-(K_W + D_W) \equiv (\gamma - t - \epsilon)H_{|W}$.*

PROOF. By Kawamata's subadjunction formula (W, D_W) is klt and

$$-(K_W + D_W) \equiv -(K_X + (t+\epsilon)H)_{|W} \equiv (\gamma - t - \epsilon)H_{|W}$$

In particular (W, D_W) is Fano and $(-K_W + D)_{|W} \equiv (\gamma - t - \epsilon)H_{|W}$ □

The nice point is that thanks to the usual vanishings and Serre's duality if the index is quite high with respect to the dimension then the fundamental divisors of klt log Fanos have sections, [**Ale**]. Therefore one can use Lemma 2.4.9 to prove that W is not in $\mathrm{Bs}\,|H|$. Coupling this with Bertini Theorem and some tricks of tie breaking one can prove the following.

THEOREM 2.4.10 ([**Me1**]). *Let X be a Mukai variety with at worst log terminal singularities. Then X has good divisors except in the following cases:*

- i) *X is a singular terminal Gorenstein 3-fold which is a "special" (see* [**Me1**]*) complete intersection of a quadric and a sextic in* $\mathbb{P}(1,1,1,1,2,3)$
- ii) *let $Y \subset \mathbb{P}(1,1,1,1,1,2)$ be a "special" complete intersection of a quadric cone and a quartic; let σ be the involution on $\mathbb{P}(1,1,1,1,1,2)$ given by $(x_0 : x_1 : x_2 : x_3 : x_4 : x_5) \mapsto (x_0 : x_1 : x_2 : -x_3 : -x_4 : -x_5)$ and let π be the map to the quotient space. Then $X = \pi(Y)$ is a terminal not Gorenstein 3-fold.*

In both exceptional cases the generic element of the fundamental divisor has canonical singularities and $\mathrm{Bs}\,|-K_X|$ *is a singular point. It has to be stressed that the generic 3-fold in i) and ii) has good divisors, but there are "special" complete intersections whose quotient has a singular point in the base locus of the anticanonical class, see* [**Me1**, Examples 2.7, 2.8] *for details.*

With similar arguments one can prove a good divisor problem for other MfS, for details and related results see [**Me2**].

Beside many applications of the Base Point Free I think it is important to recall the following conjecture.

CONJECTURE 2.4.11 ([**Ka5**]). *Let X be a complete normal variety, D an effective $\mathbb{Q}$-divisor on X such that the pair (X, B) is klt, and L a Cartier divisor on X. Assume that L is nef, and that $H = L - (K_X + B)$ is nef and big. Then* $H^0(X, L) > 0$.

The conjecture is proved for surfaces and few other special cases, note that a special case is that $2K_X$ has sections for any Gorenstein minimal model of general type.

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